COMPLEXES, GRAPHS, HOMOTOPY, PRODUCTS AND SHANNON CAPACITY

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Abstract. A finite abstract simplicial complex G defines the Barycentric refinement graph $\phi(G) = (G, \{(a,b), a \subset b \text{ or } b \subset a\})$ and the connection graph $\psi(G) = (G, \{(a,b), a \cap b \neq \emptyset\})$. We note here that both functors ϕ and ψ from complexes to graphs are invertible on the image (Theorem 1) and that $G, \phi(G), \psi(G)$ all have the same automorphism group and that the Cartesian product of G corresponding to the Stanley-Reisner product of $\phi(G)$ and the strong Shannon product of $\psi(G)$, have the product automorphism groups. Second, we see that if G is a Barycentric refinement, then $\phi(G)$ and $\psi(G)$ are graph homotopic (Theorem 2). Third, if γ is the geometric realization functor, assigning to a complex or to a graph the geometric realization of its clique complex, then $\gamma(G)$ and $\gamma(\phi(G))$ and $\gamma(\psi(G))$ are all classically homotopic for a Barycentric refined simplicial complex G (Theorem 3). The Barycentric assumption is necessary in Theorem 2 and 3. There is compatibility with Cartesian products of complexes which manifests in the strong graph product of connection graphs: if two graphs A, A' are homotopic and B, B' are homotopic, then $A \cdot B$ is homotopic to $A' \cdot B'$ (Theorem 4) leading to a commutative ring of homotopy classes of graphs. Finally, we note (Theorem 5) that for all simplicial complexes G as well as product $G = G_1 \times G_2 \cdots \times G_k$, the Shannon capacity $\Theta(\psi(G))$ of $\psi(G)$ is equal to the number f_0 of zero-dimensional sets in G. An explicit Lovasz umbrella in $\mathbb{R}^{f_0}$ leads to the Lovasz number $\theta(G) \leq f_0$ and so $\Theta(\psi(G)) = \theta(\psi(G)) = f_0$ making Θ compatible with disjoint union addition and strong multiplication.

1. Theorem 1

1.1. A finite set G of non-empty sets that is closed under the operation of taking finite non-empty subsets is called a **finite abstract simplicial complex**. It defines a finite simple **connection graph** $\psi(G)$ in which the vertices are the elements in G and where two sets are connected if they intersect. In the **Barycentric refinement graph** $\phi(G)$, two vertices are connected if and only if one set is contained in the other. It is a subgraph of the connection graph $\psi(G)$.

Theorem 1. *G can be recovered both from $\psi(G)$ or $\phi(G)$.*

1.2. The proof will be obvious, once the idea is seen. The reconstructions from $\phi(G)$ or $\psi(G)$ are identical. The reconstruction can be done fast, meaning that the cost is polynomial in $n = |G|$. In particular, no computationally hard **clique finding** is necessary in $\psi(G)$. When looking at a graph like $\psi(G)$, we of course only assume to know the graph structure and not what set each node represents. As no explicit reconstruction of G from $\psi(G)$ appears to have been written down before,

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this is done here. It will show that it is considerably simpler than the construction of a set of sets from a general graph that is enabled by the **Szpilrajn-Marczewski theorem**: any finite simple graph A can be realized as a connection graph of a finite set G of non-empty sets [41, 34]. Connection graphs are special: their adjacency matrix A has the property that $L = 1 + A$ has determinant 1 or -1 and the number of positive eigenvalues of L is the number of even-dimensional sets in G.

1.3. The **automorphism group** $\mathrm{Aut}(G)$ is the set of permutations T of G which preserve the order structure: $x \subset y$ if and only if $T(x) \subset T(y)$. The automorphism group of a graph is the automorphism group of its Whitney simplicial complex. This is the same than the automorphism of its 1-dimensional skeleton complex $G = V \cup E$ because if edges are mapped to edges then also complete graphs are mapped into complete graphs. An automorphism T of a graph is nothing else than a map from the graph to itself which is an isomorphism. A consequence of the proof is that $G, \phi(G), \psi(G)$ all have the same automorphism groups. Groups like $\mathrm{Aug}(G)$ is in the **Klein Erlanger picture** an important object of a geometry as $\mathrm{Aut}(G)$ is a **symmetry group**. **Frucht's theorem** shows that any finite group can occur for a graph. By building the Whitney complex of this graph, we see that any finite group can occur as an automorphism group of a simplicial complex and so also of a connection graph $\psi(G)$ or a Barycentric graph $\phi(G)$.

Corollary 1. *For any simplicial complex G, all $\psi(G)$ and $\phi(G)$ have the same automorphism group.*

Proof. If T is an automorphism of G, then it produces an automorphism both on $\phi(G)$ and $\psi(G)$. On the other hand, if we have an automorphism of a graph then the reconstruction allows to transport this automorphism to G: T commutes the vertices which produces a permutation of the vertex set G_0 of G. This define uniquely the permutation on G. □

1.4. In the case when G is a Barycentric refinement, then also the Lefschetz number $L(T, G)$ (the super trace of the induced map on the cohomology groups) is the same than the Lefschetz number $L(T, \phi(G))$ or $L(T, \psi(G))$ and if it is not zero, like for a contractible complex, there is at least one fixed point, because of the Lefschetz fixed point theorem $L(T, G) = \sum_{x, T(x)=x} i_T(x)$ where $i_T(x) = (-1)^{\dim(x)} \mathrm{sign}(T : x \to x)$ is the index of the fixed point x which in the case of a graph is a fixed clique. See [16]. We will also see

Corollary 2. *The Cartesian product $G_1 \times G_2$, the Stanley-Reisner product $\phi(G_1 \times G_2) = \phi(G_1) \cdot \phi(G_2)$ as well as the strong product $\psi(G_1) \cdot \psi(G_2)$ have the same automorphism group which is the product group of the automorphism groups of G_1 and G_2.*

1.5. Theorem 1 shows that one does not lose information by looking at connection graphs of simplicial complexes. Finite abstract simplicial complexes have only one axiom and still allow to explore interesting topology, like for example the topology of compact differentiable manifolds. The simple set-up for finite abstract simplicial complexes is even simpler than Euclid's axiom system about points and lines. When thinking about sets of sets, it is helpful to look at incidence and intersection graphs because sets by themselves evoke little geometric intuition. The graphs $\phi(G)$ and $\psi(G)$ help for that and build a link to the actual topology. The graph $\phi(G)$ for

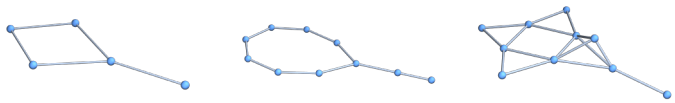

FIGURE 1. The complex $G = \{(1,2),(2,3),(3,4),(4,1),(4,5),(1),(2),(3),(4),(5)\}$, its Barycentric refinement graph $\phi(G)$ and the connection graph $\psi(G)$. The first picture visualizes G also as a graph as G is the Whitney complex of that graph. We know by Theorem 5 that $\Theta(\psi(G)) = 5$. Here also $\Theta(\phi(G)) = 5$.

example is all we need to get all the cohomology groups which are kernels of block matrices of the Hodge Laplacian $(d + d^*)^2$ invoking the incidence structure. The intersection structure is spectrally natural: because the product for connection graphs produces spectral data which multiply.

1.6. We will discuss in a moment the topology and homotopy of $\phi(G)$ and $\psi(G)$. For now, note that the graph $\psi(G)$ is topologically quite different from G already for 1-dimensional simplicial complexes. While the functor ϕ from simplicial complexes to graphs does honor the maximal dimension, (the dimension of the maximal complete subgraph), the functor ψ does not, as simple examples show. For example, if x is a set in G which intersects with n other sets, then there are $n+1$ sets which all intersect with each other, so that the graph $\phi(G)$ has a clique of dimension n. For the 1-dimensional **star complex** $G = \{\{1,2\},\{1,3\},\dots,\{1,n\},\{1\},\dots,\{n\}\}$ for example, the connection graph $\psi(G)$ already has maximal dimension n. The next theorem will show however that $\psi(G)$ is graph homotopic to G, implying that the geometric realizations of $\psi(G)$ and G are classically homotopic.

1.7. This note justifies partly some statements related to simplicial complexes and graphs [18, 17, 24]. It is a small brick in a larger building hopefully to emerge at some point. We also hope to pitch the simplest homotopy set-up in mathematics and illustrate with some lemmas how to work effectively with it (an Appendix gives an other example). We will see in particular that important deformation procedures for graphs are homotopies: examples are **edge refinements**, which serve as local Barycentric refinements, as well as global Barycentric refinements. These two deformations even preserve the manifold structure of graphs in the sense that unit spheres remain spheres and keep the dimension during the deformation. Homotopies (like $K_n \to K_{n+1}$) of course do not preserve dimension in general. We add in an appendix more about discrete spheres.

1.8. For us, it is important that we can make sense of **Cartesian products** of simplicial complexes without having to dive into other discrete combinatorial notions like **simplicial sets** or **discrete CW complexes** (this has been used in [28]) which are both combinatorial useful and categorically natural but also require more mathematical sophistication. We also don't want to use the geometric realization functor to prove things in combinatorics.

2. Theorem 2

2.1. If x is a vertex of a finite simple graph $B = (V, E)$, let $S(x)$ denote the **unit sphere** of x. It is the graph induced from the set of all the vertices connected to x. The class of **contractible graphs** is defined recursively: the 1-point graph $K_1 = \{\{1\}, \{\}\}$ is contractible. A graph (V, E) is called **contractible**, if there exists $v \in V$ such that the unit sphere $S(v)$, (the graph induced by the neighbors of v), as well as the graph $B - v$, (the graph induced by $V \setminus \{v\}$), are both contractible. Since both $S(v)$ and $B - v$ have less vertices, the inductive definition works and allows to check contractibility in polynomial time with respect to the number of vertices in the graph.

2.2. A **homotopy step** is the process of removing a vertex v for which $S(v)$ is contractible or the reverse procedure of choosing a contractible subgraph A of B and connecting every vertex in A to a new vertex v. Two graphs A, B are called **homotopic**, if one can chose a finite set of homotopy steps to get from A to B. This homotopy emerged from [11, 12, 4], is based on [43] and already appears in [9] according to [4]. It is simple and fully equivalent to the continuum homotopy and defines also a homotopy for simplicial complexes: two complexes G, H are declared to be homotopic if $\phi(G)$ and $\phi(H)$ are homotopic graphs.

2.3. An different homotopy was suggested in [3]; it is based on product graphs, closer to the continuum but harder to implement. We have used the above mentioned homotopy since [13, 20]. See [18, 17, 24] for presentations when discussing coloring problems [19, 21, 25]. In the appendix, we illustrate how homotopy defines spheres and allows to prove properties of spheres like the Euler Gem formula. The Appendix was a talk given in 2018 and gives all definitions of the two classes "spheres" and "contractible graphs". The Ljusternik-Schnirelman point of view is that contractible spaces have category 1, and spheres have category 2 with the exception of the (-1)-sphere, the empty graph which has L-S category 0.

2.4. Contractible graphs by definition are homotopic to 1 (the one-point graph K_1 with only one vertex and no edges) but the graphs like the **dunce hat** are homotopic to 1 but not contractible. It is necessary first to thicken up a graph in general before it can be contracted. The class of graphs which are homotopic to 1 form a much larger class of graphs than the set of contractible graphs and are inaccessible in the sense that it is a computational hard problem to decide whether a graph is homotopic to 1 or not. Contractibility on the other hand is decidable: as we only need to go through all vertices and check whether their unit spheres are contractible and because unit spheres have less vertices.

Theorem 2. *For G Barycentric, $\phi(G)$ and $\psi(G)$ are homotopic.*

2.5. The 1-dimensional complex

$$G = \{\{1, 2\}, \{2, 3\}, \{3, 1\}, \{1\}, \{2\}, \{3\}\}$$

is the boundary complex of a triangle K_3 and topologically a 1-sphere (circle). The graphs $\phi(G) = C_6$ and $\psi(G)$ are not homotopic because $\chi(\phi(G)) = 0$ and $\chi(\psi(G)) = 1$ and Euler characteristic χ is a homotopy invariant. For the **octahedron complex** which has six 0-dimensional vertices, the graph $\phi(G)$ has the

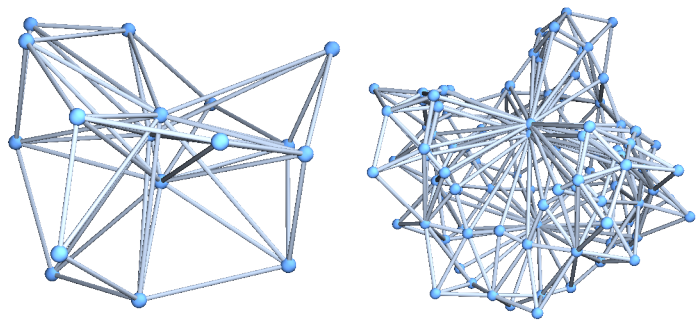

FIGURE 2. The dunce hat is a graph with 17 vertices and 52 edges and 36 triangles, which is homotopic to 1 but not contractible. All its Barycentric refinement are homotopic to 1 but not contractible.

topology of a 2-sphere while $\psi(G)$ is homotopic to a 3-sphere. The proof of Theorem 2 is not difficult, once one sees how the homotopy steps are done. We will write down the concrete graph homotopy.

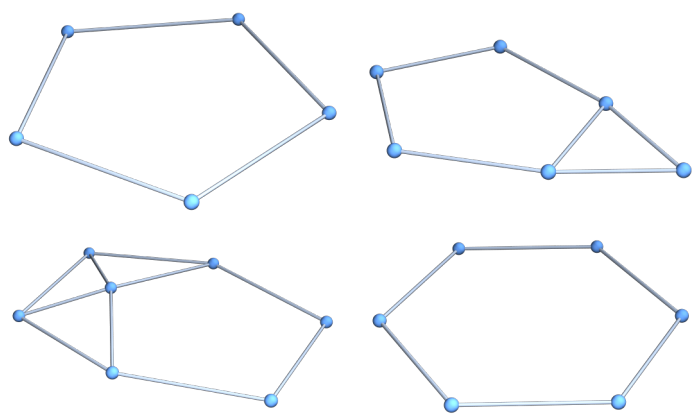

FIGURE 3. The homotopy deformation from the circular graph C_5 to C_6 needs three homotopy steps. It is not possible to contract C_6 to C_5 as the unit spheres of all vertices are 0-spheres and not contractible. The graph first needs to be thickened up at first and temporarily becomes two-dimensional.

2.6. Homotopy preserves **Euler characteristic** $\chi(A) = \sum_{x \in A} (-1)^{\dim(x)}$, summing over all complete subgraphs x of A. That homotopy is an invariant follows directly from

$$\chi(B +_A v) = \chi(B) + (1 - \chi(A))$$

if $B +_A v$ is the graph in which a new vertex is attached to a subgraph A of B. If A is contractible, then $\chi(A) = 1$ and $\chi(B) = \chi(B + v)$. This formula is a direct consequence of the **valuation property** $\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B)$

for any subgraphs A, B of a larger graph. When applying it to a build up of the complex, it implies the **Poincaré-Hopf formula** [15, 26]

$$\chi(A) = \sum_{v \in V(A)} i_f(x)$$

for a locally injective function f on $V(\Gamma)$, where $i_f(x) = 1 - \chi(S_f(x))$ is the **Poincaré-Hopf index** of f at x and $S_f(x)$ is the subgraph generated by all $y \in S(x)$, where $f(y) < f(x)$.

Corollary 3. *If G is Barycentric then $\chi(G) = \chi(\psi(G)) = \chi(\phi(G))$.*

2.7. A consequence is that all cohomology groups and their dimensions, the **Betti numbers** are invariant under homotopy deformations. This can be verified also within the discrete setup: extend any cocycle from G to $G +_A x$ and also extend every coboundary from G to $G +_A x$. Also discrete Hodge theory works: deform the kernel of the blocks $H_k(G)$ of $H(G)$ to the kernels of $H_k(G +_A x)$: just let the heat flow e^{-tH} act on a function f on G that had been harmonic and initially was extended to $G +_A x$ by assigning 0 to every k-simplex in $G +_A x$ not in G. The heat flow will deform the function to a harmonic function on $G +_A x$.

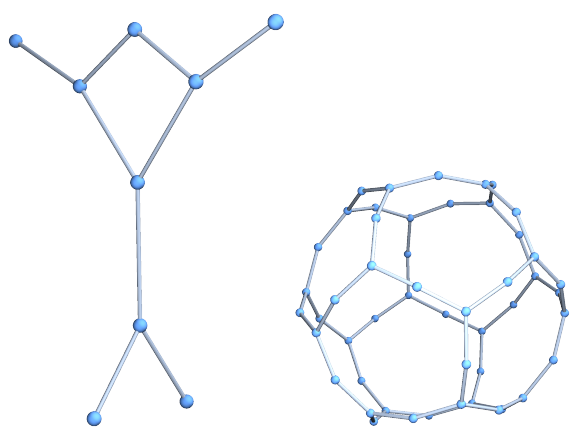

FIGURE 4. For 1-dimensional connected simplicial complexes (curves), the Betti number b_1 is the genus, the number of holes completely determines the homotopy type. Alternatively $1 - b_1 = \chi(G)$, the Euler characteristic characterizes connected "curves". All trees are contractible, the graph in the first picture has $b_1 = 1$ and is homotopic to a circle. The second 1-dimensional complex has the Betti numbers $(1, 11)$ and Euler characteristic $1-11 = -10$. We could fill the 12 holes and get a 2-sphere of Euler characteristic 2.

3. THEOREM 3

3.1. In this section, we leave combinatorics in order to illustrate the connection with topology. Mathematicians like [2] thought in terms of discrete graphs (i.e. Eckpunktgerüst) and this is still visible in modern algebraic topology [10] and especially texts which show in drawings how topologists think [6]. The geometric

Whitney realization of a graph A is a union $\gamma(A)$ of simplices in some Euclidean space $\mathbb{R}^n$ such that every complete subgraph in A is mapped into a simplex. The simplest way to do that is to see the clique complex of A as a subcomplex of the maximal simplex in the complete graph with vertices in A.

Theorem 3 (Theorem 3)**.** *If two graphs A and B are homotopic, their geometric Whitney realizations $\gamma(A), \gamma(B)$ are homotopic topological spaces.*

3.2. Because contractible and collapsible are are used differently in the literature and are easily confused, we use contractible and collapsible as a synonym and use **"homotopic to 1"** if a graph is homotopic to 1. The equivalence relation "homotopic to 1" is much harder to check than being contractible. While we can by brute force decide in a finite number of steps whether a graph is contractible (just go through all the vertices and see whether one can remove it by checking its unit sphere to be contractible), the difficulty with the wider homotopy is that we possibly have to expand the graph first considerably before we can contract.

3.3. It is already for 2-dimensional complexes known to be undecidable in the sense that there is no Turing machine which takes as an input a finite simple graph and as an output the decision whether it is homotopic to 1 or not. This work has started with Max Dehn and relates to other problems like the triviality of the fundamental group which can be related to word problems in groups. Already for 2-dimensional simplicial complexes, the problem to decide simply connectedness is algorithmically unsolvable. Lets abbreviate "Barycentric G" for "Barycentric refined G".

Corollary 4. *If G is Barycentric, then $\gamma(G), \gamma(\phi(G))$ and $\gamma(\psi(G))$ are all homotopic topological spaces.*

4. Theorem 4

4.1. The **strong product** of two graphs $(V, E), (W, F)$ is the graph $(V \times W, Q)$, where $Q = \{((a, b), (c, d)), b = d$ or $(b, d) \in F$ and $a = c$ or $(a, c) \in E\}$. It is an associative operation which together with disjoint union $+$, (the monoid is group completed to an additive group), produces the **strong ring** of graphs. It is a commutative ring with 1-element $1 = K_1$ and where 0 is the empty graph. While the Cartesian product $G \times H$ of simplicial complexes is not a simplicial complex, one has a product $\phi(G) \times \phi(H)$ on the Barycentric graph level $(G \times H, \{((a, b), (c, d)), a \subset c,$ and $b \subset d)$. One can compare this with $\psi(G) \cdot \psi(H) = (G \times H, ((a, b), (c, d)), a \cap c \neq \emptyset, b \subset d \neq \emptyset\}$. The later graph $\psi(G) \cdot \psi(H)$ is spectrally nice in that the connection Laplacians $L(G \times H)$ is the tensor product of $L(G)$ and $L(H)$. (See [28]).

Theorem 4 (Theorem 4)**.** *If A, A' are homotopic and B, B' are homotopic, then $A \cdot B$ is homotopic to $A' \cdot B'$.*

4.2. It follows that the strong ring of graphs defines also a **ring of homotopy classes** of signed graphs. The homotopy of a signed graph $A - B$ allows to deform A and B. If A, B are homotopic then $A - B$ is 0. The additive primes in the ring of the classes of connected graphs, the multiplicative primes are the homotopy classes graphs which come from finite abstract simplicial complexes. The 1-element in the ring is the class of all graphs which are homotopic to 1.

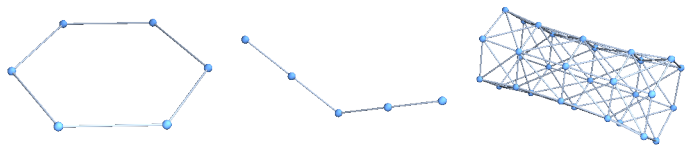

FIGURE 5. The strong product a circular graph C_6 and a linear graph L_5 is a discrete cylinder.

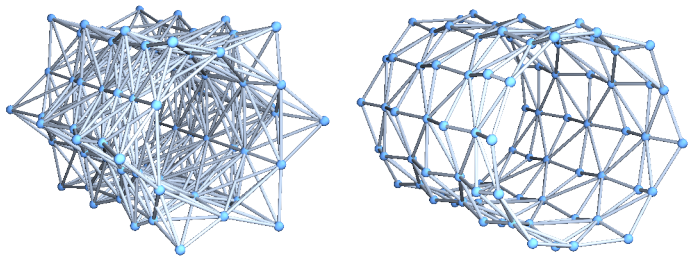

FIGURE 6. The strong product $\psi(G \times H)$ (left) of a circular complex G and linear complex H compared with the Barycentric product $\phi(G \times H)$ (right). Both have the same number of vertices and are homotopic (Theorem 2). Only $\phi(G \times H)$ is a discrete manifold with boundary but $\psi(G \times H)$ is the strong product of $\psi(G)$ and $\psi(H)$.

5. THEOREM 5

5.1. We can assign to a product of simplicial complexes $G \times H$ (the Cartesian product as sets is not a simplicial complex) the graph $\psi(G) \cdot \psi(H)$, which is the **strong product** of the two connection graphs. This product appeared already in work of Shannon [40] or Sabidussi [37]. Shannon used it when defining Shannon capacity of a graph A as the asymptotic growth of the **independence number** of the n'th power A^n of a complex. This number $\Theta(A)$ is in general difficult to compute: Shannon himself computed it for all graphs with less than 5 points and estimated $\sqrt{5} \leq \Theta(C_5) \leq 5/2$. Only 23 years later, $\Theta(C_5) = \sqrt{5}$ was proven [32] and $\Theta(C_7)$ is still unknown [33]. Lovasz introduced the Lovasz number $\theta(G)$ which is the $\sec(\theta)^2$ of the maximal angle of the opening angle of the **Lovasz umbrella**, a geometric cone shaped object obtained from an orthonormal representation of the graph, and satisfies $i(G) \leq \Theta(G) \leq \theta(G) \leq c(\overline{G})$, where $c(G)$ is the chromatic number also dubbed **sandwich theorem** [29] showing that Θ has lower and upper bounds which are NP hard but also has an upper bound $\theta(G)$ which can be computed in polynomial time.

5.2. The independence number $i(A)$ of a graph is A the clique number of the graph complement $\overline{A}$. Because clique numbers are hard to computer, also independence numbers are difficult to get. We expect especially the **Shannon capacity**

$$\log(\Theta(A)) = \lim_{n \to \infty} \frac{1}{n} \log(i(A^n))$$

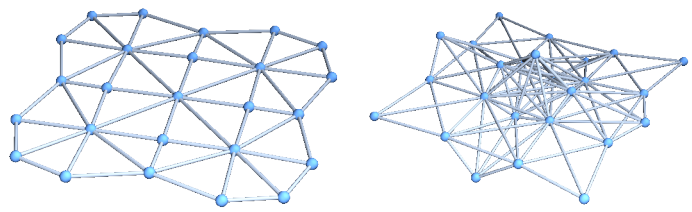

FIGURE 7. We see $\phi(G \times H)$ and $\psi(G \times H)$ for $G = H = \{(1,2),(2,3),(1),(2),(3)\}$. The Barycentric product is the Stanley-Reisner product multiplying $G = x + y + z + xy + yz$, $H = u + v + w + uv + vw$ and connecting monomials which divide each other. The right graph $\psi(G \times H)$ is homotopic to $\phi(G \times H)$ and is equal to $\psi(G) \cdot \psi(H)$. Not only cohomology [22] but also the spectral compatibility is satisfied as the connection Laplacian $L(G \times H) = L(G) \otimes L(H)$ is the tensor product. We have $\Theta(\phi(G \times H)) = \Theta(\psi(G \times H)) = f_0(G)f_0(H) = 3 \cdot 3 = 9$.

to be difficult to compute in general. Shannon used the log [40], while in modern treatments like [33] one looks at $\Theta(A)$, which we use too. Here is a bit of a surprise. For connection graphs as well as Barycentric graphs, we know the capacity explicitly.

5.3. Let $f_0(G)$ denote the number of 0-dimensional sets in G.

Theorem 5 (Theorem 5)**.** *We have* $\Theta(\psi(G)) = f_0(G)$.

5.4. So, while for some $G = C_n$, we can not compute $\Theta(G)$, we can do it for $\psi(G)$. We know the capacity for complete graphs $A = K_n$, for $A = C_{2n}$ the linear graph L_n, as well as all connection graphs $A = \psi(G_1 \times \cdots \times G_n))$ of products of simplicial complexes have the property $\Theta(A) = i(A)$. This begs for the question: which connected graphs have the property that their Shannon capacity is equal to the independence number? The above list is far from complete: for example take K_5 and remove edges to get a connected graph making sure that $i(A)$ remains 2. Then $\Theta(A) = 2$ also. These are graphs for which communication capacity can not be increased by taking products.

5.5. While $\Theta(A + B) \geq \Theta(A) + \Theta(B)$ in general, it is a bit surprising that $\Theta(A + B)$ can be strictly larger than $\Theta(A) + \Theta(B)$ as believed to be true by Shannon himself [33]. However, for connection graphs, Θ is a compatible with addition and multiplication:

Corollary 5. *If* $A = \psi(G), B = \psi(H)$ *are connection graphs, then* $\Theta(A + B) = \Theta(A) + \Theta(B)$ *and* $\Theta(A \cdot B) = \Theta(A)\Theta(B)$.

Proof. This follows directly from the fact that for connection graphs as well as products of connection graphs, the number $f_0(G)$ of 0-dimensional elements in the complex is a ring homomorphism and that $\Theta(\psi(G)) = f_0(G)$ for products of simplicial complexes. □

5.6. The Shannon capacity joins now a larger and larger class of functionals which are compatible with the ring structure: $f_0(G), |G|$, the Euler characteristic, Wu characteristic or the ζ-function $\zeta(s) = \sum_j \lambda_j^{-s}$ defined by the eigenvalues λ_j of the connection Laplacian $L = 1 + A(G)$ of G, where $A(G)$ is the adjacency matrix of the connection graph $\psi(G)$.

6. Proof of Theorem 1

6.1. Let $d(x)$ denote the **vertex degree** of $x \in G$ when x is considered to be a vertex in the connection graph $\psi(G)$. Let $\delta(x)$ denote the **minimal vertex degree of all neighboring vertices** of x. Formally, this is

$$\delta(x) = \min_{y \in S(x)} d(y) \ .$$

The following lemma shows that strict local minima of the function $d : V \to \mathbb{N}$ reveal the 0-dimensional sets, the sets x with cardinality $|x| = 1$.

Lemma 1. *If G is a complex, then* $\dim(x) = 0$ *for $x \in G$ if and only if $d(x) < \delta(x)$.*

Proof. Assume $x \in G$ is a 0-dimensional point and assume that $y \in G$ is connected to x. Because x, y intersect and x is a point, $x \subset y$. Furthermore, y intersects any set z which x intersects meaning $d(y) \geq d(x)$. Since y also intersects some point different than x, we have $d(y) > d(x)$. On the other hand, assume that x is a point which is not 0-dimensional. Then it contains a 0-dimensional point y and $d(x) > d(y)$ by what we have seen before. It therefore can not happen that the degree $d(x)$ is smaller than any neighboring degree. □

6.2. We can now prove Theorem 1: given a graph $\psi(G)$ we want to reconstruct the simplicial complex G. By the Lemma we can identify the set G_0 of 0-dimensional sets. This is an independent set already. None of them are adjacent because two different 0-dimensional sets do not intersect. The 1-dimensional sets G_1 are the vertices y in the graph which have the property that $y \in S(a) \cap S(b)$ where $a, b \in G_0$ are two different 0-dimensional points. The 2-dimensional sets G_2 are the vertices y in the graph which have the property that $y \in S(a) \cap S(b) \cap S(c)$ with three different vertices a, b, c. We see that $G_k = \{x \in V, x \in S(a_0) \cap S(a_2) \cdots \cap S(a_k)$, $a_0, a_1, \ldots, a_k \in G_0$ are all disjoint. This reconstruction only needs a polynomial amount of computation steps: we have to go through all the n vertices, compute $d(x)$ and then form intersections of unit spheres.

6.3. The same proof also establishes that the Barycentric graph $\phi(G)$ determines G. The later could also be achieved also differently: we can see the **facets** of G by looking at maximal sub-graphs which belong to Barycentric refinements of simplices. But this point of view is computationally much more costly as we have to find subgraphs which are refinements of simplices which in particular also means requires to find complete subgraphs.

7. Proof of Theorem 2

7.1. The proof of theorem 2 can serve as a nice independent introduction to graph homotopy. Doing graph homotopy steps can be seen as a game. Indeed, the **homotopy puzzle** to deform a graph homotopic to 1 to 1 is a nice game. It can be difficult, like for dunce hats or bing houses. Like for any game, it is good to know what combinations of moves can do. We will see in particular that **edge refinements** are homotopy steps.

7.2. When combining two homotopy steps we can achieve that the set of vertices does not change but that we can get rid of an edge.

Lemma 2 (Lamma A)**.** *If A is a graph and $e = (v, w)$ is an edge such that $S(v)$ and $S(v) - w$ are both contractible, then A is homotopic to $A - e$.*

Proof. $A \to B = A - v$ is a homotopy step because $S(v)$ is contractible. Now $B \to B +_A v$ is a homotopy step. And $B +_A v$ is $A - e$. □

7.3. If $e = (a, b)$ is an edge in a finite simple graph A, an **edge refinement** $A + e$ is a new graph obtained from A by replacing e with a new vertex e and connecting this new vertex to a, b as well as every vertex in $S(a) \cap S(b)$. Edge refinement is a homotopy. This is true in general for any graph. The graphs do not need to come from simplicial complexes.

Lemma 3 (Lemma B)**.** *If G is a graph and $e = (a, b)$ is an edge. Then the edge refinement $G + e$ is homotopic to G.*

Proof. Without removing the edge (a, b), add a new vertex e and connect it to a, b and $A = S(a) \cap S(b)$. Since the graph generated by $e, a, b, V(A)$ is now a unit ball with center e, it is contractible so that this is a homotopy step. Now, $S(a)$ and $S(a) \setminus b$ are both contractible in this new graph. By Lemma A, it remains contractible after removing the edge e from it. Now we have the edge refinement. □

7.4. The following Lemma is not true without the Barycentric assumption.

Lemma 4 (Lemma C)**.** *If G is a Barycentric complex and y, z are two elements with $y \cap z \neq \emptyset$, but not $y \subset z$ nor $z \subset y$, then $S(y) \cap S(z)$ is contractible in the connection graph of G.*

Proof. We know that $x = y \cap z$ is a simplex as an intersection of simplices. Case (i): Assume there are no other parts except subsets of x, then $x = S(y) \cap S(z)$ as a simplex is contractible. Case (ii): If there is an other point u, we must have u intersecting x or then u be contained in larger v containing x. Proof: Assume this is not the case, then we have y, z, u which are pairwise not contained in each other but which intersect. Pick 0-dimensional points x, a, b in the intersections. These points were already points in the original complex G from which one has taken the Barycentric refinement. The union (x, a, b) of them generates a simplex v in G which is a point in the Barycentric refinement. This v intersects all three sets x, y, z. We can do that for any choice of x, a, b so that there is a simplex v containing y, z, x. So $S(y) \cap S(z)$ contains this point v which is connected to all other points in $S(y) \cap S(z)$. So, $S(y) \cap S(z)$ is contractible. □

7.5. An example for a non-Barycentric complex, where it is false is $A = \psi(C_3)$, where $S(y \cap S(z)$ is not contractible.

7.6. Here is the proof of Theorem 2:

Proof. Assume G is a Barycentric refined complex. Let y,z be two sets in G which do intersect but which are not contained in each other. We want to show that removing this edge is a homotopy. Using Lemma B, make an edge refinement with $e = (y, z)$. By Lemma C, $S(y) \cap S(z)$ is contractible Now, $B = S(y) \cap S(z) + y + z$ is contractible. By Lemma A, we can remove the edge e and have a homotopy. Because the new vertex e has a contractible unit sphere $S(y) \cap S(z) + y + z$, we can

remove it. Overall we have removed the edge e. We can now do this construction for any connection between two sets which are not contained in each other. In the end, we reach the graph $\phi(G)$. □

7.7. Going through connection graphs also allows to see that the Barycentric refinement can be written as a homotopy: if A is a graph, then its **Barycentric refinement** A_1 is the graph in which the simplices of A are the vertices and where two such simplices are connected if one is contained in the other.

Corollary 6. *Any graph A and its Barycentric refined graph A_1 are homotopic.*

Proof. The deformation from A to A_1 can be done by edge refinements (Lemma B). First remove the vertices belonging to the highest dimensional simplices, then get to the next smaller points and edge refine them, until only the vertices of A are left as points. □

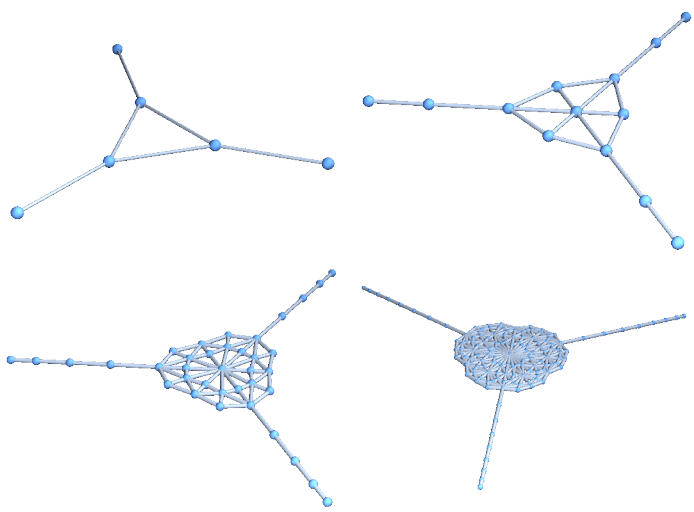

FIGURE 8. we see three Barycentric refinements. Each Barycentric refinement $A \to A_1$ is a homotopy but we can not contract A_1 to A in general directly. We need to use both contractions and expansions to do that.

8. Proof of Theorem 3

8.1. Theorem 3 leaves finite combinatorics and looks at the continuum. If $A = (V, E)$ is a finite simple graph with $|V| = n$ vertices we can embed it into $\mathbb{R}^n$. Put the vertices x_k as unit vectors e_k in $\mathbb{R}^n$. Now connect each pair (a, b) of vertices which are connected by a line segment. Then look at all 3-cliques (a, b, c) of point triples which are all pairwise connected. This produces a concrete triangle spanned by the points e_a, e_b, e_c in $\mathbb{R}^n$. Go on like this with all k-cliques. The realization $\gamma(A)$ produces a compact subset of $\mathbb{R}^n$ equipped with the Euclidean norm. We usually con realize a complex in much lower dimension.

8.2. Two functions $f_0 : X \to X$ and $f_1 : X \to X$ are **homotopic** if there exists a continuous map $F : X \times [0,1] \to X$ such that $F(x,0) = f_0(x)$ and $F(x,1) = f_1(x)$. Two topological spaces X, Y are **homotopic** if there is a pair of continuous maps $f : X \to Y$ and $g : Y \to X$ such that $g \circ f : X \to X$ is homotopic to the identity map $I(x) = x$.

8.3. The construction of a geometric realization of two homotopic graphs A,B produces two topological spaces $\gamma(A), \gamma(B)$ which are classically homotopic.

8.4. To perform the proof, one only has to check this for a single homotopy extension or its inverse. We proceed by induction. Assume the graph A has already been realized as $\gamma(A) \subset \mathbb{R}^n$. Now add a new vertex by attaching it to a contractible subgraph C of A. Take the larger space $\mathbb{R}^n \times \mathbb{R} = \mathbb{R}^{n+1}$ and place the new vertex there. Now build the pyramid over $\gamma(C)$. This produces an explicit embedding of the larger graph. The homotopy $F_t(x,h) \to (x, th)$ deforms the geometric realization of $\gamma(A)$ to $\gamma(A + x)$.

9. Proof of Theorem 4

9.1. It is enough to prove that if A' is a homotopy extension of A and B is fixed, then $A' \cdot B$ is a homotopy extension of $A \cdot B$. The general case can be done by two such steps, one for A and one for B.
Let x be a new vertex so that $A' = A +_C x$ with a contractible subgraph C of A. If V is the vertex set of A, let W the vertex set of B and let $V' = V \cup \{x\}$ the vertex set of A'. The Cartesian product $V \times W$ is the vertex set of $A \cdot B$ and $V' \times W$ is the vertex set of $A' \cdot B$.
There are $m = |W|$ copies $x_1, \dots, x_m$ of the vertex x in $A' \cdot B$. The graph $A' \cdot B$ is a $|W|$ fold homotopy extension of $A \cdot B$: start with $A \cdot B$, then add x_1 and attach it to the graph U_1 generated by the union of all sets $C_y = C \times \{y\} \subset A \cdot B$ and vertices $x \in V(A)$ connected to x_1 and summing over all $y \in V(B)$ with $(x,y) \in E(B)$.
Continue like this until all x_k are added. This works, because each U_k is contractible. At the end we have $A' \cdot B$.

9.2. The analogue result of Theorem 4 for addition (disjoint union) is clear. If A is homotopic to A' and B is homotopic to B' then the there is a deformation of $A + B$ to $A' + B'$.

10. Proof of Theorem 5

10.1. If G, H be finite abstract simplicial complexes and let $\psi(G), \psi(H)$ their connection graphs. To the product $G \times H$ belongs the strong product graph $\psi(G \times H) = \psi(G) \cdot \psi(H)$. The vertices of this graph is the Cartesian product $G \times H$ as a set of sets. Two points (a,b) and (c,d) in this product are connected if $a \cap c \neq \emptyset$ and $b \cap d \neq \emptyset$. Let G_0 denote part of the vertex set G of $\psi(G)$ consisting of 0-dimensional sets.

Lemma 5. *For any G, we have $i(\psi(G)) = |G_0|$.*

Proof. The set $\{x \in G, \dim(x) = 0\}$ is an independent set of vertices in $\psi(G)$. The reason is that two zero dimensional sets do not intersect. This shows $i(A) \leq |G|$. But assume we have an independent set I which contains a positive dimensional vertex $x = (a_1, \dots, a_k)$. But then we can replace x with a larger independent set $\{a_1, \dots, a_k\}$. □

10.2. The independence property holds also for the product:

Lemma 6. *if $A = \psi(G), B = \psi(H)$ are connection graphs, then $i(A \cdot B) = i(A)i(B) = f_0(G)f_0(H)$.*

Proof. The 0-dimensional parts $G \times H$ are the points (x, y), where both x, y are zero dimensional. There are $f_0(G)f_0(H)$ such sets in $G \times H$ and they form an independent set in $\psi(G \times H)$. For the same reason than in the previous lemma, we can not have an independent set containing one positive dimensional point because we could just replace it with the collection of its zero dimensional parts which are all independent of each other. □

10.3. We can write the Shannon capacity as

$$\Theta(A) = \lim_{n\to\infty} (i(A^n))^{1/n} .$$

It is bound below by $i(A)$ and bound above by the **Lovasz number** $\theta(A)$.

10.4. As part of the **sandwich theorem**, we have have $i(G) \leq \Theta(G) \leq \theta(G)$. In our case, we can see $\Theta(\psi(G)) \geq f_0(G) = i(G)$ also because the zero-dimensional parts remain also in the product an independent set. But also in the product $\psi(G^k)$ if we have a product point $(a_1, \ldots, a_k)$ in the independent set, where some a_j is not zero dimensional, then we can replace this point by individual points and so increase the independence number. This shows that $\Theta(\phi(G)) = \Theta(\psi(G)) = f_0(G)$. Applying the previous lemma again and again, we have $i(G^n) = f_0(G)^n$ so that $\Theta(G) = f_0(G) = m$.

10.5. An elegant proof uses the **Lovasz umbrella construction** which attaches to every vertex x in the graph a unit vector $u(x)$ such that the dot product $\langle u(x) \cdot u(y)\rangle = 0$ if (x, y) is not an edge. In our case, this means that the vectors $u(x), u(y)$ are orthogonal if $x \cap y = \emptyset$. The explicit **Lovasz representation** in $\mathbb{R}^{f_0}$ is

$$u_j(x) = \begin{cases} 1, & j \in x \\ 0, & \text{else} \end{cases} .$$

It is clear that $u(x) \cdot u(y) = 0$ if $x \cap y = \emptyset$. The **stick of the Umbrella** is the vector $c = [1, 1, \ldots, 1]/\sqrt{f_0}$. The Lovasz number is bound above by any value

$$\max_{x\in G}(u(x) \cdot c)^{-2}$$

of a choice of a Lovasz umbrella and stick (it is the $\sec^2(\alpha)$ of the maximal opening angle of the umbrella) and for our Lovasz umbrella equal to f_0. So, we know $\theta(\psi(G)) \leq f_0$. But we also have the lower bound $i(\psi(G)) = f_0$ so that $\Theta(\psi(G)) = f_0$.

10.6. We have $\Theta(\phi(G)) \geq f_0(G)$: since $\phi(G)$ is a subgraph of $\psi(G)$, we have $\Theta(\phi(G)) \geq \Theta(\psi(G)) = f_0(G)$. We have $\Theta(\psi(G)) = f_0(G)$ we have in general $\Theta(\phi(G)) < \theta(\phi(G))$.

10.7. In some cases, we can get an umbrella for $\phi(G)$ which lives in $\mathbb{R}^{f_0(G)}$ and so get $\Theta(\phi(G)) = f_0(G)$. For $G = K_3$, where $\phi(G)$ is a wheel graph with 6 spikes, we have $i(G) = f_0(G) = 3$. The umbrella in $\mathbb{R}^3$ has the stick $[1,1,1]/\sqrt{3}$ and assigns vectors $u(x) = \min_{v \in x} e_v$. The is the standard basis. In general for the graph C_{2n} or its pyramid extension W_{2n}, the wheel graph with boundary C_{2n}, we have the Lovasz system $u(\{1\}) = u(\{1,2\}) = e_1$, $u(\{2\}) = u(\{2,3\}) = e_2$ etc $u(\{n\}) - u(\{n,1\}) = e_n$ in the C_{2n} case and $u(\{1\}) = u(\{1,2\}) = u(\{1,2,n+1\}) = e_1$, $u(\{2\}) = u(\{2,3\}) = u(\{2,3,n+1\}) = e_2$ etc $u(\{n\}) - u(\{n,1\}) = u(\{n,1,n+1\}) = e_n$ in the wheel graph case. Now $W_6 = \phi(K_3)$.

10.8. Here is an example of $\phi(G)$ without a Lovasz umbrella. For the Barycentric refinement of the **figure** 8 **complex**

$$G = \{\{1,2,3,4,5,6,7,(1,2),(2,3),(3,4),(4,1),(4,5),(5,6),(6,7),(7,4)\})$$

which is a genus $b_1 = 2$ complex with Euler characteristic $b_0 - b_1 = -1$ (equivalently by Euler-Poincaré $f_0 - f_1 = 7 - 8 = -1$), we would have to attach to every edge a unit vector and since none of the edges connect in the connection graph, we would need so 8 pairwise perpendicular vectors. This can not be done in $\mathbb{R}^{f_0}$. We see that $\theta(\phi(G)) > 7$. Shannon computed the capacity for all graphs with ≤ 6 vertices except for C_5, where Lovasz eventually established $\sqrt{5}$. Here is a general observation:

10.9. If G is a one-dimensional simplicial complex with f_0 vertices and f_1 edges, then the graph $\phi(G)$ has the Shannon capacity $\max(f_0, f_1)$ because both the edge set as well as the vertex set is an independent set in $\phi(G)$. For the figure 8 complex above we have $\Theta(\phi(G)) = 8$ and $\Theta(\psi(G)) = 7$. When looking at bouquets of circles, we can arbitrary large differences between $\Theta(\phi(G))$ and $\Theta(\psi(G))$.

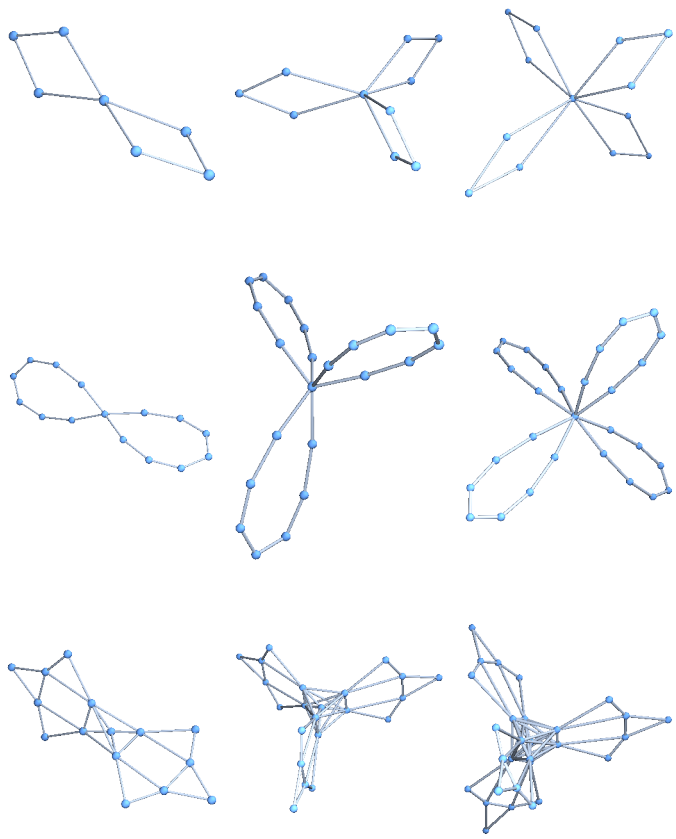

FIGURE 9. Bouquet complexes G_2, G_3, G_4 of genus $k = 2, 3, 4$ its Barycentric graphs $\phi(G_k)$ and its connection graphs $\psi(G_k)$. We have $\Theta(\phi(G_2)) = f_1(G_2) = 8, \Theta(\psi(G_2)) = f_0(G_2) = 7$, and $\Theta(\phi(G_3)) = f_1(G_3) = 12, \Theta(\psi(G_3) = f_0(G_3) = 10$, and $\Theta(\phi(G_4)) = f_1(G_3) = 16, \Theta(\psi(G_4) = f_0(G_3) = 13$,

11. Homotopy game

11.1. The concept of homotopy for graphs produces puzzles: give two graphs which are homotopic and solve the task to find a concrete set of homotopy steps which move one to the other. A simple example is to deform C_5 to C_6 or to deform the octahedron to the icosahedron. In general, one can not go with homotopy reduction steps alone but needs first to enlarge the dimension. The reason is simple: for a discrete manifold, each unit sphere is sphere and not homotopic to 1. One can not apply a single homotopy contraction step on a discrete manifold. One can do more complicated moves like edge reductions, covered in Lemma B.

11.2. The homotopy puzzle leads also to 2-player games: player Ava proposes a graph, player Bob must reduce it to a point. If Bob succeeds, he wins. If player Bob does not succeed, Ava has to reduce it to a point. If she does not succeed, player Bob wins. In the next round, the roles of player Ava and Bob are reversed. The game has a creative aspect and has the advantage that it can be played on any

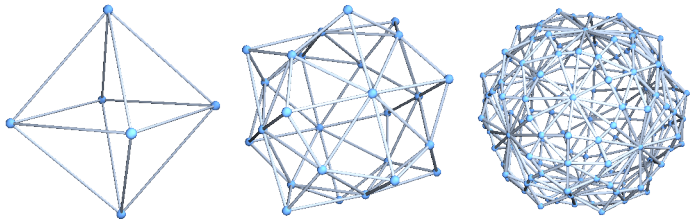

FIGURE 10. The octahedron graph and the first two Barycentric refinements.

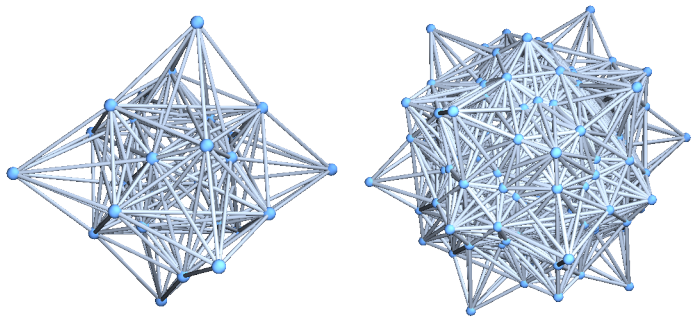

FIGURE 11. The connection graphs of the octahedron graph O and the first Barycentric refinement complex O_1. While $\psi(O)$ is homotopic to a 3-sphere (we see this by brute force computing the Betti numbers $(1,0,0,1)$), the graph $\psi(O_1)$ is homotopic to the 2-sphere with Betti numbers $(1,0,1)$.

level: two small kids can be challenged similarly as two graph theory specialists. One can play more advanced versions by giving two homotopic graphs and ask the opponent to deform one into an other. For example, one can ask to deform an octahedron to an icosahedron.

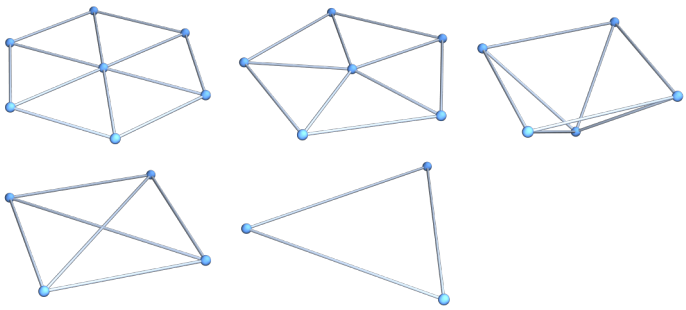

FIGURE 12. Barycentric refinement is a homotopy deformation. The refinement of a triangle can be done with first expanding to a tetrahedron, then do three edge refinements.

12. General remarks

12.1. Categorical view. Simplicial complexes form a category in which the complexes are the objects and the order preserving maps are the morphisms. A **partial order** on a complex is given by incidence $x \subset y$ of its sets. The sets in G are called simplices or faces. The complete complex $G = K_{n+1} = [0, \dots, n]$ can naturally be identified with the n-dimensional face $x = \{0, \dots, n\}$ which is a facet in G, a largest element in the partial order. Also finite simple graphs for a category in which the graph homomorphisms are the morphisms: $f : (V, E) \to (W, F)$ is a homomorphism if $f(V) = W$ and if $(a, b) \in E$, then $(f(a), f(b)) in F$. One often sees graphs as one-dimensional simplicial complexes. A more natural functor is to assigns to a graph its Whitney complex.

12.2. The continuum. We usually do not look at the geometric realization functor γ to topological spaces because this leaves combinatorics and requires stronger axiom systems. It can not hurt to compare classical homotopy with discrete graph homotopy, in particular because many textbooks treat graphs as one-dimensional simplicial complexes. Mathematicians like Euler, Poincaré or Alexandroff [1] considered graphs building higher dimensional structures (Geruest stands for scaffold). Seeing them as 1-dimensional skeleton simplicial complexes came later. Later, (like in [11]) the language of graph theory was again recognized as valuable for low dimensional topology. It is close to [43] but when simplified as in [4], the notion becomes even more accessible.

12.3. Genus spectrum. We have seen in a [23] that Barycentric refinements "smooth out" pathologies. We had seen there that the set of possible genus values $\{1-\chi(S(x))\}_{x\in G}$ is stable after one Barycentric refinement. It is therefore a combinatorial invariant of G, (which by definition is an invariant which does not change any more when doing refinements). For discrete even dimensional manifolds the genus spectrum is $\{1\}$ and for odd-dimensional manifolds, the sphere spectrum is $\{-1\}$. The sphere spectrum can be more interesting for discrete varieties. For the figure 8 complex for example, it is $\{-1, -3\}$. We have seen here an other instance where Barycentric refinement makes Shannon capacity computable.

12.4. Riemann Hurwitz. Barycentric refined complexes are also needed when looking at a Riemann-Hurwitz theorem. If G is a finite simple graph which is a Barycentric refinement and A is a finite group of order n acting by automorphisms, then $\phi(G)$ is a ramified cover over the graph $H = G/A$. The Riemann-Hurwitz formula is then $\chi(G) = n\chi(G/A) - \sum_{x\in G} e_x$, where $e_x = \sum_{a\neq 1, a(x)=x} (-1)^{\dim(x)}$ is a **ramification index**. This can be derived from [16]. The graph G is a cover of G/A. This cover is **unramified** if the ramification indices e_x are all zero everywhere. In this case, the cover $G \to H$ is a discrete **fibre bundle** with structure group A.

12.5. Finitism. Given any laboratory X, we can only do finitely many experiments each only accessing finitely many data. Every measurement defines a partition of X into sets of experiments which can not be distinguished by f. A finite set of functions generates a simplicial complex G if the smallest sets are collapsed to point. One can see this also as follows. Let (X, d) is a compact metric space modelling the laboratory. Take $\epsilon > 0$ and take a finite set V of points such that every $x \in X$ is ϵ close to a point in V. A non-empty set of points is a simplex if all points are pairwise closer than ϵ. The set of all simplices is a simplicial complex G.

12.6. π-Systems. One can also look at the category of set of non-empty sets which are closed under the operation of non-empty intersection. Any such structure G is homomorphic to a simplicial complex: just collapse every atom to a point and remove any element which does not effect the order relation. If we allow also also the empty set, then we have a π-system a set of sets which is closed under the operation of taking finite intersections. A π-system without empty set can be generalized to a **filter base** in which the intersection of two elements must contain an element in the set. Every π-system is isomorphic to a simplicial complex but the super symmetry structure given by the sign of the cardinality is not compatible. It produces inconsistent Euler characteristic for example. Euler characteristic is an invariant under isomorphisms of simplicial complexes but not an isomorphism invariant for π-systems.

12.7. The SM theorem. By the Szpilrajn-Marczewski theorem [41], every graph can be represented as a connection graph of a set of sets. This theorem is also abbreviated as SM theorem in intersection graph theory. Edward Szpilrajn-Marcewski (1907-1976) proved this in 1945. The theorem has been improved and placed into extremal graph theory by Erdös, Goodman and Posa who showed in 1964 [34] that one can realize any graph of n vertices as a set of subsets of a set V with $[n^2/4]$ elements and that for $n \geq 4$, the set of sets can be all different.

12.8. Spectral question. We still do not know whether the spectrum of the connection Laplacian determines the complex G. We know that the number of positive eigenvalues is the number of even dimensional simplices. We know also that the spectrum $\sigma(L(\psi(G \times H)))$ of the connection Laplacian of $\psi(G \times H)$ has eigenvalues $\lambda_j \mu_k$ where λ_j are the eigenvalues of $\psi(G)$ and $\mu(k)$ are the eigenvalues of $\psi(H)$. The spectrum of connection Laplacians could contain more topological information.

12.9. Characterize SM graphs. Connection graphs are special as they can be realized by on sets with n elements. This begs for the question which graphs with n vertices have the property that they can be realized with a set G of subsets of a set with n elements. The non-simplicial complex example $G = \{\{1,2\},\{2,3\},\{3,1\}\}$ with three sets belongs to a graph C_3 that can be realized on a set with $n = 3$ atoms.

Appendix: Euler's Gem

12.10. In this appendix, we give a combinatorial proof of the Euler gem formula telling that a d-sphere has Euler characteristic $1+(-1)^d$. We also classify **Platonic d-spheres**. The first part of this document was handed out on February 6, 2018 at a Math table talk "Polishing Euler's gem", a day before Euler's day 2/7/18. The section is pretty independent of the previous part and added because it gives more details about homotopy and should be archived somewhere.

12.11. A **finite simple graph** $G = (V, E)$ consists of two finite sets, the **vertex set** V and the **edge set** E which is a subset of all sets $e = \{a, b\} \subset V$ with cardinality two. A graph is also called a **network**, the vertices are the **nodes** and the edges are the **connections**. A subset W of V **generates** a subgraph (W, F) of G, where $F = \{\{a, b\} \in E \mid a, b \in W\}$. Given G and $x \in V$, its **unit sphere** is the sub graph generated by $S(x) = \{y \in V | \{x, v\} \in E\}$. The **unit ball** is the

sub graph generated by $B(x) = \{x\} \cup S(x)$. Given a vertex $x \in V$, the graph $G - x$ **with** x **removed** is generated by $V \setminus \{x\}$. We can identify $W \subset V$ with the subgraph it generates in G.

12.12. The empty graph $0 = (\emptyset, \emptyset)$ is the (-1)**-sphere**. The 1-point graph $1 = (\{1\}, \emptyset) = K_1$ is the smallest contractible graph. Inductively, a graph G is **contractible**, if it is either 1 or if there exists $x \in V$ such that both $G - x$ and $S(x)$ are contractible. As seen by induction, all **complete graphs** K_n and all trees are contractible. Complete subgraphs are also called **simplices**. Inductively a graph G is called a d**-sphere**, if it is either 0 or if every $S(x)$ is a $(d-1)$-sphere and if there exists a vertex x such that $G - x$ is contractible.

12.13. Let f_k denote the number of complete subgraphs K_{k+1} of G. The vector $(v_0, v_1, \dots)$ is the f**-vector** of G and $\chi(G) = v_0 - v_1 + v_2 - \dots$ is the **Euler characteristic** of G. Here is Euler's gem:

Theorem 6. *If G is a d-sphere, then $\chi(G) = 1 + (-1)^d$.*

12.14. To prove this, we formulate three lemmas. Given two subgraphs A, B of G, the **intersection** $A \cap B$ as well as the **union** $A \cup B$ are sub-graphs of G.

Lemma 7. *$\chi(A) + \chi(B) = \chi(A \cap B) + \chi(A \cup B)$ for any two subgraphs A, B of G.*

Proof. Each of the functions $f_k(A)$ counting the number of k-dimensional simplices in a subgraph A satisfies the identity. The Euler characteristic $\chi(G)$ is a linear combination of such valuations $f_k(G)$ and therefore satisfies the identity. □

12.15. A graph G is a **unit ball**, if there exists a vertex x in G such that the graph generated by all points in distance ≤ 1 from x is G. We write $B(x)$ and rephrase it that it is a **cone extension** of the unit sphere $S(x)$. Algebraically, one can say $B(x) = S(x) + 1$ where $+$ is the join addition.

Lemma 8. *Every unit ball B is contractible and has $\chi(B) = 1$.*

Proof. Use induction with respect to the number of vertices in $B = B(x)$. It is true for the one point graph $G = K_1$. Induction step: given a unit ball $B(x)$. Pick $y \in S(x)$ for which $B(y)$ is not equal to $B(x)$ (if there is none, then $B(x) = K_n$ for some n and $B(x)$ is contractible with Euler characteristic 1). Now, both $B(y), B(y) - x$ and $S(x)$ are smaller balls so that by induction, all are contractible with Euler characteristic 1. As both $B(x) \setminus y$ and $S(y)$ are contractible, also $B(x)$ is contractible. By the valuation formula, $\chi(B(x)) = \chi(B(x) - y) + \chi(B(y)) - \chi(S(x)) = 1 + 1 - 1 = 1$. □

Lemma 9. *If G is contractible then $\chi(G) = 1$.*

Proof. Pick $x \in V$ for which $S(x)$ and $G - x$ are both contractible. By induction, $\chi(G - x) = 1$ and $\chi(S(x)) = 1$. By the unit ball lemma, $\chi(B(x)) = 1$. By the valuation lemma, $\chi(G) = \chi(B(x)) + \chi(G - x) - \chi(S(x)) = 1 + 1 - 1$. □

12.16. Here is the proof of the Euler gem theorem.

Proof. For $G = 0$ we have $\chi(G) = 0$. This is the induction assumption. Assume the formula holds for all d-spheres. Take a $(d+1)$-sphere G and pick a vertex x for which both $S(x)$ and $G - x$ are contractible. Now, $\chi(G) = \chi(G - x) + \chi(B(x)) - \chi(S(x)) = 1 + 1 - (1 + (-1)^d) = 1 + (-1)^{d+1}$. □

12.17. We look now at regular (Platonic) d-spheres. For $d = 2$, we miss the tetrahedron (because this is a 3-dimensional simplex K_4) as well as the cube and dodecahedron because their Whitney complexes are one-dimensional. Combinatorially, one can include them using CW complex definitions. The classification of **Platonic d-spheres** is very simple. First to the **recursive definition**: a **Platonic d-sphere** is a d-sphere for which all unit spheres are isomorphic to a fixed Platonic $(d-1)$-sphere H. The **curvature** $K(x)$ of a vertex is defined as $K(x) = \sum_{k=0}(-1)^k f_k(x)/(k+1)$, where $f_k(x)$ is the number of k-dimensional simplices z which contain x.

Lemma 10 (Gauss-Bonnet)**.** $\sum_{x \in V} K(x) = \chi(G)$.

Proof. We think of $\omega(x) = (-1)^{\dim(x)}$ as a charge attached to a set x. By definition, $\chi(G) = \sum_{x \in G} \omega(x)$. Now distribute the charge $\omega(x)$ from x equally to all zero dimensional parts containing in x. □

12.18. For a triangle-free graph, $K(x) = v_0 - v_1/2 = 1 - \deg(x)/2$. For a 2-graph and in particularly for a 2-sphere, we have $K(x) = v_0 - v_1/2 + v_2/3 = 1 - \deg(x)/6$, where $\deg(x)$ is the **vertex degree**.

Theorem 7. *There exists exactly one Platonic d-sphere except for $d = 1, d = 2$ and $d = 3$. For $d = 1$ there are infinitely many, for $d = 2$ and $d = 3$ there are two.*

Proof. $d = -1, 0, 1$ are clear. For $d = 2$, the curvature $K(x) = 1 - V_0/2 + V_1/3 - V_2/3$ is constant, adding up to 2. It is either $1/3$ or $1/6$. For $d = 3$, where each $S(x)$ must be either the octahedron or icosahedron, G is the 16-cell or 600-cell. For $d = 4$, by Gauss-Bonnet, $K(x)$ add up to 2 and be of the form $L/12$. For $L = 1$, there exists the 4-dimensional cross-polytop with f-vector $(10, 40, 80, 80, 32)$. There is no 4-sphere, for which $S(x)$ is the 600-cell as the f-vector of it is $(120, 720, 1200, 600)$. We would get $K(x) = 1 - 120/2 + 720/3 - 1200/4 + 600/5 = 1$ requiring $|V| = 2$ and $\dim(G) \leq 1$. □

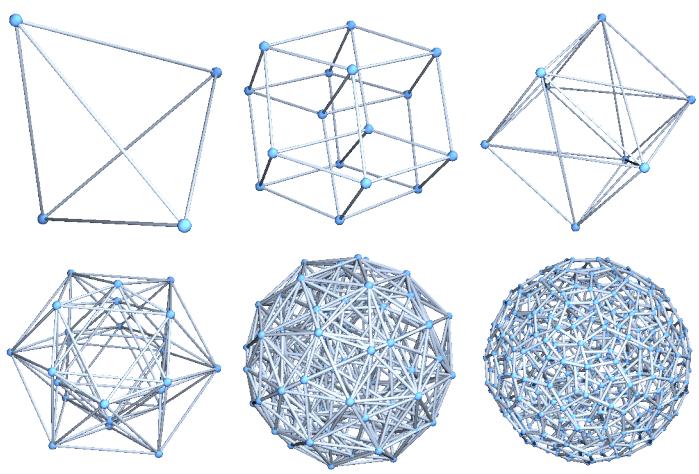

FIGURE 13. The six classical 4-polytopes are the 5-cell, 8-cell, 16-cell, 24-cell, the 600-cell and 120-cell. Only the 16-cell and the 600-cell are 3-spheres, discrete 3-dimensional discrete manifold.

12.19. In order to include all classically defined Platonic solids (including also the dodecahedron and cube in $d = 2$), we would need to use **discrete CW complexes**. Recursively, a **k-cell** requires to identify a $(k-1)$ sub sphere in the already constructed part. A CW-complex defines the Barycentric graph, where the cells are the vertex set and where two cells are connected if one is contained in the other. A CW complex is **contractible** if its graph is contractible. A CW complex G is a **d-sphere** if its Barycentric graph $\phi(G)$ is a **d-sphere**. Each cell in a complex has a dimension $\dim(x)$, which is one more than the dimension of the sphere, the cell has been attached to. One can then define a **Platonic d-sphere**, to have the property that for every dimension k, there exists a Platonic CW k-sphere H_k such that for every cell of dimension k, the unit sphere is isomorphic to H_k.

12.20. The **Zykov sum** [44] or **join** of two graphs $G = (V, E), H = (W, F)$ is the graph $G + H = (V \cup W, E \cup F \cup \{\{a, b\}|\ a \in V, b \in W\}$. For example, the Zykov sum of $S_0 + S_0 = C_4$. And $C_4 + S_0 = O$ is the octahedron graph. The sum of G with the 0-sphere S_0 is called the **suspension**. The sum of G with $1 = K_1$ is a **cone extension** and by definition always a ball. One can quickly see from the definition that under taking graph complements, the **Zykov-Sabidussi ring** with Zykov join as addition and large multiplication is dual to the **Shannon** ring, where the addition is the disjoint union and where the multiplication is the strong product.

Lemma 11. *If H is contractible and K arbitrary then the join $H+K$ is contractible.*

Proof. Use induction with respect to then number of vertices in H. It is clear for $H = K_1$. In general, there is a vertex in H for which $S(x)$ is contractible. As $S(x) + K$ is contractible also $(y + S(x)) + K = y + (S(x) + K)$ contractible. □

Lemma 12. *The join G of two spheres $H + K$ is a sphere.*

Proof. Use induction. One can use the fact that for $x \in V(H)$, we have $S_{H+K}(x) = S_H(x) + K$ which is a sphere and for $x \in V(K)$ one has $S_{H+K}(x) = H + S_K(x)$. This shows that all unit spheres are spheres. Furthermore, we have to show that when removing one vertex, we get a contractible graph. For $x \in V(H)$ one has then a sum of a contractible $H - x$ and K, for $x \in V(K)$ one has a sum of H and $K - x$ which are both contractible.
For a suspension $G \to G + P_2$, it is more direct. Use induction: if $G = H + \{a, b\}$ with $H = (V, E)$ then every unit sphere of $x \in V(H)$ becomes after suspension a unit sphere of G. The unit spheres $S(a) = S(b) = H$ are already spheres. If we take away a from G, then we have $G - a = H + \{b\}$ which is a ball and so contractible. In general: either take a away a vertex x of H or from K. Now $H - x$ is contractible and so $(H - x) + K$ contractible by the previous lemma. That means $G - x$ is contractible. □

Lemma 13. *The genus $j(G) = 1 - \chi(G)$ satisfies $j(H)j(G) = j(H + G)$.*

The set of spheres with join operation is a **monoid** with zero element 0. The **graph complement** of $G = (V, E)$ with $|V| = n$ is the graph $\overline{G} = (V, E^c)$ where E^c is the complement $E(K_n) \setminus E(G)$. Given two graphs G, H, denote by $G \oplus H$ the disjoint union of G and H. We have $(G + H)^c = G^c \oplus H^c$.

12.21. A graph G is an **Zykov prime** if G can not be written as $G = A+B$, where A, B are graphs. The primes in the Zykov monoid are the graphs for which the graph complement $\overline{G}$ is connected. As there is a unique prime factorization in the dual monoid, we have a unique additive prime factorization in the Zykov monoid. The map i which maps a sphere to its genus satisfies $j(G+H) = j(G)j(H)$. We can extend Euler characteristic to negative graphs and since $j(0) = 1$ we should define $j(-G) = j(G)$.

12.22. Examples of contractible graphs are complete graphs, star graphs, wheel graphs or line graphs. We can build a contractible graph recursively by choosing a contractible subgraph and building a cone extension over this. In general, if we make a cone extension over A, then the Euler characteristic changes by $1 - \chi(A))$. This change is called the Poincaré-Hopf index. Examples of 1-spheres are cyclic graphs. From all the classical platonic solids, the octahedron and the icosahedron are 2-spheres. From the classical 3-polytopes, as shown, only the 16 cell and the 600 cell are e-spheres.

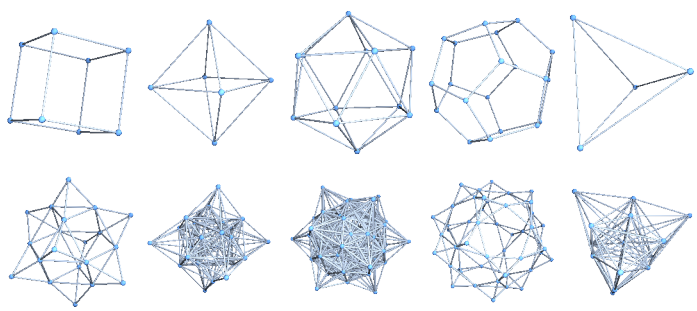

FIGURE 14. The five platonic solid complexes and their connection graphs. Only the octahedron and icosahedron are $d = 2$-spheres with $\chi(G) = 2$. The cube has Euler characteristic -4, the dodecahedron has Euler characteristic -10, the tetrahedron has Euler characteristic 1. All connection graphs are homotopic to their platonic solid graphs except for the octahedron, where the connection graph is a homology 3-sphere and has maximal dimension 8 (there are 9 complete subgraphs intersecting all each other at each vertex so that there is K_9 subgraph of the connection graph.

12.23. Results on graphs immediately go over to **finite abstract simplicial complexes** G because the Barycentric graph $\phi(G)$ of G is a graph. A simplicial complex is a d-sphere if its Barycentric refinement is a d-sphere. The class of objects can be generalized discrete CW-complexes, where cells take the role of balls but the boundary of a ball does not have to be a skeleton of a simplex. Every cell has a dimension attached. We can see the cube or the dodecahedron as a sphere. The definition of Platonic sphere must be adapted in that we require every unit sphere $S(x)$ is isomorphic to a fixed Platonic $d-1$ sphere which only depends on the dimension of x. Now the classification is the Schläfli classification $(1, 1, \infty, 5, 6, 3, 3, 3, 3, \dots)$. In the graph case, the numbers are $(1, 1, \infty, 2, 2, 1, 1, 1, 1, \dots)$.

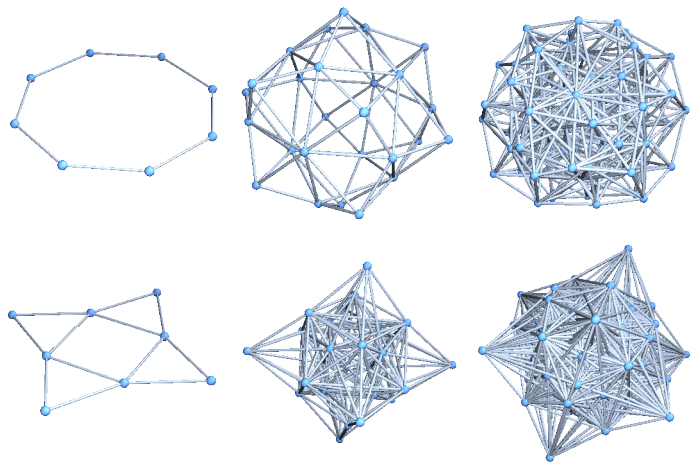

FIGURE 15. The Barycentric graphs $\phi(S^k)$ of the one,two and three dimensional spheres. The 3-sphere is the 16-cell. Below the connection graphs $\psi(S^k)$ of the 1,2,3 spheres. They are homotopic.

12.24. [42] asked to characterize spheres combinatorially. The question whether some combinatorial data can answer this without homotopy is still unknown. Robin Forman [7] defined spheres using Morse theory as classically through Reeb: spheres can be characterized as manifolds which admit a Morse function with exactly 2 critical points. The inductive definition of [11], simplified by [4] goes back to Whitehead and is equivalent [27]. The story of polyhedra is told in [36, 5]. Historically, it was [38, 39, 35].

12.25. In [21], Platonic spheres were defined d-spheres for which all unit spheres are Platonic $(d-1)$-spheres. Gauss-bonnet [14] have the classification all 1-dimensional spheres $C_n, n > 3$ are Platonic for $d = 1$, the Octahedron and Icosahedron are the two Platonic 2-spheres, the sixteen and six-hundred cells are the Platonic 3-spheres. For earlier appearances of Gauss-Bonnet theorem [14] see [12, 31, 8].

12.26. The Euler gem episode illustrates the value of precise definitions [36], especially if the continuum is involved. Already when working with 1-spheres, which are often modeled as polygons, one can get into muddy waters capturing what a polygon embedded into Euclidean space is. Are self-intersections allowed? Do polygons have to be convex? The one-dimensional case gives a taste about the confusions which triggered the "proof and refutation" dialog [30]. Being free from Euclidean realizations places the topic into combinatorics so that the results can be accepted by a finitist like Brouwer who would not even accept the existence of the real number line.

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