ON SYMMETRIES OF FINITE GEOMETRIES

OLIVER KNILL

Abstract. The isospectral set of the Dirac matrix $D = d + d^*$ consists of orthogonal Q for which $Q^* DQ$ is an equivalent Dirac matrix. It can serve as the symmetry of a finite geometry G. The symmetry is a subset of the orthogonal group or unitary group and isospectral Lax deformations produce commuting flows $d/dtD = [B(g(D)), D]$ on this symmetry space. In this note, we remark that like in the Toda case, $D_t = Q_t^* D_0 Q_t$ with $e^{-tg(D)} = Q_t R_t$ solves the Lax system.

1. Finite Geometries

1.1. Let G be a finite set with n elements. A **dimension function** $R : G \to \{0, 1, \ldots, q\}$ defines a partition $G = \bigcup_{k=0}^q G_k$ of G. A **Dirac matrix** is a symmetric **block-tri-diagonal** $n \times n$ matrix $D = d + d^* + m$ such that $L = D^2$ and $C = (d + d^*)^2$ are both **block diagonal** with respect to the decomposition $l^2(G) = \oplus_k l^2(G_k)$. In particular, $d^2 = 0$, defining so **cohomology groups** as the kernel of L_k resp C_k on $l^2(G_k)$. Their dimensions are the **Betti numbers** of G. If $m = 0$, then $d : l^2(G_k) \to l^2(G_{k+1})$ defines a standard **co-chain complex**. A particular case is if $df = \sum_k (-1)^k f(\delta_k)$, where $\delta_k : G_{k+1} \to G_k$ are the **face maps** of a **delta set** $G = \bigcup_{k=0}^q G_k$.

1.2. Let us call the triple (G, D, R) a **finite geometry**. Its **f-vector** is defined by the cardinalities $f_k = |G_k|$. Its **Betti vector** b is given by $b_k = \dim(\ker(C_k))$. The **Euler-Poincaré formula** $\sum_k (-1)^k f_k = \sum_k (-1)^k b_k$ expresses that the **Euler characteristic** $\chi(G) = \sum_{x \in G} \omega(x)$ with $\omega(x) = (-1)^{\dim(x)}$, the alternating sum of the f_k, agrees with the alternating sum of the b_k. The Euler-Poincaré identity is best shown by comparing the super trace of e^{-tL} at $t = 0$ and $t = \infty$ using the **McKean-Singer theorem** [17, 12] that $\text{str}(L^k) = 0$ for all $k > 0$ implying $\text{str}(e^{-tL})$ is independent of $t \in \mathbb{R}$.

1.3. A particular case is when G is **finite abstract simplicial complex**, a finite set G of non-empty sets x closed under the operation of taking finite non-empty subsets. In that case, D and $R(x) = |x| - 1$ are canonically defined. This is special however. For example, if G is the set of intersecting pairs of simplices in a simplicial complex, we deal with the chain complex for **quadratic cohomology**. An other example in one-dimension are **quivers**, one dimensional delta sets which produce finite geometries that are not simplicial complexes but for which there still is a natural Dirac matrix D having the property that $D^2 = L_0 \oplus L_1$ is block diagonal with 0-form Laplacian L_0 and 1-form Laplacian L_1. That L_0, L_1 are essentially isospectral (which is a special case of McKean-Singer symmetry) has been exploited by Anderson and Morley [2] to estimate the spectral radius of K_0 leading to general eigenvalue bounds for all eigenvalues [15].

Date: August 31, 2024.
Key words and phrases. Isospectral deformation, Finite Geometries, Symmetries.

1.4. A bit more general than the case of simplicial complexes is if d is the exterior derivative, coming from face maps of a **delta set**. An other example is the cohomology of **Wu characteristic** $\chi_2(G) = \sum_{(x,y) \in G} \omega(x)\omega(y)$, leading to topological invariants but no homotopy invariants. Already for chain complexes of higher characteristics, we have difficulty to frame this within the category of delta sets. [1] The formula $\chi_2(G) = \chi(G) - \chi(\delta G)$ for a manifold with boundary δG illustrates why quadratic cohomology is not a homotopy invariant. An other case going beyond the familiar simplicial complex or delta set geometry is obtained when the Dirac matrix D is deformed in an isospectral way. We should also mention the **Witten deformation** [22, 5] $d_t = e^{-tf} d e^{tf}$ which is not isospectral but preserves cohomology. Witten deformation is not part of the symmetry considered here.

1.5. Finite geometries contain the **topos of finite sets**. This is the case when R is constant 0 and where the matrix D is the zero matrix. An other example is the **topos of finite quivers**, which is the case if R takes values in $\{0, 1\}$ and $V = G_0$ or $E = G_1$ are the set of **vertices** or **edges** including loops. Quite general already is the **topos of delta sets** and in particular, the **topos of simplicial sets**. The later are delta sets with additional degeneracy maps. Delta sets are important because they allow the formation of products, quotients and level sets within the category. The most general of these topoi is the topos of delta sets but finite geometries go even beyond that, as we do not need the matrix D to come from **face maps**. Looking at **data structure** (G, D, R) rather than insisting it to emerge as a pre-sheaf, is a computer science approach and can be implemented fast.

1.6. A rich source of examples for finite geometries are given by **finite simple graphs** equipped with a **Whitney complex** G (given by the set of complete subgraphs). A finite simple graph containing a sub-graph K_{q+1} but no K_{q+2} so defines a finite q-dimensional geometry with dimension function $R(x) = |x| - 1$, taking values from 0 to q. Simplicial complexes are too narrow because there is no associative Cartesian product within simplicial complexes satisfying the Kuenneth formula and preserving discrete manifolds. The Stanley-Reisner product satisfies the later two properties but it is not associative as multiplication with 1 products the Barycentric refinement. Finite geometries defined by finite graphs are almost as general as the abstract frame work. We can always for a given finite delta set look at G as the vertex set of a graph and then connect two points $x, y \in G$ if there is a sequence of face maps getting from x to y or from y to x.

1.7. A finite geometry G can be used to define a **spectral triple** (H, D, A), where $H = l^2(G)$ is a Hilbert space and A is a not necessarily commutative sub Banach algebra of $\mathcal{B}(H)$ like $l^\infty(G)$, the set of functions on G with supremum norm. The **pseudo metric** $d(x, y) = \sup_{f, |[D,f]|_\infty = 1} |f(x) - f(y)|$ [4] defines then a **metric space** on the **Kolmogorov quotient**, where equivalence classes are points with 0 pseudo distance. This picture shows somehow plays the role of a Riemannian metric in the continuum. The simple observation that distance alone determines the metric tensor g in a Riemannian manifold is known much longer and probably first done by Schrödinger [6]. Spectral triples allow to work with discrete or non-commutative settings or both.

[1] Delta sets technically require exactly $n+1$ face maps from $G_n \to G_{n-1}$. We prefer to work with D and forget about face maps.

1.8. If the Dirac matrix D comes from face maps like in a delta set, there is a **partial order** $x \leq y$, if there is a combination of face maps getting from y to x. In a simplicial complex G, where elements are sets of sets, this **poset structure** is already given by inclusion $x \subset y$. A partial order defines then a **Alexandrov topology** [1] $\mathcal{O}$ on G: the basis for the topology is the set of **stars** $U(x) = \{y \in G, x \subset y\}$ which are the smallest open sets containing x. In positive dimensions, this topology is never Hausdorff. There is **arithmetic** on the entire category of finite geometries. Start with the monoid formed by disjoint union, extend it to a ring where the addition is first Grothendieck completed to a group, then use Cartesian product with $R((x,y)) = x + y$ and $D(G+H) = D(G) \oplus D(H)$ and $D * H = D(G) \otimes D(H)$, where the tensor products for $D * H$ and $H * D$ are identified.

2. The symmetry space

2.1. Definition: The **symmetry space** of a geometry (G, D, R) is defined as the set of orthogonal matrices $Q \in SO(n)$ such that $D' = Q^* D Q$ is again a Dirac matrix producing a finite geometry (G, D', R) which is chain-homotop. In particular, the deformed Laplacian $L' = D'^2$ has the same block diagonal structure $L' = \oplus_{k=0}^q L'_k$ than $L = \oplus_{k=0}^q L_k$ with $L_k : l^2(G_k) \to l^2(G_k)$ where $G_k = R^{-1}(k)$ and the dimensions of the kernels of L_k and L'_k agree. The orthogonal Q does not need to preserve k-forms $l^2(G_k)$. The deformed $D' = d'_t + (d'_t)^*$ still has $d_t^2 = (d_t^*)^2 = 0$ but d_t now map $l^2(G_k)$ to $l^2(G_k) \times l^2(G_{k+1})$.

2.2. The assumption implies that different geometries in the same symmetry space have the same cohomology and Euler characteristic. The terminology "symmetry space" and not "symmetry group" is chosen because this is a priori not a group, at least not with the induced group structure from $SO(n)$. [2] To see that S is not a subgroup of $O(n)$, note that already $(Q^2)^2 D Q^2$ is not a Dirac matrix in general because Q^2 is not necessarily block tri-diagonal.

2.3. The **isospectral Lax deformation** $D' = [B, D]$, where $B = g(D)^+ - g(D)^-$ produces an isospectral deformation $D_t = Q_t^* D Q_t$ with deformed exterior derivatives $d_t = Q_t^* d_0 Q_t$. But unlike d_0 which maps k-forms to $(k+1)$-forms, the image of the map d_t consists also of k-forms. We have $d_t = c_t + n_t$, where d_t does map k-forms to $(k+1)$-forms, leading to $D_t = d_t + d_t^* = c_t + c_t^* + m_t$. If we look at a geometry, we only use measurements with electromagnetic waves and only use c_t, not involving m_t. Since we don't see it, we have called it "dark matter part" of the geometry. For the simplest flow $g(D) = D$, we have $L' = (DD)' = D'D + DD' = (BD - DB)D + D(BD - DB) = BL - LB = [B, L]$. If $B = d - d^*$ and $B^2 = L$ the matrices L and B commute so that $L' = [B, L] = 0$ and L does not change. However, already for $B = g(D) = D^3$, the Laplacian will be deformed. In the case when $g(D)$ is invertible, i.e. if $g(D) = e^{h(D)}$ we will see that the deformation is explicit in terms of QR deformation.

2.4. The isospectral deformation symmetry is a continuous symmetry, unlike the **inner symmetries** $Q(D) = D(T)$, that come from a discrete set of automorphisms T of G and which

[2] There is a non-trivial group structure on it provided by the isospectral flows. It is a non-compact group of scattering paths converging to geometries that are block diagonal. The Liouville-Arnold theory [3] suggests that that it has connected components that are of cylinder type $\mathbb{T}^r \times \mathbb{R}^s$, with r being non-zero if one looks a evolutions allowing complex matrices.

induce symmetries. The inner automorphism symmetry is in general trivial. [3] [4] Certainly, if G is a simplicial complex and T is an automorphism, a bijection which is permuting elements in G then $(G, D(T), R)$ is again a finite geometry in the same symmetry than (G, D, R). Can we connect D with $D(T)$ using isospectral flows provided the corresponding orthogonal matrix has determinant 1? We would need to find a function g such that $e^{g(D)} = QR$ with $Qf(x) = f(Tx)$. In the Toda case, we used time dependent Hamiltonian systems to achieve that.

2.5. In [14, 13] we saw that when starting with $D = d + d^*$, the differential equation $D' = [B, D]$ with $D_0 = d - d^*$ produces deformed operators $c_t + c_t^* + m_t$, containing now a **block diagonal part** m_t. In the actually unrelated periodic Toda case, we saw that isospectral deformations allow to interpolate the translation symmetry on the cyclic graph. There was an explicit time dependent isospectral deformation in the isospectral set within Bäcklund transformations. The cochain complex defined by the deformed exterior derivative d_t is homotopy equivalent to the complex defined by d, similarly as the Witten deformation [22, 5] $d_t = e^{-tf} d e^{tf}$.

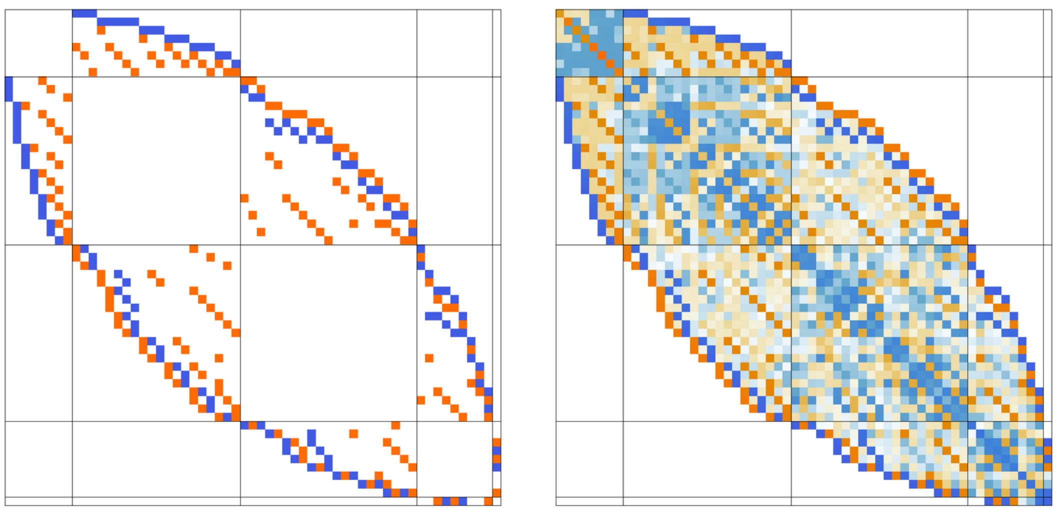

FIGURE 1. We see the Dirac matrix $D = d + d^*$ of a randomly chosen simplicial complex G and the Dirac matrix $D_t = d_t + d_t^* = c_t + c_t^* + m_t$ of a deformed complex. Unlike the initial $D = D_0$, which has only off diagonal entries, the deformed matrix D_t has a block diagonal part. If we focus in the new geometry of what we “can see” with exterior derivatives c_t mapping k-forms to $(k+1)$-forms, we consider the still equivalent geometry $(G, C_t = c_t + c_t^*, R)$. It now represents an expanded space if we measure with the Connes formula.

[3]In the periodic Toda case, which has a Z_n symmetry on C_n, we had observed an explicit deformation within the isospectral set from D to $D(T)$, the translated operator [9, 8, 10]. In the real case, because $SO(n)$ does not contain reflections, we can have disconnected isospectral parts. We would especially need connections from D to $D(T)$.

[4]Inner symmetries could be defined more broadly and not require T to be an automorphism but that T is **continuous map** from a Barycentric refinement to G and that there is an inverse of S given again by a continuous map and that TS and ST are homotop to the identity.

2.6. The isospectral deformation of the Dirac operator is motivated by the **Toda chain**, [20, 21], which is an isospectral deformation of Jacobi matrices L. It is a **Lax pair** [16] $L' = [B, L]$ with $B = a - a^*$ and tri-diagonal $L = a + a^* + b$. If L is invertible, it is possible to write $L \oplus L_1 = D^2$, where D is defined on a doubled lattice and L_1 is a Bäcklund transformation of L. [5] However, the deformation of the Dirac matrix D coming from a finite geometry is different from Toda. It does not generalize Toda and applies for general geometries [14, 13]. It is defined here even for all finite geometries, including the topos of delta sets. Toda on the other hand is just the deformation of a Schrödinger operator L on a one-dimensional circular geometry. Deformations of higher dimensional Laplacians are in general not possible. There is spectral rigidity in general in the sense that Laplacians, isospectral to a given operator, in general form a discrete set.

2.7. Let us elaborate a bit more why the geometry deformation of the Dirac matrix is different from Toda, even so also in the Toda case we could write a Jacobi matrix as $L = D^2$. Given a Dirac D matrix from an arbitrary finite geometry G and a polynomial f, then $f(D)$ for any polynomial is again of the same tri-diagonal type. This obviously is not the case for Jacobi matrices, where we have a Schroedinger operator on the graph C_n (periodic Toda lattice) or the linear path graph L_n (a scattering situation). In the Dirac deformation, we have **block tri-diagonal** matrices defined by exterior derivatives on a general geometry while isospectral deformation of higher dimensional Laplacians are in general **not possible** within differential operators. The Lax pair for Dirac matrices is also defined for any continuum geometry, like a Riemannian manifold but the deformed operators are pseudo differential operators. We consider here only the finite case.

2.8. The next observation parallels the observation known since a long time for the Toda case [18, 19]: the Toda lattice flow is related to a QR flow. A function g defines from $e^{-tg(L)} = QR$ a deformation $Q_t^* D Q_t = D_t$. This produces a solution of the Lax equation $D' = [g(D)^+ - g(D)^-, D]$. Any Lax deformation of a Dirac operator is interpolated by a QR flow.

Theorem 1. *If* $\exp(-tg(D_0)) = Q_t R_t$, *then* $D_t = Q_t^* D Q_t$ *solves* $D' = [B, D]$, *with* $B = g(D)^+ - g(D)^-$.

Proof. We verify the equivalent statement (switch t to $-t$) that $D' = [g(D)^- - g(D)^+, D]$ is realized with $D_t = Q^* D Q_t$, where $A_t = e^{tg(D_0)} = Q_t R_t$. Given a path D_t defined by $D_t = Q_t^* D Q_t$ with $A_t = e^{tg(D_0)} = Q_t R_t$. We need to show that the Q_t from this decomposition satisfies the differential equation Then $Q' = Q(g(D_t)^- - g(D_t)^+$.
If Q_t is given by the QR-decomposition, Then $Q_t' = Q_t B_t$ with anti-symmetric B_t. We need to verify that B_t is of the form $g(D)^- - g(D)^+$. The uniqueness of solutions of differential equations then assures that the QR flow and the Toda flow give the same orbits.
Since A_t is block-triangular also Q_t is block triangular as one can see when doing the Gram-Schmidt process. The key is to look at the differential equations

$$R_t' = [2g(D_t)^+ + g(D_t)^0] R_t, Q_t' = Q_t[g(D_t)^- - g(D_t)^+]$$

for the pair R, Q with $R(0) = Q(0) = 1$ coming from $e^{-0g(D)} = 1 \cdot 1$. The R_t remains upper triangular. Now differentiate $A_t = Q_t, R_t$:

$$g(D_0)A = A' = Q'R + QR' = Q_t[g(D_t)^- - g(D_t)^+]R_t + Q_t[2g(D_t)^+ + g(D_t)^0]R_t = Q_t g(D_t) R_t \; .$$

[5] Most integrable continuous time systems are equivalent to Lax pairs. An example is the oscillator $L' = [B, L]$ with L a reflection-dilation matrix and B a rotation by $\pi/2$ [7]. Also the free top $L' = [B, L]$ in $\mathbb{R}^n$ for angular momentum L and B related to angular velocity, related to L by the energy-momentum tensor.

Because $e^{tg(D_0)}$ is invertible for all t, also R_t is invertible for all t and the equation $g(D_0)Q_tR_t = Q_tg(D_t)R_t$ is equivalent to $g(D_0)Q_t = Q_tg(D_t)$. Also $Q_t^*D_0Q_t = D_t$ implies $Q_t^*g(D_0)Q_t = g(D_t)$. Having established that $Q_t' = QB_t$, we see that $D_t' = -B_tQ_t + Q_tB_t$ matching $E_t' = [g(D_t)^- - g(D_t)^+, D_t]$ Because this ordinary differential equation in the matrix algebra has global solutions $B_t = g(D_t)^- - g(D_t)^+$. [6] The statement in the theorem is the version with t replaced with $-t$. □

2.9. All these flows commute. The verification is the same as in the Toda case. See [19]. And $e^{-tg(D)}$ is independent of t if and only if $g(D) = 0$. This means that we have no equilibria for all$t$. We have to be careful however because the flow extends in the limit $t \to \infty$ to situations $D = c + c^* + m = m$, where $c = 0$ which are not geometric any more and which we do not consider to be equivalent.

2.10. The McKean Singer formula still holds also in the deformed case. The super trace any power L_t^k of $L_t = D_t^2$ is still zero implying $\text{str}(e^{-L_t}) = \chi(G)$.

2.11. The deformation can happen in over the complex field $\mathbb{C}$. To get complex solutions, look at

$$D' = [g(D)^+ - \overline{g(D)}^- + i\beta g(D)^0, D]$$

Now that at $t = 0$ and $g(D) = D$, we have $g(D)^0 = 0$ but still get an evolution. One can still associate with this with some sort of complex QR decomposition by modifying the differential equations for R, Q

$$R_t' = [2g(D_t)^+ + g(D_t)^0 - i\beta g(D)^0]R_t, Q_t' = Q_t[\overline{g(D_t)}^- - g(D_t)^+ + i\beta g(D)^0]$$

Now, $Q(t)$ is unitary and $R(t)$ block upper triangular complex. In the limit $t \to \infty$, we reach a Schrödinger wave evolution $Q_t' = im_t\beta Q_t$. This complex evolution can also be pushed to quaternions by replacing i with a unit quaternion.

2.12. With $g(D) = \log(1 + cD)$, which is defined for small c, we get $(1 + cD)^t = Q_tR_t$ which produces a discrete time evolution when restricting t to the integers. Now, take $t = m$, an integer then $(1 + cD)^m$ is a polynomial and if x is a simplex, look at $(1 + cD)^m = Q_mR_m$. It produces a deformation $D_m = Q_m^*D_0Q_m$ which has the property that properties of the geometry in distance larger than m to x do not influence the motion of D at x. We can make c time dependent for fixed t, and could look at the polynomial $(1 - tD/k)^k \to e^{-tD}$. We see that we can approximate the deformation path with an orbit of a **discrete time system** that has the **local property** simplices y of G in distance larger than some distance L are not affecting the change of entries like $D_{x,x}$

3. Discussion

3.1. Lets start with a more historical or philosophical remark. Geometers at the time of Euclid looked at symmetries like **similarities** or **congruences** of triangles or circles. Klein's Erlanger program saw symmetries in the form of **symmetry groups** defining the class of geometry under considertation. Nöther related symmetries with **conservation laws**. The special covariance principle in the form of Lorentz or Poincaré groups guided special relativity. Representations of the Poincaré group are "particles". The general covariance principle related to the diffeomorphism group of a manifold is central to general relativity. Arnold [3] noticed, motivated from the fact that the free motion of a mass point is a geodesic in the rotation group,

[6] There are global bounds on the norm of B_t and Q_t which by the way do not hold in infinite dimensions [18].

that fluid flows like the Euler equations are **geodesics in the diffeomorphism group**. Such principles suggest the following postulate: **a geometry is allowed to deform freely along geodesics in its symmetry group.** In the geometric frame work considered here, where D_t converge to block diagonal case where $c = 0$ and so $m = d + d^*$, meaning that each d preserves the set of k-forms in the limit. A consequence is that "finite geometries in general spontaneously expand when distances are measured using the electromagnetic exterior derivatives c_t".

3.2. That a geometry can float in its symmetry space is usually not spectacular. The reason is that physical properties of the geometry stay the same. If our space-time manifold would undergo a diffeomorphism change, then by the **equivalence principle**, all physical properties remain the same despite that the change of coordinates produces forces but these forces are a consequence of the equivalence principle. A rigid body released in space, if free from any external forces, will in general rotate like a free top. This is described by a Lax pair $L' = [\omega, L]$. It is integrable and produces geodesics in $SO(3)$. The probability that it does not rotate or move with respect to a reference frame is zero. One could think to go into a coordinate system where centrifugal and Coriolis forces are minimal, but such considerations are non-relativistic. The deformed Dirac matrix $D_t = d_t + d_t^* + m_t$ is a **change of coordinates** which relates it to $D_0 = d + d^*$. It is completely equivalent to the original D_0 as it is given by a linear change of coordinates. But the restriction of the Dirac matrix to its (electro-magnetic) exterior part makes the geometry to expand we we measure with electro-magnetic tools (mathematically given by exterior derivatives c_t that map k-forms to $(k+1)$-forms. The association of electro-magnetism is not that far-fetched because all distance measurements we are aware of eventually boil down to electro-magnetic forces and so the wave equation $u_{tt} = -Lu$ for the Laplacian $L = (c_t + c_t^*)^2$ of the geometry.

3.3. From a mathematical perspective, if we look at the motion of the exterior derivative c_t with $C_t = c_t + c_t^*$ alone and look the Connes formula for C_t rather than $D_t = c_t + c_t^* + m_t$, the geometric effect is visible, as the c_t shrink in general and space expands. The QR flow in general converges to a block diagonal matrix, where in the limiting case (which we never reach in finite time), we have no geometry any more. The Q which diagonalizes D is still in the symmetry group but the exterior derivative c is zero then. This is a limiting situation which the QR flow in general tends towards to. We expect a finite set of points in the symmetry group to consists of matrices $D = m$ without exterior derivative. We can such points still as geometry since the Betti numbers defined Hodge theoretically as the kernels of L_k are still the same and because we still have wave dynamics. But any Lax flow $D' = [g(D)^+ - g(D)^-, D] = 0$ does not move. We have seen above that if we allow the geometry to move in the **complex symmetry** then in the limit we still move and have a wave dynamics.

3.4. Exterior derivatives are associated to classical physics like the Maxwell equations $dF = 0, d^*F = j$ which give the electro-magnetic field $F \in l^2(G_2)$ from the charge-current $j \in l^2(G_1)$. As the diagonal part m in the Dirac matrix $D = c_t + c_t^* + m$ is not electro-magnetic, it associates with dark matter speculations. If we subscribe to the covariance principle that time evolution is just a choice of geodesic flow in the symmetry group of a finite geometry and assume that at time $t = 0$, we ahve $m_0 = 0$. During the evolution, the non-classical part m_t grows in general in time and the c_t which is used to define waves like the wave operator $u(s) = \cos((c_t + c_t^*)s)u(0)$ solving the wave equation $u_{ss} = -(c_t + c_t^*)^2 u$. Note that the solution $u(t)$ simultaneously looks at the deformation on $l^2(G)$ and especially on each form sector like **1-forms** $l^2(G_1)$ which can represent **electromagnetic potentials** $A(s) \in l^2(G_1)$, defining electromagnetic fields $F(s) = dA(s) \in l^2(G_2)$ represented as **2-forms**.

3.5. To summarize: we have seen that if a geometry given initially as $D = d + d^*$ is allowed to float freely in its symmetry, it in general expands. If we look at the electro-magnetic $c + c^*$ part of $D_t = c_t + c_t^* + m$, then what that we focus on what we can "see" using light. [7] With $c_t + c_t^*$ and neglecting m_t, we get a traditional geometry, where the exterior derivative maps k-forms to $(k+1)$-forms. But distances given by Connes formula have now expanded. Also interesting is that we can find discrete time evolutions in the symmetry which are local, meaning that signals propagate with **finite speed**. This is not the case for the wave equation $u_{tt} = -Lu$ (understanding having continuous time and discrete space). The solution $u_t = \sum_k a_k \cos(\sqrt{\lambda_k} t)\psi_k$ for the initial $u_0 = \sum_k a_k \psi_k$ decomposed in an eigen-system ψ_k uses eigenfunctions ψ_k that depend on the entire geometry. The value $u_t(x)$ can depend on $u_0(y)$ with y located arbitrarily far away. It is only in the continuum limit that we have strict finite propagation speed of the wave equation. We have once modified the wave equation for discrete operators so that we have finite propagation speed [11].

4. Code

4.1. The following Mathematica code allows to compute both the QR flow as well as the Lax deformation for an arbitrary simplicial complex G and compares how close we are. [8]

```
Generate[A_]:=If[A=={},{},Sort[Delete[Union[Sort[Flatten[Map[Subsets,A],1]]],1]]];
Whitney[s_]:=Union[Sort[Map[Sort,Generate[FindClique[s,Infinity,All]]]]]; L=Length;
sig[x_]:=Signature[x];    nu[A_]:=If[A=={},0,L[A]-MatrixRank[A]];    omega[x_]:=(-1)^(L[x]-1);
F[G_]:=Module[{l=Map[L,G]},If[G=={},{},Table[Sum[If[l[[j]]==k,1,0],{j,L[l]}],{k,Max[l]}]]];
sig[x_,y_]:=If[SubsetQ[x,y]&&(L[x]==L[y]+1),sig[Prepend[y,Complement[x,y][[1]]]]*sig[x],0];
Dirac[G_]:=Module[{f=F[G],b,d,n=L[G]},b=Prepend[Table[Sum[f[[l]],{l,k}],{k,L[f]}],0];
   d=Table[sig[G[[i]],G[[j]]],{i,n},{j,n}];{d+Transpose[d],b}];
Hodge[G_]:=Module[{Q,b,H},   {Q,b}=Dirac[G];   H=Q.Q;
   Table[Table[H[[b[[k]]+i,b[[k]]+j]],{i,b[[k+1]]-b[[k]]},{j,b[[k+1]]-b[[k]]}],{k,L[b]-1}]];
Betti[s_]:=Module[{G},If[GraphQ[s],G=Whitney[s],G=s];Map[nu,Hodge[G]]];
Fvector[A_]:=Delete[BinCounts[Map[L,A]],1];    Euler[A_]:=Sum[omega[A[[k]]],{k,L[A]}];
QR[A_]:=Module[{F,B,n,T},T=Transpose;F=T[A];n[x_]:=x/Sqrt[x.x];B={n[F[[1]]]};Do[v=F[[k]];
 u=v-Sum[(v.B[[j]])*B[[j]],{j,k-1}];B=Append[B,n[u]],{k,2,Length[F]}];{T[B],B.A}];
QRDeformation[B_,t_]:=Module[{Q,R,EE},{Q,R}=QR[MatrixExp[-t*B]];  Transpose[Q].B.Q];
PS={ColorFunctionScaling -> False};
DiracPlot[{B_,b_}]:=Module[{S1,S2,S3},n=L[B];S1=MatrixPlot[B,FrameTicks->None,PS,Frame->False];
S2=Graphics[{Thickness[0.001],Table[Line[{{0,n-b[[k]]},{n,n-b[[k]]}}],{k,L[b]}]}];
S3=Graphics[{Thickness[0.001],Table[Line[{{b[[k]],0},{b[[k]],n}}],{k,L[b]}]}];Show[{S1,S2,S3}]];
LowerT[A_]:=Table[If[i>=j,0,A[[i,j]]],{i,Length[A]},{j,Length[A[[1]]]}];
Str[B_,b_]:=Module[{},Sum[-(-1)^k*Sum[B[[b[[k]]+l,b[[k]]+l]],{l,b[[k+1]]-b[[k]]}],{k,L[b]-1}]];
RK[f_,x_,s_]:=Module[{u,v,w,q},u=s*f[x];v=s*f[x+u/2];w=s*f[x+v/2];q=s*f[x+w];x+(u+2v+2w+q)/6];
DiracDeformation[DD_,tt_]:=Module[{dt=1./10^5,d,e,B,q=DD,T}, NN=Floor[tt/dt];
   Do[d=LowerT[q];e=Transpose[d];B=e-d;FF[x_]:=B.x-x.B;q=RK[FF,1.*q,dt],{NN}];q];
ComplexDirac[DD_,tt_]:=Module[{dt=1./10^5,d,e,m,B,q=DD}, NN=Floor[tt/dt];Do[d=LowerT[q];
  e=Conjugate[Transpose[d]];m=q-d-e;B=e-d+I m;FF[x_]:=B.x-x.B;q=RK[FF,1.*q,dt],{NN}];q];
EvolvDirac[DD_]:=Module[{dt=1./10^5,d,e,m,B,q=DD,Q={}},Do[d=LowerT[q];e=Conjugate[Transpose[d]];
  m=q-d-e;B=e-d+0*I m;FF[x_]:=B.x-x.B;q=RK[FF,1.*q,dt];Q=Append[Q,Tr[m.m]],{Floor[2.0/dt]}];Q];

s=RandomGraph[{8,20}]; G=Whitney[s];{B,b}=Simplify[Dirac[G]]; n=Length[B];
V[X_]:=Max[Abs[Flatten[Table[X[[k,l]],{k,n},{l,n}]]]];
B2=DiracDeformation[B,0.12]; B1=QRDeformation[B,0.12]; Print["Difference:-",V[B2-B1]];
```

[7]Light is based on the Maxwell equations $dF = 0, d^*F = j$ assume that $F = dA$ is a 2-form and j is a 1-form. In the deformed $D_t = d_t + d_t^*$ the $d_t A$ is a mixture of 1-forms and 2-forms.

[8]Mathematica 14.1.0 produces sometimes different sign in the last diagonal entry of R in the QR decomposition. We corrected this on August 27, 2024 and programmed the QR decomposition by hand.

```
S=GraphicsGrid[{{DiracPlot[{Chop[B],b}],DiracPlot[{Chop[B1],b}]}}]

s=CompleteGraph[4]; G=Whitney[s]; {B,b}=Simplify[Dirac[G]];n=Length[B];A=EvolvDirac[B];
S=ListPlot[Table[Re[A[[k+1]]-A[[k]]],{k,Length[A]-1}],Filling->Bottom,FillingStyle->Yellow]
```

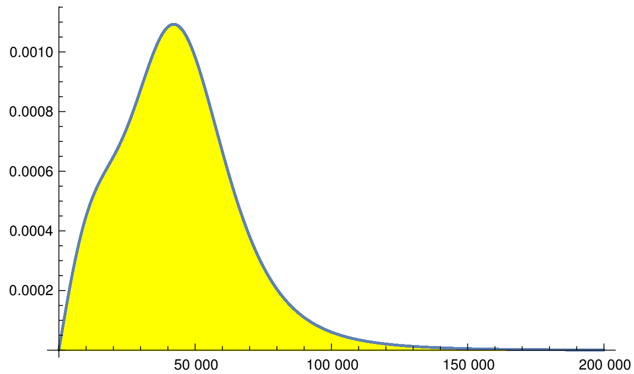

FIGURE 2. The rate of change of the norm $||m(t)||$ of the block diagonal matrix $m(t) = \oplus_{k=0}^q m_k(t)$ acting on $l^2(G) = \oplus_{k=0}^q l_2(G_k)$ shows inflation. The graph looks pretty similar for different G.

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Department of Mathematics, Harvard University, Cambridge, MA, 02138