

ON QUADRATIC ORBITAL NETWORKS

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ABSTRACT. These are some informal remarks on quadratic orbital networks over finite fields Z_p . We discuss connectivity, Euler characteristic, number of cliques, planarity, diameter and inductive dimension. We prove that for $d = 1$ generators, the Euler characteristic is always nonnegative and for $d = 2$ and large enough p the Euler characteristic is negative. While for $d = 1$, all networks are planar, we suspect that for $d \geq 2$ and large enough p , all networks are non-planar. As a consequence on bounds for the number of complete subgraphs of a fixed dimension, the inductive dimension of all these networks goes 1 as $p \rightarrow \infty$. [December 23 update: longer runs]

1. POLYNOMIAL ORBITAL NETWORKS

Given a field $R = Z_p$, we study orbital graphs $G = (V, E)$ defined by polynomials T_i which generate a monoid T acting on R . We think of (R, T) as a **dynamical system** where positive **time** T is given by the monoid of words $w = w_1 w_2 \dots w_k$ using the generators $w_k \in A = \{T_1, \dots, T_d\}$ as alphabet and where $\{T^w x \mid w \in R\}$ is the **orbit of** x . The **orbital network** [1, 3, 4] is the finite simple graph G where $V = R$ is the set of vertices and where two vertices $x, y \in V$ are connected if there exists T_i such that $T_i(x) = y$ or $T_i(y) = x$. The network generated by the system consists of the union of all orbits. As custom in dynamics, one is interested in invariant components of the system and especially forward attractors $\Omega(x)$ of a point x as well as the **garden of eden**, the set of points which are not in the image of any T_i . We are also interested in the **inductive dimension** of the network. This relates to the existence and number of cliques, which are complete subgraphs of G .

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Since most questions one can ask are already unsettled for quadratic polynomial maps, we restrict here to polynomial maps of the form $T_i(x) = x^2 + a_i$ and call the corresponding finite simple graphs **quadratic orbital networks**. These systems have been studied since a while in the case of one generator [7]. For the case $T(x) = x^2$ especially, see [8]. For a given prime p and a fixed number of generators $d \geq 1$ we have a natural probability space X_p^d of all ordered d tuples $a_1 < a_2 < \dots < a_d$ each generating an orbital network. We can now not only study properties for individual networks but the probability that some event happens and especially the asymptotic properties in the limit $p \rightarrow \infty$.

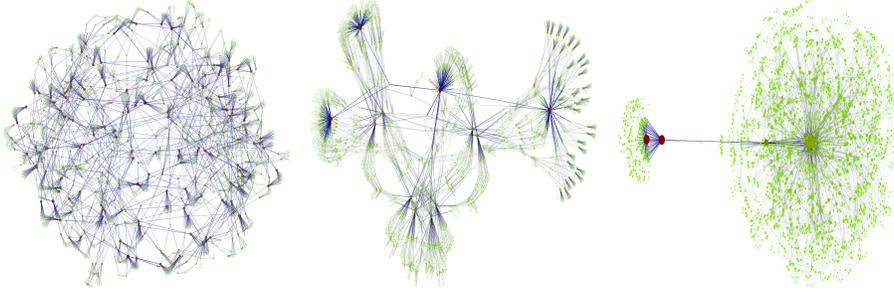


FIGURE 1. The orbital network generated by $T(x) = x^2 - 1$, $S(x) = x^2 + 1$ on Z_{2310} , on Z_{4096} and $T(x) = x^2 - 1$ and $S(x) = x^2$ on Z_{2187} . The rings Z_n are not fields. It is the smoothness of n which can lead to special structures.

Many of the questions studied here for $d = 1$ have been looked at in other contexts. The dynamical system given by $T(x) = x^2$ on the field Z_p for example was well known already to Gauss, Euler, Fermat and their contemporaries. What we call the **garden of eden** is in this case the set of **quadratic non-residues**. As Gauss knew already, half of the vertices different from 0 are there. The dynamical system has $T(x) = x^2$ has a single component if and only if p is a Fermat prime and two components $\{0\}, Z_p^*$ if and only if 2 is a **primitive root** modulo p . One believes that asymptotically 37.4... percent of the primes p have 2 as a primitive root leading to connectedness on Z_p^* . This probability is related to the **Artin constant** $\prod_p (1 - 1/(p(p-1))) = 0.3739558\dots$

We have worked with affine maps in [3]. Even very simple questions are interesting. Working in the ring Z_{p^2} for example, with linear maps

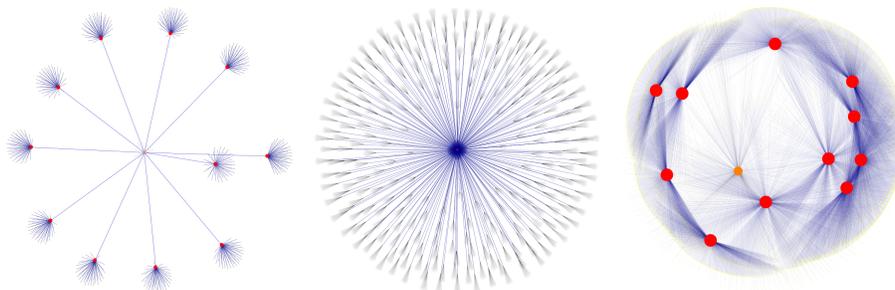


FIGURE 2. The orbital network generated by $T(x) = px^2$ on Z_{p^2} with $p = 23$ and $T(x) = p^2x^2$ on Z_{p^3} .

$T(x) = ax$ is interesting because solutions to $a^{p-1} = 1$ modulo p^2 are called **Fermat solutions** [2] There is still a lot to explore for affine maps but the complexity we see for quadratic maps is even larger.

Understanding the system $T(x) = x^2$ on the ring F_{pq} for two primes p, q is the **holy grail** of integer factorization as Fermat has known already. Finding a second nontrivial root of a number a^2 (a and $-a$ are the trivial ones) is equivalent to factorization as virtually all advanced factorization methods like the continued fraction, the Pollard rho or the quadratic sieve method make use of: if $b^2 = a^2$, then $\gcd(b - a, n)$ reveals one of the factors p or q . The fact that integer factorization is hard shows that understanding even the system $T(x) = x^2$ is difficult if n is composite. In fact, it is difficult even to find the second square root of 1. If one could, then factorization would be easy.

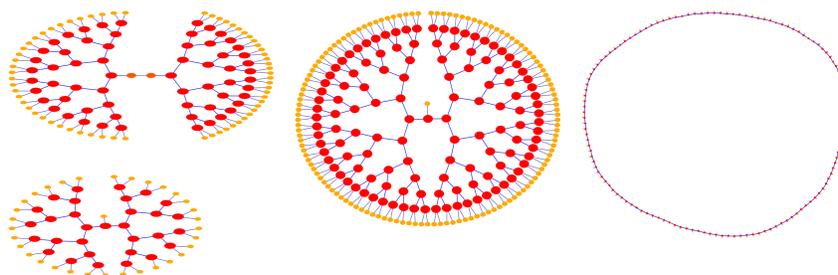


FIGURE 3. The orbital network to $T(x) = x^2$ on Z_p . In the first case, with $p = 193$, we have two connected components for $p = 257$, a Fermat prime, we have one tree. For $p = 263$, there is one ring and one isolated vertex 0. This always happens if 2 is a primitive root modulo p .

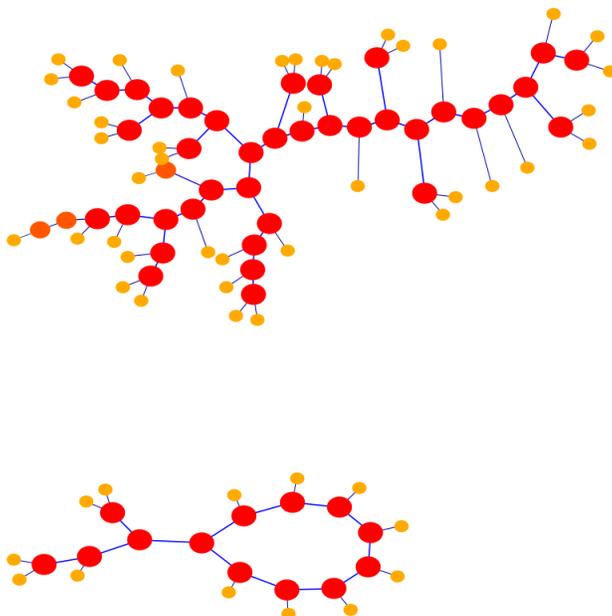


FIGURE 4. The orbital network to $T(x) = x^2 + 1$ on Z_{107} has two components. The smaller one has Euler characteristic 0 and the larger one is a tree. For any network generated by one map, every connected component contains maximally one closed loop to which many transient trees lead.

While for $d = 1$, connected quadratic orbitals become rare in the limit $p \rightarrow \infty$ and for $d = 2$, the probability goes to 1, it is difficult to find examples of disconnected quadratic orbital graphs of $d = 3$. Initial experiments made us believe that there are none. In the mean time we found one.

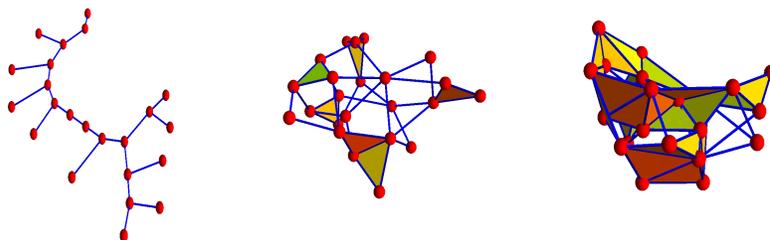


FIGURE 5. Orbital networks on Z_{23} generated by one, two or three transformations $T_k(x) = x^2 + k$.

It is still possible that there is a largest prime for which such a disconnected exist and that all quadratic graphs with 3 different generators are connected if p is large enough. Here is an example of a disconnected graph with $d = 3$:

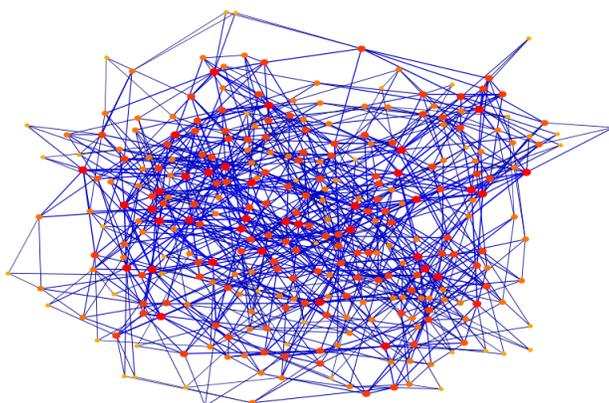


FIGURE 6. The graph $(\mathbb{Z}_{311}, x^2 + 57, x^2 + 58, x^2 + 213)$ is a quadratic graph with 3 generators over a prime field which is not connected. It consists of two different universes. One of them is small: it has a subgraph with vertices $\{77, 78, 233, 234, 79, 232\}$. It is remarkable that this "diamond" is simultaneously invariant under 3 different quadratic maps. We have not found any other one and we have so far checked until $p = 571$. Diamonds seem rare.

Question: What is the nature of these exceptions? Are there only finitely many?

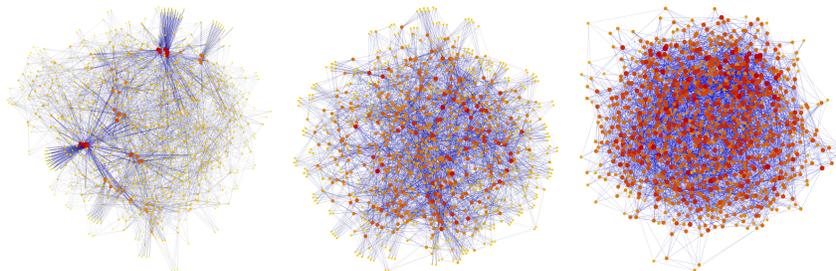


FIGURE 7. The orbital network generated by $T_k(x) = x^2 + k$ with $k = 1, \dots, 5$. on Z_{1000}, Z_{1001} and Z_{1009} . For smooth numbers we see typical rich club phenomena where a few nodes grab most of the attention and are highly connected. In the third case, where $n = 1009$ is prime, the society is more uniform.

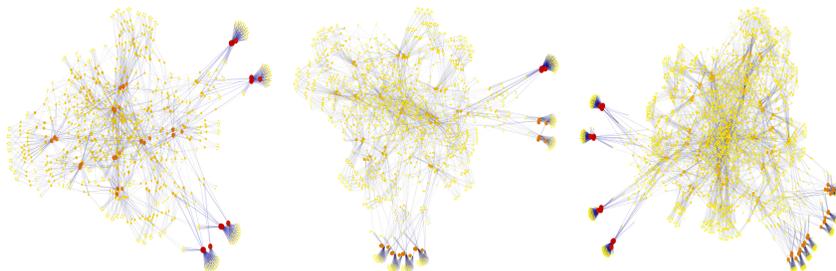


FIGURE 8. The orbital network generated by three transformations $T_k(x) = x^2 + k$ with $k = 1, \dots, 3$ on Z_{210}, Z_{211} and Z_{212} .

2. THE BRANCH GRAPH

Given a vertex x , call

$$B(x) = \{T^w x \mid w \text{ word in } T\}$$

the **branch** generated by x . It is the orbit of x under the action of the monoid T . Speaking in physics terms, it is the **future** of the vertex x because invertibility built into the monoid T produces an **arrow of time**.

Lets call the orbital graph **1-connected** if there exists $x \in R$ such that its branch is R . An orbital graph which is not connected needs several **light sources** to be illuminated completely. Define a new graph $B(G)$ on R , where two points x, y are connected, if their branches

intersect. Lets call it **k-connected**, if k is the minimal number of branches reaching R .

Lets call the orbital graph **positively connected** if for every x, y , the branches of x and y intersect. A positively connected orbital graph is connected. Is the reverse true?

No, there are counter examples as seen in Figure (9). In this case there are two **universes** or **communities** which are separated in the sense that one can not get from one to the other by applying generators. While they are connected, each community is unreachable in the future from the other community. There is however a third community which reaches both.

Obviously, G is connected if and only if $B(G)$ is connected and by definition, $B(G)$ is the complete graph if G is positively connected. Here is an orbital graph, where $B(G)$ is not the complete graph:

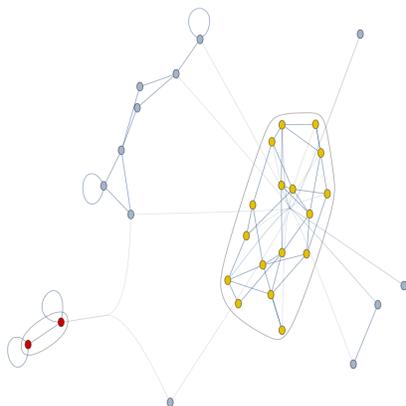


FIGURE 9. The graph on Z_{31} generated by $x^2 + 4$ and $x^2 + 7$ has the property that $B(G)$ is not connected. The branch of the point $x = 1$ has length 17 and is disjoint from the branch of the point 14 which consists only of two points 15, 17. While the graph is connected, it is not 2-connected but only 3-connected: we can not get out of the two communities by applying the quadratic maps. This situation is rather rare. In most cases, two different branches intersect or one branch already produces the entire graph.

Of course, already disconnected graphs produces examples where all branches $B(x)$ are not the entire graph.

3. EULER CHARACTERISTIC

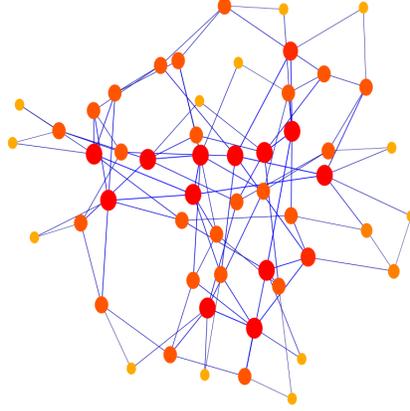


FIGURE 10. The graph $(Z_{53}, x^2 + 1, x^2 + 17)$ has minimal Euler characteristic -52 among all quadratic orbital networks on Z_{53} with two generators.

Given a graph G , denote by c_k the number of complete subgraphs K_{k+1} . The number

$$\chi(G) = \sum_{k=0}^{\infty} (-1)^k c_k$$

is called the **Euler characteristic** of G . Since in the case $d = 1$, we have no tetrahedra, the Euler characteristic is in this case given by

$$\chi(G) = v - e + f ,$$

where $v = |V| = c_0$ is the number of vertices, $e = |E| = c_1$ is the number of edges and $f = |F| = c_2$ is the number of triangles. We can also write by the Euler-Poincaré formula

$$\chi(G) = b_0 - b_1 ,$$

where b_i are the Betti numbers (for cohomology and a proof of Euler-Poincaré, see e.g [6]).

Here are some observations:

Lemma 1. *For any orbital network with $d = 1$, the Euler characteristic is nonnegative.*

Proof. The reason is that the graph comes from a directed graph which always has maximally one outgoing edge at every vertex. Every forward orbit $T^n(x)$ ends up at a unique attractor. The number of attractors

is b_1 . The number of components of the graph b_0 is clearly larger or equal than b_1 . \square

We observe that minimal Euler characteristic is constant 0 for $p > 7$ and networks $T(x) = x^2 + a$.

Lemma 2. *For $d = 2$ and p large enough, the Euler characteristic is always negative.*

Proof. There are only finitely many solutions C_1 of Diophantine equations which reduce the average degree and there are only finitely many solutions C_2 of Diophantine equations which produce triangles. The Euler characteristic is now bounded above by $p - 2p + C_1 + C_2$ which is negative for large enough p . \square

Remark. We see that for $p > 13$ the Euler characteristic is always negative.

For $d = 2$, the minimum is always very close to $-p$. For $p > 23$ we see already that the minimal Euler characteristic is always either $1 - p$ or $2 - p$. The maximum is 0 for $p = 17$ and becomes negative afterwards. For $p = 29$ for example, it is -11 . We see also the difference between the maximum and minimum seem to settle pretty much. The minimum is $-p$ or $-p + 1$ and the maximum between $23 - p$ or $25 - p$.

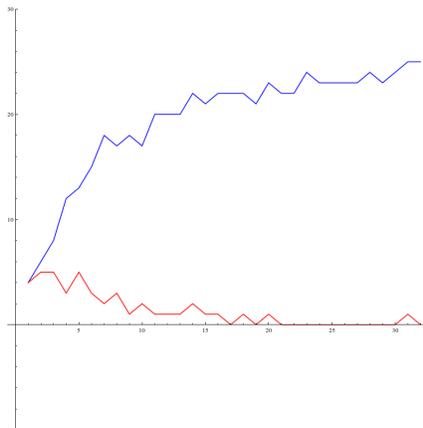


FIGURE 11. We plot $\chi(G) + p$ to illustrate the minimal and maximal Euler characteristic for all quadratic orbital graphs with two generators on Z_p as a function of primes p up to $p = 137$. The figure shows $\chi_{\min}(G(Z_p)) + p$ and $\chi_{\max}(G(Z_p)) + p$ which prompts the question whether these numbers will settle eventually or at least stay in definite intervals.

Lemma 3. *If a graph has no triangles and uniform degree 4 then the Euler characteristic is $-p$.*

Proof. The handshaking lemma telling that the average degree is $2|E|/|V|$ which means that $|E| = 2|V|$. Because there are no triangles, we know that $\chi(G) = |V| - |E| = -|V| = -p$. \square

The uniform degree and no triangle case is "generic" in the sense that it happens if there are no solutions to Diophantine equations like $T^2 = T, T^3 = Id, T^2 = Id$. The question is whether we always can find a which avoids these cases. This leads to the question:

Question: Is the minimal Euler characteristic equal to $-p$ for large p ?

Question: Is the difference between minimal and maximal Euler characteristic constant for large p ?

The minimal Euler characteristic has led for almost all larger p cases to regular graphs. They are achieved if degree reducing Diophantine equations like $T(S(x)) = x, S(T(x)) = x, T^2(x) = x, T(x) = S(x), S^2(x) = x$ have no solutions and triangle producing Diophantine equations like $TST(x) = x, STS(x) = x, S^2T(x) = x, T^2S(x) = x, TS^2(x) = x, ST^2(x) = x, S(x) = T^2(x), T(x) = S^2(x)$ have no solutions. These are all higher degree congruences. Heuristically, if each

of these K equations have no solution with probability $1/2$, then we expect for $p^2 > 2(-K)$ to have cases where none has a solution. In any case, we expect for large enough p to have cases of regular graphs with Euler characteristic $-p$. As always with Diophantine equations, this could be difficult to settle.

4. THE NUMBER OF CLIQUES

For $d = 1$ we see no tetrahedra but in general 2 triangles. This can be proven:

Lemma 4. *For $d = 1$, there can not be more than 2 triangles.*

Proof. Triangles are solutions to the Diophantine equation $T^3(x) = x$ without satisfying the Diophantine equation $T^2(x) = T(x)$ or $T^2(x) = x$. For $T(x) = x^2 + a$ we have maximally 8 solutions to

$$T^3(x) - x = a^4 + 4a^3x^2 + 2a^3 + 6a^2x^4 + 4a^2x^2 + a^2 + 4ax^6 + 2ax^4 + a + x^8 - x$$

and

$$T^2(x) - T(x) = x^4 + 5x^2 + 9.$$

There are maximally 8 solutions to the first equation. This means that we have either 1 or 2 triangles. Because triangles form cycles of T they can not be adjacent but have to consist of disjoint vertices. \square

Remarks.

- 1) With primes p larger than 19 we so far always have found a map $T(x) = x^2 + a$ on Z_p for which we have 2 triangles.
- 2) For cubic $x^3 + a$ have less or equal than $3^3/3 = 9$ triangles. For $p = 53$ and $a = 0$ we have 8 triangles. All other cases with $T(x) = x^3 + a$ on Z_p we have seen has less or equal than 2 triangles. The case $p = 53$ is special because $x^{27} - x$ has 27 solutions modulo 53 by **Fermat's little theorem**.

Lemma 5. *For $d = 1$ and prime p , there are no K_4 graphs.*

Proof. A tetrahedral subgraph would have 4 triangles. \square

Remark. Here is a second proof which works for higher order polynomials too: look at the directed graph with edges (x, y) if $T(x) = y$. The degree of each edge can not be larger than 3 and we have maximally one outgoing edge at each point. Lets look at a tetrahedron and ignore connections to it. Let $V_{\pm}(x)$ the number of out and incoming directions at a node x . We must have $4 = \sum_x V_+(x) = \sum_x V_-(x) = 8$, a contradiction showing that the tetrahedron is not possible.

For two generators $d = 2$, the questions become harder. The maximal number of triangles we have seen is 18, which happens for $p = 67$.

Lemma 6. *There are no K_6 subgraphs for $d = 3$.*

Proof. Look at such a subgraph and the directed graph. This is not possible because we have only 2 outgoing edges at each vertex. \square

Question: Can there be K_5 subgraphs in any of the spaces X_p^d ?

We have checked for all $p \leq 47$. If the answer is *no*, and for $d = 2$ there are no graphs K_5 embedded, then the Euler characteristic is $v - e + f - t$ where v is the number of vertices, e the number of edges and f the number of triangles and t the number of tetrahedra.

Question: What is the maximal number of K_4 graphs for $d = 2$?

It is very small in general. We have never seen more than 2 (which was the case for $p = 19$).

5. DIAMETER

The minimal diameter among all graphs on X_p^2 seems to be monotone in p . 3 is the largest with diameter 1, 7 the largest with minimal diameter 2, 13 the largest with diameter 3, 31 the largest with minimal diameter 4, 61 the largest with diameter 5, 127 the largest with minimal diameter 6. For 241 the largest diameter is 7. For $p = 251$ it is already 8 even so $251 < 2^8$.

While we see in experiments that diameter of a network in X_p^2 can not be smaller than $\log_2(p)$ we can only show a weaker result:

Lemma 7. *The diameter in X_p^2 can not be smaller than $\log_6(p)$.*

Proof. The maximal degree is 6. A spanning tree has diameter less than $\log_6(p)$. \square

Question: Can the diameter in X_p^2 become smaller than $\log_2(p)$?

6. PLANAR GRAPHS

We measure that for $p > 23$, quadratic graphs with two generators are not planar. We checked so far until the prime $p = 1223$.

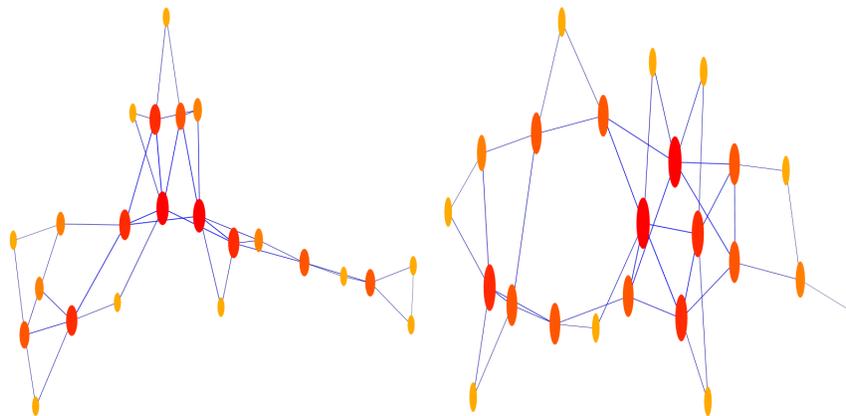


FIGURE 12. The graph $(\mathbb{Z}_{23}, x^2 + 4, x^2 + 20)$ and $(\mathbb{Z}_{23}, x^2 + 11, x^2 + 17)$ quadratic networks which are planar. They might be the largest quadratic planar graphs with two or more different generators.

The largest to us known planar quadratic graph with two generators is $p = 23$.

Question: Is there a planar quadratic orbital network in X_p^2 for $p > 23$?

7. DIMENSION

The inductive dimension for graphs [5] is formally related to the inductive Brouwer-Menger-Urysohn dimension for topological spaces. Let $S(x)$ denote the unit sphere of a vertex x . Define $\dim(\emptyset) = -1$ and inductively:

$$\dim(G) = 1 + \frac{1}{|V|} \sum_{v \in V} \dim(S(v)),$$

where $S(x)$ is the graph generated by vertices connected to x . Isolated points have dimension 0, trees with at least one edge or cycle graphs with at least 4 vertices have dimension 1, a graph K_{n+1} has dimension n .

Since there are no isolated points, the dimension of quadratic orbital graphs is always ≥ 1 with the exception of $T(x) = x^2$ on \mathbb{Z}_2 . We see that the minimum 1 is attained for all $p > 11$ and that the maximum can become larger than 2 for $p = 7$, where the maximum is $449/210$ for $a = 2, b = 3$. This seems to be the only case.

Lemma 8. *For every p, k, d there exists a constant $C = C_m(d) \leq (2d)^{m+1}$ such that the number of K_{m+1} subgraphs of $G \in X_p^d$ is smaller than C .*

Proof. The existence of a subgraph is only possible if one of finitely many systems of finitely many Diophantine equations consisting of polynomials has a solution. Given Z_n and the degree of the generating polynomials, the constant d gives an upper bound for the degree of polynomials involved. Since a polynomial f in Z_n has less or equal solutions than the degree and the degree of the polynomials involved is less or equal to 2^{m+1} and there are less than d^{m+1} polynomials which can occur. \square

Of course, this applies for any polynomial map. For non-prime p we can have more solutions. But since the number $A(n)$ of solutions to a polynomial equation $f(x) = 0 \pmod n$ is multiplicative (Theorem 8.1 in [2]), the following result should be true for all n . We prove it only for primes:

Lemma 9. *For all quadratic networks the dimension goes to 1 for $n \rightarrow \infty$ along primes.*

Proof. We have seen that for fixed n and d quadratic polynomial maps there is a bound $C_m(n, d)$ such that the quadratic orbital graph on Z_n has less or equal C_m sub graphs K_{m+1} . This means that the dimension of a vertex can be m only for a bounded number of vertices. Since for $n \rightarrow \infty$ the number of vertices goes to infinity, the inductive dimension goes to zero. \square

A modeling question is:

Question: How large does d and p have to be chosen in order to get an orbital network of given dimension?

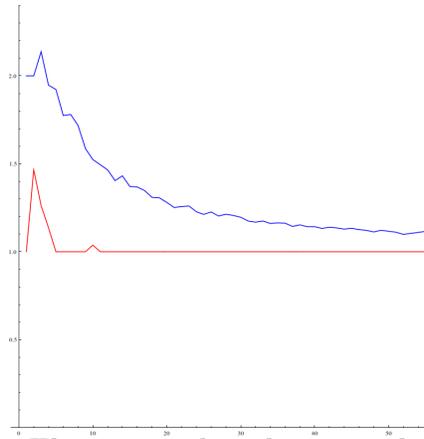


FIGURE 13. The minimal and maximal dimension of quadratic orbital graphs with two generators on Z_p for $p \leq 239$.

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