ON THE ARITHMETIC OF GRAPHS

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ABSTRACT. The Zykov ring of signed finite simple graphs with topological join as addition and compatible multiplication is an integral domain but not a unique factorization domain. We know that because by taking graph complements, it becomes isomorphic to the strong Sabidussi ring with disjoint union as addition. We prove that the Euler characteristic is a ring homomorphism from the strong ring to the integers by demonstrating that the strong ring is homotopic to a Stanley-Reisner Cartesian ring. More generally, the Kuenneth formula holds on the strong ring so that the Poincaré polynomial is compatible with the ring structure. The Zykov ring has the clique number as a ring homomorphism. Furthermore, the Cartesian ring has the property that the functor which attaches to a graph the spectrum of its connection Laplacian is multiplicative. The reason is that the connection Laplacians do tensor under multiplication, similarly to what the adjacency matrix does for the weak ring. The strong ring product of two graphs contains both the weak and direct product graphs as subgraphs. The Zykov, Sabidussi or Stanley-Reisner rings are so manifestations of a network arithmetic which has remarkable cohomological properties, dimension and spectral compatibility but where arithmetic questions like the complexity of detecting primes or factoring are not yet studied well. We illustrate the Zykov arithmetic with examples, especially from the subring generated by point graphs which contains spheres, stars or complete bipartite graphs. While things are formulated in the language of graph theory, all constructions generalize to the larger category of finite abstract simplicial complexes.

1. EXTENDED SUMMARY

1.1. Finite simple graphs extend to a class $\mathcal{G}$ of signed finite simple graphs which carry three important commutative ring structures: the **weak ring** $(\mathcal{G}, \oplus, \square, 0, 1)$, the **direct ring** $(\mathcal{G}, \oplus, \otimes, 0, 1)$ and the **strong ring** $(\mathcal{G}, \oplus, \boxtimes, 0, 1)$ [18, 6, 3]. In all three cases, the disjoint union $\oplus$ is the addition, the empty graph the zero element and the

one point graph K_1 the one element. The weak ring product $\square$ is the Cartesian product for graphs which corresponds to the tensor product of adjacency matrices, the direct ring product $\otimes$ is also known as the **tensor product** of graphs. The strong product $\boxtimes$ of Sabidussi combines edge sets of the other two. In each case, taking graph complements produces dual rings in which the addition is the Zykov join $+$ which corresponds to the join in topology, and which preserves the class of spheres. We first observe that the dual to the Zykov ring introduced in [14] is the strong ring so that also the Zykov ring was already known. The **Sabidussi unique prime factorization theorem for connected graphs** and the Imrich-Klavzar examples of non-unique prime factorization in the disconnected case or in general for the direct product and are so inherited and especially hold for the Zykov ring which is therefore, like the Sabidussi ring, an **integral domain** but **not a unique factorization domain**.

1.2. The **clique number** from $\mathcal{G} \to \mathbb{Z}$ assigning to a graph G the number $\dim(G) + 1$, where dim is the maximal dimension of a simplex in G, extends to a ring homomorphism from the Zykov ring to the integers. Inherited from its dual, also the Zykov addition, the join, has a unique additive prime factorization, where the primes are the graphs for which the dual is connected. We observe that the **Euler characteristic** χ is a ring homomorphism from the strong ring to the integers and that the **Kuenneth formula** holds: the map from the graph to its **Poincaré polynomial** $p(G)(x) = \sum_{k=0}^{\infty} b_k x^k$ is a ring homomorphism $p(G + H) = p(G) + p(H)$ and $p(G \boxtimes H) = p(G)p(H)$. To do so, we observe that the strong product graph $G \boxtimes H$ is homotopic to the graph product $G \times H$ treated in [10] and which is essentially the Stanley-Reisner ring product when written down algebraically.

1.3. We also note that the tensor product of the connection Laplacians of two graphs is the connection Laplacian L of this Stanley-Reisner product $\times$, confirming so that the **energy theorem** equating the Euler characteristic $\chi(G)$ with the sum $\sum_{x,y} g(x, x)$ of the **Green function** matrix entries of $g = L^{-1}$ in the case of simplicial complexes, extends to the full Stanley-Reisner ring generated by complexes. It follows that the spectra of the connection Laplacians $L(G)$ of a complex G satisfy $\sigma(L(G) + L(H)) = \sigma(L(G)) \cup \sigma(L(H))$ and $\sigma(L(G \times H)) = \sigma(L(G))\sigma(L(H))$. So, not only the potential theoretical energy, but also the individual energy spectral values are compatible with the arithmetic. The Zykov ring $(\mathcal{G}, +, \cdot, 0, 1)$ relates so with other

rings sharing so extraordinary topological, homological, potential theoretical and spectral properties modulo duality or homotopy.

1.4. This was added June 20: we missed one important property which is not algebraic but graph theoretic. After consulting more with literature: [1, 19, 2], we have not stated yet one property known for the strong product: the Fredholm adjacency matrices tensor. This is independent of our observation that the Fredholm connection matrices tensor if we take the Cartesian (Stanley-Reisner) product. But this can be combined and leads to $(G \times H)' = G' \boxtimes H'$, where G' is the connection graph of G. This is now a new graph theoretical statement not involving any matrices. But it is nice as $G \times H$ is not a graph, only a CW complex. We see that when going to connection graphs, the Cartesian product becomes the strong product.

2. Graph arithmetic

2.1. Let $\mathcal{G}_0$ denote the category of finite simple graphs $G = (V, E)$. With the disjoint union $\oplus$ as addition, the **weak product** $\square$, the **direct product** $\otimes$ and the **strong product** $\boxtimes$ produce new graphs with vertex set $V(G) \times V(H)$ [18]. Let $(\mathcal{G}, \oplus)$ be the group generated by the monoid $(\mathcal{G}_0, \oplus)$. The tensor product is also called direct product and the weak product the graph product. The definitions of the products are specified by giving the edge sets. We have $E(G \square H) = \{((a, b), (c, d)) | a = c, (b, d) \in E(H)\} \cup \{((a, b), (c, d)), b = d, (a, c) \in E(G)\}$ and $E(G \otimes H) = \{((a, b), (c, d)) | (a, c) \in E(G), (b, d) \in E(H)\}$ and $E(G \boxtimes H) = E(G \square H) \cup E(G \otimes H)$. The unit element in all three monoids is the one point graph K_1 called 1. A graph $G \in \mathcal{G}$ is **prime** in a ring if they are only divisible by 1 and itself. All three products have a prime factorization but no unique prime factorization [6]. All three products belong to a ring if the addition $\oplus$ is the disjoint union Grothendieck augmented to a group by identifying $A \ominus B \sim C \ominus D$ if $A \oplus C \oplus K = B \oplus D \oplus K$ for some K. Since we will see that unique additive prime factorization holds in $(\mathcal{G}, \oplus)$, the additive primes being the connected graphs, one can simplify this and write every element in the ring as an ordered pair (A, B) of graphs and simply write also $A \ominus B$ or $A - B$ and multiply the usual way like for example $(A - B) \boxtimes (C - D) = (A \boxtimes C) \oplus (B \boxtimes D) \ominus (A \boxtimes D) \ominus (B \boxtimes C)$.

2.2. The weak product seems first have appeared in the Principia Mathematica of Whitehead and Russell. The three products appeared together in the pioneering paper [18], where they were first mathematically studied. In [14], we took the Zykov join [24] $(V, E) + (W, F) =$

$(V \cup W, E \cup F \cup VW)$, where VW denotes all edges connecting any V to any W as a "sum" and constructed a compatible product $G \cdot H = (V \times W, \{((a,b),(c,d)) \mid (a,c) \in E(G) \text{ or } (b,d) \in E(H)\})$. Also $(\mathcal{G}, +, \cdot)$ produces a ring. Finally, there is a Cartesian product $\times$ on graphs which satisfies the Kuenneth formula [10]: in that product, the vertex set is $V(G_1) \times V(H_1)$, the product of the vertex sets of the Barycentric refinements G_1, H_1 of G, H and two vertices are connected if one product simplex is contained in the other. It is a ring when extended to CW-complexes or interpreted as the product in the Stanley-Reisner ring. All three rings $(\mathcal{G}, \oplus, \square, 0, 1), (\mathcal{G}, \oplus, \otimes, 0, 1), (\mathcal{G}, \oplus, \boxtimes, 0, 1)$ have the property that they are defined in $\mathcal{G}$ and that the addition of two graphs (V, E) and (W, F) has the vertex set $V \cup W$ and that the product in the ring has the vertex set $V \times W = \{(v, w) \mid v \in V(G), w \in V(H)\}$.

2.3. The theory of graph products is a rich field. The main sources for this topic [6, 3] show this. The additive operations in these works are usually the disjoint union. The mirror operation appears in the handbook [3] who mention also that there are exactly 20 associative products on graphs. In this sense, the ring considered in [14] is not new. Maybe because it is just the dual to the strong product, the Zykov product $\cdot$ appears not have been studied much in graph theory. [4] attribute the construction of the Zykov sum $+$ to a paper of 1949 [24]. It has the same properties than the join in topology which is covered in textbooks like [17, 5]. The join especially preserves spheres, graphs which have the property that all unit spheres are spheres and where removing one vertex produces a contractible graph. Examples of such Evako spheres are $P_2 + P_2 = S_4$ or $S_4 + S_4$ which is a 3-sphere. The empty graph is considered the -1-sphere and feeds the rest of the inductive definition of spheres.

2.4. We will look in more detail at the join monoid $(\mathcal{G}, +)$ which is the additive submonoid of the additive group in the Zykov ring. About spectra of the Kirchhoff Laplacian, we know already from [14] that the sum of the vertex cardinalities $|V(G)| + |V(H)|$ is an eigenvalue of $G + H$ so that $nG = K_n \cdot G = G + G + \cdots + G$ has an eigenvalue $n|V(G)|$ of multiplicity $n - 1$. We also proved about the **ground state**, the smallest non-zero eigenvalue λ_2 of a graph that $\lambda_2(G+H) = \min(|V(H)|, |V(K)|) + \min(\lambda_2(G), \lambda_2(H)$. Furthermore, the eigenvalues of the **Volume Laplacian** H_n if $H = (d + d^*)^2 = H_1 \oplus \ldots \oplus H_n$ is the **Hodge Laplacian** defined by the **incidence matrices** d have the property that $\sigma_V(G) + \sigma_V(H) = \sigma_V(G + H)$. Already well known is $\sigma(G) + \sigma(H) = \sigma(G \boxtimes H)$ [3]. In some sense, this volume Laplacian result about the join is dual to the fact that the Kirchhoff Laplacian

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L_0 (which can be seen as the Poincaré dual to the Volume Laplacian). The disjoint union $\oplus$ satisfies $\sigma(G) \cup \sigma(H) = \sigma(G \oplus H)$. The tensor and strong products have no obvious spectral properties. But we will see below that if we look at the spectrum of the **connection Laplacian**, an operator naturally appearing for simplicial complexes, then the spectrum behaves nicely for the Cartesian product of finite abstract simplicial complexes. This product corresponds to the multiplication in the Stanley-Reisner ring. The product $G \times H$ is not an abstract simplicial complex any more, which is the reason to take its Barycentric refinement, which is the Whitney complex of a graph. But as the product is homotopic to the strong product, we can stay within the category of graphs or simplicial complexes.

2.5. Here is an overview over the definitions of the three rings with $\oplus$ as addition:

Ring	Addition
Weak	$G \oplus H = (V \cup W, E \cup F)$
Tensor	$G \oplus H = (V \cup W, E \cup F)$
Strong	$G \oplus H = (V \cup W, E \cup F)$

Ring	Multiplication
Weak	$G \square H = (V \times W, \{((a,b),(a,d))\} \cup \{((a,b),(c,b))\})$
Tensor	$G \otimes H = (V \times W, \{((a,b),(c,d)) \mid (a,c) \in E \text{ and } (b,d) \in F\})$
Strong	$G \boxtimes H = (V \times W, E(G\square H) \cup E(G \otimes H)$

We will relate the strong product $\boxtimes$ to its dual product, the Zykov product $\cdot$ in which the Zykov join $+$ is the addition and then deform the multiplication from the strong to the Cartesian product (which however does not define a ring on $\mathcal{G}$ as associativity got removed by pushing the product back to a graph using the Barycentric refinement).

Ring	Addition
Zykov	$G + H = (V \cup W, E \cup F \cup VW)$
Cartesian	$G \oplus H = (V \cup W, E \cup F)$

Ring	Multiplication
Zykov	$G \cdot H = (V \times W, \{((a,b),(c,d)) | (a,c) \in E \text{ or } (b,d) \in F\})$
Cartesian	$(G \times H)_1 = (c(G) \times c(H), \{(a,b) | a \subset b \text{ or } b \subset a\})$

2.6. The next two figures illustrating the five mentioned products:
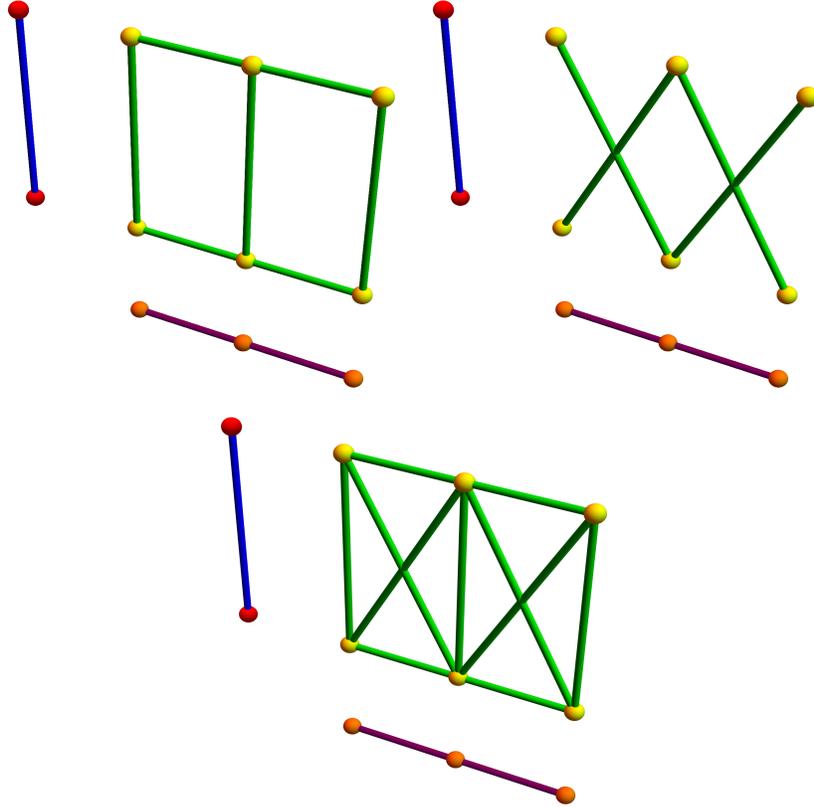

FIGURE 1. The multiplication of K_2 with L_3 in the weak, direct and strong rings.
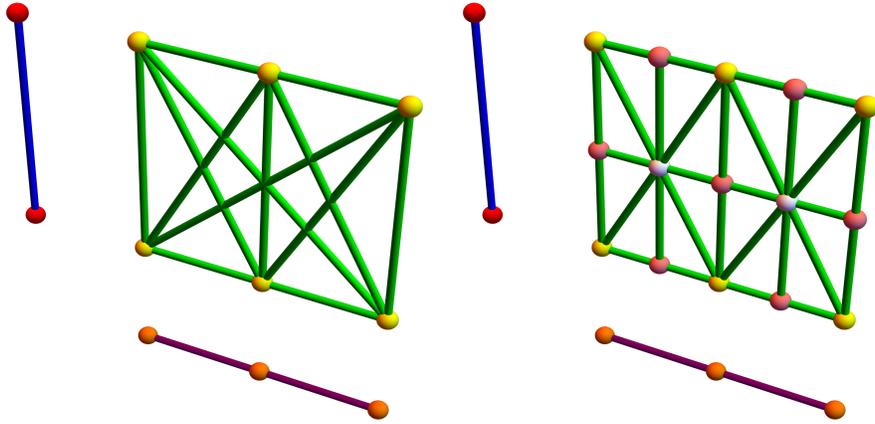
FIGURE 2. The multiplication of K_2 with L_3 in the Zykov and Cartesian simplex product.
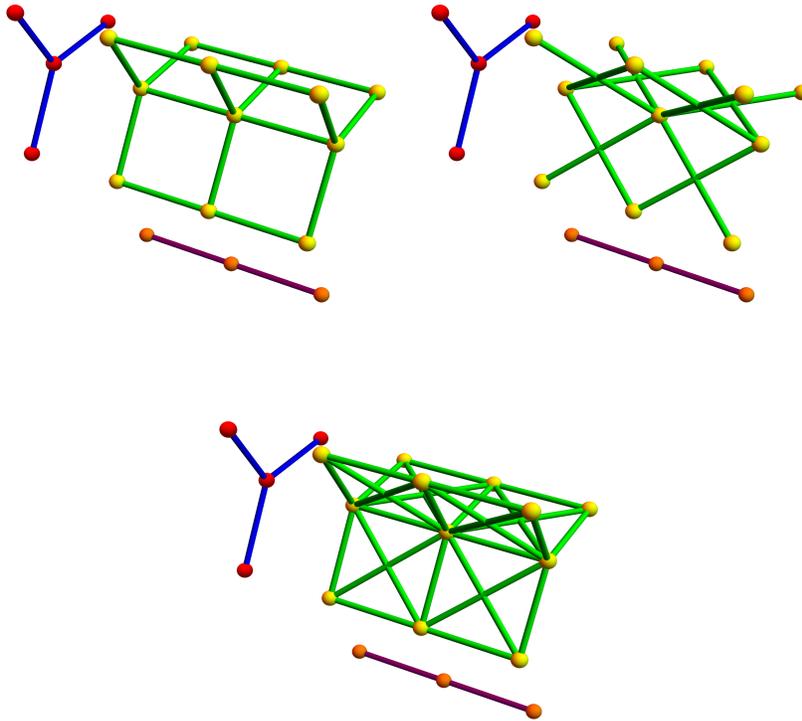

FIGURE 3. The multiplication of L_3 with S_3 is shown in the graph product, the tensor product and then the strong product.
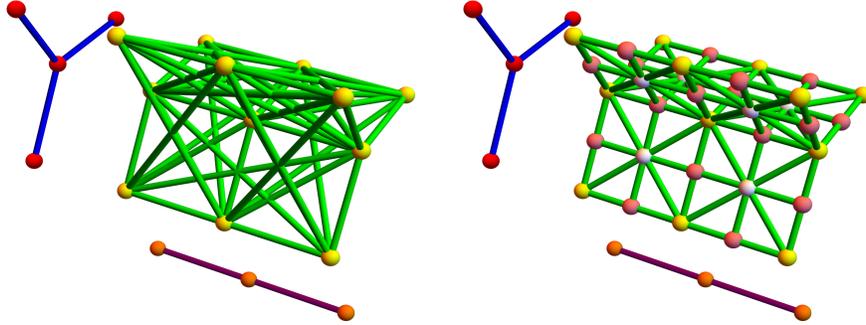
FIGURE 4. The multiplication of L_3 with S_3 is shown in the Zykov and Cartesian simplex product.
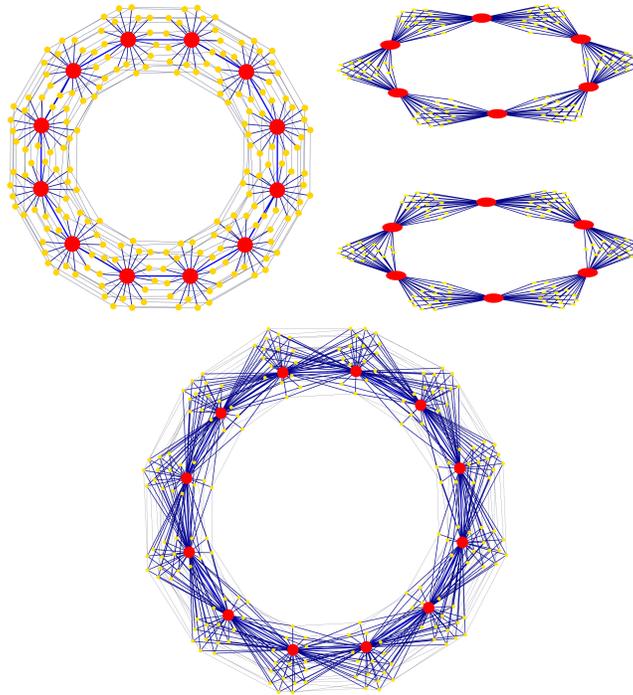

FIGURE 5. The weak, direct and strong products of the circular graph C_{12} and the star graph S_{17}. The direct product (tensor product) is not connected. It is merged with the first weak product to become the strong product. For more sophisticated visualizations, see [21].

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3. Properties

3.1. In the case of the disjoint union $\oplus$, the graphs $P_n = \{(1, 2, \ldots, n), \emptyset\}$ form a sub-monoid which extends to a subgroup which is isomorphic to the integers $\mathbb{Z}$. This is the stone age **pebble addition** which was used before cuneiforms appeared. The fact that the zero element $0 = \{\emptyset, \emptyset\}$ and the negative numbers $-G$ have been introduced much later in mathematics is paralleled in graph theory: while the empty graph is used quite frequently as a zero element, negative graphs are rarely discussed. **Valuations**, maps X from $\mathcal{G}$ to the real numbers satisfying $X(G \cup H) + X(G \cap H) = X(G) + X(H)$ obviously have to be extended in a compatible way. By the discrete Hadwiger theorem [7] it is enough to look v_k which form a basis. One defines $v_k(G - H) = v_k(H) - v_k(H)$ where $v_k(G)$ is the number of k-dimensional simplices in G. The Euler characteristic is then still $\sum_{k=0}^{\infty} (-1)^k v_k(G)$. This extension can only be done in the case of the addition $\oplus$. In the Zykov addition $+$, the generating functions $f(x) = 1 + \sum_{k=1}^{\infty} v_{k-1} x^k$ has the property that $f_{G+H} = f_G f_H$ (a product and not a sum) which means we would formally would have to require $f_{G-H} = f_G/f_H$ to extend the generating function to a group homomorphism from the Grothendieck group to the rational functions. Still we can still represent group elements in the additive Zykov groups either as ordered pairs (G, H) or $G - H$ using some equivalence relation $A - B = C - D$ if $A + D = B + D$.

3.2. In the case of the Zykov join operation $+$, the set of **complete graphs** K_n plays the role of the integers, where the negative numbers are just written as $-K_n$. We will see in a moment that one can also write any element in the additive Zykov group uniquely as $G - H$, where G, H are graphs. From the fact that $K_n + K_m = K_{n+m}$ and $K_n \cdot K_m = K_{nm}$ (especially postulating $K_{-1} \cdot K_{-1} = K_1$), we see immediately that the **clique number** $n = d + 1$ giving the largest n for which K_n is a subgraph of G is a ring homomorphism:

Proposition 1 (Clique number as ring homomorphism of Zykov ring). *For the Zykov ring, the clique number $c(G) = \dim(G) + 1$ is a ring homomorphisms.*

Proof. We have $c(0) = c(\emptyset) = 0$ as the dimension of the empty graph is -1. Also $c(1) = c(K_1) = 1$. It follows from $K_n + K_m = K_{n+m}$ that $c(G+H) = c(G) + c(H)$ and from $K_n \cdot K_m$ that $c(G \times H) = c(G)c(H)$. To extend this to the entire ring, we have to postulate $c(-G) = -c(G)$ but the definition of the group from the monoid assures that this extends to the additive group and so also to the ring. $\square$

3.3. The **Euler characteristic** of a graph G is defined as $\sum_{k=0}^{\infty}(-1)^k v_k$, where v_k is the number of k-dimensional complete subgraphs K_{k+1} in G. It can also be written as the sum $\sum_x (-1)^{\dim(x)}$ over all simplices x (complete subgraphs) in G. The Euler-Poincaré identity tells that $\chi(G) = \sum_{k=0}^{\infty}(-1)^k b_k$, where $b_k = \dim H^k(G)$ are the Betti numbers. They can be easily computed as the nullity $\dim(\ker(L_k))$, where L_k is the k'th block in the Hodge Laplacian $H = (d+d^*)^2$. It follows from the Künneth formula $H^k(G \times H) = \oplus_{i+j=k} H^i(G) \otimes H^j(G)$ that the Poincaré polynomial $p_G(x) = \sum_{k=0} b_k x^k$ satisfies $p_{G \times H}(x) = p_G(x) p_H(y)$ so that $\chi(G) = p_G(-1)$ satisfies $\chi(G \times H) = \chi(G)\chi(H)$. One can also give a direct inductive proof of the product property of χ without invoking cohomology using Poincaré-Hopf [10].

3.4. The homotopy theory of graphs and finite abstract simplicial complexes is parallel to the homotopy of geometric realizations but is entirely combinatorial. The adaptation of the Whitehead definition to the discrete have been done in the 70ies, notably by combinatorial topologists like Evako and Fisk. First define inductively what a **collapsible** graph is: a graph G is collapsible if there exists a vertex x for which the unit sphere $S(x)$ and $G \setminus x$ is collapsible. A **homotopy step** is the process of removing a vertex with contractible unit sphere or then making a cone extension over a collapsible subgraph. A graph is **contractible** if it is homotopic to K_1. The **homotopy** of abstract simplicial complexes G can then be defined through the homotopy of its Barycentric refinement G_1 which is the Whitney complex of a graph and therefore part of the graph theoretical contraction definition. The discrete description has the advantage that it can be implemented easier on a computer.

Lemma 1 (Homotopy lemma)**.** *For any finite $H, G \in \mathcal{G}$, the strong product graph $H \boxtimes G$ is homotopic to the Cartesian simplex product $H \times G$.*

Proof. The graph $H \boxtimes G$ is homotopic to its Barycentric refinement $(H \boxtimes G)_1$. Now deform each maximal simplex $x \boxtimes y$ to a simplex $x \times y$ starting with one dimensional simplices, then turning to triangles etc. These Whitehead deformation moves can best be seen in an Euclidean embedding but they can be done entirely combinatorially: first add a new vertex m in the center of $x \boxtimes y$ connecting with all vertices of the week product $x \square y$ and all interior vertices as well as all vertices connecdted to those interior vertices. Now remove all interior vertices together with their connections. We end up with $x \times y$. After doing

this for all $x \times y$ of dimension d, continue with dimension $d+1$ etc until everything is deformed. $\square$

Proposition 2 (Euler characteristic as ring homomorphism from the strong product). *Euler characteristic is a ring homomorphisms from the Zykov ring $(\mathcal{G}, \oplus, \otimes)$ to $\mathbb{Z}$.*

Proof. The Euler characteristic is a homomorphism for the Cartesian simplex ring to the integers [10]. The argument there was to factor $\chi(G \times H) = \sum_{(x,y) \subset G \times H}(-1)^{\dim(x)+\dim(y)}$ as $(\sum_{x \in G}(-1)^{dim(x)})(\sum_{y \in H}(-1)^{dim(y)})$ so so that $\chi(G \times H) = \chi(G_1)\chi(H_1)$, where G_1 and H_1 are the Barycentric refinements of G and H. But the formula $\sum_x(-1)^{\dim(x)}$ defining the Euler characteristic of G is a Poincaré-Hopf formula for the Morse function $f(x) = \dim(x)$ on the vertex set of the Barycentric refinement G_1. $\square$

Remarks.
1) Already small examples show that the other products, the weak and direct products have no compatibility whatsoever with Euler characteristic. An example is $K_2 \boxtimes K_2$.
2) The **Wu characteristic** [23, 11],

$$\omega(G) = \sum_{x \sim y} \omega(x)\omega(y)$$

with $\omega(x) = (-1)^{\dim(x)}$, where the sum is over all intersecting simplices x, y is not a homotopy invariant. We know that $\omega(G \times H) = \omega(G)\omega(H)$ for the Cartesian product, but the multiplicativity fails in general for all other products.

3.5. Here is a summary about properties

Operation	Clique number	Euler characteristic	index
Union $\oplus$	-	additive	
Join $+$	additive		multiplicative

For the graph Cartesian product $\times$ or the tensor product $\otimes$ only the already mentioned additivity of Euler characteristic for disjoint union or the trivial max-plus property of dimension for disjoint addition holds.

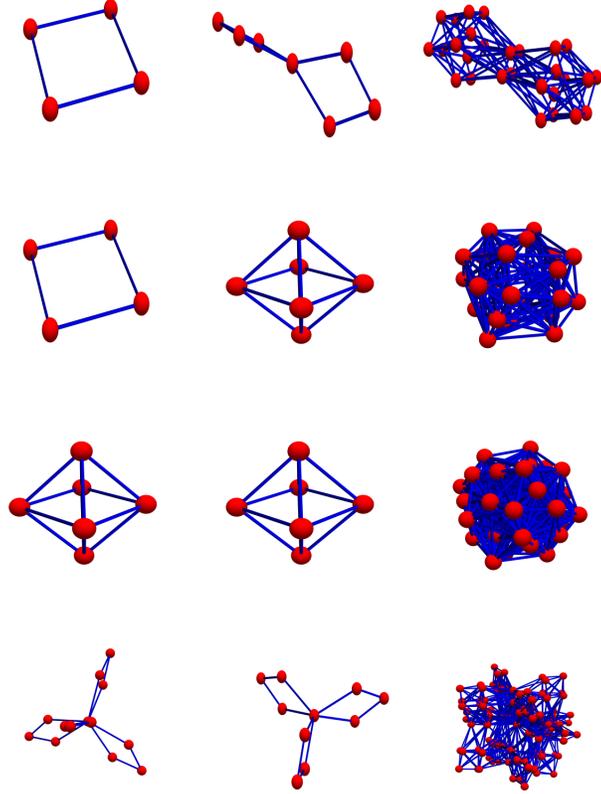| Operation	maximal dimension	Clique number	Euler characteristic
$\cdot$	-	multiplicative	
$\times$	additive	-	multiplicative
$\boxtimes$	additive	-	multiplicative
$\otimes$	-	-	-
$\square$	-	-	-

FIGURE 6. Let B_k is a bouquet with k flowers and O is the octahedron. We see first the strong product of C_4, B_2 where the Poincaré polynomial identity is $(1 + x)(1 + 2x) = 1 + 3x + 2x^2$, then the strong product of C_4, O with $(1+x)(1+x^2) = 1+x+x^2+x^3$, then the strong product of O, O with $(1+x^2)(1+x^2) = 1+2x^2+x^4$ and finally the strong product B_2, B_3 with $(1 + 3x)(1 + 4x) = 1 + 7x + 12x^2$. In the last case, the f-vector is $(130, 700, 768, 192)$ and the Euler characteristic $130 - 700 + 768 - 192 = 6$ which matches $b_0 - b_1 + b_2 = 1 - 7 + 12 = 6$ as the general Euler-Poincaré formula shows.

3.6. Small examples:

1) $K_2 \oplus K_2 = P_2 \times P_2$
2) $K_2 + K_2 = K_4$
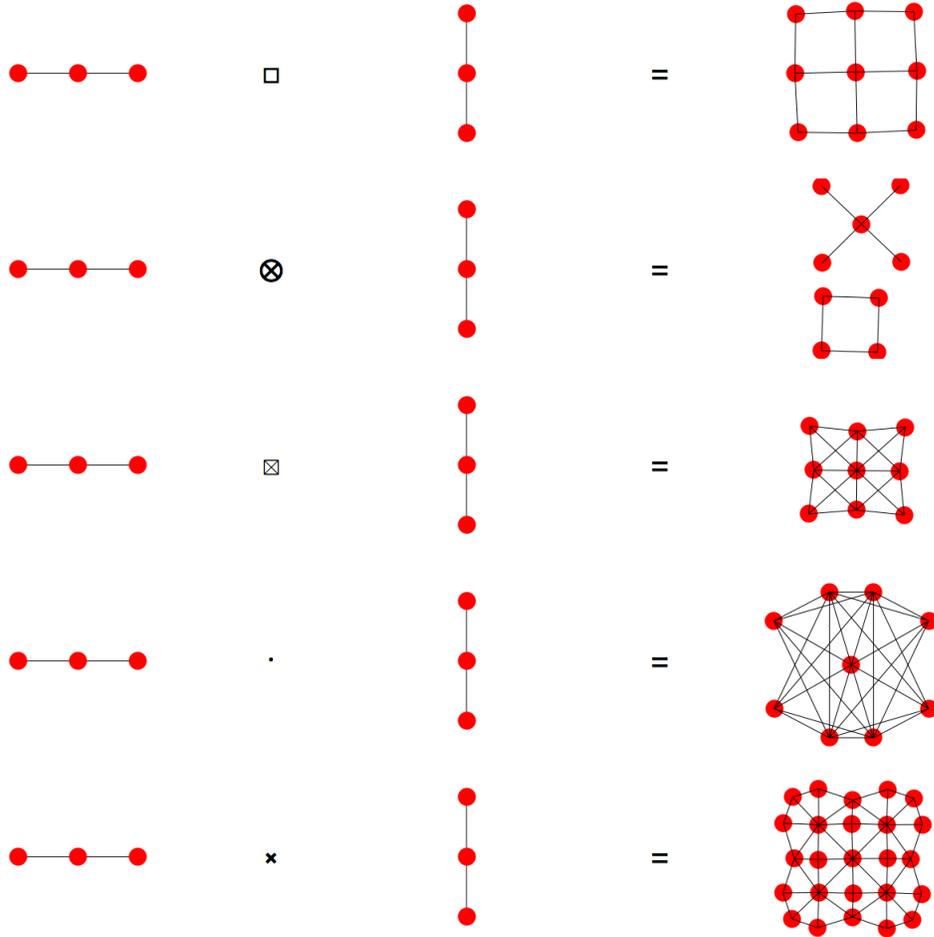**3)** $K_2 \cdot K_2 = K_4$

FIGURE 7. The five multiplications of L_3 with L_3. Only in the Cartesian ring case do we have a topological disk of dimension 2 and Euler characteristic 1. But as $G \times H$ is a homotopic to Barycentric refinement of the strong product, we have also Euler characteristic multiplicative.

4) $K_2 \otimes K_2 = C_4$
5) $K_2 \times K_2 = W_6$
6) $K_2 \square K_2 = C_4$
7) $K_2 \boxtimes K_2 = K_4$
1) $K_2 \oplus K_3$
2) $K_2 + K_3 = K_5$
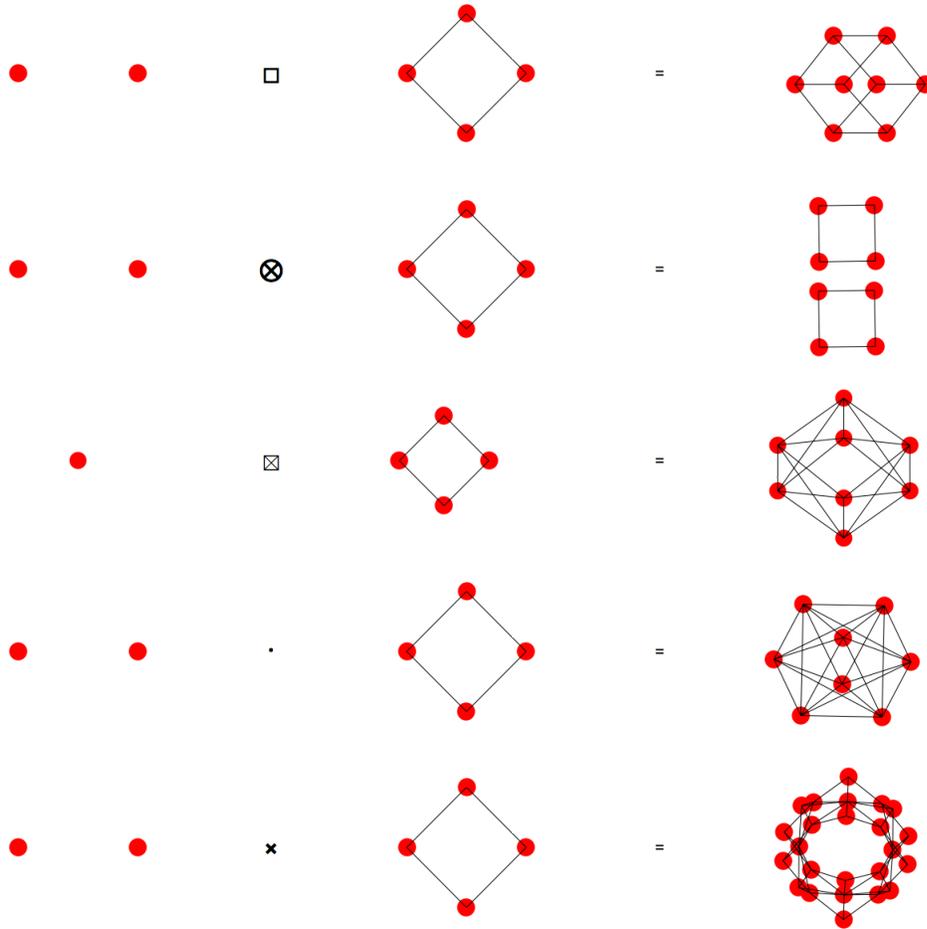**3)** $K_2 \cdot K_3 = C_6$
4) $K_2 \otimes K_3$

FIGURE 8. The five multiplications of K_2 with C_4. Only in the Cartesian product case, we see a cylinder of dimension 2 and Euler characteristic 0. The strong product is also a cylinder but has larger dimension.

5) $K_2 \times K_3$
6) $K_2 \square K_3$
7) $K_2 \boxtimes K_3 = K_6$

3.7. There are two classes of subgraphs which can play the role the integers. One is the class P_n of graphs without edges if the addition is the disjoint union $\oplus$, the second is the class K_n of complete graphs if the addition is the join $+$. The ring homomorphisms are the obvious maps $n \to P_n$ or $n \to K_n$. Lets call the first class of graphs $\mathbb{Z}_P$ and the second class of graphs $\mathbb{Z}_K$.
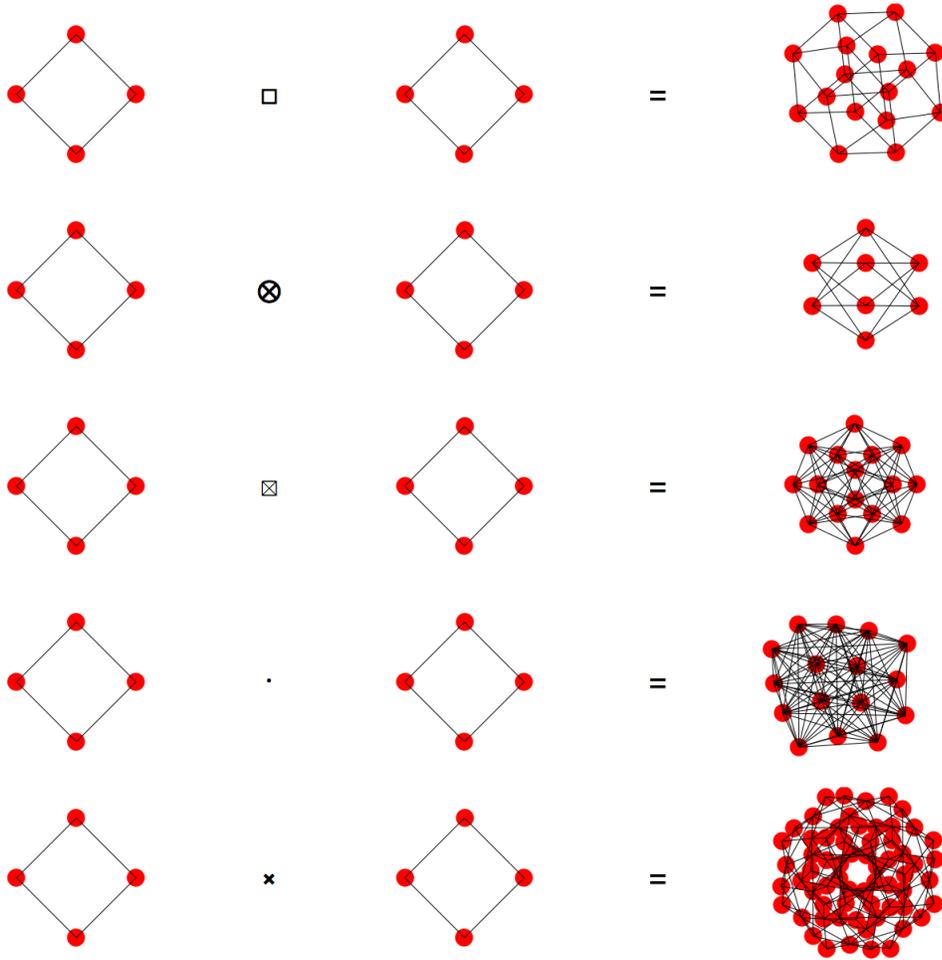
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FIGURE 9. The multiplication of C_4 with C_4 in the four rings. Only in the Grothendieck case do we see a torus of dimension 2 and Euler characteristic 0.

Proposition 3. *The weak, tensor and the strong rings contain the subring $\mathbb{Z}_P$. Their dual rings and in particular the Zykov ring contains the subring $\mathbb{Z}_K$.*

3.8. The upshot is that the Zkyov ring has a compatible extension of the dimension functional and that its dual has a compatible extension of Euler characteristic. The Zykov ring also has mixed additive-multiplicative compatibility with Euler characteristic. The reduced Euler characteristic of the sum is the product of the reduced Euler characteristics of the summands. The dual is compatible with cohomology but also has (when suitably deformed) a mixed multiplicative-additive
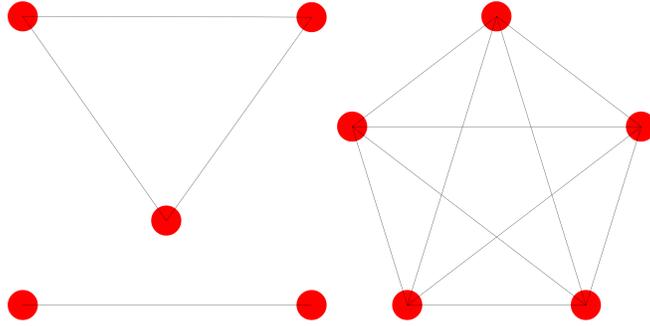
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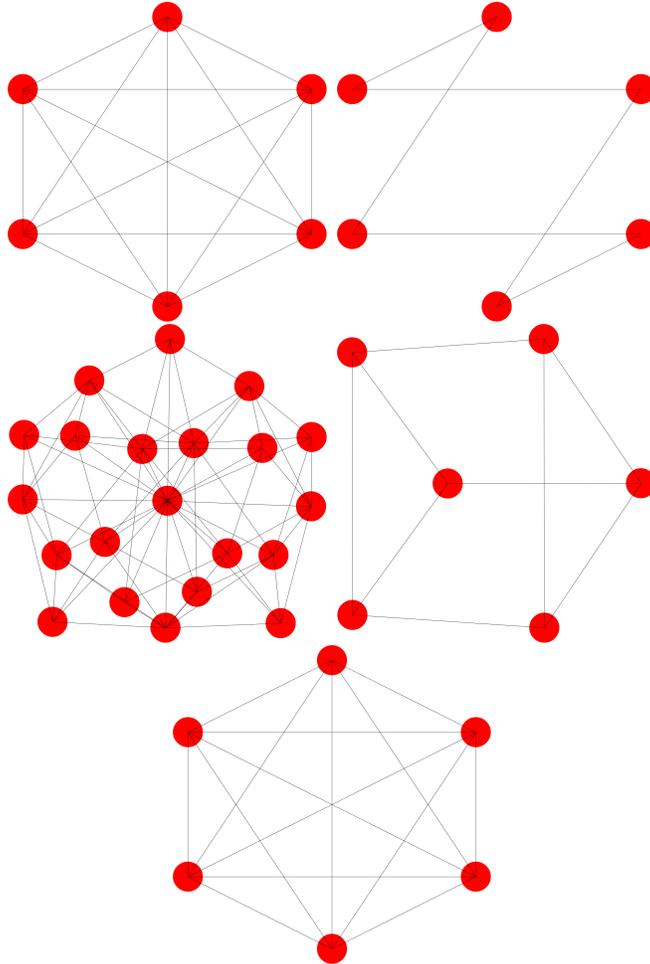FIGURE 10. The two graph additions combining K_2 with K_3.

FIGURE 11. The five graph multiplications K_2 with K_3.

compatibility of dimension which is familiar from the continuum: the dimension of the product is then the sum of the dimensions of the factors.

4. Zykov addition

4.1. Given two finite simple graphs $G = (V, E), H = (W, F)$, the **Zykov addition** is defined as

$$G + H = (V \cup W, E \cup F \cup \{(a, b) \ a \in V, b \in W\}) .$$

As it is commutative and associative and the empty graph $0 = (\emptyset, \emptyset)$ is the zero element, it is a monoid. The set of equivalence classes of the form $A - B$ is defined as follows: define $A - B \sim C - D$ if there exists K such that $A + D + K = B + C + K$. These equivalence classes now form a commutative group. The addition $+$ corresponds to the **join operation** in topology. It has been introduced in 1949 by Zykov. The general construction of a group from a monoid has its roots in arithmetic but has been formalized abstractly first by Grothendieck.
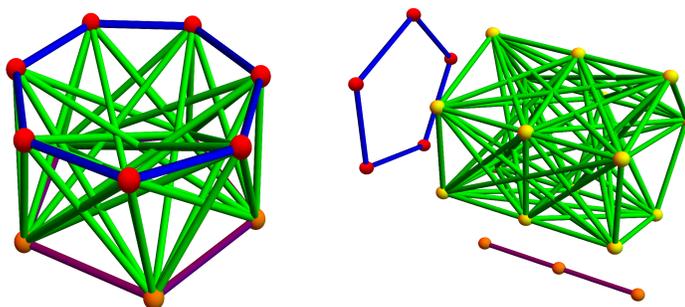

FIGURE 12. Addition and multiplication in the Zykov ring.

4.2. For the additions $+$ and $\oplus$ it is no problem to extend the operation to the larger class of simplicial complexes. A **finite abstract simplicial complex** is a finite set of non-empty sets closed under the operation of taking non-empty subsets. The disjoint union of complexes produces a monoid. Also the Zykov addition can be extended to finite abstract simplicial complexes by defining

$$G + H = G \cup H \cup \{x \cup y \mid x \in G, y \in H\} .$$

The empty set is the **zero element**. The operation is obviously commutative and associative. One again gets a group containing elements

of the form $A - B$ and identifies $A - B \sim C - D$ if there exists K such that $A + D + K = B + C + K$. If G, H are finite simple graphs then the Whitney complex of the graph $G + H$ is the sum of the Whitney complexes of G and H. The Zykov group of graphs therefore can be considered to be a sub-group of the group of simplicial complexes.

4.3. In order to do computations in the additive Zykov group, lets look at the graphs P_n,the n-vertex graph with no edges, K_n the **complete graph** with n vertices, C_n the **circular graph** with n vertices and W_n the **wheel graph** with n spikes so that for the central vertex x, the unit sphere $S(x)$ is C_n. Let $K_{n,m}$ denote the **complete bipartite graph** of type n, m and let S_n the **star graph** with n rays. Since K_n plays the role of the integer n in $\mathbb{Z}$, we write also n for K_n.

Examples.
1) $P_n + P_m = K_{n,m}$ complete bipartite graph
2) $K_n + K_m = K_{n+m}$ integer addition
3) $K_1 + G$ is the cone over G
4) $P_2 + G$ is the suspension over G
5) $P_2 + P_2 + P_2$ octahedron
6) $P_2 + P_2 + P_2 + P_2$ 16-cell, three sphere
7) $K_1 + C_n = W_n$ wheel graph
8) $K_1 + P_n = S_n$ star graph
9) $P_3 + C_n$ triple suspension of circle

4.4. Similarly as with numbers one can define classes of graphs algebraically. For example, we can look at all graphs which can be written as sums of P_k graphs. Lets call them P-graphs. If $G = P_{k_1} + \cdots + P_{k_d}$, then this graph has dimension $d - 1$. Examples are the wheel graph $P_1 + P_2 + P_2$ of dimension 2, the complete bipartite graph $P_n + P_m$ of dimension 1, the complete graph $P_1 + P_1 + \cdots + P_1 = dP_1$ of dimension $d - 1$ or the **cross polytopes** $P_2 + P_2 + \cdots + P_2 = dP_2$ which is a sphere of dimension $d - 1$.

4.5. We already know some thing about the addition: the functional $i(G) = 1 - \chi(G)$ is multiplicative. It can be interpreted as a genus or as a reduced Euler characteristic. If $f(G) = (v_0(G), v_1(G), \ldots v_{d(G)}(G))$ is the f-vector of G with dimension $d(G)$, then the clique number $c(G) = d(G) + 1$ is additive. The **volume** $V(G) = v_{d(G)}(G)$ is multiplicative. The Fermi functional $\prod_x (-1)^{\dim(x)}$ equal to the deteriminant of the connection Laplacian of G which is the Fredholm determinant of the adjacency matrix of the connection graph G' of G. We also have $f_{G+H} = f_G f_H$, where $f_G(x) = 1 + \sum_{k=0} v_k x^{k+1}$ is the f-generating
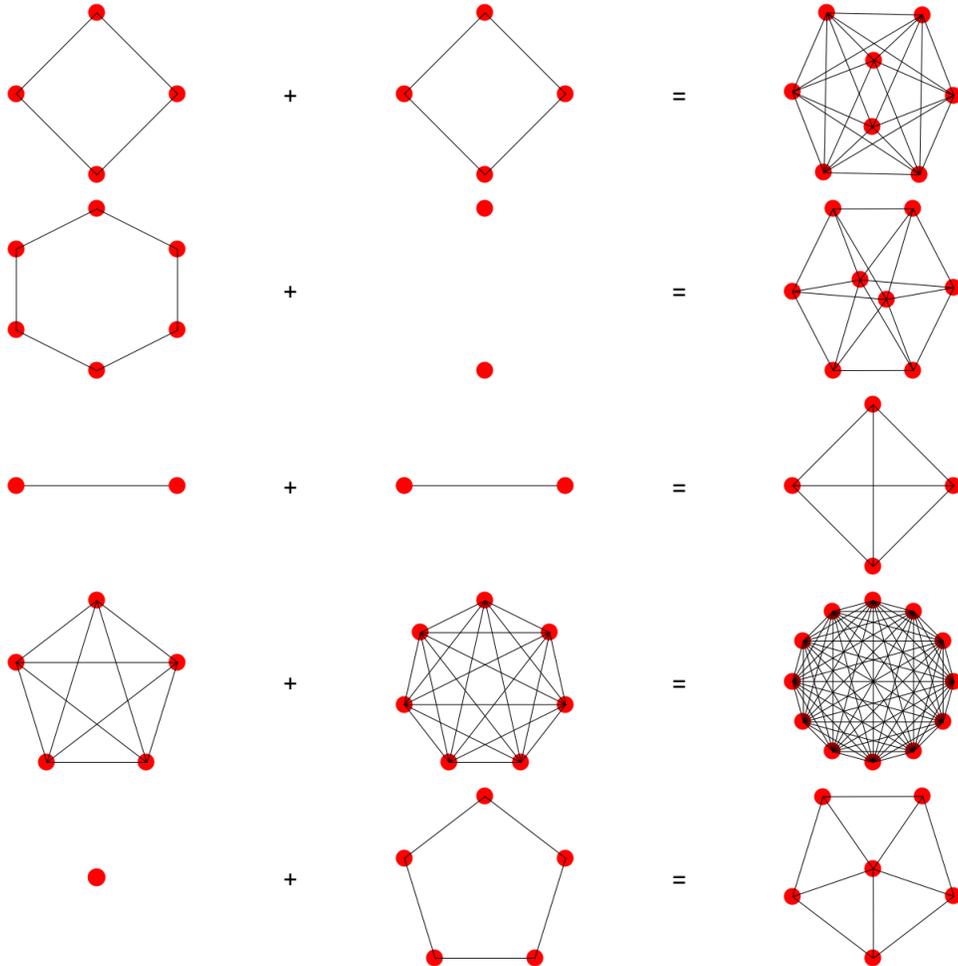
FIGURE 13. Some examples of Zykov additions.

function of G. This implies $\chi(G + H) = \chi(G) + \chi(H) - \chi(G)\chi(H)$ which is equivalent to the multiplicative property of $i(G)$. The Kirchhoff Laplacian of $G + H$ has an eigenvalue $v_0(G) + v_0(H)$. We also know that the second eigenvalue $\lambda_2(G)$ of the Kirchhoff Laplacian satisfies $\lambda_2(G + H) = \min(v_0(G), v_0(H)) + \min(\lambda_2(G), \lambda_2(H))$. Finally, if $D = d + d^*$ be the Dirac operator of G then the Hodge Laplacian $H = D^2 = (d + d^*)^2$ splits into blocks H_k for which the nullity is the k'th Betti number $b_k(G) = \dim(\ker)(H_k)$ [9, 8]. The spectrum $\sigma_{d(G)}(G)$ of $L_{d(G)}$ is the volume spectrum of G. We know that the volume eigenvalues of $G + H$ are of the form $\lambda + \mu$, where $\lambda \in \sigma_{d(G)}(G)$ and $\mu \in \sigma_{d(H)}(H)$.

4.6. Given a finite simple graph $G = (V, E)$, the **graph complement** $\overline{G} = (V, \overline{E})$ is defined by $\overline{E}$, the complement of the edge set E of G in the edge set of the complete graph on V. This produces an involution on the set of all finite simple graphs. The disjoint join operation $\oplus$ and the join $+$ are conjugated by this duality:

Lemma 2. $G + H = \overline{\overline{G} \oplus \overline{H}}$.

Proof. Let e be first an edge in G. Then it is not in $\overline{E}$ and also not in $\overline{G} \oplus \overline{H}$. So, it is in $\overline{\overline{G} \oplus \overline{H}}$. The same holds if e is an edge in H. Let now e be an edge in the Zykov sum $G + H$ which connects vertices from different graphs. Now since e is not in $\overline{G} \oplus \overline{H}$, it is in $\overline{\overline{G} \oplus \overline{H}}$. $\square$

We can derive again:

Corollary 1. *The additive Zykov monoid has the unique factorization property.*

Proof. The monoid obtained by taking the disjoint union has the unique factorization property. The duality functor carries this to the join addition. $\square$

Remark.
The unique factorization property extends to the group $\mathcal{G}$. This is similar as in the integers where -3 can be considered a prime in $\mathbb{Z}$ as it is only divisible by itself or a unit (which are $\{-1, 1\}$ in that case). What are the units in the Zykov ring? These are the graphs which have a multiplicative inverse. $G \cdot H = K_1$ however is only possible if the graph has one vertex. This means that G is either K_1 or $-K_1$. The **units** in the Zykov ring are $\{1, -1\} = \{K_1, -K_1\}$ as in $\mathbb{Z}$. One could look also at Gaussian ring over the Zykov ring where the units would be $\{1, -1, i, -i\}$. We are not aware that one has looked at **ring extensions** like $\mathcal{G}[i]$ for networks.

For the disjoint addition $\oplus$, the primes are all the connected graphs. This means that the graph complement of a connected graph in a complete graph is a prime for $+$.

Examples:
1) C_4 is not prime. But $\overline{C_4} = K_2 \oplus K_2$ is neither.
2) C_5 is prime. And $\overline{C_5} = C_5$ is also.
3) C_6 is prime. And $\overline{C_6} = K_3 \times K_2$ is too.
4) C_7 is prime. And $\overline{C_7}$ is a discrete Moebius strip.
5) C_8 is prime. And $\overline{C_8}$ is already a three dimensional graph.

6) $W_4 = P_2 + P_2 + P_1$. Indeed $\overline{W_4} = K_2 \oplus K_2 \oplus K_1$.
7) $W_5 = C_5 + P_1$. And $\overline{W_5} = C_5 + K_1$.

5. ZYKOV MULTIPLICATION

5.1. Having an addition like the Zykov addition $+$, it is natural to look for multiplications which satisfies the **distributivity law**. We were led to such a multiplication in the winter of 2016 in [14] after doing a systematic search for such a product. The compatible multiplication later turned out to be the dual of the strong graph multiplication. We call it now the **Zykov product**.

5.2. Given two finite simple graphs $G = (V, E), H = (W, F)$, define the **Zykov product**

$$G \cdot H = (V \times W, E \cup F \cup \{((a,b),(c,d)) \mid (a,c) \in E \text{ or } (b,d) \in F \} .$$

It is commutative and associative and has as the 1-element the graph K_1. Furthermore, we have $0 \cdot G = 0$ for all G. We also have $G \cdot H = 0$ if and only if one of them is the empty graph. The ring obtained is an integral domain. Also in this multiplicative monoid, we can try to extend the operation to a group getting a field. We will look at this later on.

5.3. Here is an extension of the Zykov product $\cdot$ to the larger class of simplicial complexes, (finite sets of sets closed under the operation of taking non-empty subsets). Assume G is a set of subsets of X and H is a set of subsets of Y. Define the projections π_k from the set theoretical Cartesian product $X \times Y$ to X or Y. Define

$$G \cdot H = \{A \subset X \times Y \mid \pi_1(A) \in G \text{ or } \pi_2(A) \in H\} .$$

This means $G \cdot H = G \times P(Y) \cup P(X) \times H$, where $P(X)$ is the set of all subsets of X. Again the distributivity law $G \cdot (H + K) = G \cdot H + G \cdot K$ holds. One can also get this by defining a **complementary simplicial complex** $\overline{G}$ of G. It can be defined as the set of subsets of $X = \bigcup_A A \in G$ which have the property that it does not contain any $A \in G$ **of positive dimension**. Now one has again $G + H = \overline{\overline{G} \oplus \overline{H}}$.

5.4. For any two graphs G, H, the **strong product** $G \boxtimes H$ contains the **tensor product** graph $G \otimes H$ and the **weak Cartesian product** graph $G \square H$ as subgraphs. For the weak product $\otimes$ we know that if $A(G)$ is the adjacency matrix of G then $A(G) \otimes A(H) = A(G \otimes H)$, where the former product is the tensor product of adjacency matrices.

5.5. Lets return back to the Zykov product:

Lemma 3 (Distributivity). *Multiplication is compatible with addition:*
$G \cdot (H + K) = G \cdot H + G \cdot K$.

Proof. Both $G \cdot H + G \cdot K$ as well as $G \cdot (H + K)$ have as the vertex set the product sets of the vertices. As edges in the sum $H + K$ consist of three types, connections within H, connections within K and any possible connection between H and K, two points $(a, b), (c, d)$ are connected if either (a, c) is an edge in H, or (b, d) is an edge in K or then if either a, c or b, d are in different graphs. $\square$

This lemma follows also by complementary duality. If we know that the graph tensor product and graph Cartesian product both are compatible with the disjoint union operation $\oplus$, then also their union is.
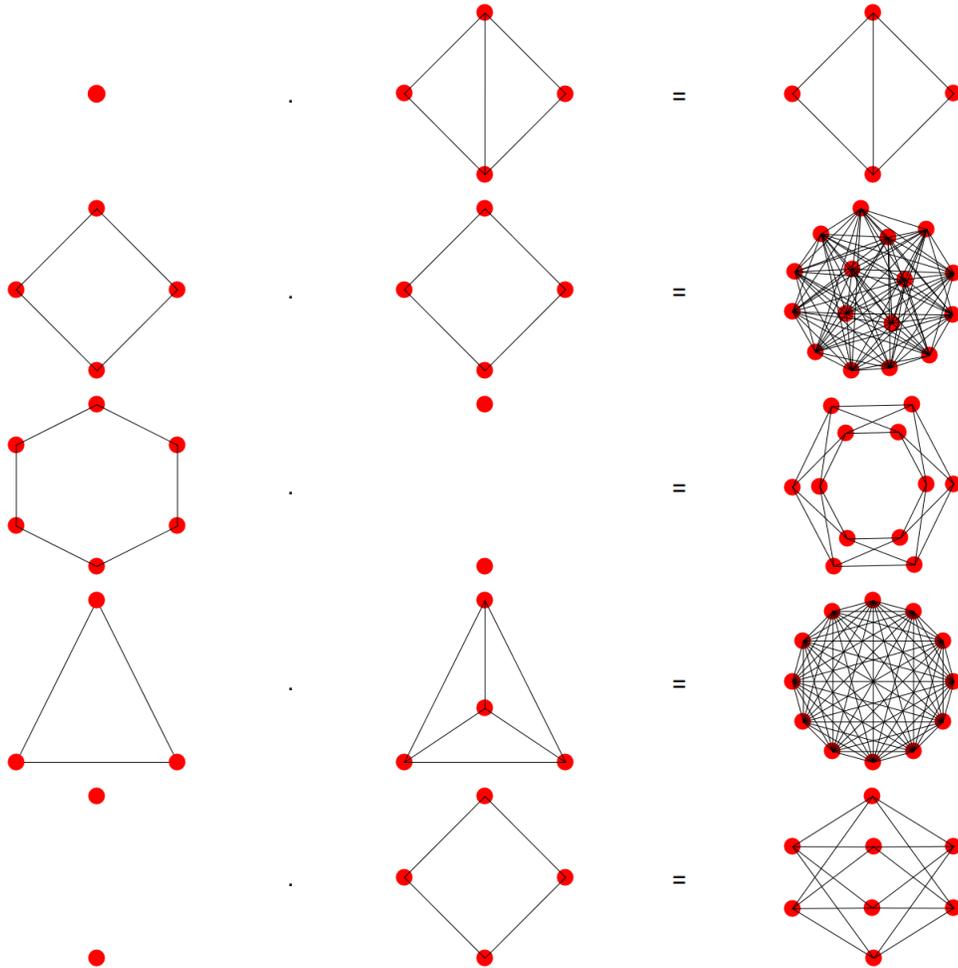
FIGURE 14. Some examples of Zykov multiplications of graphs.

6. COMPUTATIONS IN THE ZYKOV RING

6.1. Both the Zykov addition $+$ as well as the Zykov multiplication $\cdot$ produce **monoid structures** on the category of graphs $\mathcal{G}_0$ or signed graphs $\mathcal{G}$ or on the more general category of abstract simplicial complexes. For the addition, the empty graph or empty complex is the **zero element**. For the multiplication the graph K_1 or one point simplicial complex is the **one element**. A general construct of Grothendieck allows to produce a group from a monoid: in the additive case we look at all pairs $G - H$ of complexes and call $G - H = U - V$ if there exists K such that $G + V + K = U + H + K$. In the multiplicative case, look

at all pairs G/H of complexes and call $G/H = U/V$ if there exists K such that $G \cdot V \cdot K = U \cdot H \cdot K$.

6.2. Does the multiplicative monoid have the **cancellation property**? If the multiplicative monoid in a ring has the cancellation property, then the ring is called a **domain**. If there is no possibility to write $x * y = 0$ without one of the factors being zero, the ring is called an **integral domain**. Many rings are not integral domains: like Z_6 in which $2 * 3 = 0$ or the product ring $\mathbb{Z}^2$ where $(0, 1) * (1, 0) = (0, 0)$. The ring of diagonal 3×3 matrices is an example of a ring which does not have the cancellation property as $G \cdot K = H \cdot K$ does not imply $G = H$ in general as the case when G, H, K be the projections on the x-axes, y-axes and z-axes shows.

6.3.

Lemma 4. *While not unique factorization domains, the weak and strong rings have the cancellation property. They are also integral domains. Consequently the Zykov ring is an integral domain but not a unique factorization domain.*

Proof. This is covered in section 6.5 of [3]. The proof given there defines a ring homomorphism from $\mathcal{G}$ to an integral domain of polynomials. Having the property for the strong ring gives the property for the Zykov ring by the complement duality. $\square$

6.4. Lets for a moment go back to the additive monoid. Before seeing the duality connection, we searched for a direct proof of the unique prime factorization property for the Zykov monoid $(\mathcal{G}, +)$, where $+$ is the join. The unique prime factorization property can be illustrated that we know about f-generating functions $f(G) = 1 + \sum_{k=0} v_k x^{k+1}$ of G and $f(H) = 1 + \sum_{k-0} w_k x^{k+1}$ of H are the same, as $Z[x]$ is a unique factorization domain. This means that the f-vectors of G and H agree. The cancellation property $G + K = H + K \Rightarrow G = H$ implies especially that if $G' = G + P_2$ is a suspension of G and $H' = H + P_2$ is a suspension of H and G', H' are isomorphic graphs, then G and H are isomorphic graphs. In the simpler case of cone extensions $G + K_1 = H + K_1$ implying $G = H$, one can use that every simplex containing the new point x and every isomorphism T from $G + K_1$ to $H + K_1$ induces a permutation on the facets. Assume that the isomorphism T maps x to y, the other case where the isomorphism maps x to x is similar. The isomorphism of dimension $k+1$ simplices induces after removing x and y an isomorphism of k-simplices from G to H. Now, as all k, especially an isomorphism of $dim = 1$ simplices which is a graph homomorphism.

6.5. We have now a ring of networks in which we can do some computations. Lets look at some examples. Since $K_n \cdot K_m = K_{nm}$, one can abbreviate nG for $K_n \cdot G$.

a) $P_n \cdot P_m = P_{nm}$
b) $K_n \cdot K_m = K_{nm}$
c) $3P_2$ octahedron
d) $P_2 \cdot G$ doubling
e) $4C_4 = K_4 \cdot (P_2 + P_2) = 8P_2 = S^7$
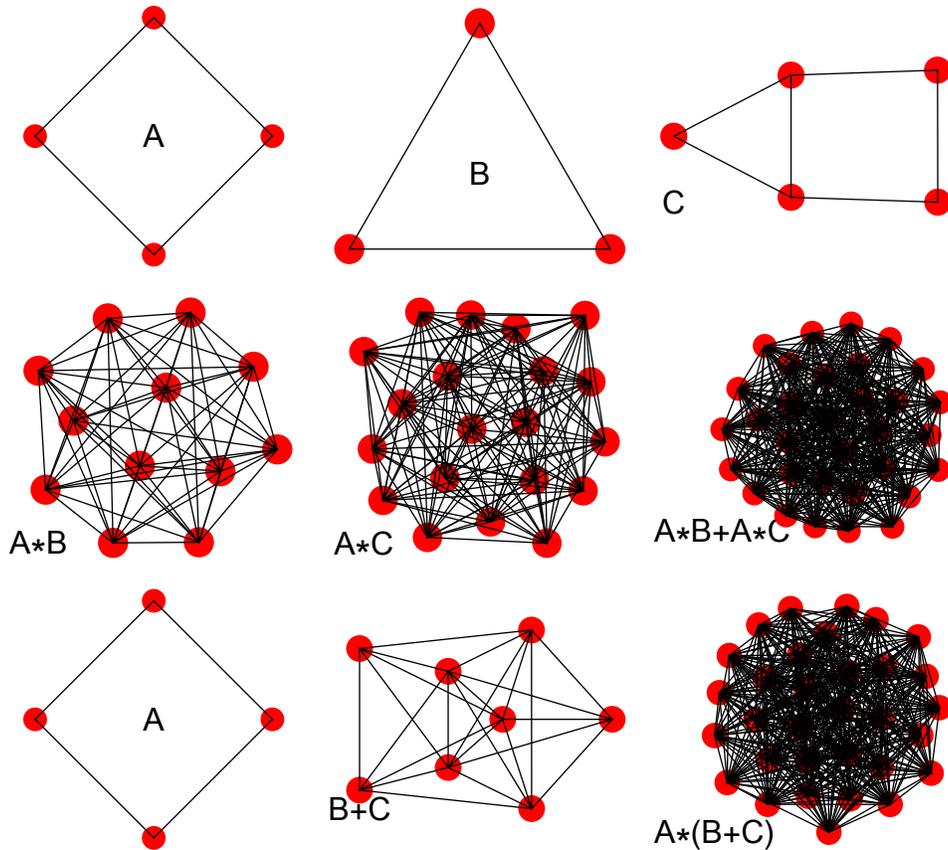f) $K_2 \cdot C_n = C_n + C_n$ three sphere
g) $K_2 \cdot P_2 = 2P_2$ kite graph

FIGURE 15. Illustrating the distributivity in the Zykov ring $(\mathcal{G}, +, \cdot, 0, 1)$.

6.6. With the disjoint union $\oplus$ of simplicial complexes as addition, one can look at the simplex Cartesian product $\times$ as a multiplication. The set theoretical Cartesian product of two simplicial complexes is not a

simplicial complex in general and the Cartesian product defined in [15] is not associative as $G_2 = (G \times K_1) \times K_1 \neq G_1 \times (K_1 \times K_1) = G_1$. In order to get a ring structure, one can define the product only on refinements of complexes $G_1 \times H_1 = G \times H$. The problem is that the product is then no more a refinement of a complex but when restricting to the set of graphs which are expressible as $G_1(1) \times G_1(2) \times \cdots \times G_1(k)$ we are fine. This is more convenient than building a data structure of CW complexes and essentially means to look at the structures in the Stanley-Reisner ring. We used that ring in [15] without being aware of the Stanley-Reisner picture. All objects $G = (G_1 \times \cdots \times G_n)$ are equipped with a CW structure so that the Euler characteristic is the product. There is a corresponding connection graph L' for which the energy is the Euler characteristic of G. When extending Euler characteristic functional to the ring, it becomes a ring homomorphism to Z. There is a relation between Cartesian and tensor product: the connection graph of the Cartesian product is the tensor product of the connection graphs.

6.7. The tensor product has some relation to the Grothendieck product: the connection graph Laplacians tensor if the product of the simplicial complexes is taken. This is an algebraic statement.
Its not true in general however that the Fredholm connection Laplacian of $G \times H$ is the tensor product of the Fredholm connection Laplacian of G and H. It is only that the Fredholm connection matrices tensor.
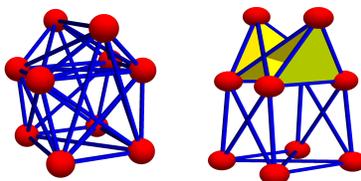

FIGURE 16. Two Zykov products $P_2 \dot{G}$ and $2G = K_2 \cdot G$. Can you guess the graph G is? On a serious side, the problem to identify the factors of a given Zykov product $G \cdot H$ might be tough. The factorization problem is equivalent to find the factors of the strong product.

7. MORE EXAMPLES

7.1. Lets do a few example computations in the Zykov ring. The main building blocks are the **point graphs** $P_n = (\{1, 2, 3, \ldots, n\}, \emptyset)$, the **complete graphs** $K_n = \overline{P_n}$, the **cycle graphs** C_n (assuming

$n \geq 4$), the **star graphs** $S_n = P_n + 1$, (with the understanding that $S_2 = P_2 + 1 = L_2$ is the linear graph of length 2 and $S_1 = P_1 + 1 = L_2$). Then we have the **wheel graphs** $C_n + 1$, the **complete bipartite graphs** $K_{n,m} = P_n + P_m$. Furthermore, we look at the **kite graph** $K = P_2 + K_2$, the **windmill graph** $W = P_3 + K_2$ as well as the **cross polytopes** $S^n = (n+1) \cdot P_2$ which are n-spheres, where especially $S^0 = P_2, S^1 = C_4$ and $S^2 = O$, the **octhahedron** graph are $0, 1$ and 2-dimensional spheres. (There should be no confusion as we do not use a graph with name S as S^d is not a power in the Zykov ring but a sum of zero dimensional spheres) Most of these graphs are contractible with trivial cohomology $b(G) = (1, 0, 0, \dots)$. The ones which are not, have the Betti numbers $b(P_n) = (n, 0, 0, ..), b(C_n) = (1, 1, 0, 0, ..), b(S^n) = (b_0, \dots, b_n) = (1, 0, 0, \dots, 1), b(K_{n,m}) = (1, (n-1)(m-1), 0, \dots)$.

7.2. Example A: the square of a circle. Lets compute the square of the circular graph C_4 and show

$$\boxed{C_4^2 = 2K_{4,4}.}$$

We use that $P_n \cdot P_m = P_{nm}$ and $P_n + P_m = K_{n,m}$ to get

$$\begin{aligned} C_4^2 &= (2P_2)^2 = 4P_2^2 = 4P_4 \\ &= 2(P_4 + P_4) = 2(K_{4,4}) \ . \end{aligned}$$

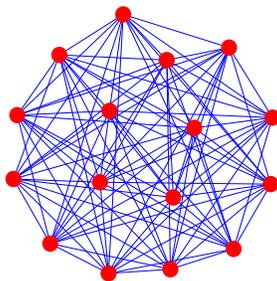It follows for example that $W_4^2 = (1 + C_4)^2 = 1 + 4P_2 + 2K_{4,4}$.

FIGURE 17. The Zykov square of the circle C_4 is $K_{4,4} + K_{4,4}$.

7.3. Example B: the square of a kite. Lets compute the square of the kite graph K and show

$$\boxed{K^2 = P_4 + K_4 + S^3.}$$

We use $P_n \cdot P_m = P_{nm}$ and $K_n \cdot K_m = K_{nm}$ and $2C_4 = C_4 + C_4 = 4P_2 = S^3$ as well as $K_2 \cdot P_2 = P_2 + P_2 = C_4$.

$$
\begin{aligned}
K^2 &= (P_2 + K_2)^2 = (P_2 + K_2) \cdot (P_2 + K_2) \\
&= (P_4 + K_4 + 2K_2 \cdot P_2) \\
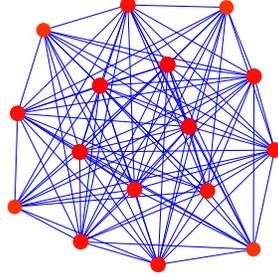&= P_4 + K_4 + 4P_2 = P_4 + K_4 + S^3 \ .
\end{aligned}
$$

FIGURE 18. The Zykov square of the kite graph is $P_4 + K_4 + S^3$.

7.4. Example C: the square of a star.

$$\boxed{S_n^2 = S_{n^2} + K_{n,n}.}$$

We use that $S_n = P_n + P_1$ and $P_n \cdot P_m = P_{nm}$ and $P_n + P_m = K_{n,m}$ to get

$$
\begin{aligned}
S_n^2 &= (P_1 + P_n)^2 = P_1^2 + P_{n^2} + 2P_1 \cdot P_n \\
&= P_{n^2} + P_1 + 2P_n == S_{n^2} + (P_n + P_n) \\
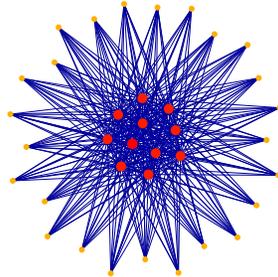&= S_{n^2} + K_{n,n} \ .
\end{aligned}
$$

FIGURE 19. The Zykov square of a star graph is the sum of a star graph and bipartite graph.

28

7.5. Example D: the square of the windmill.

$$\boxed{W^2 = P_9 + 4S_3.}$$

We use the definitions $W = P_3 + K_2$ and $S_n = P_1 + P_n$ and $P_n \cdot P_m = P_{nm}$ as well as $K_n \cdot K_m = K_{nm}$ to get

$$
\begin{aligned}
W^2 &= (P_3 + K_2)^2 = P_9 + K_4 + 2P_3 K_2 \\
&= P_9 + K_4 + 4P_3 = P_9 + K_4(K_1 + P_3) \\
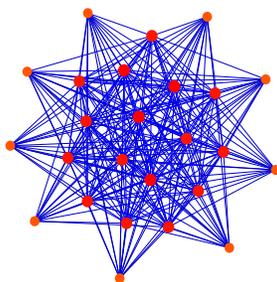&= P_9 + 4S_3 \ .
\end{aligned}
$$

FIGURE 20. The Zykov square of the windmill graph is $P_9 + 4S_3$.

7.6. Example E: the square of a complete bipartite graph.

$$\boxed{K_{n,m}^2 = K_{n^2,m^2} + K_{nm,nm}.}$$

By definition, we have $K_{n,m} = P_n + P_m$ and so $K_{n,m} + K_{n,m} = P_{2n} + P_{2m} = K_{2n,2m}$. But now to the product:

$$
\begin{aligned}
K_{n,m}^2 &= (P_n + P_m)^2 = P_n^2 + 2P_n \cdot P_m + P_m^2 \\
&= P_{n^2} + P_{m^2} + (P_{nm} + P_{nm}) \\
&= P_{n^2} + P_{m^2} + K_{nm,nm} \\
&= K_{n^2,m^2} + K_{nm,nm} \ .
\end{aligned}
$$

In the special case where $n = m$, we have

$$\boxed{K_{m,m}^k = 2^k P_m^k.}$$

Just write $(P_n + P_n)^k = (2P_n)^k$.

We see that complete subgraphs are multiplicatively closed. This can also be seen by diagonalization: $\overline{K_{n,m}} = K_n \oplus K_m$. So that $\overline{K_{n,m} \boxtimes K_{k,l}} =$

$(K_n \oplus K_m) \boxtimes (K_k \oplus K_l)$ which can now be factored out and more generally identities like $K_{n,m} \cdot K_{k,l} = K_{nk,nl} + K_{mk,ml}$ hold.
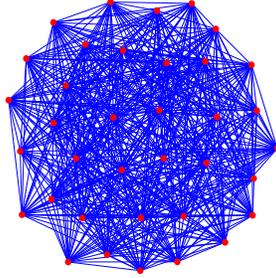
FIGURE 21. The Zykov square of a bipartite graph $K_{n,m}$ is a sum of complete bipartite graphs.

7.7. Example F: the square of the 3-sphere.

$$\boxed{(S^3)^2 = 4^2 P_4.}$$

The definition is $S^d = (d+1)P_2 = P_2 + \cdots + P_2$ so that $(S^d)^2 = (d+1)^2 P_2^2 = (d+1)^2 P_4$ Especially

$$(S^3)^2 = K_{(3+1)^2} \cdot P_4 = 16 P_4 .$$

In the same way, we can compute the square of the octahedron, the 2-sphere, as

$$\boxed{O^2 = 9 P_4.}$$

FIGURE 22. The Zykov square of a three sphere.

7.8. Example G: Subtracting a star from a sphere

$$\boxed{(C_4 - S_4) \cdot (K_{4,4} - S_{16}.}$$

The computation

$$(C_4 - S_4) \cdot (C_4 + S_4) = C_4^2 - S_4^2 = 2K_{4,4} - (S_{16} + K_{4,4}$$

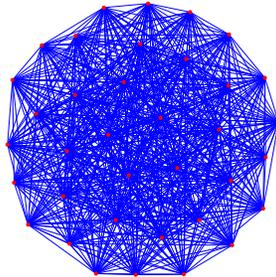follows from A) and B). We mention that as it is clear if we see somewhere an $A - A$, then it can be reduced to 0.

7.9. Most of these examples are P-**graphs**, graphs generated by the point graphs P_n. Every P-graph has the form $\sum_{k=1}^{n} a_k P_k$, where $a_k \in \mathbb{Z}$. Here is a summary of some small "numbers":

- $n = K_n$ complete graphs
- $P_k P_n = P_{kn}$ point graphs
- $P_k + P_n = K_{k,n}$ complete bipartite graphs
- $(d+1)P_2 = S^d$ spheres
- $2P_2 = C_4$ circle graph
- $1 + 2P_2 = W_4$ wheel graph
- $2 + 2P_2$ three ball
- $2 + P_2$ kite graph
- $1 + P_2$ linear graph
- $1 + P_n$ star graphs
- $1 + 2P_2$ wheel graph
- $2 + P_3$ windmill graph

7.10. As $P_k + P_n = K_{k,n}$, the complete bipartite graphs naturally extend the class of graphs without edges. One could define more generally

$$W_{a_1, a_2, \ldots, a_n} = \sum_{k=1}^{n} a_k P_k \ .$$

When looking at such sums and extends the factors a_k to become rational numbers $p/q = K_p/K_q$ one is led to the question of defining a distance between such rational networks and study questions about **limits**. This leads then to a **calculus** of networks. Defining such a calculus is easier when restricting to a small ring first like the field generated by the ring of P-graphs, where the **distance** between two P_n, P_k naturally should be $n - k$. It is clear that one can get through such power series to **irrational networks** using a cardinality argument. There are only countably many rational elements in the smallest field but **convergent power series** even with rational coefficients produce uncountably many new "networks".
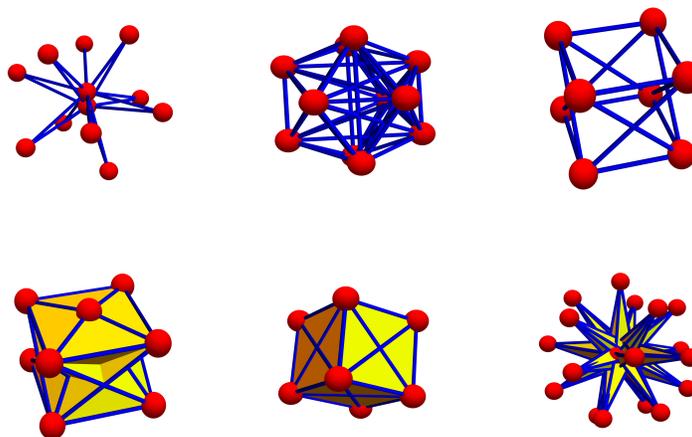
FIGURE 23. Some graphs in the subring of P-graphs. We see first $S_6 + P_2$, then $6 + P_2$, then $2P_2^2$, then $2P_2^2 + P_2$, then $4P_2$ and finally $P_2 + (1 + P_6)P_4$.

8. ADDITIVE ZYKOV PRIMES

8.1. There are two type of primes in the Zykov ring of complexes. The **additive primes** for the monoid $(\mathcal{G}, +)$ and then the **multiplicative primes** in the monoid $(\mathcal{G}, \cdot)$. For the integers $\mathbb{Z}$, the additive prime factorization is trivial as the only additive prime 1. In the Zykov ring $(\mathcal{G}, +, \cdot)$, the additive primes are a bit more interesting but still not difficult to characterize. Here again, $\overline{G}$ is the complement of the graph G so that $\overline{P_n} = K_n$ and $\overline{K_n} = P_n$.

Lemma 5. *A graph G is an additive Zykov prime if and only if $\overline{G}$ is connected.*

Proof. The complement operation is a ring isomorphism from the Zykov ring $(\mathcal{G}, +\cdot)$ to the strong ring $(\mathcal{G}, \oplus, \boxtimes)$, mapping 0 to 0, 1 to 1 and satisfying $\overline{H \cdot G} = \overline{H} \boxtimes \overline{G}$ and $\overline{H + G} = \overline{H} \oplus \overline{G}$, both additive and multiplicative primes in one ring correspond to additive and multiplicative primes in the other ring. In the additive monoid $(\mathcal{G}, \oplus)$, the primes are the connected graphs. $\square$

Remarks.
1) The volume of a graph is the number of facets, complete subgraphs K_{d+1}, where d is the maximal dimension. We first thought that the primality of the volume assures that a graph is an additive prime. This is not the case since there are primes of volume 1. An example

is $G = \overline{L_3 \oplus C_4}$, where L_3 is the linear graph of length 3 and C_4 the cyclic graph of length 4. Now G has prime volume 2 but it is obviously not prime. Indeed, one of the factors $\overline{L_3} = K_2$ has volume 1.

2) The property of being prime is not a topological one. There are spheres like the octahedron which can be factored $O = P_2 + P_2 + P_2$, and then there circles like C_5 which are prime and can not be factored. But as in the additive integer case, the "fundamental theorem of additive network arithmetic" is easy, also with a direct proof:

Theorem 1. *The additive Zykov monoid has a unique additive prime factorization.*

Proof. The disjoint union has a unique additive prime factorization. Now dualize.

Direct proof. Assume $G = A + B = C + D$, where A is prime. Following the Euclid type lemma, we prove that either $A = C$ or $A = D$.

If that is not true then $A \cap C$ and $A \cap D$ are both non-empty. Lets look at the four intersections $A \cap C$, $A \cap D$, $B \cap C$ and $B \cap D$. Because every element in $A \cap C$ is connected to $A \cap D$, we have $A = A \cap C + A \cap D$. But this contradicts that A is prime. $\qquad\square$

Corollary 2. *Every element in the additive Zykov group of networks can be written as $G = U - V$, where U, V are two graphs.*

Proof. The reason is that the cancellation property holds. $U + K = V + K$ implies $U = V$.

For integers we can always write an integer as either n or $-n$. The reason is that the unit 1 is the only additive prime. But for networks, we do not have an overlap in general. For example $C_5 - C_7$ can not be written as a single network similarly as $5/7$ can not be simplified. Sometimes we can like $K_5 - K_7 = -K_2$ as in the ring we have $K_5 = K_1 + K_1 + K_1 + K_1 = 5K_1$ and $K_5 - K_7 = 5K_1 - 7K_1 = -2K_1$.

9. MULTIPLICATIVE PRIMES

9.1. Proving the fundamental theorem of algebra for rational integers $\mathbb{Z}$ was historically a bit more convoluted. As André Weyl pointed out, there is a subtlety which was not covered by Euclid who proved however an important lemma, which comes close. Examples of number fields like $\mathbb{Z}[\sqrt{5}]$ show that factorization is not obvious. We pondered the problem of unique prime factorization for the Zykov product on our own without much luck. We finally consulted the handbook of graph products and got relieved as the answer is known for the strong ring

and realizing that this Sabidussi ring is dual to the Zykov ring. Lets look at the primes.

Lemma 6. *Every graph $G = (V, E)$ for which $|V|$ is prime is a multiplicative prime in any of the three rings $(\mathcal{G}, \oplus, \square), (\mathcal{G}, \oplus, \otimes), (\mathcal{G}, \oplus, \boxtimes)$ as well as dual rings $(\mathcal{G}, +, \diamond), (\mathcal{G}, \oplus, *), (\mathcal{G}, +, \cdot)$.*

Proof. For all ring multiplications, the cardinalities of the vertices multiplies. If one of the factors has 1 vertex only, then this factor is $1 = K_1$. The reason is that there is only one graph for which the vertex cardinality is 1.
Remark. It is the last part of the proof which fails if we look at the volume in the Zykov addition, for which volume is multiplicative. In that case, there are graphs with volume 1 which are not equal to the unit $1 = K_1$.

9.2. We have seen that the Zykov sum $G + H$ of two graphs G, H is always connected. Also the Zykov product has strong connectivity properties but the example $P_n \cdot P_m = P_{mn}$ shows that we don't necessarily have connectivity in the product.

Lemma 7. *If either G or H is connected, then $G \cdot H$ is connected.*

Proof. Given two points $(a, b), (c, d)$ and assume G is connected. We can connect $a_0 = a, ..., a_n = c$ with a path in G. Now, for any choice $b_1, ..., b_n$ in H, we have a connection connect $(a_0, b_0) \ldots (a_n, b_n)$. $\square$

9.3. As we will see below that there is a unique prime factorization for multiplication $\cdot$ as long as the graph $\overline{G}$ is connected. Here is a lemma which tells that for complete graphs, we have a unique prime factorization. This is not so clear as we had to extend the arithmetic. We have learned for example for Gaussian primes that primes like $p = 5$ Z do no more stay primes in number field and that we can have non-unique prime factorization like $2 * 3 = (1 + i\sqrt{5})(1 - i\sqrt{5})$.

Lemma 8. *The graph K_n uniquely factors into $K_{p_1} \cdot \cdots \cdot K_{p_m}$ where $n = p_1 \cdot p_m$ is the prime factorization of n.*

Proof. **Using duality.** The graphs P_n in the strong ring $(\mathcal{G}, \oplus, \boxtimes)$ are multiplicative primes if and only if n is a prime. By duality, this is inherited by the ring $(\mathcal{G}, +\cdot)$.
Direct. If not, then $U \times V = x = K_n$ is a factorization of K_n, where one of the U is not a complete graph. We can assume without loss of generality that it is U. There are then two vertices v, w for which the edge (v, w) is missing in U. But now also the vertex $((v, a), (v, a))$ is missing in the product. $\square$

9.4. Remark. We could prove like this uniqueness for graphs of the form $K_p \cdot H$, where H is prime: the reason is that $K_p \cdot H = H + H + \cdots + H$. Assume we can write it as AB, where A has p vertices, then A is a subgraph of K_p and so also of the form K_d.

9.5. Can we give a formula for the f-vector of the Zykov product $\cdot$ and so get criteria for primality? Just for a start, if we write $(v_1, e_1, ...)$ for the f-vector of G_1 and $(v_2, e_2, ...)$ for the f-vector of G_2, then the f-vector $(v, e, \dots)$ of $G_1 \cdot G_2$ satisifies $v_1 v_2 = v$ and $e = e_1 v_2^2 + e_2 v_1^2 - 2e_1 e_2$. Now this is a Diophantine problem for the unknowns e_1, e_2. If that problem has no solution, then we have a prime.

9.6. The following example is in [6, 3] for the strong ring. By duality we get:

Proposition 4. *The Zykov multiplication $\cdot$ does not feature unique prime factorization. Complements of connected graphs have a unique prime factorization for the Zykov product $\cdot$.*

Proof. This follows from Sabidussi's factorization result and by duality. The example of [6] seen below works. $\qquad\square$

We had not been able to get counter example by random search. The smallest example might be small. But an example given in [6] for the Cartesian graph product works also for the Zykov product.
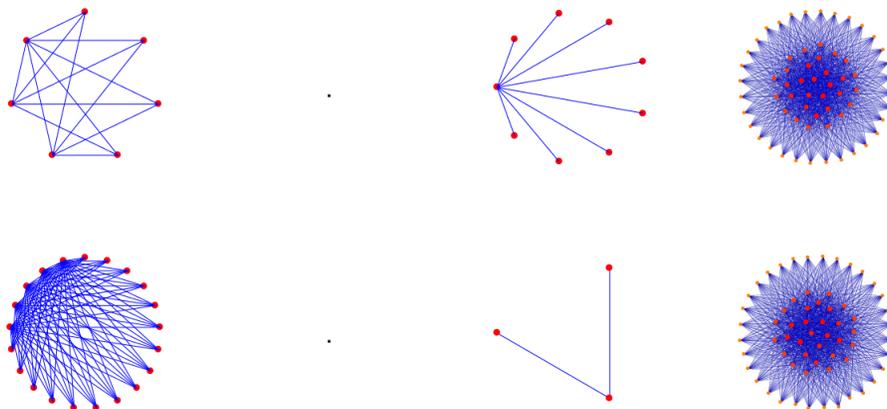

FIGURE 24. Non-uniquness of factorization in the Zykov ring. The graph which gives no uniqueness is 5 dimensional and has f-vector $(63, 1302, 11160, 41664, 64512, 32768)$.

Remarks.
1) As pointed out in [6], the example is based on the fact that $\mathbb{N}[x]$ has no unique prime factorization: $(1+x+x^2)(1+x^3) = (1+x^2+x^4)(1+x)$. Clarification March 15, 2023: since $\mathbb{Z}[x]$ is a unique factorization domain, one can factor this in the Zykov ring. (Thanks for Baian Liu for the question) **2)** Geometric realizations of the corresponding Whitney complexes produce non-unique prime factorizations of manifolds in the Grothendieck ring of manifolds, where the disjoint union is the addition and the Cartesian product is the multiplication.

9.7. By looking at geometric realizations, this immediately shows that the Grothendieck ring of simplicial complexes with disjoint union and Cartesian (topological product) has no unique prime factorization. And since the examples produce manifolds with boundary:

Corollary 3. *The Grothendieck ring of manifolds with boundary with disjoint union as addition and Cartesian product as multiplication has no unique prime factorization.*

For related but much more subtle examples of the Grothendieck ring of varieties see [16].

9.8. A Theorem of Sabidussi tells that connected graphs have a unique prime factorization with respect to the Cartesian product [18]. An other proof of this Sabidussi's theorem is given in [22]. The tensor product has a non-unique prime factorization even in the connected case as shown in [3].
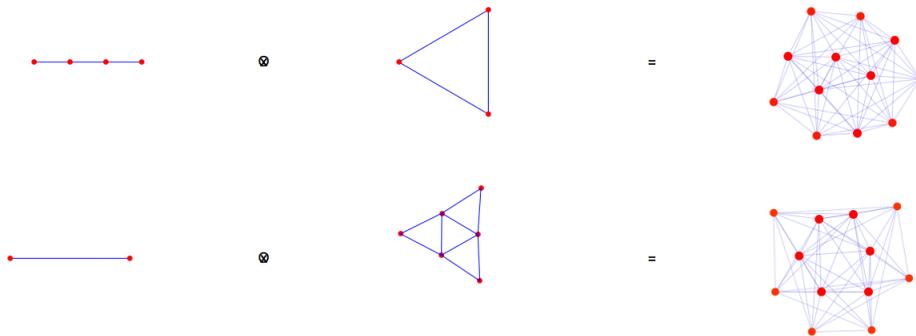

FIGURE 25. Non-uniqueness of factorization in the tensor ring. The example is given in [3].

10. Energy

10.1. A finite abstract simplicial complex G defines a **connection matrix** L. If G contains n sets, then L is a $n \times n$ matrix with $L_{xy} = L(x, y) = 1$ if the faces x, y intersect $x \cap y \neq \emptyset$ and $L(x, y) = L_{xy} = 0$ else. The connection matrix L is always unimodular [12] so that its inverse g integer valued. We also know the energy theorem $\sum_{x,y} g(x, y) = \chi(G)$ [13]. In the context of arithmetic, one can look at the product $G \times H$ of simplicial complexes. It is the set of all pairs (A, B) with $A \in G, B \in H$. The product $G \times H$ is not a simplicial complex as it is not closed under the operation of taking finite subsets. Still, one can look at the **connection matrix** of $G \times H$. Define $L(X, Y) = 1$ if the two sets $X = (x, y)$ and $Y = (a, b)$ intersect and $L(X, Y) = 0$ else. Let here $\mathcal{G}$ denote the ring of all simplicial complexes generated by simplicial complexes, disjoint union as addition and with the just defined Cartesian product as multiplication. The Euler characteristic of an element in this ring is still defined as $\chi(G) = \sum_x (-1)^{\dim(x)}$. It follows almost by definition that $\chi(G \times H) = (\sum_{x \in G} (-1)^{\dim(x)})(\sum_{y \in H} (-1)^{\dim(y)}) = \chi(G)\chi(H)$. The adjacency matrices tensor if the weak product of graphs is taken. There is an analogue for connection Laplacians:

Lemma 9 (Tensor connection lemma). $L_{G \times H} = L_G \otimes L_H$.

Proof. In the natural basis $e_i \times e_j$, the connection Laplacian of the product is a matrix containing the matrix L_H at the places where L_G has entries 1 and 0 matrices else. $\square$

Linear algebra gives

Corollary 4. $\sigma(L(G \times H)) = \sigma(L(G))\sigma(L(H))$.

The energy theorem follows

Corollary 5. $\sum_{x,y} g_{G \times H}(x, y) = \chi(G \times H)$.

Proof. As the connection Laplacian L tensors under multiplication also its inverse g tensors. The energy as a sum over all matrix entries is therefore multiplicative. Also the Euler characteristic is multiplicative. $\square$

Remarks.
1) Classically, for the **Laplace Beltrami operator** L of a manifold G, we have $\sigma(L(G \times H)) = \sigma(L(G)) + \sigma(L(H))$ which is a sum and not a product. Already in the discrete, the connection Laplacian has special and different features than the Hodge Laplacian $H = D^2 = (d + d^*)^2$.

First of all, it behaves more like the Dirac operator $D = d + d^*$ in that there is negative spectrum, but it does not have the symmetry $\sigma(D) = \sigma(-D)$ of the Dirac operator.

2) Already the Kirchhoff Laplacian $L_0 = A - B = d_0^* d_0$ with adjacency matrix A and diagonal vertex degree matrix B, which is the scalar part L_0 of the Hodge Laplacian $H = (d + d^*)^2 = \oplus_{k=0}^d L_k$ has different properties with respect to the product. An already known compatibility applies for the adjacency matrix $A(G)$: it tensors under the multiplication of the weak product (graph Cartesian product). So, also there, the eigenvalues multiply.

3) One still has to explore the relevance of the connection Laplacian in a physics context. Both the Hodge Laplacian $H = (d + d^*)^2$ as well as the connection Laplacian L are operators on the same finite dimensional Hilbert space. The block entry H_1 obviously has relations to electromagnetism as $LA = j$ is equivalent to the Maxwell equation $dA = F, dF = 0, d^*F = j$ in a Coulomb gauge. It might well be that the connection Laplacian define some internal gravitational energy. The invertibility of L is too remarkable to not be taken seriously. Furthermore, if we let the Dirac operator evolve freely in an isospectral way, this nonlinear dynamics also produces a unitary deformation of the connection Laplacian, of course preserving its spectrum. But the energy values, the Green function entries $g(x, y)$ change under the evolution preserving the total energy, the Euler characteristic. After the deformation, the connection Laplacian is no more integer valued so that also its inverse is no more integer valued in general.

11. A Field of networks

11.1. For any of the rings under consideration we can now also look at the **smallest field** generated by this ring. This corresponds to the construction of $\mathbb{Q}$ from $\mathbb{Z}$. As the cancellation property holds for the addition, we can represent elements in that field by $(a - b)/(c - d)$. As we have no unique prime factorization for the multiplication, we have to keep the equivalence relation.

This is not that strange as we know that rational numbers can be equivalent even so it is not obvious. Especially if we take $pq/(rp)$ for large primes p, q, u, v, where we can not factor the products easily.

11.2. There has been some work on the complexity of factorization in multiplicative network rings [3]. For the weak product as well as for the direct product, one can get the factors fast. What about for the Zykov product? Can we define a graph version for modular arithmetic analogous how $\mathbb{Z}_p$ is obtained from $\mathbb{Z}$? Maybe just identify vertices?

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This would be an other way to get a field structure and could have cryptological applications.

11.3. Here are some illustrations of how one can represent elements in the Zykov ring:
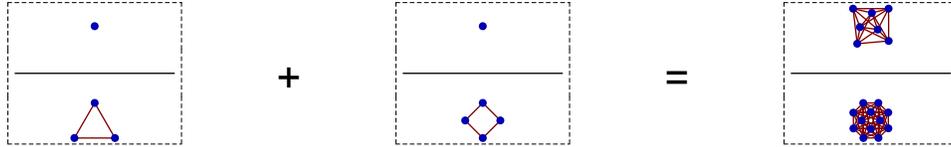

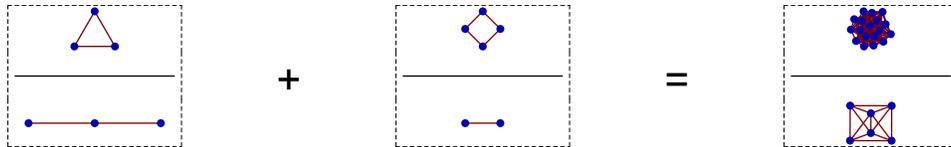FIGURE 26. A computation $1/K_3 + 1/C_4 = (C_4 + K_3)/(3C_4)$ in network arithmetic.

FIGURE 27. A computation $K_3/S_3 + C_4/K_2 = (5 + S_3C_4)/(2(1 + P_3))$ in network arithmetic. As $S_3C_4 = (1+P_3)2P_2 = 2P_2 + 2P_6$ this is the P-fraction $(5+2P_2+2P_6)/(2+2P_3)$.

11.4. Lets just prove an Euclid type lemma. Despite that we have no unique factorization domain, the argument still works:

Lemma 10. *The square root of 2 is irrational in a completion of the Zykov ring.*

Proof. Assume $\sqrt{2} = p/q$, then $2q^2 = p^2$. This means $q^2 + q^2 = p^2$. Since $q^2 + q^2$ is a sum, it is connected and features a unique prime factorization by Sabidussi's theorem. Now the proof is the same as in the classical case as 2 is a multiplicative prime and the number of prime factors on the left and right are not the same modulo 2. $\square$

11.5. More generally we can prove that for connected primes the square root is irrational:

Lemma 11. *For any connected graph G which is a multiplicative prime in the Zykov ring, the square root of G is irrational.*

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Proof. It is the same classical argument. Writing $G \cdot Q^2 = P^2$ and noticing that for connected G, the product $G \cdot Q^2$ is connected, we have a unique prime factorization on both sides. Again the number of prime factors of G modulo 2 is different on the left and right hand side. $\qquad \square$

Remark.

We currently do not know of any disconnected multiplicative prime P for which the square root is rational.

12. Code

12.1. Here is Mathematica code and example computations as shown in examples A-F in this text.

```
NormalizeGraph[s_]:=Module[{r,v=VertexList[s],e=EdgeRules[s]},
 r=Table[v[[k]]->k,{k,Length[v]}];UndirectedGraph[Graph[v /.r,e /.r]]];

ZykovAdd[s1_,s2_]:=Module[{v,w,o,p,f,g,V,e,q,r,L=Length},
 v=VertexList[s1];    o=L[v]; f=EdgeList[s1]; q=L[f];
 w=VertexList[s2]+o; p=L[w]; g=EdgeList[s2]; r=L[g]; V=Union[v,w];
 f=If[L[f]==0,{},Table[f[[k,1]]-> f[[k,2]],{k,q}]];
 g=If[L[g]==0,{},Table[g[[k,1]]+o->g[[k,2]]+o,{k,r}]];
 e=Flatten[Union[{f,g,Flatten[Table[v[[k]]->w[[l]],{k,o},{l,p}]]}]];
 NormalizeGraph[UndirectedGraph[Graph[V,e]]]];          ZA=ZykovAdd;

ZykovProduct[s1_,s2_] :=Module[{v,w,f,g,n,o,p,V,e,q,r,A,L=Length},
 v=VertexList[s1]; n=L[v]; f=Union[EdgeList[s1]]; q=L[f];
 w=VertexList[s2]; o=L[w]; g=Union[EdgeList[s2]]; r=L[g];
 V=Partition[Flatten[Table[{v[[k]],w[[l]]},{k,n},{l,o}]],2];
 f=Table[Sort[{f[[k,1]],f[[k,2]]}],{k,q}];
 g=Table[Sort[{g[[k,1]],g[[k,2]]}],{k,r}]; e={}; A=Append;
 Do[e=A[e,{f[[k,1]],w[[m]]}->{f[[k,2]],w[[l]]}],{k,q},{m,o},{l,o}];
 Do[e=A[e,{v[[m]],g[[k,1]]}->{v[[l]],g[[k,2]]}],{k,r},{m,n},{l,n}];
 NormalizeGraph[UndirectedGraph[Graph[V,e]]];          ZP=ZykovProduct;

s1=RandomGraph[{9,20}];s2=RandomGraph[{9,20}];s3=RandomGraph[{9,20}];
A1=ZykovProduct[s1,s2];A2=ZykovProduct[s1,s3];B2=ZykovAdd[s2,s3];
A=ZykovAdd[A1,A2];B=ZykovProduct[s1,B2];Print[IsomorphicGraphQ[A,B]];

KK[n_]:=CompleteGraph[n];KK[n_,m_]:=CompleteGraph[{n,m}];CC=CycleGraph;
SS[n_]:=StarGraph[n+1];WW[n_]:=WheelGraph[n+1]; KI=ZA[KK[2],PP[2]];
LL[n_]:=LinearGraph[n+1]; PP[n_]:=UndirectedGraph[Graph[Range[n],{}]];
OO=ZP[KK[3],PP[2]]; S3=ZP[KK[4],PP[2]]; WM=ZA[PP[3],KK[2]];
ZP[s_1,s2_,s3_]:=ZP[ZP[s1,s2],s3]; ZA[s1_,s2_,s3_]:=ZA[s1,ZA[s2,s3]];

(*   Example A                                                    *)
Print[IsomorphicGraphQ[ZP[CC[4],CC[4]]  , ZA[KK[4,4],KK[4,4]]]];
Print[IsomorphicGraphQ[ZP[CC[4],CC[4]]  , ZP[KK[2],KK[4,4]]]];
(*   Example B                                                    *)
Print[IsomorphicGraphQ[ZP[KK[2],CC[4]]   , S3]];
Print[IsomorphicGraphQ[ZP[KI,KI]  ,    ZA[PP[4],KK[4],S3]] ];
(*   Example C                                                    *)
Print[IsomorphicGraphQ[ZP[SS[5], SS[5]]  , ZA[ SS[25], KK[5,5]] ]];
(*   Example D                                                    *)
Print[IsomorphicGraphQ[ZP[WM, WM ]  , ZA[ PP[9]  , ZP[KK[4],SS[3]] ]]];
(*   Example E                                                    *)
Print[IsomorphicGraphQ[ZA[PP[5], PP[7]]  , KK[5,7] ]];
Print[IsomorphicGraphQ[ZP[KK[2,3],KK[2,3]],ZA[PP[4],PP[9],KK[6,6]]]];
(*   Example  F                                                   *)
Print[IsomorphicGraphQ[ZP[S3,S3 ]  , ZP[ KK[16], PP[4] ] ]];
Print[IsomorphicGraphQ[ZP[OO,OO ]  , ZP[ KK[9],  PP[4] ] ]];
```

12.2. And here is example code illustrating that the Euler characteristic is multiplicative on the strong ring. The Euler characteristic computation uses the Poincaré-Hopf theorem allowing to reduce it to Euler characteristic computations of part of unit spheres.

```
UnitSphere[s_,a_]:=Module[{b=NeighborhoodGraph[s,a]},
  If[Length[VertexList[b]]<2,Graph[{}],VertexDelete[b,a]]];
EulerChi[s_]:=Module[{vl,n,sp,u,g,sm,ff,a,k,el,m,q,A},
  vl=VertexList[s]; n=Length[vl];
  ff=Range[n]; el=EdgeList[s]; m=Length[el];
  g[b_]:=ff[[Position[vl,b][[1,1]]]];
  If[n==0,0,If[n==1 || m==Binomial[n,2],1,If[m==0,n,
  u=Table[ A=g[vl[[a]]]; sp=UnitSphere[s,vl[[a]]];
  q=VertexList[sp]; sm={};
  Do[If[g[q[[k]]]<A,sm=Append[sm,q[[k]]]],{k,Length[q]}];
  If[Length[sm]==0,1,(1-EulerChi[Subgraph[sp,sm]])],{a,n}];
  Sum[u[[k]],{k,n}]]]]]];

NormalizeGraph[s_]:=Module[{r,v=VertexList[s],e=EdgeRules[s]},
  r=Table[v[[k]]->k,{k,Length[v]}]; UndirectedGraph[Graph[v /.r, e /.r]]];
ZykovProduct[s1_,s2_] :=Module[{v,w,f,g,n,o,p,V,e,q,r,A,L=Length},
  v=VertexList[s1]; n=L[v]; f=Union[EdgeList[s1]]; q=L[f];
  w=VertexList[s2]; o=L[w]; g=Union[EdgeList[s2]]; r=L[g];
  V=Partition[Flatten[Table[{v[[k]],w[[l]]},{k,n},{l,o}]],2];
  f=Table[Sort[{f[[k,1]],f[[k,2]]}],{k,q}];
  g=Table[Sort[{g[[k,1]],g[[k,2]]}],{k,r}]; e={}; A=Append;
  Do[e=A[e,{f[[k,1]],w[[m]]}->{f[[k,2]],w[[l]]}],{k,q},{m,o},{l,o}];
  Do[e=A[e,{v[[m]],g[[k,1]]}->{v[[l]],g[[k,2]]}],{k,r},{m,n},{l,n}];
  NormalizeGraph[UndirectedGraph[Graph[V,e]]]];           ZP=ZykovProduct;
StrongProduct[s1_,s2_]:=Module[{t1,t2,t},
  t1=GraphComplement[s1]; t2=GraphComplement[s2]; t=ZykovProduct[t1,t2];
  NormalizeGraph[GraphComplement[t]]];

Do[
s1=RandomGraph[{10,13}];s2=RandomGraph[{12,9}]; ss=StrongProduct[s1,s2];
{a,b,c}={EulerChi[s1],EulerChi[s2],EulerChi[ss]};
Print["a=",a,",b=",b,",c=",c," ___a*b=c_is_",a*b==c],{10}]
```

12.3. Finally, here is example code illustrating that the connection Laplacian tensors when taking the Cartesian product and that the energy is the Euler characteristic. Also this code block is self contained and can be grabbed by looking at the LaTeX source on the ArXiv.

```
CliqueNumber[s_]:=Length[First[FindClique[s]]];
ListCliques[s_,k_]:=Module[{n,t,m,u,r,V,W,U,l={},L},L=Length;
  VL=VertexList;EL=EdgeList;V=VL[s];W=EL[s]; m=L[W]; n=L[V];
  r=Subsets[V,{k,k}];U=Table[{W[[j,1]],W[[j,2]]},{j,L[W]}];
  If[k==1,l=V,If[k==2,l=U,Do[t=Subgraph[s,r[[j]]];
  If[L[EL[t]]==k(k-1)/2,l=Append[l,VL[t]]],{j,L[r]}]]];l];
W[s_]:=Module[{F,a,u,v,d,V,LC,L=Length},V=VertexList[s];
  d=If[L[V]==0,-1,CliqueNumber[s]];LC=ListCliques;
  If[d>=0,a[x_]:=Table[{x[[k]]},{k,L[x]}];
  F[t_,l_]:=If[l==1,a[LC[t,1]],If[l==0,{},LC[t,l]]];
  u=Delete[Union[Table[F[s,l],{l,0,d}]],1]; v={};
  Do[Do[v=Append[v,u[[m,l]]],{l,L[u[[m]]]}],{m,L[u]}],v={}];v];
Fvector[s_]:=Delete[BinCounts[Length /@ W[s]],1];dim[x_]:=Length[x]-1;
Euler[s_]:=Module[{w=W[s],n},n=Length[w];Sum[(-1)^dim[w[[k]]],{k,n}]];
F[A_,z_]:=A-z IdentityMatrix[Length[A]]; F[A_]:=F[A,-1];

ConnectionLaplacian[s1_,s2_]:=Module[{c1=W[s1],c2=W[s2],n,A,c},
  n = Length[c1]*Length[c2]; c = {}; A = Table[1,{n},{n}];
  Do[c=Append[c,{c1[[k]],c2[[l]]}],{k,Length[c1]},{l,Length[c2]}];
  Do[ If[( DisjointQ[c[[k,1]],c[[l,1]]]||DisjointQ[c[[k,2]],c[[l,2]]]) ,
    A[[k,l]] = 0],{k,n},{l,n}]; A];
ConnectionGraph[s_] := Module[{c=W[s],n,A},n=Length[c];
  A=Table[1,{n},{n}];Do[If[DisjointQ[c[[k]],c[[l]]]||
  c[[k]]==c[[l]],A[[k,l]]=0],{k,n},{l,n}]; AdjacencyGraph[A]];
ConnectionLaplacian[s_]:=F[AdjacencyMatrix[ConnectionGraph[s]]];

MyTensorProduct[A_,B_]:=Module[{n1=Length[B],n2=Length[A],n},
  n = n1*n2; Table[B[[1+Mod[k-1,n1],1+Mod[l-1,n1]]]*
  A[[1+Floor[(k-1)/n1],1+Floor[(l-1)/n1]]],{k,n},{l,n}]];

s=RandomGraph[{7,10}];t=RandomGraph[{7,10}];
A=ConnectionLaplacian[s]; B=ConnectionLaplacian[t];
L=ConnectionLaplacian[s,t];
M=MyTensorProduct[A,B]; Print[L==M];
Energy=Total[Flatten[Inverse[L]]]; Energy==Euler[s]*Euler[t]
a=Eigenvalues[1.0*A]; b=Eigenvalues[1.0*B];
c=Sort[Eigenvalues[1.0*M]];
d=Sort[Flatten[Table[a[[k]]*b[[l]],{k,Length[A]},{l,Length[B]}]]];
Print[Total[Abs[c-d]]]
```

A Mathematica demonstration project, featuring the three graph products $\square, \otimes, \boxtimes$ can be seen in [20].

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