

# A SARD THEOREM FOR GRAPH THEORY

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ABSTRACT. The zero locus of a function  $f$  on a graph  $G$  is defined as the graph for which the vertex set consists of all complete subgraphs of  $G$ , on which  $f$  changes sign and where  $x, y$  are connected if one is contained in the other. For  $d$ -graphs, finite simple graphs for which every unit sphere is a  $d - 1$ -sphere, the zero locus of  $(f - c)$  is a  $(d - 1)$ -graph for all  $c$  different from the range of  $f$ . If this Sard lemma is inductively applied to an ordered list functions  $f_1, \dots, f_k$  in which the functions are extended on the level surfaces, the set of critical values  $(c_1, \dots, c_k)$  for which  $F - c = 0$  is not a  $(d - k)$ -graph is a finite set. This discrete Sard result allows to construct explicit graphs triangulating a given algebraic set. We also look at a second setup: for a function  $F$  from the vertex set to  $R^k$ , we give conditions for which the simultaneous discrete algebraic set  $\{F = c\}$  defined as the set of simplices of dimension  $\in \{k, k + 1, \dots, d\}$  on which all  $f_i - c_i$  change sign, is a  $(d - k)$ -graph in the barycentric refinement of  $G$ . This maximal rank condition is adapted from the continuum and the graph  $\{F = c\}$  is a  $(n - k)$ -graph. While now, the critical values can have positive measure, we are closer to calculus: for  $k = 2$  for example, extrema of functions  $f$  under a constraint  $\{g = c\}$  happen at points, where the gradients of  $f$  and  $g$  are parallel  $\nabla f = \lambda \nabla g$ , the Lagrange equations on the discrete network. As for an application, we illustrate eigenfunctions of  $d$ -graphs and especially the second eigenvector of 3-spheres, which by Courant-Fiedler has exactly two nodal regions. The separating nodal surface of the second eigenfunction  $f_2$  always appears to be a 2-sphere in experiments. By Jordan-Schoenflies, both nodal regions would then be 3-balls and the double nodal curve  $\{f_2 = 0, f_3 = 0\}$  would be an un-knotted curve in the 3-sphere. Graph theory allows to approach such unexplored concepts experimentally, as the corresponding question are open even classically for nodal surfaces of the ground state of the Laplacian of a Riemannian 3-sphere  $M$ .

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## 1. INTRODUCTION

We explore vector-valued functions  $F$  on the vertex set of a finite simple graph  $G = (V, E)$ . Most of the notions introduced here are defined for general finite simple graphs. But as we are interested in Lagrange extremization, Morse and Sard type results in graph theory as well as questions in the spectral theory of the Laplacian on graphs related to Laplacians of Riemannian manifolds, we often assume  $G$  to be a  $d$ -graph, which is a finite simple graph, for which all unit spheres are  $(d - 1)$ -spheres in the sense of Evako [20]. In a first setup, more suited for Sard, for all except finitely many choices of  $c \in \mathbb{R}^d$ , the graph  $\{F = c\} = \{f_1 = c_1, \dots, f_k = c_k\}$  is a  $(d - k)$ -graph, in a second setup, closer to classical calculus, we need to satisfy locally a maximal rank condition to assure that the graph representing the discrete algebraic set  $\{F = c\}$  is a  $(d - k)$ -graph.

The first part of the story parallels classical calculus and deals with the concept of level surface. It starts with a pleasant surprise when looking at a single function: there is a strong Sard regularity for a  $d$ -graph  $G$ : given a hyper-surface  $G_f(x) = \{f = c\}$  defined as the graph with vertex set consisting of all simplices on which  $f - c$  changes sign, the graph  $\{f = c\}$  is a  $(d - 1)$ -graph if  $c$  is not a value taken by  $f$ . The topology of  $G_f(c)$  changes only for parameter values  $c$  contained in the finite set  $f(V)$ . This observation was obtained when studying coloring problems [14, 18, 16], as a locally injective function on the vertex set is the same than a vertex coloring. Before, in [9, 8], we just looked at the edges, on which  $f$  changes sign and then completed the graph artificially. Now, we have this automatic. We hope to apply this to investigate nodal regions of eigenfunctions of the Laplacian of a graph, where we believe the answers to be the same for compact  $d$ -dimensional Riemannian manifolds or finite  $d$ -graphs. With the context of level surfaces one has the opportunity to look at nodal surfaces of eigenfunctions, which are also known as Chladni surfaces bounding nodal regions.

With more than one function, the situation changes as the singular set typically becomes larger in the discrete. This is where the story splits. In the commutative setup, where we look at the zero locus of all functions simultaneously, Sard fails, while in the setup, where an ordered set of functions is considered, Sard will be true: as in the classical Sard theorem [22, 23], the set of critical values has zero measure. The difference already is apparent if we take two random functions on a discrete 3-sphere for example. Then simultaneous level set  $\{f = 0, g = 0\}$  is

a graph without triangles but it is rarely a 1-graph, a finite union of circular graphs. The reason is that the tangent space is a finite set and the probability of having two parallel gradients at a vertex does not have zero probability. However, if we look at  $g = 0$  on the two dimensional surface  $f = 0$ , we get a finite union of cyclic graphs. In general, we salvage regularity and Sard by defining the algebraic set  $\{F = c\}$  in a different way by recursively building hyper surfaces: start with the hyper surface  $\{f_1 = c_1\}$ , then extend  $f_2$  to the new graph and look at  $\{f_2 = c_2\}$  inside  $\{f_1 = c_1\}$ . Now a vector value  $c \in R^k$  is always regular if  $c$  is not in the image of  $F$  applied to recursively defined graph. This is the discrete analog of the multi-dimensional Sard theorem in classical analysis. The order in which the functions are taken, matters in the discrete. But this is not a surprise as we refine in each step the graph and therefore have to extend the functions to the barycentric refinements. Its only for sufficiently smooth functions like eigenfunctions of low energy eigenvalues of the graph that the answer can be expected to be independent of the ordering.

The graph theoretical approach is useful for making experiments: take a 3-graph  $G = (V, E)$  for example and take two real valued functions  $f, g$  on the vertex set  $V$ . If  $c$  is not in the image of  $f$ , we can look at the level surface  $f = c$  and extend  $g$  to a function there (vertices are now simplices and we just average the function value  $g$  to extend the function). If  $d$  is not in the image of  $g$ , then the 1-graph  $Z = \{g = d\}$  is a subgraph of  $\{f = c\}$ . It is a finite set of closed curves in  $G_2$ . In other words, each connected component is a knot in the 3-sphere. Unlike in the continuum, we do not have to worry about cases, where the knot intersects or self intersects. The Sard theorem assures that this will never happen in the discrete. We only have to assure that the value  $c$  is not in  $f(V)$  and  $d$  is not in  $g(V)$ .

The second setup, which defines discrete algebraic sets  $Z = \{f_1 = c_1, \dots, f_k = c_k\}$  in the Barycentric refinement  $G_1$  of  $G$  requires a few definitions. We need conditions under which these sets are nonsingular in the sense that they again form a  $(d - k)$ -graph, graphs for which the unit sphere  $S(x)$  is a sphere at every vertex. In general, the situation is the same as in the continuum, where varieties are not necessarily manifolds. The definition is straightforward: define  $\{F = c\}$  as the set of simplices of dimension in  $\{k, \dots, n\}$  in  $G$  on which all functions  $f_j$  change sign simultaneously. The graph  $\{F = c\}$  is a subgraph of the Barycentric refinement  $G_1 = G \times 1$  of  $G$ . Conditions for regularity could be formulated locally in terms of spheres in spheres. In

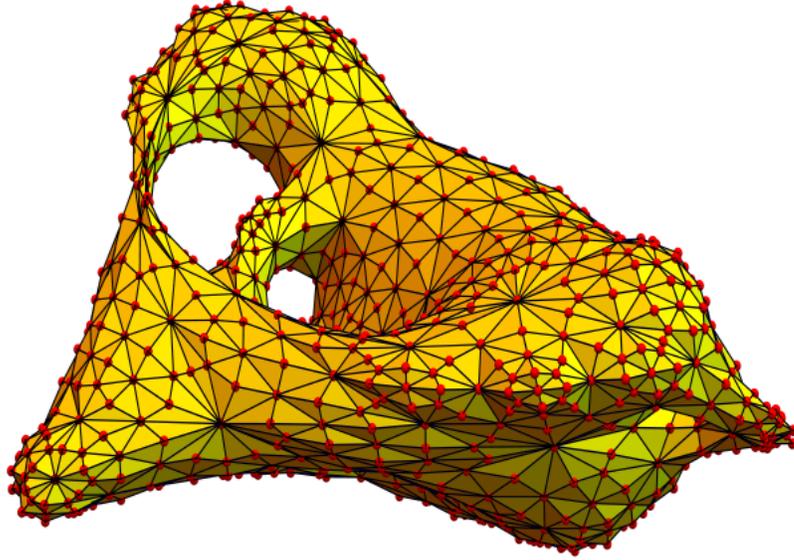


FIGURE 1. We see an example of a level surface  $G_f(c) = \{f = c\}$  in a 3-sphere  $G$ . The graph  $G$  is a Barycentric refinement of a 3D octahedron with 8 vertices with simplex cardinalities  $\vec{v} = (80, 464, 768, 384)$  satisfying  $\chi(G) = \sum_i (-1)^i v_i = 0$ . The zero Euler characteristic is a property shared by all 3-graphs like 3-manifolds. The graph  $G$  admits an involution  $T$  such that  $G/T$  is a discrete projective 3-space. The algebraic level set  $G_f(c)$  displayed here is a 2-graph, a two-dimensional surface within the Barycentric refinement  $G_1$  of  $G$ . The ambient sphere  $G_1$  already has 1696 vertices and 10912 edges and 18432 triangles, 9216 tetrahedra. The surface  $\{f = c\}$  is a discrete analogue of a 2-dimensional projective variety. By the Sard lemma, the level sets  $G_f(c) = \{f = c\}$  are always nonsingular if  $c \notin f(V)$ . Furthermore the surface  $G_f(c)$  is 3-colorable.

some sense, a locally projective tangent bundle with discrete projective spaces associated to unit spheres replaces the tangent bundle. It is still possible to define a vector space structure at each point and define a discrete gradient  $\nabla f$  of a function. If  $G$  is a  $d$ -graph,  $x$  is a complete  $K_{d+1}$  subgraph, then  $\nabla f(x, x_0)$  is defined as the vector  $\nabla f = \langle \sigma(f(x_1) - f(x_0)), \dots, \sigma(f(x_d) - f(x_0)) \rangle$ , where  $\sigma(x) = 0$  for  $x > 0$  and  $\sigma(x) = 1$  for  $x < 0$ . The local injectivity condition means that  $\nabla f(x, x_0)$  is not zero for all  $d$ -simplices containing  $x_0$ . This leads to the Sard Lemma (1). But now, when changing  $c$ , we also need to look at the values  $c_i = f(x_i)$  taken by the function  $f$  and investigate whether they are critical. What happens if  $c$  passes a value  $c_i = f(x_i)$ ? If  $f < c_i$  is homotopic to  $f \leq c_i$ , then the set  $\{f = c\} \cap S(x)$  is a  $(d - 1)$ -sphere which by Jordan-Brouwer-Schoenflies [20] divides  $S(x)$  into two complementary balls  $S^-(x)$ ,  $S^+(x)$ . The Poincaré-Hopf index  $1 - \chi(S^-(x)) = i_f(x)$  [7] is then zero. At a local minimum of  $f$  for example, the graph  $S^-(x)$  is empty so that the index is 1. At a local maximum,  $S^+(x)$  is empty and the index is either 1 or  $-1$  depending on whether the dimension of  $G$  is even or odd. In two dimensions, where we look at discrete multi-variable calculus, maxima and minima have index 1 and saddle points have index  $-1$ .

The set of hypersurfaces  $M_j = \{f = c_j\}$  form a contour map for which the individual leaves in general have different homotopy types. Topological transitions can happen only at values  $f(V)$  of  $f$  on the vertex set  $V$  of the graph. We can relate the symmetric index  $j_f(x) = [i_f(x) + i_{-f}(x)]/2$  at a vertex  $x$  of  $G$  with the  $(d - 2)$ -dimensional graph  $\{f = c_k\}$  in the unit sphere  $S(x)$ . We have called this the graph  $B_f(x)$  and showed that the graph can be completed to become a  $(d - 2)$ -graph. This completion can be done now more elegantly. As pointed out in [13], for  $d = 4$  for example, we get for every locally injective function  $f$  a 2-dimensional surface  $B_f$ , the disjoint union of  $B_f(x)$ . We see that a function  $f$  not only defines  $(d - 1)$ -graphs  $G_f(c) = \{f = c\}$  but also  $(d - 2)$ -graphs  $B_f(x)$  called “central surfaces” for every vertex  $x$ .

If we have more than one function as constraints, the singularity structure of  $\{f_1 = c_1, \dots, f_k = c_k\}$  is more complicated and resembles the classical situation, where singularities can occur. Also the Lagrange setup, where we maximize or minimize functions under constraints, is very similar to the classical situation. The higher complexity entering with 2 or more functions is no surprise as also for classical algebraic sets defined as the zero locus of finitely many polynomials, the case of several functions is harder to analyze. In calculus, when studying

extrema of a function  $f$  under constraints  $g$ , following Lagrange, one is interested in critical points of  $f$  and  $g$  as well as places, where the gradients are parallel. Lets assume that we have two functions  $f, g$  on the vertex sets of a geometric graph. The intersection  $\{f = c, g = d\}$  with  $S(x)$  is then a sphere of co-dimension 2 (a knot). If we have 2 constraints and  $G$  is 4-dimensional for example, then  $S(x)$  is a 3-sphere and  $S(x) \cap \{F = 0\}$  is a knot inside  $S(x)$ . This can already be complicated. By triangulating Seifert surfaces associated by a knot, one can see that any knot can occur as a co-dimension 2 curve of a graph. Briskorn sphere examples allow to make this explicit in the case of torus knots. With more than one function, Poincaré-Hopf indices form a discrete 1-form valued grid because changing any of the  $c_j$  can change the Euler characteristic. This allows to express the Euler characteristic as a discrete line integral in the discrete target set of  $F$ . There are now many Poincaré-Hopf theorems, for every deformation path, there is one.

If we look at the set  $F = c$ , where all functions change sign, singular values for  $F = (f_1, \dots, f_k)$  are vectors  $c = (c_1, \dots, c_k)$  in  $R^k$  for which the graph  $\{F = c\}$  is not a  $(d - k)$ -graph or values  $c = F(x)$  taken on by  $F$  on vertices  $x$ . Lets look at the very special example, where all  $f_j$  are the same function. Now, near the diagonal  $c_i = c$ , there is an entire neighborhood of parameter values, where regularity fails. We see that the Sard statement does not hold in this commutative setting. We therefore also look at the non-commutative setup, where we fix  $f_1 = c_1$  first, then look at the level surface  $\{f_2 = c_2\}$  on the surface  $\{f_1 = c_1\}$  and proceed inductively. Sard is now more obvious, but the set  $\{F = 0\}$  depend on the order, in which the functions  $f_j$  have been chosen. This order dependence is no surprise in a quantum setting, if we look at  $f_j$  as observables. It simply depends in which order we measure and fix the  $f_j$ .

There are analogies to Morse theory in the continuum: in the case of one single function, we can single out a nice class of graphs and functions, which lead to a discrete Morse theory. The geometric graph  $G$  as well as functions on vertices are the only ingredients. One can now assign a Morse index  $m(x)$  at a critical point and have  $i_f(x) = (-1)^{m(x)}$ . The requirement is an adaptation of a reformulation of the definition of being Morse means in the continuum that for a small sphere  $S(x)$ , the set  $S(x) \cap \{f(y) = f(x)\}$  is a product  $S_{m-1} \times S_{d-1-m}$ . For example for  $m = 1, d = 2$ , we have a saddle point, where  $S(x)$  intersects the level surface in 4 points forming  $S_0 \times S_0$ . In the enhanced picture, where we look at the graph  $G_1$ , we have this regularity more likely. One can

extend the function to the new graph and repeat until one get a Morse function.

We can use graphs described by finitely many equations in order to construct examples of graphs to illustrate classical calculus like Stokes or Gauss theorem or surfaces to illustrate classical multivariable calculus. The notion of  $d$ -graphs has evolved from [6, 9, 13, 14] to [18], where it reached its final form. While finishing up [20] a literature search showed that spheres had been defined in a similar way already by Evako earlier on.

The enhanced Barycentric refinement graph  $G_1 = G \times K_1$  is a regularizes graph which helps to study Jordan-Brouwer questions in graph theory [20] and introduce a product structure on graphs which is compatible with cohomology [21]. It also illustrates the Brouwer fixed point theorem [11], as the fixed simplices on  $G$  can be seen as fixed points on  $G_1$ . The simplex picture is also useful as the Dirac operator  $D = d + d^*$  on a graph builds on it [12, 10]. The graph spectra of successive Barycentric refinements converges universally, only depending on the size of the largest complete subgraph [17].

An other application of the present Sard analysis is a simplification of [8] which assures that the curvature  $K$  is identically zero for  $d = (2m + 1)$ -dimensional geometric graphs: write the symmetric index  $j_f(x) = i_f(x) + i_{-f}(x)$  in terms of the Euler characteristic of the  $(d - 2)$ -dimensional graph  $G_f(x)$  obtained by taking the hypersurface  $\{y \mid f(y) = f(x)\}$  in the unit sphere  $S(x)$ . For odd-dimensional geometric  $d$ -graphs, this symmetric index is  $-\chi(B_f(x))/2 = j_f(x)$ , while for even-dimensional graphs, it is  $1 - \chi(B_f(x))/2 = j_f(x)$ . One of the corollaries given here is that if  $f$  is locally injective, then  $B_f(x)$  is always a geometric  $(d - 2)$ -graph if  $G$  is a  $d$ -graph. In [9], we called the uncompleted graphs polytopes and showed that one can complete them. Now, since the expectation of  $j_f(x)$  is curvature, [9, 15], this gives an immediate proof that odd-dimensional  $d$ -graphs have constant 0 curvature. Zero curvature follows for odd-dimensional graphs. The regularity allows to interpret Euler curvature as an average of two-dimensional sectional curvatures. Euler characteristic written as the expectation of these sectional curvature averages is close to Hilbert action. It shows that the quantized functional ‘‘Euler characteristic’’ is not only geometrically relevant but that it has physical potential.

## 2. LEVEL SURFACES

The study of level surfaces  $\{f = c\}$  in a graph is not only part of discrete differential topology. It belongs already to discretized multivariable calculus, where surfaces  $\{f(x, y, z) = 0\}$  in space or curves  $\{f(x, y) = 0\}$  in the plane are central objects. Some would call calculus on graphs “quantum calculus”. We want to understand under which conditions a sequence of  $k$  real valued functions  $F = (f_1, \dots, f_k)$  on the vertex set of a graph leads to co-dimension  $k$  graphs  $\{F = 0\}$ . We start with the case  $k = 1$ , where we have a level surface  $\{f = c\}$ .

**Definition 1.** Given a finite simple graph  $G = (V, E)$ , define the **level hyper surface**  $\{f = c\}$  as the graph  $G' = (V', E')$  for which the vertex set  $V'$  is the set of simplices  $x$  in  $G$ , on which the function  $f - c$  changes sign in the sense that there are vertices in  $x$  for which  $f - c < 0$  and vertices in  $x$  for which  $f - c > 0$ . A pair of simplices  $(x, y) \in V' \times V'$  is in the edge set  $E'$  of  $G'$  if  $x$  is a subgraph of  $y$  or if  $y$  is a subgraph of  $x$ . In the case  $c = 0$ , the graph  $\{f = 0\}$  is also called the **zero locus** of  $f$ .

**Remarks.**

- 1) Instead of taking a real-valued function, we could take a function taking values in an ordered field. We need an ordering as we need to tell, where a function “changes sign”.
- 2) Most of the time we will assume  $c$  is not a value taken by  $f$ . An alternative would be to include all simplices which contain a vertex on which  $f = 0$ .

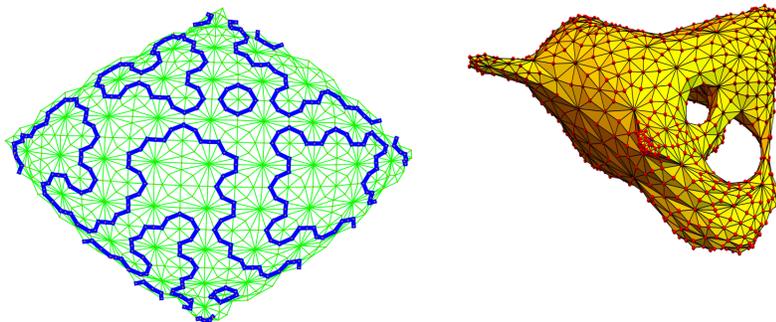


FIGURE 2. A level curve in a planar graph. A level surface in a 3-sphere.

**Examples.**

- 1) Let  $G$  be a 2-sphere like for example an icosahedron. Let  $f$  be 1 on exactly one vertex  $x$  (a discrete Dirac delta function) and 0 everywhere else. Now  $\{f = 1/2\}$  consists of all edges and triangles containing  $x$ . They form a circular graph and  $\{f = 0\} = C_{2\deg(x)}$ .
- 2) If  $G$  is a  $d$ -graph and  $x$  is a vertex, let  $f(x)$  be the geodesic distance to the vertex  $x$ . Now  $\{f = 1/2\}$  is the Barycentric refinement of the unit sphere  $S(x)$ . In the previous example,  $S(x)$  was a circular graph and  $\{f = 1/2\}$  a circular graph with twice the size.
- 3) Let  $G = C_n \times C_n \times C_n$  be a discrete 3-torus and let  $f$  be a function which is 1 on a circular closed graph and 0 else. Then  $\{f = 1/2\}$  is a 2-dimensional torus.

## 3. THE SARD LEMMA

The following definitions are recursive and were first put forward by Evako. See [20] for our final version.

**Definition 2.** A  $d$ -**sphere** is a finite simple graph for which unit sphere  $S(x)$  is a  $(d - 1)$ -sphere and such that removing a single vertex from the graph renders the graph contractible. Inductively, a graph is **contractible**, if there exists a vertex such that both  $S(x)$  and the graph generated with vertices without  $x$  are contractible.

**Definition 3.** A  $d$ -**graph**  $G$  is a finite simple graph for which every unit sphere  $S(x)$  is a  $(d - 1)$ -sphere.

**Examples.**

- 1) A 1-graph is a finite union of circular graphs, for which each connectivity component has 4 or more vertices.
- 2) The icosahedron and octahedron graphs are both 2-graphs. In the first case, the unit spheres are  $C_5$ , in the second case, the unit spheres are  $C_4$ .
- 3) In [19] we have classified all **Platonic d-graphs** using Gauss Bonnet [5]. Inductively, a  $d$ -graph  $G$  is called Platonic, if there exists a  $(d - 1)$ -graph  $H$  which is Platonic such that all unit spheres of  $G$  are isomorphic to  $H$ . In dimension  $d = 3$ , there are only two Platonic graphs, the 16 and 600 cell. In dimensions  $d \geq 4$ , only the cross polytopes are platonic.

The following Sard lemma shows that we do not have to check for the geometric condition if we look at level surfaces: it is guaranteed, as long as we avoid function values in  $f(V)$ . Not even local injectivity is needed:

**Lemma 1** (Sard lemma for real valued functions). *Given a function  $f : V \rightarrow \mathbf{R}$  on the vertex set of a  $d$ -graph  $G = (V, E)$ . For every  $c \notin f(V)$ , the level surface  $\{f = c\}$  is either the empty graph or a  $(d - 1)$ -graph.*

*Proof.* We have to distinguish various cases, depending on the dimension of  $x$ . If  $x$  is an edge, where  $f$  changes sign, then  $f$  changes sign on each simplex containing  $x$ . The set of these simplices is a unit sphere in the Barycentric refinement  $G_1$  of  $G$  and therefore a  $(d - 1)$  sphere. If  $x$  is a triangle, then there are exactly two edges contained in  $x$ , on which  $f$  changes sign. The sphere  $S(x)$  in  $\{f = c\}$  is a suspension of a  $(d - 2)$ -sphere: this is the join of  $S_0$  with  $S_{d-2}$ . In general, if  $x$  is a complete subgraph  $K_k$  then the unit sphere  $S(x)$  is a join of a  $(k - 2)$ -dimensional sphere and a  $(d - k)$ -dimensional sphere, which is a  $(d - 2)$ -dimensional sphere. As each unit sphere  $S(x)$  in  $\{f = c\}$  is a  $(d - 2)$ -sphere, the level surface  $\{f = c\}$  is a  $(d - 1)$ -graph.  $\square$

**Examples.**

1) If  $d = 2$ , and  $f$  changes sign on a triangle  $K_3$ , then it changes sign on exactly two of its edges. If  $f$  changes sign on an edge, then it changes sign on exactly two of its adjacent triangles. We see that the level surface  $\{f = c\}$  is a graph for which every vertex has exactly two neighbors. In other words, each unit sphere is the 0-sphere.

2) If  $d = 3$ , and  $f$  changes sign on a tetrahedron  $x = K_4$ , then there are two possibilities. Either  $f$  changes sign on three edges connected to a vertex in which case we have 3 edges and 3 triangles in the unit sphere of  $K_4$  with 4 vertices. A second possibility is that  $f$  changes on 4 edges and 2 triangles in which case the unit sphere consists of 6 vertices. Now look at a triangle  $x = K_3$ . It is contained in exactly two tetrahedra and contains two edges. The unit sphere is  $C_4$ . Finally, if  $x = K_2$  is an edge, then all triangles and tetrahedra attached to  $x$  form a cyclic graph of degree  $C_{2n}$  where  $n$  is the number of tetrahedra hinging on  $x$ .

The Sard lemma can be used for minimal colorings for which the number of colorings is exactly known:

**Corollary 2.** *If  $G$  is a  $d$ -graph and  $c$  is not in the range of  $f$ , then the surface  $H = \{f = c\}$  is a  $(d - 1)$ -graph which is  $d$ -colorable. The chromatic polynomial  $p(x)$  of these graphs satisfies  $p(d) = d!$ .*

*Proof.* To every vertex of  $G_1$ , we can attach a “dimension” which is the dimension of the simplex in  $G$  it came from. This dimension is the coloring. It remains a coloring when looking at subgraphs.  $\square$

**Examples.**

1) For a level surface  $f = 0$  on a  $d$ -dimensional graph, we get a  $(d - 1)$  graph which is  $(d + 1)$ -colorable. For example, for  $d = 2$ , the graph can be colored with 3 colors. This is minimal as any triangle already needs 3 colors. It implies the graph is Eulerian: the vertex degree is even everywhere.

2) If  $\Omega$  is the set of colorings with minimal color of  $G$ , we can for every vertex  $x$  look at the **index**  $i_f(x) = 1 - \chi(S_f^-(x))$  where  $S_f^-(x)$  is the set of vertices  $y$  on the sphere  $S(x)$ , where  $f(y) < f(x)$ . Given a geometric graph  $G$  of dimension  $d$ , then  $G_1$  is  $d + 1$  colorable. The number of colorings is  $(d+1)!$ . We can look at the list of indices which are possible on each point and call this the index spectrum. The set of vertices  $x$  where  $\{f < c\}$  changes the homotopy type are called **critical points** of  $f$ . If the index is nonzero, then we have a critical point because the Euler characteristic is a homotopy invariant. But there are also critical points with zero index, as in the continuum. Finding extrema of  $f$  can be done by comparing the function values of all vertices where  $i_f(x)$  is not zero or more generally, where  $S_f^-(x)$  is not contractible. At a local minimum, the graph  $S^-f(x)$  is empty.

## 4. THE CENTRAL SURFACE

Besides the surface  $G_f(c) = \{f = c\}$ , there is for every vertex  $x$  a central surface of co-dimension 2 which is obtained by looking at level surfaces  $B_f(x)$  obtained by looking at  $\{y \mid f(y) = (f(x))\}$  inside the unit sphere  $S(x)$ . This object was introduced in [8, 13] for 4-graphs  $G$ , where the central surface is a 2-dimensional graph, a disjoint union of 2-dimensional subgraphs  $B_f(x)$  of the 3-dimensional unit spheres  $S(x)$ .

**Definition 4.** A real-valued function  $f$  on the vertex set of a graph  $G = (V, e)$  is called **locally injective**, if  $f(x) \neq f(y)$  for  $(x, y) \in E$ . An other word for a locally injective function is a **coloring**.

**Definition 5.** Given a  $d$ -graph  $G$ , a locally injective function  $f$  and a vertex  $x_0$ , define the **central surface**  $B_f(x_0)$  as the level surface  $\{f = f(x_0)\}$  in  $S(x_0)$ . It is a  $(d - 2)$  graph. The graph consists of all simplices in  $S(x_0)$  for which  $f$  takes values smaller or larger than  $f(x_0)$ .

Each of these surfaces are subgraphs of their unit sphere  $S(x_0)$ . We have one surface for each vertex  $x_0$ . On each sphere  $S(x, y) = S(x) \cap S(y)$ , we can look at the intersection of  $B_f(x)$  and  $B_f(y)$ . It consists of all simplices in  $S(x, y)$  where both  $f(z) - f(y)$  and  $f(z) - f(x)$  change sign. Sometimes they can be joined together along a circle. For example, given a 3-graph  $G$ , then the union  $B_f$  of all  $B_f(x)$  consists of all

edges and triangles so that the max and min on each larger tetrahedron are attained in the edge or triangle. The following definition was first done in [7]:

**Definition 6.** Given a finite simple graph  $G$  and a locally injective real-valued function  $f$  on the vertex set  $V$ , the **Poincaré-Hopf index** is defined as  $i_f(x) = 1 - \chi(S_f^-(x))$ , where  $S_f^-(x)$  is generated by  $\{y \in S(x) \mid f(y) < f(x)\}$ . The **symmetric index** is defined as  $j_f(x) = (i_f(x) + i_{-f}(x))/2$ .

The Poincaré-Hopf theorem [7] tells that  $\sum_{x \in V} i_f(x) = \chi(G)$ . Since this also holds for the function  $-f$ , we have

$$\sum_{x \in V} j_f(x) = \chi(G) .$$

The following remark made in [8] expresses  $j_f(x)$  as the Euler characteristic of a central surface, provided the graph is geometric:

**Proposition 3.** *Given a  $d$ -graph  $G$  and a locally injective function  $f$ . If  $B_f(x)$  is the central surface, then for odd  $d$ , we have*

$$j_f(x) = -\chi(B_f(x))/2 ,$$

for even  $d$ , we have

$$j_f(x) = 1 - \chi(B_f(x))/2 .$$

*Proof.*  $B_f(x)$  is a  $(d-2)$ -graph in  $S(x)$ , which by assumption is a  $(d-1)$ -graph. Since  $f$  is locally injective, the function  $g(y) = f(y) - f(x)$  does not take the value 0 on  $S(x)$ . By the Sard lemma, the graph  $B_f(x)$  is a  $(d-2)$ -graph.  $\square$

**Corollary 4.** *For a  $d$ -graph  $G$  with odd dimension  $d$ , the curvature*

$$K(x) = 1 - \frac{S_0(x)}{2} + \frac{S_1(x)}{3} - \frac{S_2(x)}{4} + \dots$$

*(where  $S_k(x)$  are the number of  $K_{k+1}$  subgraphs of the unit sphere  $S(x)$ ) has the property that it is constant zero for every vertex  $x$ .*

*Proof.* The expectation  $j_f(x)$  is curvature [9, 15]. Since  $j_f(x)$  is identically zero as the Euler characteristic of an odd dimensional  $(d-2)$ -graph, also curvature is identically zero.  $\square$

Note that in the continuum, the Euler curvature is not even defined for odd dimensional manifolds as the definition involves a Pfaffian [2]. Having the value 0 in the discrete is only natural.

**Definition 7.** A function  $f$  on the vertex set is called a **Morse function** if it is locally injective and if at every critical point, there is a positive integer  $m$ , such that  $B_f(x) = \{f(y) = f(x)\}$  within  $S(x)$  is a product  $S_{m-1} \times S_{d-1-m}$  or the empty graph if  $m = 0$  or  $m = d$ . The integer  $m$  is called **Morse index** of the critical point  $x$ .

Depending on whether  $m$  is odd or even, we have  $\chi(B_f(x)) = 4$  or  $0$  so that the index  $1 - \chi(B_f(x))/2 = -1$  if  $m$  is odd and  $1$  if  $m$  is even. When adding a critical point, this corresponds to add a  $m$ -dimensional handle. It changes the Euler characteristic by  $(-1)^m$  and changes the  $m$ 'th cohomology by  $1$ .

## 5. LAGRANGE

In this section we try to follow some of the standard calculus setup when extremizing functions with or without constraints. But it is done in a discrete setting, where space is a graph. As school calculus mostly deals with functions of two variables, we illustrate things primarily for 2-dimensional graphs, even so everything can be done in any dimensions.

There are three topics related to critical points in two dimensions: A) extremizations without constraints, B) equilibrium points of vector fields and C) extremization problems with constraints which are called Lagrange problems. In the case A), can look at extrema of a function  $f$  on the vertex set of a 2-graph, in the case B) we look at equilibrium points of a pair  $F = (f, g)$  of functions on the vertex set of a 2-graph, and finally in the case C), we look at extrema of a function  $f$  on the vertex set under the constraint  $g = c$  on a 2-graph.

Lets first look at the “second derivative test” on graphs. Recall that a vertex  $x$  in a graph is a critical point of a function  $f$ , if  $\{f < f(x)\}$  and  $\{f \leq f(x)\}$  are not homotopic. This is equivalent to the statement that  $S_f^-(x) = \{y \in S(x) \mid f(y) < f(x)\}$  is a graph which is not contractible. The analogue of the discriminant  $D$  is the Poincaré-Hopf index  $i_f(x) = 1 - \chi(S_f^-(x))$ . Here is the analogue of the second derivative test:

**Proposition 5** (Second derivative test). *Let  $G$  be a  $d$ -graph and assume  $f$  is locally injective and  $x$  is a critical point. There are three possibilities:*

- a) *If  $S_f^-(x)$  is a  $(d - 1)$ -sphere, then  $x$  is a local maximum.*
- b) *If  $i_f(x)$  is positive and  $S_f^-(x)$  is empty then  $x$  is a local minimum.*
- c) *If  $d = 2$  and if  $i_f(x)$  is negative, then  $x$  is a type of saddle point.*

*Proof.* For a 2-dimensional graph, the index is nonzero if and only if  $x$  is a critical point because a subgraph of a circular graph has Euler characteristic 1 if and only if it is a contractible graph. It has Euler characteristic 0 if and only if it is either empty or the full circular graph. In all other cases of subgraphs of a circular graph, the Euler characteristic counts the number of connectivity components.

In higher dimensions, there are cases of graphs  $S_f^-(x)$  having Euler characteristic 1 but not being contractible. In higher dimensions,  $S_f^-(x)$  can have negative Euler characteristic so that the index  $i_f(x)$  can become larger than 1.  $\square$

**Example.**

1) For  $d = 2$ , the standard saddle point is  $i_f(x) = -1$ . The function  $f$  changes sign on 4 points. A discrete “Monkey saddle” has index  $i_f(x) = -2$ . It is obtained for example at a vertex  $x$  for which the unit ball is a wheel graph with  $C_6$  boundary such that  $f(y)$  is alternating smaller or bigger than  $f(x)$ .

2) If  $d$  is odd and  $f$  has a local maximum,  $i_f(x) = -1$ . This is analogue to the continuum, where  $D$  is the determinant of the Hessian.

For simplicity, we restrict to a simple 2-dimensional situation, where we have two functions  $f, g$  on the vertex set  $V$  of a 2-graph  $G = (V, E)$ . We can think of  $F = (f, g)$  it as a vector field and see the equilibrium points are the intersection of null-clines as in the continuum. Classically, the critical points of  $f$  under the constraint  $g = c$  are the places, where these null-clines are tangent or are degenerate in that one of the gradients is zero.

**Definition 8.** Let  $F = (f, g)$  be two functions on the vertex set of a 2-graph. The set of equilibrium points  $\{(x, y) \mid F(x, y) = (0, 0)\}$  is a zero-dimensional graph given by the set of triangles, where  $f$  and  $g$  both change sign.

In more generality we have defined the set  $G_F = \{f_1 = 0, \dots, f_k = 0\}$  as the graph whose vertex set consists of the set of simplices of dimension in  $\{k, \dots, d\}$  on which all functions  $f_j$  change sign and where two simplices are connected, if one is contained in the other. It is possible for example that for a  $d$ -graph, the set  $G_F$  consists of **all**  $d$ -dimensional simplices in  $G$ . This is still a 0-graph, a graph with no edges because no  $d$ -simplex is contained in any other  $d$ -simplex.

**Definition 9.** Let  $G$  be a 2-graph and let  $f$  be a locally injective function on the vertex set of  $G$ . Define the **continuous gradient** of  $f$  in a triangle  $t = (xyz)$  with origin  $x$  as  $df = \langle f(y) - f(x), f(z) - f(x) \rangle$ .

This assigns to each triangle in  $G$  a vector with two real components. By taking signs, the vector  $\nabla f$  becomes an element in the finite vector space  $Z_2^2$ . This is the **discrete gradient** on the triangle  $t$  rooted at the vertex  $x$ .

**Example.**

1) If  $G$  is a 2-graph and  $x$  belongs to a triangle  $t$  containing two edges which both contain  $x$ , then  $\nabla f(x, t) = \langle 1, 1 \rangle$ . Two functions have a parallel gradient in  $t$ , if and only if the sign changes in  $t$  happen on the same edges of the triangle.

**Remarks:**

1) A function  $f$  on the vertex set  $V$  is a 0-form. The exterior derivative  $df$  is a function on the edges of the graph. The exterior derivative depends on a choice of orientations on complete subgraphs. For 1 forms in particular, where edges have been ordered at first, the situation at a vertex  $x$  defines then  $df((x, y)) = f(y) - f(x)$ . When restricting to a  $d$ -simplex, we get  $d$  real numbers, we as in the continuous forms the **gradient**.

2) The ordering of the coefficients of the gradient vector depends on the orientation of the triangle. This is similar to the classical case, where a triangle in a triangulation of a surface defines a normal vector at  $x$ , once a vertex  $x$  of the triangle and an orientation of the triangle is given.

3) The analogue of rank  $d(f, g) = 2$  means that simultaneous sign changes happen on one edge of a triangle only, so that the two sign change is in the same direction on that edge. Geometrically this implies that the spheres  $f = 0, g = 0$  in  $S(x)$  intersect transversely. Two discrete gradients in  $Z_2^d$  are parallel if and only  $\nabla f = \pm \nabla g$ .

The classical Lagrange analysis shows that the critical points of  $F$ , the place where the Jacobean  $dF$  has rank 0 or 1 are candidates for maxima of  $f$  under the constraint  $g = c$ . In the same way as in the continuum, we can write down Lagrange equations for any number of functions on a  $d$ -graph. The most familiar case for two functions:

**Proposition 6.** *Let  $G$  be a  $d$ -graph. If the discrete gradients  $\nabla f, \nabla g$  are nowhere parallel, then  $F = \{f = 0, g = 0\}$  is a  $(d - 2)$ -graph.*

It should be possible to estimate the measure of the set of global critical values if  $F = (f_1, \dots, f_k)$  are functions on the vertex set of a finite simple graph  $G$ . Instead of doing so, we will look later at a Sard setup

for which the critical values have zero measure.

Lets look at the example of two functions  $f, g$  on a 3-graph. The set  $\{F = 0\}$  is the set of simplices, where both  $f, g$  change sign. The condition  $dF$  having maximal rank means that the surface  $f = 0$  on  $S(x)$  and the sphere  $g = 0$  in  $S(x)$  intersect in a union of 1-spheres. Here is the Lagrange setup with two functions in three dimensions:

**Proposition 7.** *If  $F = (f, g)$  are two functions on the vertex set of a 3-graph and if  $dF$  has maximal rank at every  $x$  and  $(c, d)$  is not in  $F(V)$ , then  $F = c$  is either empty of a 1-graph, a finite collection of circles.*

*Proof.* The set  $\{F = c\}$  is a graph whose vertices are the triangles ( $K_3$  subgraphs of  $G$ ) or tetrahedra ( $K_4$  subgraphs of  $G$ ), where both  $f$  and  $g$  change sign. Let  $x$  be a tetrahedron in  $\{F = c\}$ . The maximal rank condition prevents parallel gradients like  $\nabla f = \langle 1, 1, 1 \rangle = \nabla g = \langle 1, 1, 1 \rangle$  so that it is impossible to have 3 triangles in  $\{F = c\}$  inside  $x$ . Assume a single triangle is present where both  $f, g$  change sign, then there is a common edge, where both  $f, g$  change sign and a second triangle must also be in  $\{F = c\}$ . We see that there are exactly two triangles  $y, z$  present on which both  $f, g$  change sign. This means the vertex  $x$  has exactly two neighbors  $y, z$ . Each of the triangles  $y, z$  has two neighbors, the tetrahedra attached to them. We see that  $\{F = c\}$  is a graph with the property that every unit sphere is the zero sphere  $S_0$ . Therefore, it is a finite collection of circular graphs.  $\square$

The Lagrange problem extremizing  $f$  under the constraint  $g = c$  is the following:

**Proposition 8** (Lagrange). *Given two functions  $f, g$  on a 2-graph. Extremizing  $f$  under the constraint  $g = c$  happens on triangles, where  $df$  and  $dg$  are parallel (which includes the case when  $df = 0$  or  $dg = 0$ ). In other words, extrema happen on Lagrange critical points.*

The following maximal rank condition is the same as in the continuum. It tells that the graph formed by the simplices having dimension in  $\{k, \dots, d\}$  on which all functions  $f_k$  change sign simultaneously is a  $(n - k)$ -graph if the rank of the set of gradients in  $Z_2^k$  is maximal:

**Theorem 9** (Regularity in the commutative setup). *Assume  $G$  is a  $d$ -graph and  $F$  is a  $R^k$ -valued function on the vertex set. If  $\nabla F$  has maximal rank on every  $d$ -simplex, then  $\{F = c\}$  is a  $(d - k)$ -graph.*

*Proof.* For a fixed vertex  $x$  and  $d$ -simplex  $X$ , we have a tangent space  $Z_2^d$  for which every function  $f_j$  contributes a vector  $\nabla f_i$  telling on which

edges emanating from  $x$  inside  $X$ , the sign of  $f_i$  changes. By assumption, the  $k$  vectors  $\nabla f_i$  are linearly independent.

Use induction with respect to  $d$ : if  $k = d$ , there is nothing to show because by definition, the set  $\{F = c\}$  is the set of  $d$ -simplices, where all functions change sign and this is a 0-graph as there are no edges. We claim that in the unit sphere  $S(x)$ , the functions  $f_1, \dots, f_k$  induce  $k$  linearly independent gradients  $\nabla f_j$  on the  $(d-1)$ -simplex  $Y = X \setminus x$ . Indeed, on each triangle, the sum of the  $df_j$  values is zero (as  $\text{curl}(\text{grad}(f)) = 0$ ). A nontrivial relation between the gradients on  $Y$  would induced a nontrivial additive relation between the gradients on  $X$ . Now, by induction, the  $k$  functions on  $S(x)$  define a  $(d-1-k)$ -graph.  $\square$

**Remark.** One could also try induction with respect to  $k$  and consider  $\{f_1 = c_1\}$  which is a  $(d-1)$ -graph, by the Sard lemma. The problem is that one has to extend  $f_2, \dots, f_d$  in such a way on the simplices so that one has still maximal rank condition. The problem is that  $\{f_1 = c_1\}$  has now a different vertex set than  $G$  and that linking things is difficult.

### Examples:

- 1) The case  $d = 3, k = 2$  was discussed in Proposition 7.
- 2) In the case  $d = 4, k = 2$ , we want the two functions both to be locally injective and the gradients of the two functions  $f, g$  not to be parallel. The set  $F = c$  consists of all tetrahedra  $K_4$  and hypertetrahedra  $K_5$  on which both  $f$  and  $g$  change sign. The gradients restricted to the tangent space on  $S(x)$  are not parallel and we can apply the analysis of the previous case to each unit sphere which shows that in  $S(x)$  the set  $F = c$  is a collection of circular graphs. This shows that the unit spheres of  $F = c$  have the property that each unit sphere there is a circular graph. Therefore  $F = c$  is a geometric 2-graph, a surface if  $\nabla f, \nabla g$  are nowhere parallel.
- 3) In the case  $d = 4, k = 3$ , it is the first time that the maximal rank condition is not just a parallel condition. Given a vertex  $x$  and a tetrahedron  $t = (x, y_1, y_2, y_3)$ . The discrete gradient is  $\langle \sigma(f(y_1) - f(x)), \sigma(f(y_2) - f(x)), \sigma(f(y_3) - f(x)) \rangle$ . where  $\sigma(x) = \arg(\log(\text{sign}(x)))/\pi$  is equal to 0 if  $x > 0$  and 1 if  $x < 0$ . An example of a violation of the maximal rank condition for three functions  $f, g, h$  would be  $\nabla f = \langle 1, 0, 1 \rangle, \nabla g = \langle 1, 1, 0 \rangle, \nabla h = \langle 0, 1, 1 \rangle$ . They are pairwise not parallel but  $\nabla f + \nabla g + \nabla h = 0$ .
- 4) The non-degeneracy condition is not always needed: for  $d$  functions  $F = (f_1, \dots, f_d)$  on a  $d$ -graph, we look at the simplices on which

all functions change sign. The condition  $dF \neq 0$  implies that the set of  $d$ -dimensional simplices on which all functions  $f_j$  change sign are isolated.

## 6. SARD THEOREM

Since the set of critical values can have positive measure if we look at the simultaneous solution, we change the setup and look at hypersurfaces in hypersurfaces. This will lead to a Sard theorem as in the continuum. We will have to pay a prize: the order with which we chose the hypersurfaces within hypersurfaces now will matter.

**Definition 10.** A function  $f$  on the vertex set is called **strongly injective** if all function values  $f(x_i)$  are rationally independent. It is **strongly locally injective** if in each complete subgraph, the values are rationally independent. A list of functions  $f_1, \dots, f_k$  is called **strongly injective** if the union of all function values  $f_j(x_i)$  are rationally independent.

Strongly injective functions are generic from the measure and Baire point of view: given a finite simple graph  $G$  with  $n$  vertices. Look at the probability space  $\Omega$  of all functions from the vertex set to  $[-1, 1]$ , where the probability measure is the product measure on  $[-1, 1]^n$ .

**Lemma 10.** *For any  $k$ , with probability one, a random sample  $f_1, \dots, f_k$  in  $\Omega^k$  is strongly locally injective.*

*Proof.* There are  $v = v_0 + v_1 + \dots + v_d$  complete subgraphs in  $G$ . They define the  $\sum_i i \cdot v_i$  numbers  $c_{ij} = f_i(x_j)$ . There is a countably many rational independence conditions  $\sum_{ij} a_{ij} c_{ij} = 0$  to be avoided, where  $a_{ij}$  are integers. The complement of a countable union of such hyperplane sets of zero measure in  $[-1, 1]^n$  and consequently has zero measure.  $\square$

**Definition 11.** Given an ordered list of functions  $f_1, f_2, \dots, f_k$  from the vertex list  $V$  of a finite simple graph  $G = (V, E)$  and  $c_1, \dots, c_k$  be  $k$  values. Let  $\bar{f}_1 = f_1$ . Denote by  $\bar{f}_2$  the function  $f_2$  extended to  $\{\bar{f}_1 = c_1\}$  and by  $\bar{f}_3$  the function  $f_3$  extended to  $\{\bar{f}_1 = c_1, \bar{f}_2 = c_2\}$  etc, always assuming that  $\bar{f}_{j+1}$  defined on  $\{\bar{f}_1 = c_1, \dots, \bar{f}_j = c_j\}$  does not take the value  $c_{j+1}$ . We call the sequence  $c_1, \dots, c_k$  **compatible** with  $f_1, \dots, f_k$  if a sequence  $\bar{f}_j$  can be defined so that none of them are not constant.

**Theorem 11** (Discrete Sard for an ordered set). *Given  $k$  strongly injective functions  $f_1, \dots, f_k$ . For all except a finite set of vectors*

$(c_1, \dots, c_k)$  the sequence  $c_1, \dots, c_k$  is compatible and the set  $\{f_1 = c_1, \dots, f_k = c_k\}$  is a geometric  $(d - k)$ -graph.

*Proof.* This follows inductively from the construction. In each step, only a finite set of  $c$  values are excluded.  $\square$

The assumption is stronger than what we need. It sometimes even works in the extreme case of two identical functions: Let  $f$  be the function on the octahedron given the values  $f_1(1) = 13, f_1(2) = 15, f_1(3) = 17, f_1(4) = 19$  on the equator and the value  $f_1(5) = 1$  on the north pole and the value  $f_1(6) = 31$  on the south pole. Lets take  $c_1 = 2$ . Now,  $\{f_1 = c_1\}$  is the cyclic graph with vertices  $\{(51), (512), (52), (523), (53), (534), (54), (541)\}$ . The function  $\bar{f}_2$  takes there the values  $\bar{f}_2(51) = (f_2(5) + f_2(1))/2 = (1 + 13)/2 = 7, \bar{f}_2(52) = (f_2(5) + f_2(2))/2 = (1 + 15)/2 = 8, \bar{f}_2(53) = (f_2(5) + f_2(3))/2 = (1 + 17)/2 = 9, \bar{f}_2(54) = (f_2(5) + f_2(4))/2 = (1 + 19)/2 = 10, \bar{f}_2(512) = (f_2(5) + f_2(1)f_2(2))/2 = (1 + 13 + 15)/3 = 29/3, \bar{f}_2(523) = (f_2(5) + f_2(2)f_2(3))/2 = (1 + 15 + 17)/3 = 35/3, \bar{f}_2(534) = (f_2(5) + f_2(3)f_2(4))/2 = (1 + 17 + 19)/3 = 37/3, \bar{f}_2(523) = (f_2(5) + f_2(2)f_2(3))/2 = (1 + 19 + 13)/3 = 11$ . Now for example, for  $c_2 = 8.5$  the set  $\{f_2 = c_2\}$  is a 0-graph.

As an example, lets look at the **double nodal surface**  $f_3 = 0$  in  $f_2 = 0$ , where  $f_3$  is the third eigenvector. By Sard, we know:

**Corollary 12.** *If  $G$  is a  $d$ -graph and the eigenfunctions  $f_2, f_3$  are strongly injective not having the value 0, then the double nodal surface is a  $(d - 2)$ -surface in  $G_2$ .*

**Example:**

1) For the octahedron, the smallest 2-sphere, the spectrum is of the Laplacian is

$$\{0, 4, 4, 4, 6, 6\}$$

The ground state space is the eigenspace to the eigenvalue 4 and three dimensional and spanned by  $(-1, 0, 0, 0, 0, 1), (0, -1, 0, 0, 1, 0), (0, 0, -1, 1, 0, 0)$ . The eigenspace to the eigenvalue 6 is spanned by  $(1, 0, -1, -1, 0, 1), (0, 1, -1, -1, 1, 0)$ . In both eigenspaces, there are injective functions  $f_2 = (-1, -2, -3, 3, 2, 1), f_3 = (1, 2, -3, -3, 2, 1)$  in the eigenspace. The graph  $\{f_2 = 0\}$  is the cyclic graph  $C_{12}$  while the graph  $\{f_3 = 0\}$  is  $C_8 \cup C_8$ . The graph  $\{f_3 = 0\}$  within  $\{f_2 = 0\}$  is a two point graph. The graph  $\{f_2 = 0\}$  within  $\{f_3 = 0\}$  is not defined as  $f_2$  extended to the simplex set in a linear way produces a lot of function values 0.

It leads to a generalization of a result we have shown for 2-graphs:

**Corollary 13.** *Any compact  $d$ -manifold  $M$  has a finite triangulation which is  $(d + 1)$ -colorable.*

*Proof.* By Nash-Tognoli, any compact manifold can be written as a variety  $F = c$  in some  $R^d$ . Now just rewrite this in the discrete as the zero locus of  $F = c$ .  $\square$

**Corollary 14.** *The curvature at a vertex  $x$  of such a graph triangulation can be written as the expectation of  $d!$  Poincaré-Hopf indices.*

If  $\{f = 0\}$  be the zero locus of  $f : G \rightarrow R$ . It would be nice to find a smaller homeomorphic graph which represents this set.

## 7. NODAL SETS

To illustrate a possible application, let's look at the problem of nodal sets of the Laplacian  $L$  of a graph. Understanding the **Chladni patterns** of the Laplacian on a manifold with or without boundary is a classical problem in analysis. The nodal region theorem of Courant in the discrete also follows from the min-max principle [4, 3, 24]. For a general graph  $G$ , and an eigenfunction  $f$ , one looks at the number of connectivity components of  $Z_f^+ = \{\pm f \geq 0\}$ . Let  $v_k$  be the  $k$ 'th eigenvector. Since both for compact Riemannian manifolds as well as for finite graphs, the zero eigenvalues are not interesting as harmonic functions are locally constant, we look primarily at the second or third eigenvalue. The Fiedler nodal theorem assures then that the graph generated by  $\vec{v}_k > 0$  has maximally  $k - 1$  components. Especially, the second eigenvector, the "ground state", always has exactly two nodal components. If an eigenfunction  $f$  of the Laplacian is locally injective and has no roots, we can look at its nodal surfaces  $f = 0$  separating the nodal regions. If  $f$  does not take the value 0, then  $\{f = 0\}$  is defined even if  $f$  is not locally injective. As in the continuum, one can ask how big the set  $\{v \in V \mid f(v) = 0\}$  can become.

If  $G$  is a 2-sphere, then the two nodal surface is a simple closed curve in  $G$ . How common is the situation that the ground state does not take the value 0 and is locally injective? The situation that 0 is in the range of the ground state appears to be rare. For random 2-spheres (of the order of 500 vertices generated by random edge refinements from platonic and Archimedean solids) we get a typical ground state energy in the order of 0.08 and the third eigenvalue in the order 0.2.

[25] note the following **eigenfunction principle**: any eigenfunction to an eigenvalue  $0 < \lambda < n$  takes the value 0 on every vertex of degree  $n - 1$ , if  $n$  is the number of vertices: Proof: let  $f$  be the

eigenfunction to an eigenvalue  $\lambda$ . The function  $f$  is perpendicular to the harmonic constant function so that  $\sum_{x \in V} f(x) = 0$ . From  $Lf(v) = (n-1)f(v) - \sum_{x \neq v} f(x) = \lambda f(v)$ , we get  $nf(v) = \lambda f(v)$  which by assumption implies  $f(v) = 0$ .

**Example.**

For a wheel graph with  $n$  vertices, there is one eigenvalue  $n$  with eigenvector  $(1-n, 1, 1, \dots, 1)$ , an eigenvalue  $0$  with eigenvector  $(1, 1, \dots, 1)$ . All eigenfunctions to eigenvalues between take the zero value somewhere. For example, in the case  $n = 7$ , the eigenvalues are  $\{0, 2, 2, 4, 4, 5, 7\}$  with eigenvectors  $[0, -1, -1, 0, 1, 1, 0]$ ,  $[0, 1, 0, -1, -1, 0, 1]$ ,  $[0, -1, 1, 0, -1, 1, 0]$ ,  $[0, -1, 0, 1, -1, 0, 1]$ ,  $[0, -1, 1, -1, 1, -1, 1]$  and  $[-6, 1, 1, 1, 1, 1, 1]$ .

Let  $G$  be a  $d$ -graph. Let  $f$  be the ground state, the eigenvector to the first nonzero eigenvalue  $\lambda$ , the spectral gap. Let  $d$  denote the exterior derivative. There will be no confusion with  $d$  also denoting the dimension of  $G$ . We assume that the eigenvalue  $\lambda$  is simple, that  $f$  has no roots and that  $df$  has no roots. This assures that  $Z = 0$  is a geometric  $(d-1)$ -graph. As  $L = dd^*$  on  $0$  forms, we get from  $d^*df = \lambda f$  that  $dd^*df = \lambda df$  so that  $df$  is an eigenfunction to the one form Laplacian  $L_1 = dd^* + d^*d$ . The set  $\{df = 0\}$  of all triangles and simplices where both  $f$  and  $df$  changes sign is the same than  $f = 0$ .

What is the topology of the hypersurface  $Z_2 = \{f_2 = 0\}$ ? For a 2-sphere  $G$ , we know that there are two components so that the principal nodal curve  $Z = \{f = 0\}$  has to be a circle. Is the nodal curve to the second eigenvalue a  $(d-1)$ -sphere, if  $G$  is a  $d$ -graph? We believe that the answer is yes and robust in the following sense: if  $f$  should have some root, we can add a small random function  $g$  with  $|g(x)| \leq \epsilon$  without roots. We expect that for sufficiently small  $\epsilon > 0$  and almost all  $g$ , the surface  $Z' = \{f+g = 0\}$  has the same topology than  $Z = \{f = 0\}$ . The reason for this intuition is that while  $Z'$  is a modification of  $Z$  the modification should not be strong enough to change the topology of the graph.

Does the topology of the nodal manifold to the second eigenvalue depend on the topology of  $G$  only?

We think the answer is yes as  $Z$  has to be a connected surface and that going from a genus  $k$  to a genus  $k+1$  surface can not happen so easily. More generally, we expect: if  $G$  and  $H$  are homotopic graphs of

the same dimension, then the second nodal manifolds of  $G$  and  $H$  are homotopic.

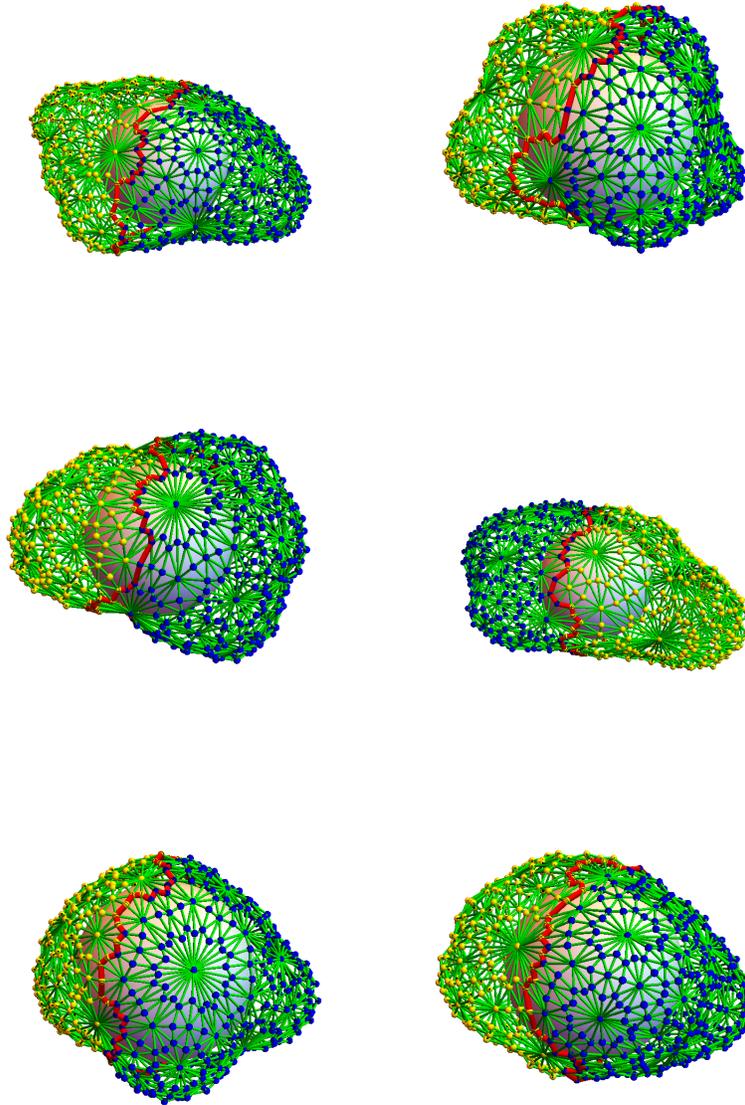


FIGURE 3. Level surfaces of the second eigenfunction  $f_2$  of the Laplacian on  $G_2$ , where  $G$  is a 2-sphere. The dividing surface is then always a Jordan curve.

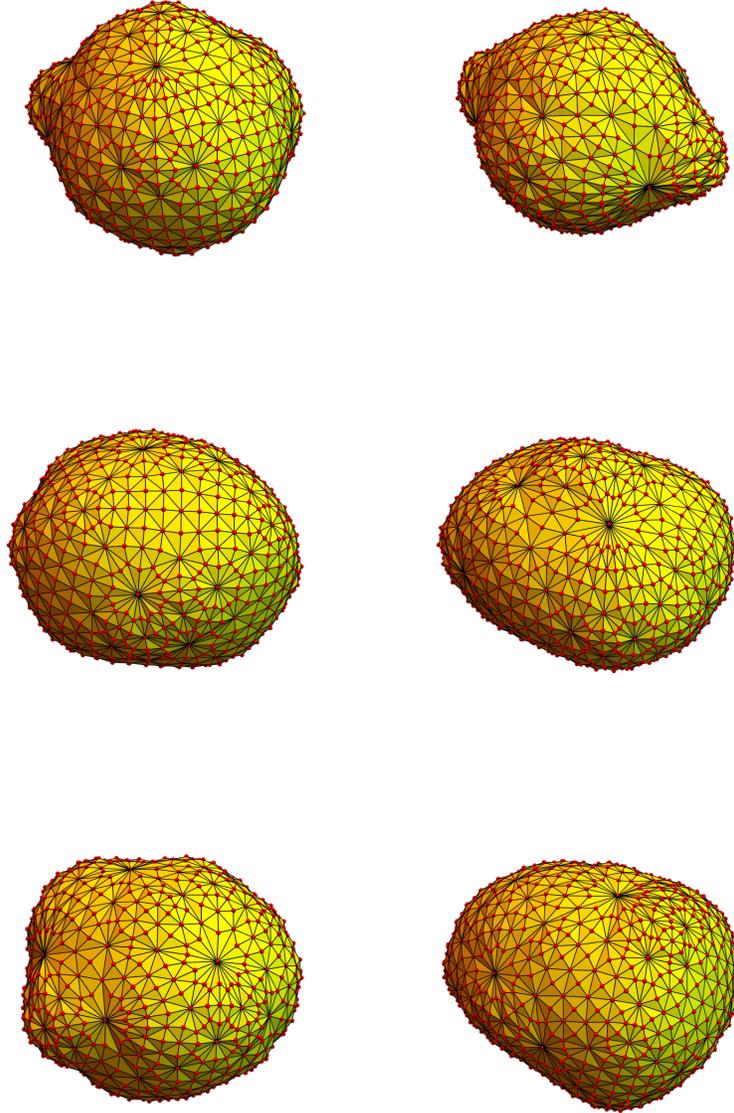


FIGURE 4. Level surfaces of the second eigenfunction of the Laplacian, where  $G$  are 3-spheres. The dividing surface is then a 2-graph. We believe that  $\{f_2 = 0\}$  is always a 2-sphere, if  $G$  is a 3-sphere and  $f_2$  is the eigenvector to the smallest nonzero eigenvalue. A more general question is whether two whether the nodal region  $\{f_2 > 0\}$  is always simply connected if  $G$  is a  $d$ -sphere.

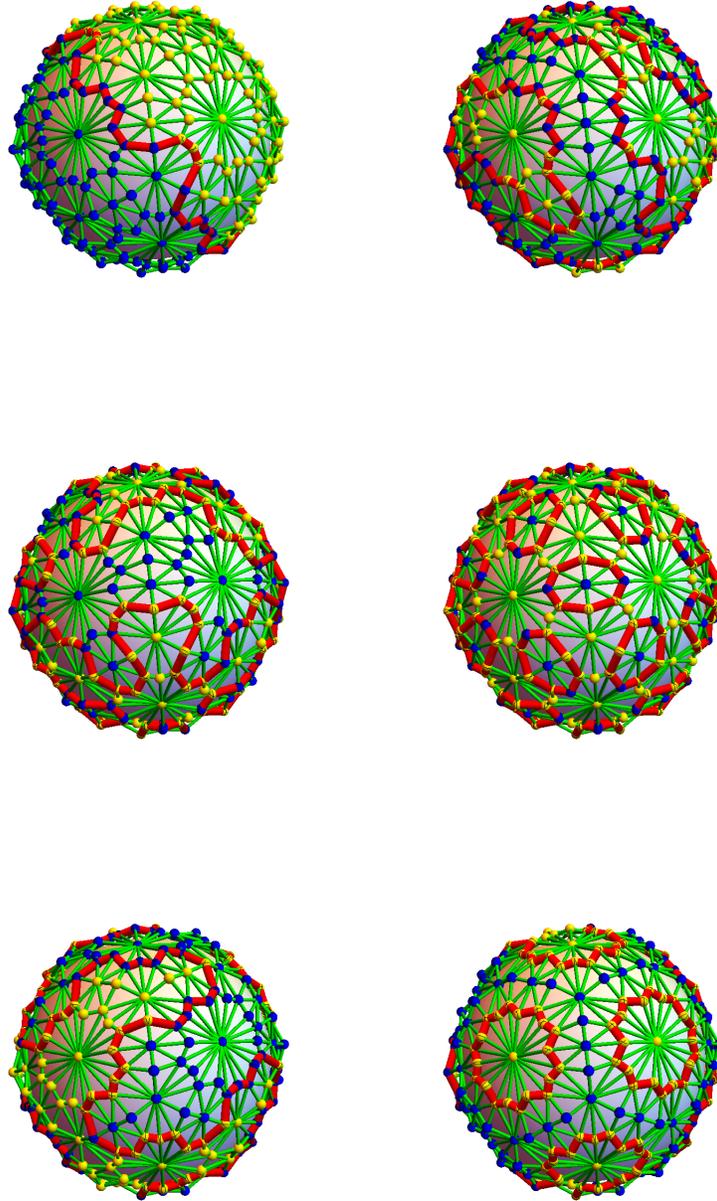


FIGURE 5. Level surfaces of eigenfunctions of the Laplacian on a graph  $G_1$ , where  $G$  is the icosahedron. The Courant-Fiedler nodal theorem tells that the number of positive nodal regions of  $\lambda_n$  is  $\leq n$ . It is confirmed in the pictures seen. The first picture shows the second eigenvector. The first one is constant and has only one region.

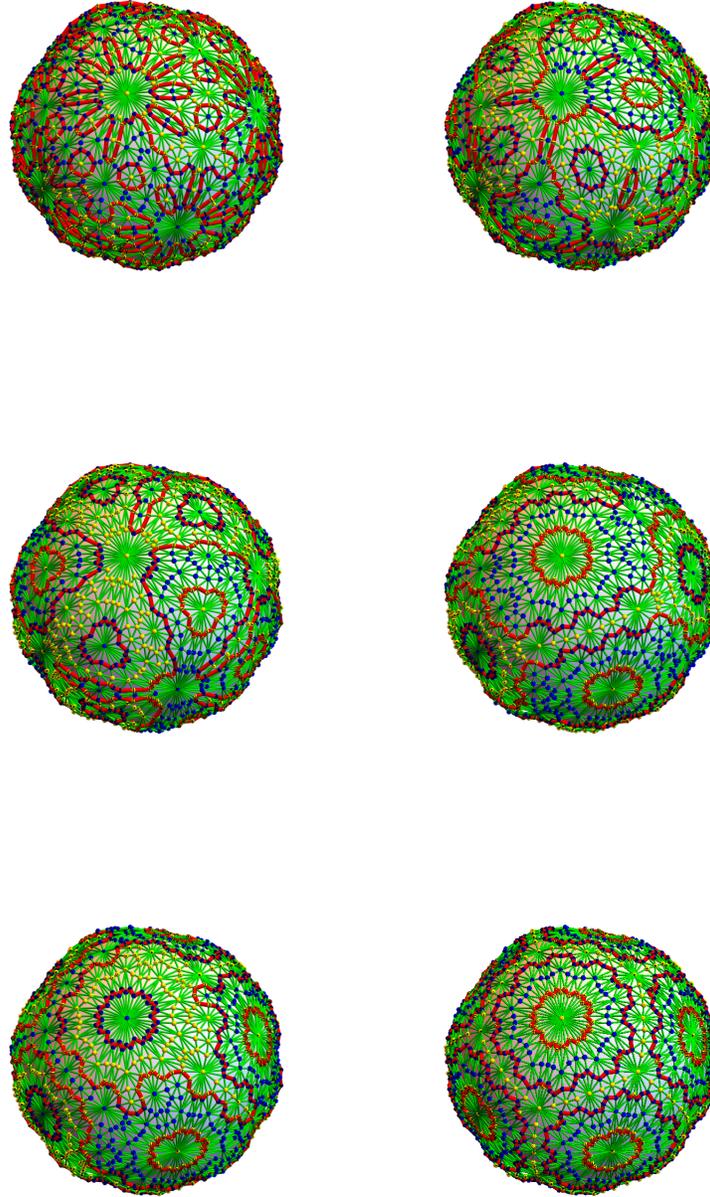


FIGURE 6. Level surfaces of eigenfunctions of the Laplacian on  $G_2$ , where  $G$  is the icosahedron. Also this illustrates the Courant-Fischer theorem. The first picture shows the second eigenvector for which the surface  $Z_2$  divides the graph into two regions. The first one is constant and has only one region.

## 8. SUMMARY

A  $d$ -graph is a finite simple graph for which every unit sphere is a  $(d-1)$  sphere. Given  $k$  locally injective real-valued functions  $f_1, \dots, f_k$  on the vertex set of such a  $d$ -graph, we can define the zero locus  $Z = \{F = 0\}$  of  $F$  in  $G$  as the graph  $(V, E)$  with vertex set  $V$  consisting of complete subgraphs in  $G$  of dimension  $\{k, \dots, d\}$ , on which all the functions  $f_j$  change sign and for which two vertices are connected if one is a subgraph of the other. The graph  $Z$  is a subgraph of the barycentric refinement  $G_1$  but like varieties in the continuum, the graph might be singular and not be a  $(d-k)$ -graph. We can define a discrete tangent bundle on  $G_1$  so that the projection of  $\nabla f$  in  $T_x G = Z_2^d$  to  $T_x H$  is zero, establishing the graph analogue of “gradients are perpendicular to level surfaces”. If every unit sphere  $S(x)$  in  $G_1$  has the property that  $Z \cap S(x)$  is a  $(d-k-1)$ -sphere in the  $(d-1)$ -sphere  $S(x)$ , then  $F = c$  is a  $(d-k)$ -graph  $H$ . The later plays the role of  $\{f_1 = c_1, \dots, f_k = c_k\}$  for  $k$  differentiable functions  $f$  on a smooth manifold  $M$  for which the maximal rank condition  $\text{rank}(dF)(x) = d - k$  for  $F = (f_1, \dots, f_k), x \in Z$  is satisfied. Discrete Lagrange equations help to maximize or minimize  $f$  under a constraint  $g = c$ : as in the continuum, critical points need parallel gradients  $\nabla f, \nabla g$ .

If the **ordered zero locus** is defined by looking at hypersurfaces on successive barycentric refinements, we never run into singularities if the functions are strongly locally injective in the sense that the function values of  $f_j$  on the vertex set are rationally independent on each complete subgraph. The **Sard theorem** states then that the graph  $\{f_1 = c_1, \dots, f_k = c_k\}$  is a  $(d-k)$ -graph inside the  $d$ -graph  $G_k$  for all  $\vec{c}$  not in a finite set. An application is the observation that any  $d$ -dimensional projective algebraic set admits an approximation by manifolds with a triangulation which is a minimally  $(d+1)$ -colorable graph, using an explicitly constructed  $d$ -graph determined by the equations defining  $V$ .

The possibility to define level surfaces in discrete setups can be illustrated in the case of eigenfunctions of the Laplacian. Pioneered by Chladni, the geometry of nodal surfaces is important for the physics of networks. Of special interest is the **ground state**  $f_2$  of a compact Riemannian manifolds or finite graph. The nodal surface  $\{f_2 = 0\}$  is a  $(d-1)$ -dimensional hypersurface and the double nodal surface  $f_2 = 0, f_3 = 0$  is generically a codimension 2 manifold. If the manifold or graph is a 3-sphere, the nodal surface is two dimensional and the

double nodal surface is a collection of closed curves in  $S^3$ . We asked here whether  $f_2 = 0$  is always a sphere and whether  $f_2 = 0, f_3 = 0$  can be a nontrivial knot. Graph theory allows to investigate this experimentally. The answers for Riemannian manifolds and graphs are expected to be the same.

Lets look at some **multi-variable calculus terminology** in a 2-graph translated to graph theory:

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Critical point $x$	$S_f^-(x)$ is not contractible
Discriminant $D$	Poincaré-Hopf index $i_f(x) = 1 - \chi(S_f^-(x))$
$D > 0, f_{xx} < 0$	$i_f(x) = 1, S_f^-(x) = S_f(x)$
$D > 0, f_{xx} > 0$	$i_f(x) = 1, S_f^-(x) = \emptyset$
$D < 0$	$i_f(x) < 0$
$D = 0$	not locally injective
Level curve $f(x, y) = 0$	zero locus $f = 0$ in $G_1$ .
$T_x M$	maximal simplex $t$ in $G$ containing $x$
Tangent vector $T_x M$	$\nabla f = \sigma(f(y) - f(x)) \mid y \in t$
Lagrange equations	$\nabla f = \lambda \nabla g$ or $\nabla g = (0, 0)$

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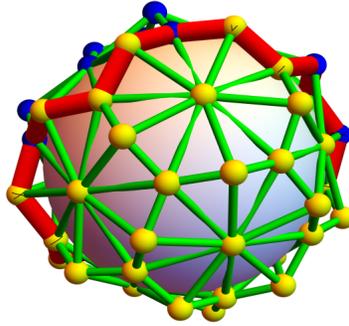


FIGURE 7. The nodal hypersurface  $\{f = 0\}$  of the ground state. The length of the surface is 20, the two nodal domains have 30 triangles each. The Cheeger number (see [1] for a discrete version),  $c(f) = |C(f)|/\min(|A(f)||B(f)|)$  satisfies the Cheeger inequality.

## 9. QUESTIONS

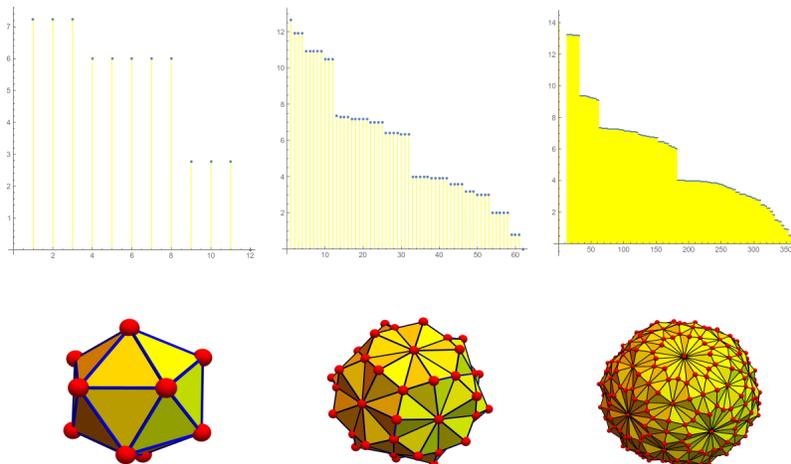


FIGURE 8. The spectrum of refinements  $G_0, G_1, G_2$  of the icosahedron graph  $G = G_0$ . We noticed that the spectral integrated density of states converges universally to a function which only depends on the dimension of the maximal complete subgraph of  $G$ . While for graphs without triangles, the convergence of  $F_G(x) = \lambda_{[kx]}$  to the limiting function  $4 \sin^2(\pi x/2)$  is immediate, in higher dimensions, the limiting function appears to be non smooth. See [17].

A) In the commutative case, where  $F = c$  is a subgraph of the barycentric refinement  $G_1$ , the set of critical values can have positive measure. It would be nice to find an upper bound on the measure of constants  $c$  for which the graph  $\{F = c\} \subset G_1$  can be singular.

B) In the context of graph colorings, the following question is analog to the **Nash embedding problem**: Is there a  $k = k(d)$ , such that every  $d$ -graph is a subgraph of a barycentric refinement  $G_m$  of  $G_0 = K_k$ ? A yes answer would give a bound  $k + 1$  for the chromatic number of  $G$ . A special case  $k = 3$  is related to the 4-color theorem. If every 2-sphere could be embedded in a Barycentric refinement of  $K_4$ , one could see why 4 colors suffice. We approached this problem by writing the sphere as embedded in a 3-sphere then making homotopy deformations to render the sphere Eulerian and so 4-colorable. While

a Whitney embedding of a graph is possible if we aim for a homeomorphic image, we don't have yet a sharp discrete analogue of the classical Whitney embedding theorem, fixing the dimension. Realizing a graph as a subgraph of a product of linear graphs is analogue to an isometric embedding of a Riemannian manifold in some Euclidean space. It is a discrete Nash problem.

C) If  $f, g$  are two strongly locally injective function on some  $d$ -graph. Under which conditions is it true that the level set  $g = d$  in  $f = c$  is topologically equivalent to the level set  $g = d$  in  $f = c$ ? Random examples in a 3-sphere show that the answer is "no" in general. It might therefore be possible that for eigenfunctions,  $f_2 = 0$  in  $f_3 = 0$  is different that  $f_3 = 0$  in  $f_2 = 0$  but we have not yet an example, where the two  $(d - 2)$  surfaces are different.

D) Assume  $G$  is a 3-sphere and the ground state  $f_2$  (eigenvector to the smallest positive eigenvalue) does not take the value 0. Is the nodal hyper surface  $\{f_2 = 0\}$  always a 2-sphere? The case where 0 is in the image of the ground state  $f_2$  only appears in very rare cases. As long as the 0's are isolated, we can randomly change the value on such places and not change the topology of the nodal surface  $\{f_2 = 0\}$ . It would be nice to verify this in general for  $d$ -spheres and check whether in general  $\{f_2 = 0\}$  is a  $(d - 1)$  sphere without having to assume 0 is not in the range of  $f_2$ .

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