DISCRETE ALGEBRAIC SETS IN DISCRETE MANIFOLDS

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ABSTRACT. A discrete d-manifold is a finite simple graph $G = (V, E)$ where all unit spheres are $(d-1)$-spheres. A d-sphere is a d-manifold for which one can remove a vertex to make it contractible. A graph is contractible if one can remove a vertex with contractible unit sphere to get a contractible graph. We prove a discrete Morse-Sard theorem: if $G = (V, E)$ is a d-manifold and $f : V \to \mathbb{R}^k$ an arbitrary map, then for any $c \notin f(V)$, a level set $\{f = c\}$ is always a $(d-k)$-manifold or empty. While a priori open sets in the simplicial complex of G, they are sub-manifolds in the Barycentric refinement of G. Level sets are orientable if G is orientable. Any complex-valued function ψ on a discrete 4-manifold M defines so level surfaces $\{\psi = c\}$ which are except for $c \in f(V)$ always 2-manifolds or empty.

1. INTRODUCTION

1.1. We continue to work on the Morse-Sard theme [27, 30] in a finite graph theoretical setting. Graphs come with a simplicial complex in which the set of vertex sets G of complete sub-graphs are the elements. An Alexandrov topology on G [1] is obtained by taking the simplicial sub-complexes as closed sets [21]. While [18] looked at one hyper-surface at a time, needing the k'th Barycentric refinement to define $H = \{f_1 = 0, \ldots, f_k = 0\}$, we define here arbitrary discrete co-dimension k varieties in a discrete d- manifold: for all d-manifolds G and all $f : G \to \mathbb{R}^k$, and all $c \notin f(G)$, the discrete algebraic set H is either a $(d-k)$ manifold or empty.
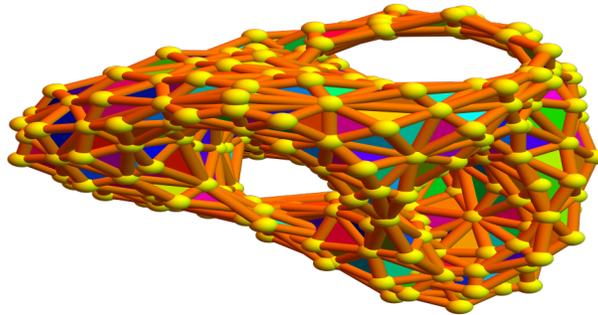

FIGURE 1. We see a random level surface of a function $f : G = \mathbb{RP}^4 \to \mathbb{C}$. The host complex $\mathbb{RP}^4$ has the f-vector $(16, 120, 330, 375, 150)$, the Betti vector $(1, 0, 0, 0, 0)$ and the Euler characteristic 1. Unlike in an orientable host manifold, the surface $H = \{f = c\}$ can be non-orientable. In this case, we have hit a Klein bottle of genus 2. It has the f-vector $(76, 174, 94)$, Betti vector $(1, 5, 0)$ and Euler characteristic -4.

1.2. Our new discrete Morse-Sard statement does not need to impose any conditions on the functions f_i. The theorem holds for the vector space of all functions $V \to \mathbb{R}^k$. In the discrete, we have explicit control about the set of regular values. In comparison, for manifolds in the continuum, the classical Morse-Sard theorem only implies that it is a set of zero measure. Consider for example a spin system $f : G \to \{-1, 1\}$ on a discrete m-manifold. The discrete Morse-Sard theorem tells that $\{f = 0\}$ is either a $(m - 1)$-manifold or empty. When taking several functions, we can so generate small models of higher-dimensional manifolds and so get probability spaces of such structures. Level sets are given at first as delta sets and are in general much smaller than their simplicial complex implementations. We can visualize the Barycentric refinements as simplicial complexes of graphs.

1.3. Among discrete geometry settings, the language of **finite simple graphs** is accessible and is a built-in data structure in many computer algebra implementations. The code section below in this article shows some computer algebra implementation of which illustrates this. Also more general structures like simplicial complexes or **delta sets** can be visualized as graphs. A **finite abstract simplicial complex** for example is a finite set of non-empty sets, closed under the operation of taking non-empty subsets. [4]. Taking these sets as nodes and connecting two if one is contained in the other produces a graph with the same topological features than the complex. It can be useful to extend the frame work slightly and work on **delta sets**, a structure that still has an elementary singular chain complex and is more robust with respect to essential operations: **1)** taking open sets in simplicial complexes or **2)** taking quotients to see the host as a branched cover **3)** taking products of simplicial complexes and **4)** taking level set $S = \{f = 0\}$ of maps. For the later, we take the sets on which the functions change sign and impose some non-degeneracy condition, replacing the notion of the Jacobean df having maximal rank. While the level sets in a simplicial complex is an open set at first, their graph realizations define again closed sets.

1.4. The face maps of a delta set allow to compute the cohomology groups directly using kernels of concrete Hodge Laplacian matrix defined without a Barycentric refinement. This is important because the computation of the delta set representing a d-manifold takes a factor of at least $(d + 1)!$ less time than computing it for the simplicial complex of the graph. For cohomology, the time improvement is even more dramatic, as we would need compute with much larger matrices, leading to a change of complexity that is polynomial in $(d + 1)!$. We still want to realize the level set S as a graph because we want to "see" it as a d-manifold and we have the above natural definition of manifold only for graphs or simplicial complexes. Also the delta set version of the Cartesian product of two manifolds is a manifold, once realized as a graph. The cohomology of the product of two simplicial complexes can be computed faster without realizing it as a graph. In [17], we had used a product which is non-associative and related to the Stanley-Reisner ring. It was the product of delta sets followed by a graph realization.

1.5. An example of a symmetry for manifolds are Dehn-Sommerville relations. They are very general [32, 12, 19, 20]. Here is an example: the f-vector of any 4-manifold is perpendicular to the **Dehn-Sommerville vector** $(0, -22, 33, -40, 45)$. The later is an eigenvector of the transpose of the Barycentric refinement matrix. Dehn Sommerville helped us to see that an odd dimensional manifold has zero curvature. Let us look at an example: if G is a 2-sphere with f-vector $(6, 12, 8)$, the f-vector of the delta set and 4-manifold $G \times G$ is the convolution $(6, 12, 8)*(6, 12, 8)$, which is $(36, 144, 240, 192, 64)$. This is not perpendicular to the above Dehn-Sommerville vector. The f-vector of the graph representing the 4-manifold $S^2 \times S^2$ however is $(676, 8928, 28992, 34560, 13824)$ which is perpendicular. To compute the cohomology, the

Hodge matrix of the delta set implementation of $S^2 \times S^2$ is a 676×676 matrix, while the Hodge matrix belonging to the simplicial complex of the graph is a 86980×86980 matrix. This is a significant reduction of complexity. By starting with delta set representations, the size can be even much smaller.

1.6. An other place where co-dimension 2 surfaces matter is when rewriting **curvature** $K(x)$ of an even-dimensional manifold in terms of expectations of the Euler characteristics of discrete algebraic set in $S(x)$ [15, 16]. Thanks to Sard, it can conveniently be related to the Euler characteristic of a co-dimension 2 manifold. For odd-dimensional manifolds M, this immediately gives that the curvature is constant zero, something which bothered us still in 2011 [13] where we only had experimental evidence yet. The Dehn-Sommerville symmetries allowed to see this. An other path to verify this was through index expectation. The symmetric Poincaré-Hopf index $j_f(x) = i_f(x) + i_{-f}(x)$ can be written as $-\chi(M_f(x))$, where $M_f(x)$ is a level surface contained in the unit sphere of x. The symmetric Poincaré-Hopf index are therefore constant zero for odd-dimensional manifolds and curvature $K(x) = \mathrm{E}[j_f(x)]$ given as expectation is then zero too.

1.7. The co-dimension two picture generalizes what we know classically for Riemannian manifolds. Let us explain this in dimension 2, where it is elementary in the discrete and goes back to Eberhard [6] in the discrete. It took longer in the continuum (see [2] for history). Let M is a classical 2-dimensional Riemannian manifold, meaning now a continuum manifold, not a discrete manifold. The index $j_f(x) = 1 - \{y \in S_r(x), f(y) = f(x)\}/2$ at a point $x \in M$ is for small enough positive r and for a generic smooth f is a **divisor**. This means that it is an **integer-valued function** on M, which is non-zero only at finitely many points. These points are located on the singularity set of f on M. For **Morse functions** f on M, one has $j_f(x) = -1$ at saddle points and $j_f(x) = 1$ at maxima and minima. Poincaré-Hopf tells that the number of maxima and minima minus the number of saddles is the Euler characteristic. When averaging $j_f(x)$ over all Morse functions one gets the traditional Gaussian curvature provided the probability measure on Morse functions is locally homogeneous. This can be achieved by placing M into an ambient Euclidean space and taking a rotationally symmetric Haar measure on the probability space of all linear functions.

1.8. When Nash-embedding a general compact Riemannian manifold into an Euclidean space E [23] and taking the expectation $K(x) = \mathrm{E}[i_f(x)]$ over the probability space of all linear functions with measure invariant under all rotations in the Euclidean space E we get the normalized **Gauss curvature** satisfying $\int_M K(x)\ dV$, if dV is the normalized Riemannian volume measure. The same works for any even-dimensional manifold and the index expectation is the Euler integrand in the **Gauss-Bonnet-Chern** result for even-dimensional Riemannian manifolds. For discrete Gauss-Bonnet, see [13, 20].

delta set most general (topos!)

Finite set of sets X with map $\dim : X \to \{0,1,2,\ldots\}$ and face maps $d_i : X_{n+1} \to X_n$ for $i = 0,\ldots,n+1$, satisfying $d_i d_j = d_{j-1} d_i$ with $X_n = \{x, \dim(x) = n\}$.

$V = X, E = \{(a,b), a \subset b \text{ or } b \subset a\}$
where $a \subset b$ means $a = d_{i_1} d_{i_2} \cdots d_{i_m} b$

$d_i(x) = d_i(x_0, \ldots, x_n) = x \backslash \{x_i\}$
$\dim(x) = |x| - 1$

finite simple graph most intuitive (see it!)

(V,E), finite vertex set V and edge set
$E \subset \{\{a,b\}, a \in V, b \in V, a \neq b\}$

simplicial complex most elegant (one axiom!)

Finite set X of non-empty sets closed under the operation of taking finite non-empty subsets.

$X = \{x, x \subset V, x$
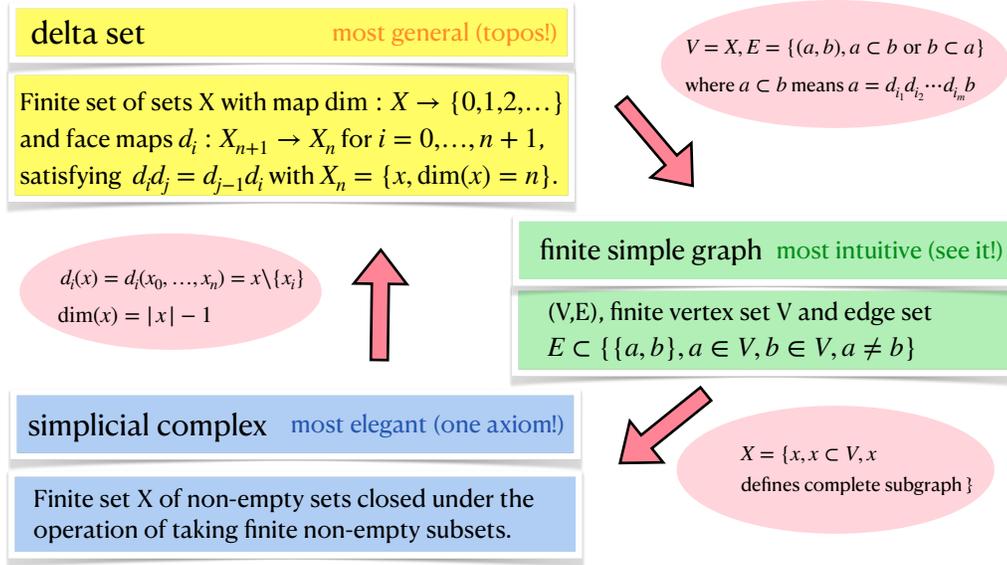defines complete subgraph $\}$

FIGURE 2. Delta sets, simplicial complexes or graphs form a **trinity of structures**, where each structure has advantages. We see it aligned with the Unix philosophy "Simplicity, Clarity and Generality" [11]. Going around once in this circle produces a Barycentric refinement.

2. DISCRETE MORSE SARD

2.1. We start with giving the definition of a manifold. We have used these notions for many years now. They were motivated by notions in "digital topology" (in particular [7, 9]) and "discrete Morse theory" (which takes a slightly different approach like defining spheres through critical points [8]). The history of manifolds in the discrete and continuum is interest [24, 31, 5, 29]. We start with the important notion of **unit sphere** $S(v)$ of a vertex v in a graph $G = (V, E)$. The graph $S(v)$ is the sub-graph generated by the set W of vertices directly attached to v, meaning that $S(x) = (W, E_W)$, where $E_W \subset E$ consists of all edges (a, b) in E for which both a, b are in W. The name "unit sphere" is justified because it is the unit sphere in the metric space obtained by taking the geodesic distance as metric on the graph.

2.2. The inductive definitions are close to piecewise linear (PL) geometry. It is however constructive in the sense that we can find out in finitely many computation steps whether a graph is a manifold or not: a finite simple graph $G = (V, E)$ is a d-**manifold** if every unit sphere $S(v), v \in V$ is a $(d-1)$-sphere. A d-manifold is called a d-**sphere** if the graph $G - v$ is contractible for some $v \in V$. A graph is **contractible** if both $S(v)$ and $G - v$ are contractible for some $v \in V$. The empty graph 0 is the (-1)-sphere, the 1-point graph 1 is contractible.

2.3. The reason to stick with "contractible" rather than "homotopic to 1" is because recognizing spheres is decidable as defined, while recognizing spheres using "1-homotopy" is not [28, 3, 10]. While the set-up does not matter for mathematical theorems and most results in this area can easily be bent to the larger non-constructive version of manifolds, it matters when working with concrete models as a computer scientist, where we consider it important that we can decide whether we deal with a manifold or not.

2.4. Most theorems proven for the just defined constructive class obviously also hold for the more general case. The definitions essentially covers **PL geometry**, a category of geometries known to be much different from the category of **topological manifolds**. For example, there

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are complexes like the double suspension of a homology sphere which has a geometric realization that is homeomorphic to a sphere (by the double suspension theorem) but which is not a sphere in the above sense because in the above definition, unit spheres in unit spheres are spheres, while for a double suspension of a homology sphere, there are 2 vertices v, w, for which the unit sphere $S_{S(v)}(w) = S(w) \cap S(v)$ in $S(v)$ is a homology sphere.

2.5. The following is a discrete version of the classical Morse-Sard theorem [27, 30]. The classical theorem assured that for any differentiable function f on a manifold, almost all c are regular values in the sense that $f^{-1}(c)$ is a $(d-1)$-manifold (see [26]). In the discrete, the exception set is now known to be contained in $f(V)$. The set $\{f = 0\}$ is defined as the set of all simplices in G on which f changes sign. It could be formulated differently: a simplicial map from a simplicial complex G representing a d-manifold to $K_2 = \{\{1\}, \{2\}, \{1, 2\}\}$ has the property that $f^{-1}(\{1, 2\})$ represents a $(d-1)$-manifold.

2.6. If G is the simplicial complex of the geometry, the level set is at first just an open set $U \subset G$ in the Alexandrov topology. When saying $U = \{f = 0\}$, we think of it as a **graph** in which the sets of U are the vertices and where two are connected if one is contained in the other. We let this graph represent what we mean with $\{f = 0\}$. The level set $\{f = 0\}$ is the same when we replace f with $\text{sign}(f)$, a function taking values in $\{-1, 1\}$.

Theorem 1 (Morse Sard for one function). *If G is d-**manifold** and $f : V \to R$ is a real function, then the level surface $\{f = c\}$ for $c \notin f(V)$ is either a $(d-1)$-manifold or empty.*
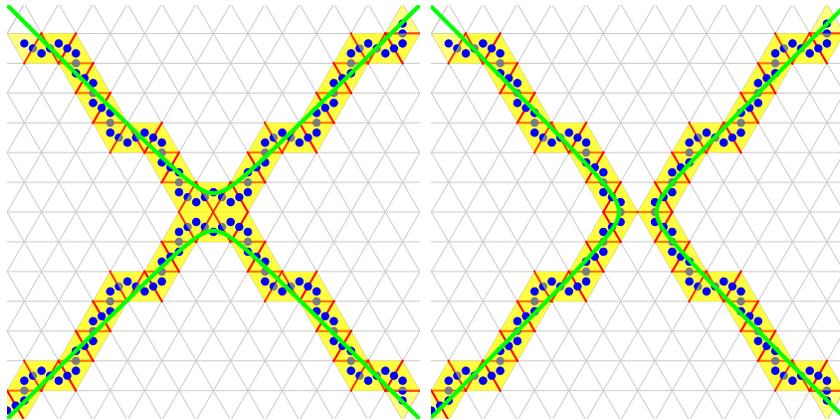

FIGURE 3. In this figure, we see part of a flat discrete 2-torus. It is embedded in the plane to conveniently attach x, y coordinates to each vertex and $f(x, y) = x^2 - y^2$. We see the contours $\{f = 0.3\}$ and $\{f = -0.3\}$. These level sets $\{f = c\}$ consist of all edges and triangles on which $f - c$ changes sign. It is an open set in a simplicial complex but becomes a graph with simplices as vertices and where two are connected, if one is contained in the other.

2.7. The graph $K = \{f = c\}$ is a sub-graph of the **Barycentric refinement graph** $G_1 = (V_1, E_1)$ in which the complete sub-graphs x of G are the vertices and where two such vertices are connected if one is contained in the other. The graph K has as vertices all complete sub-graphs of G on which $f - c$ changes sign. While the classical Morse-Sard lemma [27, 30] tells that the critical values have measure zero, the discrete case more explicitly states that **all** values different from $f(V)$ are regular. As in the continuum case, also values not reached are considered regular in the sense that they are not considered singular values.

2.8. The proof of the discrete Morse-Sard theorem used that the **Barycentric refinement** G_1 of a graph G has a natural **hyperbolic structure** for the **dimension function** $\dim(x)$. This function dim is naturally a discrete Morse function in the sense that both $S_f^-(x)$ and $S_f^+(x)$ are spheres at critical points. For the dimension function, every x is a critical point and the dimension $m = \dim(x)$ is the **Morse index** and $(-1)^{\dim(x)} = 1 - \chi(S_f^-(x))$ is the Poincaré-Hopf index. By definition, the unit sphere $S(x)$ of a vertex x in G_1 consists of all simplices $S^-(x)$ which are either strictly contained in x or the set of simplices $S^+(x)$ which strictly contain x. The simplicial complex $S^-(x)$ is a $(m-1)$-sphere in the sense that its Barycentric refinement is a $(m-1)$-sphere graph. The set $S^+(x)$, also called **star** of x, is an open set, but it is not a simplicial complex in general. It is a $(d-k-1)$-sphere if we look at its Barycentric refinement, where the elements are the vertices and a connection holds if one is contained in the other. The Barycentric refinement graphs of $S^-(x)$ and $S^+(x)$ are both spheres if G is a manifold and the unit sphere $S(x)$ is equal to $S^-(x) \oplus S^+(x)$, where $\oplus$ is the **Zykov join** [33]. The join operation is a monoid structure on graphs and spheres are a sub-monoid. The (-1)-sphere 0 is the zero element. To see that $K = \{f = c\}$ of G_1 is a $(d-1)$-manifold note that because f changes sign on x it also changes sign on any y which contains x so that $S_G^+(x) = S_K^+(x)$. The open set $S_K^-(x)$ is a hyper-sphere in $S_G^-(x)$. It is obtained from $S_G^-(x)$ So, $S_K(x) = S_K^-(x) \oplus S_K^+(x)$ is a $(d-2)$-sphere and K is a $(d-1)$-manifold.

3. Two functions

3.1. So far, we covered essentially what was done already in [18]. Here is a new approach to the case when we have 2 or more functions. If f, g are two functions on G, then the set of simplices on which both f, g change sign is not a manifold in general. It is much too "fat" (indeed, we will see it soon as a union of manifolds). In 2015, we opted to extend the function g to the Barycentric refinement G_1 of G and to look at the level surface $\{g = d\}$ in the sub-graph $\{f = c\}$ of G_1. While it works, it was unsatisfactory for two reasons, explained in the next paragraph.

3.2. First of all, $\{f = c, g = d\}$ was not the same than $\{g = d, f = c\}$. Second, we needed to extend g to the Barycentric refinement so that the graph $\{f = c, g = d\}$ is a sub-graph of the second Barycentric refinement of G. The number of vertices grows exponentially with every refinement. The f-vector $(f_0, f_1, \ldots, f_d)$ counting the number of complete sub-graphs gets mapped into $A^2 f$, with the Barycentric operator $A_{ij} = S_{ji} i!$ involving **Stirling numbers of the second kind**. If G is the smallest possible 4-manifold with $f = (10, 40, 80, 80, 32)$, then the second Barycentric refinement G_2 already has the f-vector $A^2 f = (24482, 303840, 970560, 1152000, 460800)$. When experimenting with co-dimension 2 level surfaces in a 4-manifold, we would need to deal with graphs of at least 24 thousand nodes. The first Barycentric refinement G_1 has a reasonable size because $Af = (242, 2640, 8160, 9600, 3840)$. Morse-Sard dealing with several functions at once therefore is a relief in the sense that it reduces complexity.

3.3. Let us first consider the case of two functions f, g on a 4 manifold G. A motivation for the case of two functions is **quantum mechanics** because **waves** in quantum mechanics are modeled by complex-valued functions. The usual notion for such functions is ψ. Considering $\psi(v) = f(v) + ig(v)$ is equivalent to have two real valued functions f, g. The reason for taking dimension 4 for the host is that this is the smallest dimension, where level surfaces $\{\psi = 0\}$ become interesting; in dimension 2 we would just get points and in dimension 3 just get unions of cyclic graphs. If G is 4-dimensional real manifold, we can also think of it as a 2-dimensional **complex surface**. The complex curve, a 1-dimensional complex manifold, $\{\psi = 0\}$ is according

to the classical Morse-Sard a real manifold. In the finite, where we can look at any function as "analytic", we are allowed to see level surfaces as complex curves. But what do we mean with $\{\psi = 0\}$?

3.4. Definition: Define $\{\psi = c\}$ as the sub-graph of the Barycentric refinement G_1 of G, generated by the vertices which are simplices in G which contain a triangle x on which $\psi - c = f + ig$ has the property that
(i) $\text{sign}(f + ig)$ takes at least 3 different values on x
(ii) $(f + ig)(x)$ must includes the values $e = \{-1 + i, 1 - i\}$. This generalizes the real function case, where we had defined $\{f = c\}$ as the set of simplices which contain an edge x on which $f - c$ changes sign.

3.5. There is a choice involved. We could for example have forced that $e = \{-1 - i, 1 + i\}$ is included, a case which would have corresponded to the wave $\overline{\psi}$, the **complex conjugate**. In the continuum, $\{\psi = 0\}$ and $\{\overline{\psi} = 0\}$ are the same. In the discrete, these are most of the time different surfaces with different features. Only if ψ is sufficiently smooth, the two surfaces agree topologically. Indeed, for small G, the difference between $\{\psi = 0\}$ and $\{\overline{\psi} = 0\}$ is significant. It raises many interesting questions. The two cases were not the only ones. With k functions, there are $2^{k-1}(2^k - 1)$ cases which is 6 for $k = 2$. Every of the 6 edges e in the complete graph U with 4 vertices $\{\pm 1, \pm 1\}$ produces a different surface $\{\psi = 0\}_e$ in general.

Theorem 2 (Morse-Sard for complex functions on a d-manifold). *If $G = (V, E)$ is a d-manifold and $\psi : V \to \mathbb{C}$ is a complex-valued function. Then, for all $c \notin \psi(V)$, the level surface $K = \{\psi = c\}$ is a $(d - 2)$-manifold or empty.*

Proof. The proof is very similar to the case with one function. If $K = \{\psi = 0\}$ is not empty, we need to show that every unit sphere $S_K(x) \subset S_G(x)$ of x is a $(d-3)$ sphere. Define $m = \dim(x)$. To see that $S_K(x)$ is a $d - 3$ sphere, use the decomposition $S_G(x) = S_G^-(x) \oplus S_G^+(x)$, where $S^-(x)$ is a $m - 1$ sphere and $S^+(G)$ is a $d - m - 1$ sphere. We still have $S_K^+(x) = S_G^+(x)$ and that $S_K^-(x)$ is a co-dimension 2-manifold in the $S_G^-(x)$. Because $S^-(G)$ is the boundary sphere of a simplex for which every sub-manifold is a sphere, $S^-(K)$ is a $(m - 3)$-dimensional sphere. Now $S_K(x) = S_K^-(x) \oplus S_K^+(x)$ is the join of $(k-3)$-sphere and $(d-k-1)$-sphere which produces a $(d - 3)$-sphere. $\qquad\square$
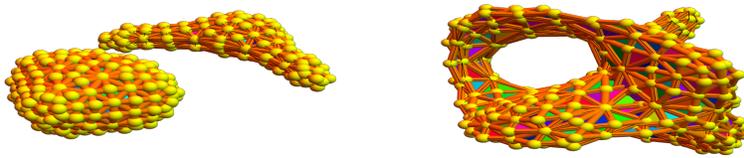

FIGURE 4. For a fixed ψ, we display the level surfaces $\{\psi = 0\}$ and $\overline{\psi} = 0\}$ in a small $\mathbb{RP}^4$. The complex K_4 (for the 4 possible sign values) 6 edges leading to 6 different type of surfaces. They can differ topologically only in the small. If G is larger and ψ is smooth enough, then they have the same topology.

3.6. Here is a more general set-up for co-dimension-2 manifolds: look at the 3-dimensional simplex $U = K_4$ with vertices $\{(-1, -1), (-1, 1), (1, -1), (1, 1)\}$. Given one of the 6 edges e in U, there are two triangles t_1, t_2 containing e. Define

$$\{\psi = 0\}_e = \{x \in G, \text{sign}(\psi(x)) \in \{t_1, t_2\}\},$$

where $\mathrm{sign}(a + ib) = \mathrm{sign}(a) + i\,\mathrm{sign}(b)$. Each of these 6 level sets $\{\psi = 0\}_e$ is a co-dimension 2-manifold.
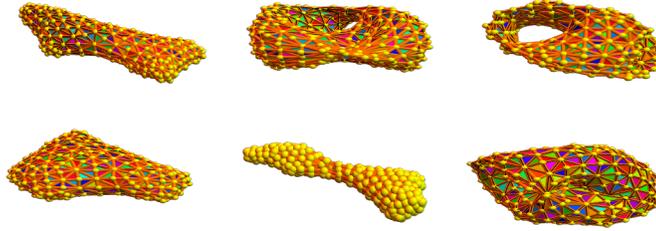

FIGURE 5. Chose an edge e in the 3-simplex U defined by the corners $\{\pm 1, \pm 1\}$. Define $\{\psi = 0\}_e = \{x \in G, \psi(x)$ contains at least one triangle containing $e\}$. This produces in general 6 different surfaces for 2 functions and $(2^k - 1)2^{k-1}$ if we have k functions. If $p = k$ and $2^p - 1$ are prime, then we have a "perfect number" of surfaces.

3.7. What happens if we take the inverse image of one of the 4 triangles t in U If $\psi : G \to \mathbb{C}$ does not take the value 0, define $\Psi(x) = \mathrm{sign}(\psi(x))$. The following result allows us to produce manifolds with boundary.

Theorem 3. *If G is a d-manifold and t is a triangle in U, then $\Psi^{-1}(t)$ is a $(d-2)$-manifold with boundary.*

Proof. If $t = t_1$ is a triangle in U and t_2 is an other triangle in U which shares with t_1 an edge e. By the previous result

$$H_1 \cup H_2 = \Psi^{-1}(t_1) \cup \Psi^{-1}(t_2)$$

is a $(d-2)$ manifold H without boundary. Obviously $H_1 = \Psi^{-1}(t_1)$ and $H_2 = \Psi^{-1}(t_2)$ are both part of this manifold H. Their intersection $H_1 \cap H_2$ consist of all x such that $\Psi(x)$ is both in t_1 and t_2. This means that $\Psi(x)$ must reach all of U. This must be a $(d-3)$ manifold (next section) as it can be seen as the level surface $\{f = 0, g = 0, h = 0\}$ where h is a constant function not taking the value 0. Since $H_1 \cap H_2$ is a $d - 3$ manifold and $H_1 \cup H_2$ is a $d - 2$ manifold, both H_1, H_2 are $(d-2)$-manifolds with boundary. $\square$
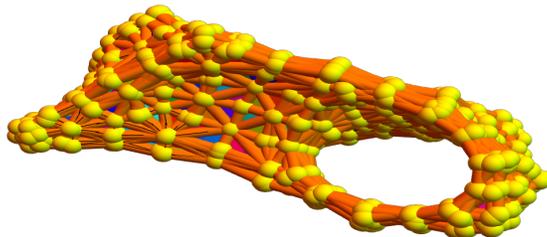

FIGURE 6. As an application, we can produce solid 3 manifolds with boundary by taking a constructing a co-dimension 2 manifold $\psi^{-1}(t)$ in a 5-manifold for a given triangle. In this case we took for G a suspension of RP^4 and got a solid 2-torus.

3.8. Here is a possible reformulation. We hope to generalize this much more in a future work.

Corollary 1. *If G is a d-manifold and if t_1, t_2 are two triangles in U intersecting in an edge e, then $\{\psi = 0\}_e$ is obtained by gluing $\Psi^{-1}(t_1)$ and $\Psi^{-1}(t_2)$ along a $d - 3$ manifold.*

3.9. This works already in dimension $d = 3$ and a complex function ψ. The set $\psi^{-1}((1+i, i-1, -1-i))$ consists of a finite set of path graphs. Also $\psi^{-1}((1+i, -1+i, -1-i))$ consists of a set of path graphs. The union is a choice of $\{\psi = 0\}$ and is a 1-manifold, a union of cyclic graphs obtained by gluing the two parts along 0-dimensional manifold, a set of isolated points.
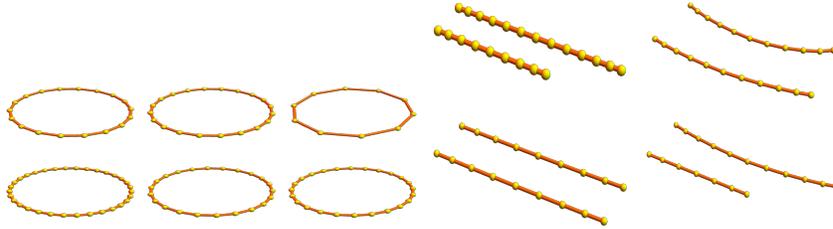

FIGURE 7. We first look at the level surfaces of a random complex-valued function in a 3-dimensional manifold G. The second picture shows the manifolds H with boundary obtained by just taking the inverse of a simplex.

3.10. We can rephrase this that a continuous map from a d-manifold G to K_3 has $f^{-1}(1, 2, 3)$ as the empty set or an open set becomes in the Barycentric refinement a $(d-2)$-manifold with boundary. One dimension lower, a continuous map from G to K_2 has $f^{-1}(\{1, 2\})$ either an empty set or a $(d-1)$-manifold with boundary. For more on the Alexandrov topology for simplicial complexes, see [22]. This does not yet like being the most general set-up even with 2 functions. We will try to extend this to general simplicial maps from a d-manifold M to a 3-manifold with a hyper surface N. The inverse $f^{-1}(N)$ appears then to be a $d-2$ manifold but much larger in general than M. Every $y \in N$ has as an inverse image either a $d-2$ manifold with boundary or a $d-3$ manifold gluing these $d-2$ patches together to a $d-2$ manifold.

4. ARBITRARY MANY FUNCTIONS

4.1. Definition: given k functions $f_1, \ldots, f_k$ in a d-manifold G, define $\{f_1 = 0, \ldots, f_k = 0\}$ as the set of simplices x in G for which the image $\text{sign}(f)(x)$ has at least $k+1$ elements and such that it includes a fixed set of k values like for example $(-1, 1, \ldots, 1), (1, -1, \ldots, 1), (1, 1, \ldots, -1)$.

4.2. As in the case $k = 2$, there is more than one surface. This is one of the Binomial$(2^k, 2) = 2^{k-1}(2^k - 1)$ different choices we can do. If $p = k$ is prime and $2^p - 1$ are prime (the later is then called a **Mersenne prime**), then $n = 2^{k-1}(2^k - 1)$ is a **perfect number** in the sense of number theory: the sum of proper divisors of n is equal to n. (One of the oldest problem in mathematics asks whether there are odd perfect numbers). All these surfaces can differ for small G or random f but for large G and smooth enough f, all have the same topology.

Theorem 4. *For any d-manifold (V, E), and any set of functions $f_1, \ldots, f_k$, and any $c \notin f(V)$, the level surface $\{f = c\}$ is a $(d - k)$-manifold or empty.*

Proof. The unit sphere $S_K(x) = S_K^-(x) \oplus S_K^+(x) = S_K^-(x) \oplus S_G^+(x)$ is a sphere if $S_K^-(x)$ is a sphere. But $S_K^-(x)$ is a level surface in the $(m - 1)$-sphere $S_G^-(x)$. $\qquad \square$

4.3. Morse-Sard boils down to a local statement about level surfaces in boundary spheres of simplices. We only have to assure that the level surface $\{f = 0\}$ in such a sphere is a sphere. Let $x = (x_0, \ldots, x_m)$ be a m-dimensional simplex and $f_1, \ldots, f_k : x \to \{-1, 1\}$. Assume $(f_1, \ldots, f_k)$ takes at least $m + 1$ values including $(-1, 1, \ldots, 1), (1, -1, \ldots, 1), \ldots (1, 1, \ldots, 1, -1)$. Then $\{f = 0\}$ is a $(m - 1 - k)$-sphere. If $k = 0$, then $\{f = 0\}$ is the Barycentric refinement of the boundary sphere of x. If $k = 1$, then $\{f = 0\}$ is the set of simplices on which f changes sign. This is always the join of two spheres, the sphere where f is positive and the sphere, where f is negative, as long as there are two values.

5. INTEGRAL GEOMETRY

5.1. This section is a review of a result [15] formulated in 2012 and [16] in 2013, when no discrete Sard was available yet. In [14] we gave the Poincaré-Hopf index $i_f(v) = 1 - \chi(S_f^-(v))$ for a graph $G = (V, E)$ and coloring f, where $S^-(v)$ is generated by the vertices w of $S(v)$, where $f(w) < f(v)$. Then $i_{-f}(v) = 1 - \chi(S_f^+(v))$. The complement of the disjoint union $S^-(v) \cup S^+(v)$ in $S(v)$ is now the level set $\{f = c\}$ with $c = f(w)$ in $S(v)$. Because the unit sphere $S(v)$ can be seen as a level set for the distance function to v, we actually deal with a contour surface of co-dimension 2. If G was a manifold, where $\chi(S(v)) = (1 + (-1)^{d-1})$, we got $j_f(v) = 1 - \chi(\{f = f(v)\}_{S_v})/2$ for even-dimensional manifolds and $j_f(v) = -\chi(\{f = f(v)\}_{S_v})/2$ for odd-dimensional manifolds.

5.2. The symmetric Poincaré-Hopf theorem $\chi(G) = \sum_{v \in V} j_f(v)$ therefore related the Euler characteristic $\chi(G)$ of an even-dimensional manifold G equipped with a color function $f : G \to \mathbb{R}$ with the sum of the symmetric indices $j_f(v) = 1 - \chi(\{y \in S(v), f(y) = f(v)\})/2$. The **curvature** $K(v) = E_f[j_f(v)]$ was given as the expectation of $1 - E[\{f = f(x)\}]/2$ on the odd-dimensional unit sphere $S(x)$. The probability space is a space of colorings on G:

Theorem 5. *For an even-dimensional manifold $K(x) = 1 - E[\chi(\{f = f(x)\})]/2$. For an odd-dimensional manifold, $K(x) = -E[\chi(\{f = f(x))\}]/2$.*

5.3. This immediately implied:

Corollary 2. *For an odd-dimensional manifold, the curvature is constant $K(x) = 0$.*

5.4. We now can write Euler characteristic as $\chi(G) = 1 - E[X]/2$, where X is the Euler characteristic of a "random co-dimension 2 surface. The probability space however is rather restricted. If we write it as a probability space of functions $\psi = f + ig$, we require that for fixed f, we have $g(w) = 1 - 1_v(w)$ and that on the set $S(v) = \{g = 0\}$ the function $f(v)$ is different from all values on $S(v)$. It begs the question what happens if we take the expectation of $\chi(\{f + ig = 0\})$ when averaging over all possible functions $f + ig$ on the graph taking values in $\{1 + i, 1 - i, -1 + i, -1 - i\}$.

6. REMARKS

6.1. The discrete Morse-Sard result illustrates that moving from continuum geometries to finite quantum geometries becomes not only more regular but also more rich. While the translation of the objects from Euclidean manifolds to discrete manifolds is not obvious at first, the results are completely analog. A point x is a **critical point** of a function f, if $M(x) = \{f = f(x)\}$ in $S(x)$ is not a $(d - 2)$-sphere. The symmetric Morse index $j(x) = ((1 + (-1)^d) - \chi(M(x)))/2$ quantifies that. The sum $\sum_{x \in V_1} j(x)$ is the Euler characteristic of G. This is analog to the continuum. A smooth function f has a critical point x on a smooth Riemannian d-manifold, if

for small radius $r > 0$, the set $M_r(x) = S_r(x) \cap \{f = f(x)\}$ is not a $(d-2)$-sphere. An example is a maximum or minimum of f, where $S(x) \cap \{f = f(x)\}$ is empty.

6.2. Given two functions f, g on a d-manifold we can look at the **discrete Lagrange problem** where f is given on the **constraint** $\{g = 0\}$. We would like to find the extrema of f on the level surface using Lagrange equations similarly as in the continuum. Critical points should be simplices in $g = 0$, where the center manifold $\{f = f(x)\}$ in $S(x)$ is not a $(d-3)$-sphere. For example, on a 2-manifold, then $g = 0$ is a 1-manifold and f can be extremized at points y on this manifold $\{g = 0\}$, where $S(y)$ is empty. This is what happens for a coloring f on a 1-dimensional manifold? A point x could be called critical point if $\{f = f(x)\}$ is not a zero sphere and a regular point if $\{f = f(x)$ is a 0-sphere.

6.3. Let us look at a triangle x and two functions f, g such that f, g both change sign on x. Define the **gradient** of f as the structure of the signs of f. The tail of the arrow is the set where f is negative. The head is the arrow where f is positive. For a simplex $x = (a, b, c)$ in the intersection $\{f = c, g = d\}$ the values $(f(a), f(b), f(c))$ and $(g(a), g(b), g(c))$ must have the properties that the gradients have the same type but are not the same. The two possible cases (up to rotation) are $(1, -1, -1), (-1, 1, -1)$ and $(1, -1, 1), (-1, 1, 1)$. If the sign structure is $(1, 1, -1), (-1, -1, 1)$ for example, we do not intersect even so both functions change sign on x.

6.4. Instead of starting with a graph, we could have begun with a finite abstract simplicial complex G, a finite set of sets closed under the operation of taking finite non-empty subsets. It carries a topology generated by the stars $U(x) = \{y \in G, x \subset y\}$ which are the smallest open sets. The closed sets are then the sub-complexes. The boundary $S(x)$ of $U(x)$ is the unit sphere. As it is $S(x) = \overline{U}(x) \setminus U(x)$ it is closed as the complement of an open set in a closed set. A complex is a d-**manifold** if all unit spheres $S(x)$ are $(d-1)$-spheres. A d-manifold is a d-**sphere** if there exists x such that the complex $G \setminus U(x)$ is contractible. A complex G is contractible if there exists x such that both the complexes $S(x)$ and $G \setminus U(x)$ are contractible. These inductions start with $0 = \{\}$ being the (-1)-sphere and $1 = \{1\}$ being contractible.

6.5. Here is an application of co-dimension 2 manifolds. We can visualizations of eigenfunctions of Laplacians on 3-manifolds $\{f = 0\}$ in 3-manifolds. Let H be a 3-manifold $H = \{f = 0\}$ in a 4-manifold G. We can now look at the Laplacian L on H and an eigenvector f to an eigenvalue λ. Then f is a real-valued function on H. For every original vertex v of G which happens to be in at least one vertex x of H (so that x is a simplex in G), then define the function $g(v) = \sum_{x, v \in x} \psi(v)$. Now, we can look at the manifold $\{f = 0, g = 0\}$ which is a nodal surface of the Laplacian. In other words, we can study **Cladni surfaces** for Laplacians on surfaces.

7. Code

7.1. The following self-contained Mathematica code allows to compute level surfaces of real or complex functions of arbitrary simplicial complexes and compute its cohomology (as an open set). The closed manifold is obtained by adjusting the dimension. We encode a delta set as (G, D, b), where G is a set of sets G, D is a Dirac matrix and b a dimension function. Two 3-manifolds $\mathbb{RP}^3$ and the homology sphere are from [25].

```
Generate[A_]:=Sort[Delete[Union[Sort[Flatten[Map[Subsets,A],1]]],1]];
Whitney[s_]:=Generate[FindClique[s, Infinity, All]];  sig[x_]:=Signature[x]; L=Length;
F[G_]:=Module[{ l=Map[L,G]}, If[G=={},{},Table[Sum[If[l[[j]]==k,1,0],{j,L[l]}],{k,Max[l]}]]]];
sig[x_,y_]:=If[SubsetQ[x,y]&&(L[x]==L[y]+1),sig[Prepend[y,Complement[x,y][[1]]]]*sig[x],0];
nu[A_]:=If[A=={},0,L[NullSpace[A]]];
Dirac[G_]:=Module[{ f=F[G]},b,d,n=L[G]},b=Prepend[Table[Sum[f[[l]],{l,k}],{k,L[f]}],0];
 d=Table[sig[G[[i]],G[[j]]],{i,n},{j,n}]; {d+Transpose[d],b}];
Beltrami[G_]:=  Module[{B=Dirac[G][[1]]},B.B];
Hodge[G_]:=Module[{Q,b,H},{Q,b}=Dirac[G];H=Q.Q;Table[Table[H[[b[[k]]+i,b[[k]]+j]],
 {i,b[[k+1]]−b[[k]]},{j,b[[k+1]]−b[[k]]}],{k,L[b]−1}]];
Betti[s_]:=Module[{G},If[GraphQ[s],G=Whitney[s],G=s];Map[nu,Hodge[G]]];
Fvector[s_]:=Module[{G},If[GraphQ[s],G=Whitney[s],G=s]; Delete[BinCounts[Map[L,G]],1]];
ToGraph[G_]:= UndirectedGraph[n=L[G]; Graph[Range[n],
  Select[Flatten[Table[k−>l,{k,n},{l,k+1,n}],1],(SubsetQ[G[[#[[2]]]],G[[#[[1]]]]])&]]];
Barycentric[s_]:=If[GraphQ[s],ToGraph[Whitney[s]],Whitney[ToGraph[s]]];
Suspension[s_] :=Module[{v=VertexList[s],e=EdgeRules[s],m},m=Max[v];
 Do[e=Union[e,{v[[k]]−>m+1,v[[k]]−>m+2}],{k,L[v]}]; UndirectedGraph[Graph[e]]];
USphere[s_,v_]:= VertexDelete[NeighborhoodGraph[s,v],v];    Rnd[x_]:=x −> RandomChoice[{−1,1}];
RFunction[s_]:=Module[{G},If[GraphQ[s],G=Whitney[s],G=s];Map[Rnd,Union[Flatten[G]]]];
LSurface[s_,f_]:=Module[{G,H={},K},If[GraphQ[s],K=Whitney[s],K=s];G=Select[K,L[#1]>1 &];
  Ch[b_]:=Module[{},A=Union[Table[Sign[b[[k]]],{k,L[b]}]];SubsetQ[A,{1,−1}]&&L[A]>=2];
  Do[If[Ch[G[[k]] /. f],H=Append[H,G[[k]]]],{k,L[G]}]; H];
LSurface[s_,{f_,g_}]:=Module[{G,H={},K},If[GraphQ[s],K=Whitney[s],K=s];G=Select[K,L[#1]>2 &];
  Ch[b_,d_]:=Module[{},A=Union[Table[Sign[{b[[k]],d[[k]]}],{k,L[b]}]];
    MemberQ[A,{1,−1}] && MemberQ[A,{−1,1}] && L[A]>=3];
  Do[If[Ch[G[[k]] /. f,G[[k]] /. g],H=Append[H,G[[k]]]],{k,L[G]}];H];

G = Whitney[Suspension[Suspension[PolyhedronData["Icosahedron", "Skeleton"]]]];
f=RFunction[G];g=RFunction[G];  gm=Table[g[[k,1]]−>−g[[k,2]],{k,L[g]}];
H1=LSurface[G,{f,g}];H2=LSurface[G,{f,gm}];
Print[{ Betti[H1], Betti[H2]}]; {GraphPlot3D[ToGraph[H1]], GraphPlot3D[ToGraph[H2]]}

RP3=Generate[{{1,2,3,4},{1,2,3,5},{1,2,4,6},{1,2,5,6},{1,3,4,7},{1,3,5,7},{1,4,6,7},
{1,5,6,7},{2,3,4,8},{2,3,5,9},{2,3,8,9},{2,4,6,10},{2,4,8,10},{2,5,6,11},{2,5,9,11},
{2,6,10,11},{2,7,8,9},{2,7,8,10},{2,7,9,11},{2,7,10,11}, {3,4,7,11},{3,4,8,11},
{3,5,7,10},{3,5,9,10},{3,6,8,9},{3,6,8,11},{3,6,9,10}, {3,6,10,11},{3,7,10,11},
{4,5,8,10},{4,5,8,11},{4,5,9,10},{4,5,9,11}, {4,6,7,9},{4,6,9,10},{4,7,9,11},{5,6,7,8},
{5,6,8,11},{5,7,8,10},{6,7,8,9}}];f=RFunction[RP3];H=LSurface[RP3,f];U=GraphPlot3D[ToGraph[H]]

G=Generate[{{0,1,5,11},{0,1,5,17},{0,1,7,13},{0,1,7,19},{0,1,9,15},{0,1,9,21},{0,1,11,13},
{0,1,15,19},{0,1,17,21},{0,2,6,12},{0,2,6,18},{0,2,7,13},{0,2,7,19},{0,2,10,16},{0,2,10,22},
{0,2,12,13},{0,2,16,19},{0,2,18,22},{0,3,8,14},{0,3,8,20},{0,3,9,15},{0,3,9,21},{0,3,10,16},
{0,3,10,22},{0,3,14,22},{0,3,15,16},{0,3,20,21},{0,4,5,11},{0,4,5,17},{0,4,6,12},{0,4,6,18},
{0,4,8,14},{0,4,8,20},{0,4,11,12},{0,4,14,18},{0,4,17,20},{0,11,12,13},{0,14,18,22},
{0,15,16,19},{0,17,20,21},{1,2,5,11},{1,2,5,17},{1,2,8,14},{1,2,8,20},{1,2,10,16},{1,2,10,22},
{1,2,11,14},{1,2,16,20},{1,2,17,22},{1,3,6,12},{1,3,6,18},{1,3,7,13},{1,3,7,19},{1,3,8,14},
{1,3,8,20},{1,3,12,20},{1,3,13,14},{1,3,18,19},{1,4,6,12},{1,4,6,18},{1,4,9,15},{1,4,9,21},
{1,4,10,16},{1,4,10,22},{1,4,12,16},{1,4,15,18},{1,4,21,22},{1,11,13,14},{1,12,16,20},
{1,15,18,19},{1,17,21,22},{2,3,5,11},{2,3,5,17},{2,3,6,12},{2,3,6,18},{2,3,9,15},{2,3,9,21},
{2,3,11,15},{2,3,12,21},{2,3,17,18},{2,4,7,13},{2,4,7,19},{2,4,8,14},{2,4,8,20},{2,4,9,15},
{2,4,9,21},{2,4,13,21},{2,4,14,15},{2,4,19,20},{2,11,14,15},{2,12,13,21},{2,16,19,20},
{2,17,18,22},{3,4,5,11},{3,4,5,17},{3,4,7,13},{3,4,7,19},{3,4,10,16},{3,4,10,22},{3,4,11,16},
{3,4,13,22},{3,4,17,19},{3,11,15,16},{3,12,20,21},{3,13,14,22},{3,17,18,19},{4,11,12,16},
{4,13,21,22},{4,14,15,18},{4,17,19,20},{11,12,13,23},{11,12,16,23},{11,13,14,23},{11,14,15,23},
{11,15,16,23},{12,13,21,23},{12,16,20,23},{12,20,21,23},{13,14,22,23},{13,21,22,23},
{14,15,18,23},{14,18,22,23},{15,16,19,23},{15,18,19,23},{16,19,20,23},{17,18,19,23},
{17,18,22,23},{17,19,20,23},{17,20,21,23},{17,21,22,23}}];f=RFunction[G];H=LSurface[G,f];
Print[Betti[H]];  GraphPlot3D[ToGraph[G]]
```

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