ON THE ARBORICITY OF MANIFOLDS

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Abstract. The arboricity of a discrete 2-sphere is always 3. The arboricity of any other discrete 2-dimensional surface is always 4. For d-manifolds of dimension larger than 2, the arboricity can be arbitrary large and must be larger than d.

1. Introduction

1.1. A finite simple graph G is a d-manifold if every **unit sphere** $S(v)$ is $(d-1)$-sphere. It is a d-**sphere**, if $G-v$ is contractible for some v. A graph G is **contractible** if both $S(v)$ and $G-v$ are contractible for some v. These inductive assumptions start with the empty graph 0 being the (-1)-sphere and the 1-point graph $K_1 = 1$ being contractible.

1.2. The **arboricity** of graph G is the minimal number of forests that are needed to partition the edge set of G. It is determined by the maximum of $|E_W|/(|W|-1)$ where (W, E_W) runs over all induced subgraphs of G with more than one element. The **Nash-Williams theorem** assures that $\text{arb}(G)$ is the smallest integer larger or equal than this number.

1.3. For d-manifolds, the vertex degree $\deg(v)$ is larger or equal than $2d$ because this is the vertex cardinality of the smallest $(d-1)$-sphere. By the **Euler handshake formula**, $2E/V = \sum_v \deg((v)/V \geq 2d$, meaning $E/V \geq d$ and $E/(V-1) > d$ so that the arboricity is at least $d+1$. The case, where G is a cross polytop, the smallest d-sphere, shows that this lower bound can indeed happen, at least for spheres.

1.4. As we will show here, in the cases $d=1$ and $d=2$, we understand everything: in the case $d=1$, where a manifold a disjoint union of circular graphs, we always have arboricity 2. In dimension $d=2$, spheres have arboricity 3 and all other 2 dimensional surfaces have arboricity 4. In higher dimensions, the arboricity of d-spheres can take any value in $\{d+1, d+2, \dots, \}$. For any d-manifold type which is not a sphere, there are examples with arbitrary high arboricity. We have no information yet about the lower bounds for manifold types different from spheres. Already the case of a d-torus is open for $d > 2$. We have seen 3 tori of arboricity 5 but no example of arboricity 4 yet.

Theorem 1. *For a d-manifold the arboricity is larger or equal than $d+1$. For all d, there are d-spheres with arboricity $d+1$. For any manifold type and dimension larger than 2, there are examples with arbitrary large arboricity. For 2-spheres the arboricity is 3 for all other 2-manifold the arboricity is 4.*

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2. Three forests suffice

2.1. The **arboricity** $\text{arb}(G)$ of a finite simple graph $G = (V, E)$ is the minimal number of forests partitioning the edge set E. A **2-manifold** is a finite simple graph such that every **unit sphere** $S(v)$, the graph induced by all direct neighbors of v, is a cyclic graph with 4 or more vertices. We simply call a **2-manifold** or a **surface**. For a connected graph, the arboricity is the same than the minimal number of trees covering E. By the **Nash-Williams theorem** [20], $\text{arb}(G)$ is the smallest integer k such that $E' \leq k(V'-1)$ for any induced sub-graph (V', E') with a vertex subset $V' \subset V$ of more than one element. [1] The **Euler handshake formula** $2E = \sum_{v \in V} \deg(v)$ shows that the arboricity is close to half the maximal average **vertex degree** $\deg(v)$ which an induced sub-graph G' of G can have.

2.2. The **Nash-Williams functional** $\phi(G) = E/(V-1) = \sum_{v \in V} \deg(v)/(2V-2)$ is defined for all graphs with $V \geq 2$ vertices. Again using notation overload, the **area** of the graph is denoted by F, as is the set of triangles. The **Euler characteristic** of a K_4 graph is defined as $X = V - E + F$. A 2-manifold satisfies the **Dehn-Sommerville** relation $3F = 2E$ because every edge (a, b) is contained in exactly 2 triangles abc, abd, where $\{c, d\} = S(a) \cap S(b)$ and every triangle contains exactly 3 edges. For surfaces, the functionals V, E, F and so ϕ, X are spectral properties for the Kirchhoff Laplacian L because $V = \text{tr}(L^0), E = \text{tr}(L)/2, F = \text{tr}(L)/3, X = \text{tr}(L^0 - L^1/6)$.

Lemma 1. *A surface G of area F and Euler characteristic X has the Nash-Williams functional $\phi(G) = 3F/(X - 1 + F)$.*

Proof. We use the linear equations $3F = 2E$ and $X = V - E + F$ and choose F and X as free variables. The solution is the **f-vector** $(V, E, F) = (X + F/2, 3F/2, F)$. ☐

2.3. The following result is exercise 21.4.6 in [4] and some sort of "folklore". It generalizes as the **Edmond covering theorem** in matroid theory [21] and has been used as a tool to show that the star arboricity is less or equal to 6 [2]. We deduced it earlier from having acyclic chromatic number 4 for all non-prismatic 2-spheres and using that for an acyclic coloring with c colors, all the $c(c-1)/2$ Kempe chains are forests and that if c is even, one can form groups of $c/2$ chains as one forest producing $c - 1$ forests.

Theorem 2 (Algor-Alon)**.** *The arboricity of a planar graph is 3 or less.*

Proof. Given a planar graph $G = (V, E)$. Because the arboricity of a disjoint union of graphs is the maximum of the arboricity of each component, we can assume that G is connected. Because removing a leaf (a $v \in G$ with vertex degree $\deg(v) = 1$) does not change the arboricity, we can also assume that G has no leafs. A leafless planar graph can be triangulated by adding edges to make it **maximally planar** meaning that adding an other edge would no more have it planar. Adding edges without changing the set of vertices increases the Nash-Williams functional $\phi(G) = E/(V-1)$. In other words, ϕ as a function on induced sub-graphs without leafs takes for surfaces its maximum on G. Every 4-connected component of a maximal planar graph is by a result of Whitney [23] (see Appendix) either K_4 or a triangulation of a 2-sphere, a surface of Euler characteristic 2. The graph K_4 can be covered with 3 forests. That a 2-sphere can not have arboricity 2 follows from the fact that $3F = 2E, V - E + F = 2$ implies $V - 1 = 1 + F/2$ and so

$$\phi(G) = E/(V-1) = (2E)/(2F) = 3F/(2+F) .$$

[1] We denote the cardinalities of sets with the name of the set itself.

Setting the ratio $\phi(G)$ to 2 gives $3F = 4 + 2F$, so that $F \leq 4$, which is an impossibility for a 2-sphere. Setting the ratio $\phi(G) = 3$ gives $F/(2 + F) = 1$ which is not possible for any F. $\square$

2.4. This proof shows that any subgraph G of a 2-manifold for which all leafs have been recursively removed, has the property that any polygon face (a locally minimal polygon) can be completed. This then leads to a 2-manifold. The arboricity is smaller or equal than the arboricity of the full triangulation. This means that if we have a global bound $\phi(G) \leq c$ for all manifolds of a specific topological type, then $\phi(G) \leq c$ for all sub-graphs of such manifolds. More generally, for any surface $\phi(H)$ as a functional on this class of sub-graphs takes the **global maximum** on G. We have never seen an example, where an induced subgraph H of G has $\phi(H) > \phi(G)$.

2.5. Let us look a bit more at the functional $\phi(G) = E/(V - 1)$. The Nash-Williams theorem states that the maximum over all $\phi(W, E_W)$ with induced subgraphs (W, E_W) determines the arboricity. For 2-manifolds, already the single number $\phi(G)$ determines the arboricity. Lets us formulate this and also note that we do not know whether this holds in higher dimensions. The use of **critical graphs**, minimal graphs for which the arboricity is larger than expected has been used already by Nash-Williams.

Corollary 1 (ϕ on G determines)**.** *a) All 2-spheres have arboricity* 3 *if and only if* $2 \leq \phi(G) < 3$ *for all 2-spheres.*
b) All 2-manifold different from spheres have arboricity 4 *if and only if* $3 \leq \phi(G) < 4$ *for all such 2-manifolds.*

Proof. a) As we have seen, that arboricity 3 implies $2 \leq E/(V - 1) < 3$ is a direct consequence of the Nash-Williams theorem. That checking $E/(V - 1) < 3$ is enough, follows from the fact that we can look at the smallest subgraph $G_W = (W, E_W)$ generated by W for which $E_W/(W - 1) \geq 3$ and so arboricity being equal to 4. This critical G can not have leaves as otherwise, we could take away a leaf and have no minimality. Now we have polygonal faces. Triangulate those by adding edges. This can only increase the ratio. But then we end up with a 2-manifold or K_4.
b) Take the 2-manifold G of smallest genus for which we have failure and $\phi(G) = 4$. Then, take the smallest induced subgraph H_W for which the Nash-Williams bound fails. This graph must be leafless. Now add edges until we have a triangulation. That triangulation however either is a 2-manifold with $E/(V - 1) \geq 4$ or a manifold of smaller degree with $E/(V - 1) \geq 4$ which was excluded by the minimality on the degree. $\square$

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2.6. We derived the planarity result first from a result about acyclic chromatic number of 2-spheres which relies on the 4-color theorem. We learned that from Chat GPT who did not provide reference that the result must be known. A literature search then led to the exercise in [4]. **"Google Bard"** gave us, when pressed, as reference [20] which however does not mention this result on planar graphs. Antropic's "Claude" gave the following proof: *removing an edge reduces the number of edges and faces by one (which is not true for* K_2*) and claims that removing* $E - (V - 2)$ *edges to get a forest leading to* $E \leq 3(V - 2)$ *(also not true for* K_2*).* So far, we have not found a reference which states and proves the result in detail and so can label it as a "folklore result". [2] states the $E \leq 3(V - 2)$ bound too, which however fails for K_2. The Algor-Alon article however is the first we could locate, where the result is stated and proven in a way that a reader can fill the details.

3. Surfaces

3.1. A finite simple graph for which every unit sphere $S(v)$ is a circular graph with 4 or more elements, is a **two-dimensional discrete manifold**. We simply call this also a **surface**. If (a,b) is an edge, then $S(a) \cap S(b)$ is a unit sphere in a cyclic graph and so a zero-dimensional sphere (a graph with 2 vertices (c,d) and no edge). There are now two triangles (abc) and (abd).

3.2. The Euler characteristic of a surface is $X = V - E + F$, where F is the number of **triangles**, K_3 sub-graphs of G. A connected surface of Euler characteristic is 2 is called a **2-sphere**. The class of 2-spheres together with K_4 are known by Whitney to agree with the class of maximal planar graphs that are 4-connected (see Appendix). The graph K_4 is the only planar graph which is 4-connected and contains K_4. Any connected surface which is planar and has Euler characteristic 2, must be a sphere.

3.3. The main result in this note is:

Theorem 3 (Surface arboricity)**.** *The 2-sphere has arboricity 3. All other surfaces have arboricity* 4.

3.4. In general, if G is a graph and H is a sub-graph, then the arboricity of H is smaller or equal than the arboricity of G because a sub-graph of a forest is again a forest. The obvious monotonicity also follows from the Nash-Williams formula. In general however, the Nash-Williams ratio for an induced sub-graph H could be larger for a sub-graph. For example, if we remove a single isolated vertex, then the ratio goes up. The number $E/(V-1)$ alone does not determine the arboricity of $G = (V,E)$ in general, even so we suspect that it does so for manifolds.

3.5. Let F denote the set of triangles. [2] The Euler characteristic $X = V - E + F = 2$ and the Dehn-Sommerville equation $3F = 2E$ show that the **f-vector** $(f_0, f_1, f_2) = (V, E, F)$ encoding the cardinalities of all simplices in G is determined by the **area** F alone:

Lemma 2. *For a 2-manifold with Euler characteristic X, we have $(V,E,F) = (X + \frac{F}{2}, \frac{3F}{2}, F)$.*

Proof. The second equation is the Dehn-Sommerville $2E = 3F$, the first is the Euler gem formula. ☐

Corollary 2 (Algor-Alon)**.** *If G is a 2-sphere, then $E < 3V - 6$ and so $E/(V-1) < 3$.*

Proof. From the lemma, we see that $3F/2 < 2 + F/2 - 6$ which holds for $F > 4$ But every 2-sphere satisfies this inequality. ☐

Corollary 3 (Lower bounds)**.** *Let G be a connected 2-manifold of Euler characteristic X.*
a) If $X = 2$, then the arboricity is at least 3.
b) If $X \leq 1$, the arboricity is at least 4.

Proof. a) The above lemma gives for the entire graph the Nash-Williams ratio $E/(V-1) = 3F/(2+F)$. This is ≥ 2 for $F \geq 4$ and so by Nash-Williams, we have arboricity larger than 2. b) For $X = 1$, which forces a connected surface to be a projective plane, we have $E/(V-1) = 3$ so that the arboricity is already 4. For $X \leq 0$, we have $E/(V-1) > 3$. ☐

[2] Again write the cardinalities of V, E, F without the absolute signs.

4. Upper bound

4.1. In order to get upper bound for the arboricity, we will need to estimate how many faces F are needed for a given Euler characteristic X. If G is a surface G, define $K(x) = 1 - \deg(x)/6$ as the **curvature** of G at the point x.

Lemma 3 (Gauss Bonnet). $\sum_{x \in V} K(x) = X$.

Proof. The Euler characteristic is $\sum_{x \in V \cup E \cup F} \omega(x)$, where $\omega(x) = (-1)^{\dim(x)}$. Leave the value $\omega(x) = 1$ on v for $v \in V$, move the value $\omega(x) = -1$ from each $x \in E$ to the two vertices and each value $\omega(x) = 1$ for $x \in F$ to its three vertices. This leads to $K(v) = 1 - \deg(v)/2 + \deg_2(v)/3$, where $\deg_2(v)$ are the number of triangles containing v. In the manifold case, $\deg_2(v) = \deg(v)$ so that $K(v) = 1 - \deg(v)(1/2 - 1/3) = 1 - \deg(v)/6$. □

4.2. It is an interesting challenge to find the minimal area for a 2-manifold of a certain type. We do not need to know the minimum but can find some lower bounds. We think that $a), b), c)$ are sharp and that c) is the sharp discrete Loewner's inequality:

Corollary 4. *Assume G is a connected 2-manifold of Euler characteristic X.*
a) If $X = 2$, then G has at least 8 faces. This is the minimum.
b) If $X = 1$, then G has at least 14 faces
c) If $X = 0$, then G has at least 20 faces.
d) If $X < 0$, then G has at least $12 - 14X$ faces.

Proof. We have $(V, E, F) = (X + F/2, 3F/2, F)$ and $V = X + F/2$ gives

$$F = 2(V - X) \ .$$

a) The curvature can not be larger than $1/3$, so that we need at least $2 * 3 = 6$ vertices which gives $F = 2(V - X) = 8$.
b) Since we need to have a closed loop changing the orientation of a triangle, we need at least 7 vertices. In order to make this a manifold, we need at least one vertex more so that $V \geq 8$. This implies $F = 2(V - X) = 2(8 - 1) = 14$.
c) Look at the two shortest generators of the fundamental group. These are two curves of length 4 which intersect a point a. This gives 7 points. By Gauss-Bonnet two of the points must have at least 6 neighborhood this gives 3 new points at least so that $V \geq 10$. This implies $F \geq 2(V - X) = 2V = 20$.
Given any triangulation of a 2-torus, we can get a triangulation of a Klein Bottle and vice versa by reversing the identification along one of the loops, we again have $F \geq 20$.
d) Since $X = 2 - b_1$ or $X = 1 - b_1$, depending on orientability, we have $b_1 = 2 - X$ or $b_1 = 1 - X$. There must be at least $3 * 2 * b_1 = 12 - 6X$ or $6 - 6X$ vertices. With $V \geq 6 - 6X$ we have $F = 2(V - X) \geq 2(6 - 7X) \geq 12 - 14X$ vertices. □

4.3. All these estimates could certainly be improved: the **systole** of a discrete 2-torus G is the length of the shortest non-contractible curve in G. It is 4. A discrete version of **Loewners inequality** will give $4^2 \leq F/2$. This gives $F \geq 32$.

4.4. The next proposition shows that the **projective plane** is the threshold for arboricity. We should point out that with "discrete projective plane" we do of course not mean finite projective planes over finite fields but finite graphs which are 2-manifolds and have Euler characteristic 1. In any example, the geometric realization (which we do not look at) would be a classical 2-dimensional projective plane.

Proposition 1. *a) If $X = 2$, then $E/(V-1) < 3$.*
b) If $X = 1$, then $E/(V-1) = 3$.
c) If $X \leq 0$, then $E/(V-1) < 4$ meaning $F > 8 - 8X$.

Proof. We have $V = X + F/2, E = 3F/2$. For $F \to \infty$ the ration $E/(V-1)$ converges to 3. We need to show that in the case $X = 2$ it is always smaller than 3 and that in the case $X \leq 1$, it is always smaller than 4. This produces lower bounds for the area F.
a) $E/(V-1) = 3F/(2+F) < 3$. This is always the case. We do not need any conditions. b) In the case $X = 1$ we have $E/(V-1) = 3$. The Nash-Williams functional is independent of the surface! We can not get lower nor higher. The arboricity is 4.
c) Lets look at the equation $E/(V-1) = 3F/(2X+F-2) = 4$. This gives us an upper bound for F for which we possibly could have Nash-Williams ratio 4. This means $3F = 8X + 4F - 8$. This means $F = 8 - 8X$. ☐

Corollary 5. *There is no surface of arboricity* 5.

Proof. (i) For $X = 2$, we have seen the arboricity is 3.
(ii) For $X = 1$, we have the arboricity at least 4 because of the previous proposition b).
(iii) For $X = 0$ we need at least 28 faces.
(iv) Lets look at $X < 0$. But there $F \leq 8 - 8X$ is incompatible with $F \geq 10 - 14X$. ☐

5. Remarks

5.1. The **Barycentric refinement** of a graph G takes the complete subgraphs as the new vertex set and connects two if one is contained in the other.

Proposition 2. *a) For $X = 2$, the 2-spheres, the Barycentric refinement increases the ratio $E/(V-1)$. b) For $X = 1$, the projective plane the refinement leaves the ratio invariant. c) For $X \leq 0$, the refinement reduces the ratio.*

For given Euler characteristic X, the area F alone determines both V and E and so the Nash-Williams fraction. We have seen $\phi(G) = E/(V-1) = 3F/(X-2+F)$ We have $V' = V+E+F = X - 2E = X + 3F, E' = 2E + 6F = 9F$, $F' = 6F$. Therefore $E'/(V'-1) = 9F/(X-1+3F)$. It follows that in the Barycentric limit, $E/(V-1)$ converges to 3.

5.2. Given any surface type. For which surface do we maximize or minimize the Nash-Williams functional $E/(V-1)$ on sub-graphs. Is there an example of a surface for which $\phi((V,E)) = E/(V-1)$ evaluated on a subgraph generated by a subset of the vertex set is smaller than the functional evaluated on the manifold itself?
Conjecture: for any d-manifold (a graph for which every unit sphere $S(v)$ is a $(d-1)$-sphere), the Nash-Williams functional has its global maximum at G. This fails for non-manifolds like $G = K_2 + K_1$ of $H = K_2$ and K_1 for example where $\phi(K_2 + K_1) = 1/3$ but $\phi(H) = 1/2$ which is larger than $\phi(G)$. For surfaces, if $\phi(G)$ determines the arboricity, it would be a spectral property.

5.3. For d-manifolds, graphs for which every unit sphere $S(v)$ is a $(d-1)$-sphere, with $d \geq 3$ the situation is much different:

Theorem 4. *For any d-manifold type for $d \geq 3$, there are examples of arbitrary large arboricity.*

Proof. It is enough to look at a 3-manifold because for a $d > 3$ manifold, we can look at intersections of unit spheres which are 3-manifolds. Assume we want to reach the target arboricity n. First refine edges to render an edge degree larger than n. We call this graph H. Note that this can increase the vertex degrees also. Now, refine the edge m times. Each such refinement adds one vertex and $n+1$ edges. So, after m steps, (always doing it with the same edge) and m large enough, we have the fraction E/V (for the graph H with at least one large vertex degree larger than n) change to

$$\frac{E}{V} \to \frac{E+m(n+1)}{V+m} .$$

In the limit $m \to \infty$, this gets closer and closer to $n+1$. We will therefore eventually be larger than n. By the Nash-Williams theorem, also the arboricity of that refined graph will be larger than n. $\square$

5.4. The arboricity of a d-sphere can take values $d+1$ or more higher. We first believed that there is no d-manifold G with arboricity larger than c_d, the Perron-Frobenius bound given in the Barycentric limit. See [19].

5.5. Here are two questions.
1) Is the arboricity of a graph a spectral property for graphs or at least for d-manifolds? The later would be true if the above stated conjecture would hold.
2) The minimal possible arboricity of a d-manifold type is a topological invariant by definition. Can one use this to distinguish manifolds? The minimum for d-spheres is $d+1$. What is the minimum for d-tori?

Appendix: the theorem of Whitney of 1931

5.6. The argument of reducing the 4-color problem to planar graphs which are triangulations goes back to Arthur Cayley [5]. Hassler Whitney [23] proves a variant of the lemma below (we used it in [10]). A graph $G = (V, E)$ is **maximally planar** if one can not increase the edge set E (while keeping the vertex set V fixed) without losing the planar property. Planarity is by Kuratowski a finite combinatorial notion not needing the continuum: the class of planar graphs are the graphs which do not contain any edge refined version of K_5 nor $K_{3,3}$.

5.7. A graph $G = (V, E)$ is called **4-connected**, if removing 3 or less vertices keeps the graph connected. We always assume G to be connected = 1-connected. We make the choice to call the empty graph 0 not connected and K_1 1-connected by not 2-connected and K_2 2-connected by not 3-connected and K_3 3-connected but not 4-connected and K_4 4-connected but not 5-connected. Every 2-sphere is 4-connected, as we will see. A wheel graph with boundary C_n with $n \geq 4$ is never 4-connected as we can separate any of the boundary points v by removing its unit sphere $S(v)$ which is a path graph with 3 vertices. Graphs with more than 4 elements for which some vertex degree $\deg(v) \leq 3$, is not 4-connected. The octahedron graph is a 2-sphere. It is 4 connected but not 5-connected.

5.8. The next statement is not in the literature as such, because the notion of a 2-sphere is usually defined differently, like using notions of triangulations of Euclidean spheres. The statement on page 389 of the paper [23] comes close. The argument of reducing the 4-color problem to planar graphs which are triangulations goes back to Arthur Cayley and is part of "folklore", meaning often referred to without bothering to reference or prove it. According to [22] (work from 1975 having the bad luck to be in the shadow of the computer assisted proof

[3] from 1976), almost everybody who has thought about the 4-color theorem has been led to ideas like reducing to 4-connected and maximally planar graphs. We first believe that coloring 2-spheres could be easier than coloring planar graphs as there are more tools like the Sard theorem [12], telling that for a locally injective function f on a d-manifold, the level curves are all $(d-1)$-spheres or empty. Barycentric refinement changes chromatology however. The level curves are in the Barycentric refinement G_1 of G a graph G of maximal dimension 2, the chromatic number of G_1 is 3 as $f(x) = \dim(x)$ is a color. For 2-manifolds already, the chromatic number can be 5 and is conjectured to be 5 or less by a **conjecture of Albertson and Stromquist** of 1982 [1] (using slightly different definitions for manifolds).

Lemma 4 (Whitney)**.** *A graph with at least one edge is both* 4*-connected and maximal planar, if and only if is a* 2*-sphere or* K_4*.*

Proof. $\boxed{\textbf{(i)} \Leftarrow}$ First assume that G is K_4 or a 2-sphere.
a) If $G = K_4$, then it is maximally planar and 4-connected: The maximal planarity is clear since K_4 is planar (Kuratowski theorem) and because all edges are drawn. To see 4-connectedness, note that removing 3 vertices keeps a single vertex, which is by definition a connected graph. Assume from now on in part (i) of this proof that G is a 2-sphere:
b) G must be **tetrahedral-free** because $S(v)$ can not contain K_3 by definition of 2-spheres.
c) To see that G is 4-connected, we use contraposition and assume that it is now. There are then 3 vertices a, b, c which when removed, split the graph into two disconnected parts A, B. Case (1) If these three vertices are not all connected to each other, then it is a path graph abc of length 2 in the sphere which cuts the sphere into two parts. But this means a and c are boundary points. Case (2) If the three vertices form a triangle then necessarily there is a path around the triangle. This is the union of all balls $B(a), B(b), B(c)$ minus a, b, c. But this implies that removing the three vertices does not separate the graph. Any previous path connecting two points x, y which passes through a or b or c can be rerouted to take the detour.
d) G is planar: as G is a 2-sphere, removing one vertex in G produces a **contractible** graph H, meaning recursively that it is either K_1 or that there exists a vertex v for which both $S(v)$ or $G \setminus v$ are both contractible. H is now a graph with interior or boundary: the interior has circular graphs $S(v)$, the boundary has path graphs $S(v)$. That such a graph is planar can be seen by by induction, starting the smallest of this kind, the wheel graphs which do not contain any possibly refined $K_{3,3}$ nor K_5. Every extension step which adds a vertex and keeps the property of being a ball can be drawn in the plane and at every stage the boundary of the graph is a cyclic graph. Each extension step is a pyramid extension over a path subgraph of the boundary.
e) G is maximal planar: adding an other edge (x, y) with y different from $S(x)$ is not possible because the edge would have to cross the circle $S(x)$ which by the Jordan curve theorem can not happen. [The reference to the Jordan curve theorem uses the classical Jordan curve theorem. Alternatively, if x, y are connected in $S(x)$ intersected with $S(y)$ consists of exactly two points a, b which are not adjacent. The unit sphere $S(a)$ is not connected as it contains part of $S(x)$ and part of $S(y)$ which are not connected.]

$\boxed{\textbf{(ii)} \Rightarrow:}$ Now assume that G is a 4-connected, maximally planar graph G different from K_4. We want to show that it is a 2-sphere. By assumption, G has at least one edge. Define a **face** in G is a closed path in the graph which encloses a connectivity component. $A \subset \mathbb{R}^2$ of the complement of a planar embedding $\tilde{G}$ of G. [Alternatively if we do not want to refer to the

continuum, it can be defined as a minimal non-contractible loop in G. This loop can be also a triangle.] If there is no face, then there is no closed loop in the graph the graph is a tree, which is not 4-connected. So there is at least one face.
a) Every face of G is a triangle: this follows from maximal 4-connected planarity: if we had an n-gon face with $n > 3$, we could add additional diagonal connections, without violating either 4-connectivity nor planarity.
b) The graph G can not contain K_4: this would contradict 4-connectivity or imply that G itself is K_4.
c) A unit sphere $S(x)$ is triangle-free: otherwise we had a tetrahedral unit ball $B(x)$ contradicting b).
d) The unit sphere $S(x)$ is connected: if it were disconnected, then an additional connection in $S(x)$ could be added. This would violate the maximal 4-connected planarity.
e) The vertex degree $\deg(y)$ of every $y \in S(x)$ within $S(x)$ is larger than 1: assume $y \in S(x)$ was a leaf, having only one neighbor a in $S(x)$. As removing both x, y keeps the graph connected, a is connected via a path to an other point $b \in S(x)$ and so directly connected to a neighboring point $y \in S(x)$.
f) The vertex degree of y in $S(x)$ can not be 3: removing x, y, b with not interconnected (a, b, c) neighboring $y \in S(x)$ would render G disconnected as a path $(a, \dots, h, \dots, c)$ would lead to a homeomorphic copy of $K_{3,3}$ containing vertices (h, y, x, a, b, c) inside G contradicting the Kuratowski theorem.
g) Each unit sphere is cyclic: from steps c)-f) follows that $S(x)$ is cyclic with $n \geq 4$. The reason is that we have a graph for which every vertex y in $S(x)$ has exactly 2 disconnected neighbors. This means that $S(x)$ is a cyclic graph C_n with $n \geq 4$.
h) If we remove one vertex, the graph $H = G - v$ is contractible: we can assume that G is larger than K_4. The graph has interior points, the vertices which were not connected to v and so have circular unit spheres as well as boundary points, vertices which were connected to v. As we have seen already that all unit spheres $S(v)$ in G were circular graphs, the set of boundary points of H forms a circular graph. Take a boundary point w away. The set of boundary points still forms a circular graph. Now add again v and connect to the new boundary. This procedure is technically an edge collapse. We might have generated a K_4 subgraph. But then we either have K_4 or can remove the vertex with $S(v)$ being a triangle. We have made the graph smaller and can see by induction that H was contractible. $\square$

Appendix: about higher dimensional manifolds

5.9. For every k-simplex $x = (x_0, \dots, x_k)$ in a d-manifold (meaning a K_k subgraph), the intersection of all unit spheres $S(x_j)$ is a $(d-k-1)$-dimensional sphere. For $k = d-2$, it is 1-sphere. Its length is the **degree** $\deg(x)$ of that simplex x. For $d = 2$, this is the usual vertex degree. The f-vector $(f_0, \dots, f_d)$ counting the sub-simplices is in small dimension renamed with letters like (V, E, F) for surfaces. The quantities $V, E, F, C, H, \dots$ counting the number of vertices, edges, faces, chambers, hyper-chambers etc are all **valuations** meaning that $X(A \cup B) = X(A) + X(B) - X(A \cap B)$. The edge degrees $\deg_k(x)$ in dimension k divided by $k+1$ are the **curvatures** of f_k. Formulas like $\sum_{x \in F} \deg(x) = 4C$ are **Gauss-Bonnet formulas** for these valuations. Here are some lower dimensional cases:

dim	f-vector	Dehn-Sommerville	degree	X
1	(V,E)	$2E=2V$		$V-E$
2	(V,E,F)	$3F=2E$	$\sum_{x\in V}\deg(x)=2E$	$V-E+F$
3	(V,E,F,C)	$4C=2F,\ X=0$	$\sum_{x\in E}\deg(x)=3F$	$V-E+F-C$
4	(V,E,F,C,H)	$5H=2C$,$22E+40C=33F+45H$	$\sum_{x\in F}\deg(x)=4C$	$V-E+F-C+H$

The relation $22E+40C=33F+45H$ is the only surprising one up to dimension 4. See [11, 13].

5.10. A **3-manifold** is a finite simple graph G for which every unit sphere $S(v)$ is a 2-sphere. A **2-sphere** S is a graph for which every unit sphere is a 1-sphere, a circular graph with 4 or more elements such that $S-w$ is contractible for any w. The f-vector (f_0,f_1,f_2,f_3) of a 3-manifold is also denoted (V,E,F,C), where V is the number of vertices, E the number of edges, F the number of triangular faces and C the number of chambers, tetrahedral sub-graphs. The f-function $f(t)=1+\sum_k f_k t^{k+1}$ for a 3-manifold is the quartic polynomial $1+Vt+Et^2+Ft^3+Ct^4$.

5.11. The smallest 3-manifold is the 3-sphere $S=C_4\oplus C_4$, where $C_4=S_0\oplus S_0$ is the cyclic graph with 4 vertices and $H\oplus G$ is the Zykov join obtained by taking two disjoint copies $H+G$ of H and G, then connecting all vertices of H with all vertices of G. The join of a k-sphere and an l-sphere is in general a $k+l+1$-sphere. The f-functions of joins multiply $f_{H\oplus G}(t)=f_H(t)f_G(t)$. So, $f_S(t)=f_{S_4}(t)^2=(1+4t+4t^2)^2=1+8t+24t^2+32t^3+16t^4$. The **Euler characteristic** $X=V-E+F-C$ can also be written as $1-f_G(-1)$.

5.12. The Euler characteristic of any 3-manifold is zero. One can see this from the **general Gauss-Bonnet formula**

$$f_G'(t)=\sum_{v\in V} f_{S(v)}(t)\ .$$

See [14, 15]. For a 3-manifold, where every unit sphere is a 2-sphere, we have $f_{S(v)}(-1)=-1$ and $f_{S(v)}$ is odd around the center $t=-1/2$, then $f_G'(t)$ is odd around the center $t=-1/2$ and so $f_G(t)=1+\int_0^t f'(s)\,ds$ is even around the center $t=-1/2$ which means $f_G(-1)=f_G(0)=1$ so that the Euler characteristic is 0.

5.13. There are various other ways to see that an odd dimensional manifold has zero Euler characteristic. The most recent one is [18] for simplicial complexes. A very early approach is via Poincaré-Hopf [7], leading in the manifold case to an index formula [8] based on index expectation [9]. We tried in the past to use index expectation to understand better the Hopf conjectures [17, 16] stating that for even dimensional manifolds, positive sectional curvature implies positive Euler characteristic.

5.14. The Dehn-Sommerville relations for 3-manifolds is $4C=2F$. It complements the other Dehn-Sommerville relation $V-E+F-C=0$. Of interest is the **average vertex degree** $2E/V$ and the **average edge degree** $6C/E=3F/E$. The product of the average vertex degree and edge degree is $12C/V$ which is an **average chamber density**. In the Barycentric limit, the quantities $E/V, C/E$ or C/V become **universal**. They can be read off from the Perron-Frobenius eigenvector of the Barycentric refinement operator A (the eigenvector to the largest eigenvalues 24). See [19].

5.15. What happens for 3-manifolds or 3-dimensional **Dehn-Sommerville spaces** (manifold like spaces where each unit sphere is a Dehn-Sommerville graph), is that the arboricity number E/V determines all quantities (V, E, F, C) up to scale:

Proposition 3. *For 3-manifolds $C/V = E/V - 1$ and $F/V = 2E/V - 2$.*

Proof. Given E and V, we have $E - V = F - C$ and since $F = 2C$ we have $C = E - V$ and $F = 2E - 2V$. $\square$

This means $E/V = 1 + C/V$ so that $1 + C/V$ gives a lower bound on arboricity.

5.16. There is a 3-torus which has $E/V = 7$ so that the arboricity can already be 8 for 3-tori. For 3-spheres, we can construct graphs of arboricity 4. An example with arboricity 4 is the cross polytop with $(8, 24, 32, 16)$.

5.17. A 4-manifold is a finite simple graph such that every unit sphere $S(v)$ is a 3-sphere. Now there are two interesting Dehn-Sommerville invariants for the f-vector (V, E, F, C, H), where H is the number of **hyper-chambers**, 4-dimensional units of space. We have $5H = 2C$, the usual one and the more interesting identity

$$-22E + 33F - 40C + 45H = 0 \ . \tag{1}$$

They are derived from 2 eigenvectors of A^T, where A is the Barycentric refinement operator

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 6 & 14 & 30 \\ 0 & 0 & 6 & 36 & 150 \\ 0 & 0 & 0 & 24 & 240 \\ 0 & 0 & 0 & 0 & 120 \end{bmatrix} .$$

5.18. The curvature of the 5-manifold at a vertex v is

$$K(v) = 1 - \frac{V(v)}{2} + \frac{E}{3} - \frac{F}{4} + \frac{C}{5} - \frac{H}{6} \ .$$

The equations $H = 2C/5, E = (-40C + 33F + 45H)/22, V - E + F - C + H = 2$ simplify this to $K(v) = 0$ for all v. We see from this computation that the **zero curvature condition** $K(x) = 0$ is equivalent to the nontrivial Dehn-Sommerville equation. When writing [6], we had known only experimentally that for odd dimensional manifolds, the curvature is always zero. Both index expectation and Dehn-Sommerville then confirmed this picture.

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