

THE TREE-FOREST RATIO

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ABSTRACT. The number of rooted spanning forests divided by the number of spanning rooted trees in a graph G with Kirchhoff matrix K is the spectral quantity $\tau(G) = \det(1 + K)/\det(K)$ of G by the matrix tree and matrix forest theorems. We prove that that under Barycentric refinements, the tree index $T(G) = \log(\det(K))/|G|$ and forest index $F(G) = \log(\det(1 + K))/|G|$ and so the tree-forest index $i = F - T = \log(\tau(G))/|G|$ converge to numbers that only depend on the size of the maximal clique in the graph. In the one-dimensional case, all numbers are known: $T(G) = 0, F(G) = i(G) = 2 \log(\phi)$, where ϕ is the golden ratio. The convergent proof uses the Barycentral limit theorem assuring the Kirchhoff spectrum converges weakly to a measure μ on $[0, \infty)$ that only depends on dimension of G . Trees and forests indices are potential values $i = U(-1) - U(0)$ for the subharmonic function $U(z) = \int_0^\infty \log|x - z| d\mu(x)$ defined by the Riesz measure $d\mu = \Delta U$ which only depends on the dimension of G . The potential $U(z)$ is defined for all z away from the support of $d\mu$ and finite at $z = 0$. Convergence follows from the tail estimate $\mu[x, \infty] \leq Ce^{-a_d x}$, where the decay rate a_d only depends on the maximal dimension. With the normalized zeta function $\zeta(s) = \frac{1}{|V(G)|} \sum_k \lambda_k^{-s}$, we have for all finite graphs of maximal dimension ≥ 2 the identity $i(G) = \sum_{t=1}^\infty (-1)^{s+1} \zeta(s)/s$. The limiting zeta function $\zeta(s) = \int_0^\infty x^{-s} d\mu(x)$ is analytic in s for $s < 0$. The Hurwitz spectral zeta function $\zeta_z(s) = U_s(z) = \int_0^\infty (x - z)^{-s} d\mu(x)$ complements $U(z) = \int_0^\infty \log(x - z) d\mu(x)$ and is analytic for z in $\mathbb{C} \setminus \mathbb{R}^+$ and for fixed z in $\mathbb{C} \setminus \mathbb{R}^+$ is an entire function in $s \in \mathbb{C}$.

1. IN A NUTSHELL

1.1. We define here the **tree forest ratio** $\tau(G)$ of a connected finite simple graph $G = (V, E)$ as the number of rooted spanning forests in G divided by the number of rooted spanning trees in G . By the tree

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and forest matrix theorems, this is $\tau = \text{Det}(K + 1)/\text{Det}(K)$, where Det is the pseudo determinant and K the Kirchhoff Laplacian of G . In other words, it is $\tau = \prod_{\lambda \neq 0} (1 + 1/\lambda)$, where λ runs over the non-zero eigenvalues of K . An upper bound is $(1 + \lambda_2)^{-1}|V|$, where λ_2 is the **ground state**, the smallest non-zero eigenvalue of K . An expansion of the logarithm and a basic bound on the product gives $\log(1 + \zeta(1)) \leq \log(\tau) = \sum_{s=1}^{\infty} (-1)^{s+1} \zeta(s-1)/s \leq \zeta(1)$, where $\zeta(s)$ is the Kirchhoff spectral zeta function of G .

1.2. The same functional could be considered for other operators like the connection matrix L or the Hodge matrix H of a graph, even so a tree or forest interpretation lacks then. For the connection matrix which has negative eigenvalues in general, we would use L^2 to have no ambiguity when defining zeta functions. As we have an equivalent zeta function expression in the Kirchhoff case, the function $\tau(G)$ can now make sense even for manifolds G which do not feature spanning trees or spanning forests. The reason is that the Laplacian on the manifold defines a Minakshisundaram - Pleijel zeta function and a Ray-Singer determinant. For $M = \mathbb{T} = \mathbb{R}/\mathbb{Z}$, where ζ is the **classical Riemann zeta function**, we have $\tau_G = \sinh(\pi)/\pi$.

1.3. Potential theory comes in when seeing the logarithm of the **tree number** as a potential value $V(0) = \log(\text{Det}(L))$. Also the logarithm of the **forest number** $V(-1) = \log(\text{Det}(1 + L))$ is a **potential value** of the potential $V(z) = \log(\text{Det}(L - zI)) = \int \log(x - z) dk(x)$, where $dk = \sum_j \delta_{\lambda_j}$ is a finite pure point measure defined by the eigenvalues of L . When normalizing dk to become a probability measure $d\mu$, then $U(z) = \int \log(x - z) d\mu(x)$ has a subharmonic real part and $d\mu$ is the **Riesz measure** $d\mu = \Delta U$, a probability measure with support on the positive real axes. The Barycentral limit theorem assures that $d\mu_n$ converges in the Barycentric limit weakly to a measure $d\mu$ which only depends on the maximal dimension of the initial graph. The potential value $U(z)$ is defined for all z away from the support of $d\mu$ and finite at $z = 0$ because it is there the limiting growth rate of trees which is boxed in by 0 and the growth rate $U(-1)$ of forests. In the 1-dimensional case, it is the arc-sin distribution on $[0, 4]$ which is the potential theoretical equilibrium measure on that interval.

1.4. A major point we want to make here is to see that the limiting universal measure $d\mu$ has an exponentially decaying **tail distribution** $\mu[x, \infty) \rightarrow 0$ as $x \rightarrow \infty$. This implies that both the forest and tree numbers converge in the Barycentral limit because the potential values $U(-1)$ and $U(0)$ exist. Since by nature of tree and forest numbers $0 \leq$

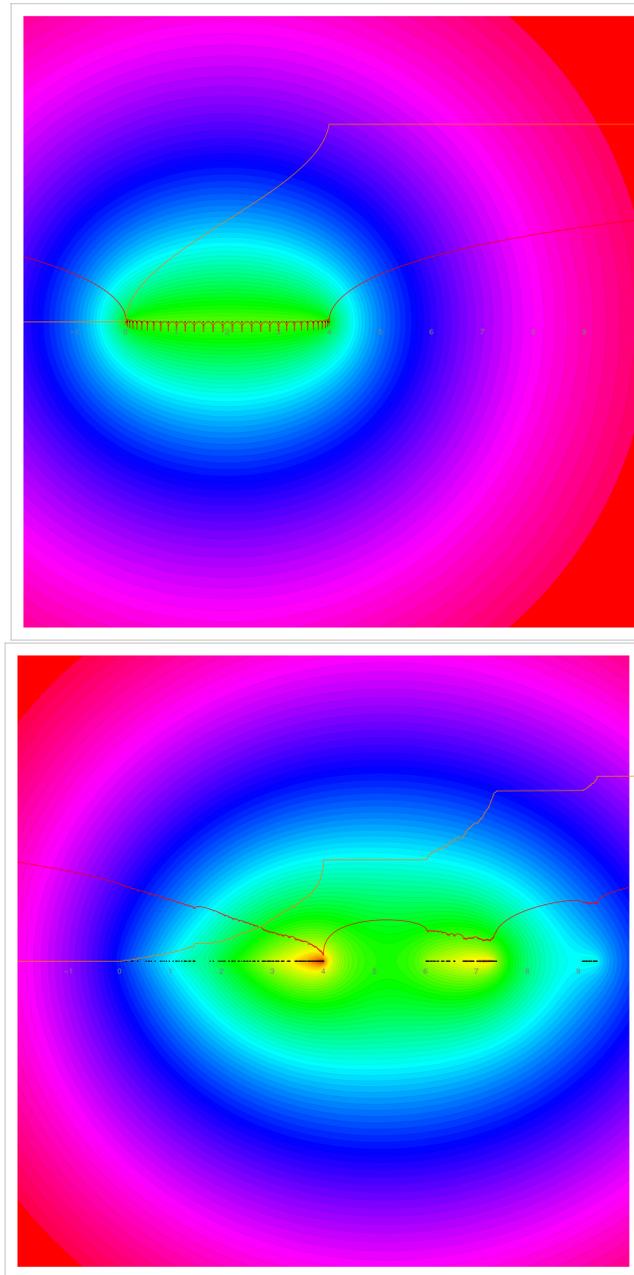


FIGURE 1. The real and imaginary part of the potential in the 1-dimensional case is seen to the left. The support of the density of states is $[0, 4]$. In the 2-dimensional case, where the support of the density of states is non-compact, we see gaps and the support of the spectrum could be a Cantor set.

$U(0) \leq U(-1)$, the convergence at the bottom of the spectrum $z = 0$ is not a problem. The tail estimate also shows that for z away from the positive real axes, the **Hurwitz zeta function** $\zeta_z(s) = U_s(z) = \int_0^{\infty} (x-z)^{-s} d\mu(x)$ is an entire function in s . For $z = 0$, we only know that $\zeta_0(s) = \zeta(s)$ is analytic for $s < 0$.

1.5. We will use some linear algebra to estimate the integrated density of states near ∞ in terms of the vertex degree distribution of the graph. We need a bit more than the **Schur-Horn inequality** or the **Gershgorin circle theorem**: we use $\lambda_k \leq 2d_k$, where d_k is the k 'th vertex degree. This appears to be a new result we have written down separately and which follows directly from the Cauchy interlace theorem. We have looked at the zeta function $\zeta(s)$ of circular graphs $G = C_n$ in [11] and looked at the limiting behavior of roots and then in [18] for the connection graphs of circular graphs, where the connection Laplacian is invertible [16, 19] and the Zeta function satisfies a functional equation.

1.6. For small graphs like complete graphs K_n or the graph complements C'_n of cyclic graphs C_n , the tree-forest ratio converges to the **Euler constant** $e = 2.718\dots$ in the limit $n \rightarrow \infty$. For K_n , one can see that directly from the eigenvalues $\lambda_0 = 0, \lambda_k = n$ because $\tau(G) = (1 + 1/n)^{n-1} \rightarrow e$. For Barycentric refinements of a graph $G = G_0$ the tree forest index $i(G_n) = \log(\tau(G_n))/|V(G_n)|$ of G_n converges to a universal constant that only depends on d . Since $\sum_{k=0}^d f_k(G_{n-1}) = |G_n| = f_d \leq \sum_{k=0}^d f_k(G_n)$ we could also take the limit, where $|V(G)|$ is replaced by any f_k like for example the volume f_d . The result is a direct consequence of **spectral Barycentral limit theorem** [14, 15]. In the Barycentric limit, we have $\log(\tau(G_n))/|G_n| \rightarrow \int_0^\infty \log(1+1/x) d\mu(x)$, where μ is the limiting density of states. To show that this is a finite number, we only need to show that the density of states dk decays fast. We needed something stronger than the Gershgorin circle theorem [6, 26] or the Schur inequality in matrix theory [3] and got $\lambda_k \leq d_k$. For $d = 1$, we can compute this limiting constant as $\lim_{n \rightarrow \infty} i(G_n) = 2 \log(\phi)$, where $\phi = (1 + \sqrt{5})/2$ is the golden ratio. When looking at Schur for the sum of the first n eigenvalues and the Cheeger inequality for the ground state λ_1 , it is natural to conjecture that the **linear graph** with n vertices and $(n - 1)$ edges strictly maximizes $\tau(G)$ among all connected graphs with n vertices.

1.7. Having made more experiments with the limiting density of states in the case $d = 2$ in particular, it is numerically indicative that [4, 6] is the **largest gap** in the Barycentric spectral measure for $d = 2$. If

we take a triangle $G = K_3$ and make Barycentric refinements, then after 2 or more Barycentric refinements, exactly half of the positive eigenvalues are in the interval $[0, 4]$ and half of the positive eigenvalues are in $[6, \infty)$. It is natural to ask whether there is a **gap labeling**: the **integrated density of states** $\int_0^x d\mu(x)$ is a rational number for x in a gap of the spectrum. But already in the case $d = 2$ for the relatively large gap containing the point 8, we don't see an obvious small rational number attached. Our best guess for the integrated density of states is $\int_0^8 dk(x) = 137/164$. Gap labeling results are prototyped by p-periodic Jacobi matrices, where the integrated density of states is j/p with integer j . For classes of non-periodic operators like almost periodic operators, the gap labeling works too. For early sources, see [21, 5, 25, 1]. For a recent development interesting for graph theory is the case of periodic Jacobi matrices on universal covers of leaf-free one-dimensional graphs [20], where one observes only point or absolutely continuous spectrum. In that respect, we have absolutely no idea yet what the spectral type of the Barycentric density of states dk is in dimension $d \geq 2$. Observing that the potential $U(z)$ (the analogue of the Lyapunov exponent in the case of Schrödinger operators) appears to be positive almost everywhere on the support of dk if the dimension is larger than 1.

1.8. In the case of Jacobi matrices, one is interested in the spectral measures of random operators $L(\omega)$ parametrized by a point in a probability space Ω which in the almost periodic case is a compact topological group with Haar measure. We believe that there is a non-Abelian compact topological group in all dimensions. It is hidden for $d \geq 2$ still. There should be Laplacian on this group such that dk_d is the density of states of a “random” Laplacian. In the case $d = 1$, where things are Abelian and understood, the group \mathcal{G}_1 is the **dyadic group of integers** and the limiting operator is the extension of $Hf(n) = 2 - f(n+1) - f(n-1)$ from the dense set \mathbb{Z} to its completion \mathcal{G}_1 . By Fourier theory, H is diagonalized and unitarily equivalent to $\hat{H}f(x) = 2 - 2\cos(\pi x) = 4\sin^2(\pi x/2)$ on the **Prüfer group** $\hat{\mathcal{G}}_1$ of all dyadic rationals on \mathbb{T} . While $dk = 1/(\pi\sqrt{4(4-x)})$ on $[0, 4]$ is clearly absolutely continuous, the operator H has dense pure point spectrum as an operator on $L^2(\mathcal{G}_1)$. Unlike in the Pontryagin pair $\mathbb{Z} \leftrightarrow \mathbb{T}$ where the Dirac points δ_x in \mathbb{T} are distributions and not functions in $L^2(\mathbb{T})$, the Dirac points $\delta_x \text{ in } L^2(\hat{\mathcal{G}}_1)$ are actual functions as the Prüfer group is discrete. In the Jacobi case, the operator was diagonal on the compact, non-discrete group of the Pontryagin pair, in the dyadic case, the operator is diagonal on the discrete-non-compact case.

2. INTRODUCTION

2.1. Investigating the relation between spectral data of a geometry and observable quantities like topological invariants is a classical theme in geometry [2]. It was popularized by Mark Kac whether one can “hear the geometry” [7]. Historically, a prototype result is **Weyl’s law** [27] relating the number of Dirichlet eigenvalues n of the Dirichlet Laplacian on G below λ with the volume $|G|$ of a domain G as $n \sim C_d \lambda^{\frac{d}{2}} |G|$ with $C_d = \frac{|B_d|}{2\pi^d}$. Herman Weyl looked in [27] at the planar case and proved $n/\lambda_n = |G|/(4\pi)$ using the explicit eigenvalues $\lambda_{m,n} = \pi^2(m^2 + n^2)/|G|$ for a square G . Kac’s question addresses the problem to what degree the spectral data determine space and how large the set of objects is that are isospectral. Because the zeta function encodes the spectrum, this also means to what degree the spectral zeta function determines the geometry.

2.2. Spectral questions are also related to inverse problems like to find the isospectral set in classes of geometries. While in one-dimensional cases, one can deform geometry in an isospectral way, in higher dimensions, isospectral rigidity appears. Isospectral sets tend to be discrete as isospectral deformations of Riemannian geometries in general do not exist any more [23]. For finite simple graphs, the question of how spectral data relate to a given invariant is largely parallel to the continuum theory [4] and can be investigated experimentally. A typical spectral result is that the pseudo determinant $\text{Det}(K)$ of the Kirchhoff Laplacian of a graph is the number of rooted spanning trees and the determinant of the Fredholm modification is the number of rooted spanning forests. We look here at this ratio. In this context, we can also remind about the spectral properties of connection matrices, where the number of positive eigenvalues minus the number of negative eigenvalues is the Euler characteristic [17].

2.3. The **tree forest ratio** is picked up here also because it belongs to a larger quest to investigate **functionals on graphs**, that is assigning numerical values to a graph. Considering critical points of functionals on geometries has been successful in physics, like for geodesics, area minimizing surfaces, total curvature minimizing. In graph theory, one can look at “packing numbers” like the chromatic number or the independence number manifolds. In graph theory, quantities that can be counted are the average simplex cardinality [9], the number f_k of k -dimensional simplices or the Euler characteristic $\chi(G) = \sum_k (-1)^k f_k$ or other length-related functionals [13]. Interesting examples of **spectral quantities** is the ground state energy, the smallest positive eigenvalue

of the Laplacian or then the **analytic torsion** which involves spectral data of the Hodge Laplacian $L = (d + d^*)^2$.

3. TREES AND FORESTS

3.1. Given a **finite simple graph** G , the collection of **rooted spanning trees** and the set of **rooted spanning forests** inside G are of interest. For forests, “rooted” means that every tree in the forest has an assigned vertex, which is interpreted as the **root** of the tree. Rooted trees in a forest include also seeds, which are one-point graphs. Each rooted tree can be seen as pointed topological space without closed loops nor triangles. It is a directory organization of the vertices and a stratification of space by giving the distance from the root. For any spanning tree T or spanning forest, the induced graph is the full space G because T spans G : the vertex set of T and G is the same. The ratio of rooted spanning forests and trees is the same than the ratio of spanning forests and trees.

3.2. By the **Kirchhoff matrix tree theorem**, the number of rooted spanning trees in G is $\text{Det}(K)$, where K is the **Kirchhoff matrix** of G and Det is the **pseudo determinant** of K , the product of the non-zero eigenvalues of K . Since every spanning tree has n possible roots if the graph has n vertices, the number $\text{Det}(K)/n$ is the number of spanning trees in G .

3.3. By the **Chebotarev-Schamis forest theorem** [22, 24, 10], the number of rooted spanning forests in G is $\det(1 + K)$, the **Fredholm determinant** of the Kirchhoff matrix K . See [12]. We introduce here the **tree-forest ratio**

$$\tau(G) = \frac{\det(1 + K)}{\text{Det}(K)}$$

and the **tree forest index**

$$i(G) = \log(\tau(G))/|G|$$

where $|G|$ is the number of vertices. In general $\tau(G) \geq 0$ because there are more forests than trees.

3.4. In dimension larger than 1, both the number of forests as well as the number of trees grow exponentially when doing Barycentric refinements. The reason is that in the two dimensional case already, we can see this geometrically. This is different for one-dimensional graphs, where the number of trees grows polynomially under Barycentric subdivisions while the number of forests grows exponentially. The precise

polynomial degree of the tree growth rate in the one-dimensional case depends on the genus b_1 of the graph.

3.5. For graphs with multiple connected components, where spanning trees naturally need to be disconnected, one can define the number of rooted spanning trees as the product $\prod_k \text{Det}(K_k)$, where K_k is the Kirchhoff matrix block belonging to the k 'th component. For the number of rooted forests, we naturally have the product $\prod_k \text{Det}(K_k + 1)$ already because forests do not need to be connected. With this extension of the definition for not necessarily connected graphs G , the ratio $\tau(G) = \text{Det}(K + 1)/\text{Det}(K)$ can in general be understood as the tree-forest ratio, whether G is a connected or a disconnected graph. To make the tree-forest ratio functional defined on all graphs, we define $\tau(0) = 1$ and $i(0) = 0$ if 0 is the empty graph.

3.6. Since all rooted spanning trees are also rooted spanning forests, we have $\tau(G) \geq 1$ and so $i(G) = \log(\tau(G))/|G| \geq 0$. It is easy to check that $\tau(G) = 1$ happens if and only if G has no edges. The reason is that already the presence of one single edge produces more forests than trees. For the complete graph K_2 for example, we have $\tau(K_2) = 3/2$ because there are 3 rooted forests and 2 rooted trees in K_2 .

3.7. When taking graph limits with larger and larger trees, the tree forest ratio diverges in general exponentially fast with the diameter. For the cycle graph C_n for example, there are n^2 rooted spanning trees and n spanning trees. In the same circular graph C_n , the number of spanning forests is the alternate **Lucas number** $L(n)$ given by the recursion $L(n + 1) = 3L(n) - L(n - 1) + 2, L(0) = 0, L(1) = 1$. The number of forests grows exponentially, while the number of trees grows polynomially. We have

$$\tau(C_n) = n^2/L(n)$$

and

$$\log(\tau(C_n))/n \rightarrow 2 \log(\phi) = \text{arccosh}(3/2) ,$$

where ϕ is the **golden ratio**.

3.8. For any one-dimensional graph G with $|G| = |E|$ edges, the **Barycentral limit measure** $d\mu$ exists explicitly. The limiting density of states measure is the arcsin distribution. Potential theoretically, the fact that that the number of trees grows polynomially while the number of forests grows exponentially is reflect in $\int_0^1 \log(4 \sin^2(\pi x)) dx = 0$

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which is not totally obvious. In terms of the density of states $dk(x) = \frac{1}{\pi\sqrt{x(4-x)}}$ we have

$$U(0) = \int_0^4 \frac{\log(x)}{\pi\sqrt{x(4-x)}} dx = 0$$

and

$$U(-1) = \int_0^4 \frac{\log(1+x)}{\pi\sqrt{x(4-x)}} dx = \log(\phi^2) = \operatorname{arccosh}(3/2) = 0.962424 .$$

3.9. Here is an other class, of graphs where we have a chance to compute the limit. One can look at **circulant graphs** $C_{n,A}$, the Cayley graph of a generating set $A \subset V(C_n)$. If the diameter of the graph is larger than 2, then the tree-forest ratio is going to explode for such graphs. The radius of a circulant graph is 2 if the set $A + A$ in \mathbb{Z}_p covers the entire set \mathbb{Z}_p . An extreme case is if $A = V(C_n)$, in which case we have the complete graph. An other example is the graph complement C'_n of C_n in which case the diameter is always 2. If the set $A + A$ for the generating set of the Cayley graph does not cover the graph, meaning that there are relations in the finitely generated group with words 3 or longer, then the tree-forest ratio of G_n diverges exponentially as $n \rightarrow \infty$. In some cases, we can compute the limit. For the graph complement \overline{C}_n of cyclic graphs C_n , which are graphs of diameter 2, one has:

Proposition 1. $\tau(\overline{C}_n) \rightarrow e$.

Proof. The eigenvalues of \overline{C}_n are explicitly known as

$$\lambda_{k,n} = \sum_{m=2}^{n-2} 1 - \cos(2\pi m \frac{k}{n}) = \sum_{m=2}^{n-2} 2 \sin^2(\pi m \frac{k}{n}) .$$

We have $2\lambda = 5 - 2 + 2 \cos(2\pi k/n) - \frac{\sin(k(2m-3)\pi/n)}{\sin(k\pi/n)}$. □

Proposition 2. $\tau(K_n) \rightarrow e$.

Proof. The non-zero eigenvalues are all n . We have $\tau(K_n) = \operatorname{Det}(1 + K) / \operatorname{Det}(K) = (1 + 1/n)^{n-1}$. □

4. THE KIRCHHOFF SPECTRAL ZETA FUNCTION

4.1. The **Kirchhoff spectral zeta function** of a finite simple graph $G = (V, E)$ is defined as

$$\zeta(s) = \sum_{\lambda \neq 0} \frac{1}{\lambda^s} ,$$

where λ ranges over the eigenvalues of the **Kirchhoff Laplacian** K of G . We see immediately from the definitions that

$$\tau(G) = \prod_{\lambda \neq 0} \left(1 + \frac{1}{\lambda}\right).$$

When taking the Barycentral limit, we need to normalize the function, usually by scaling with a factor $1/n$, where n are the number of vertices in the graph.

4.2. It follows from the definition that we can estimate the **spectral gap** λ_1 , which is the smallest eigenvalue of K from below.

Proposition 3. $\lambda_1 \geq 1/(\tau(G) - 1)$.

Proof. We have $(1 + 1/\lambda_1) = \tau(G) / \prod_{k \geq 2} (1 + 1/\lambda_k) \leq \tau(G)$. So $\lambda_1 \geq 1/(\tau(G) - 1)$. \square

4.3. Since $h(G) \geq \lambda_2/2$, where $h(G)$ is the **Cheeger constant** we also have

Proposition 4. $h(G) \geq 1/2(\tau(G) - 1)$.

4.4. Also in reverse, a very small spectral gap λ_1 produces a large τ because $\tau(G) \geq 1 + 1/\lambda_1$.

4.5. In general, we have for a finite simple graph with (not yet normalized) zeta function:

Proposition 5. $\log(\tau(G)) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\zeta(k)}{k}$.

Proof. Taking logs gives

$$\log(\tau(G)) = \sum_{\lambda \neq 0} \log\left(1 + \frac{1}{\lambda}\right).$$

Using the expansion $\log(1+x) = x - x^2/2 + x^3/3 + \dots$ and Fubini, we see

$$\log(\tau(G)) = \zeta(1) - \zeta(2)/2 - \zeta(3)/3 + \dots$$

\square

4.6. The Kirchhoff Laplacian K is one possible choice which comes natural when looking at trees or forests. We could also look at the **Hodge Laplacian** $L = D^2$ with **Dirac operator** $D = d + d^*$ of a graph G . The later has symmetric spectrum $-\lambda_n, \dots, -\lambda_1, 0, \lambda_1, \lambda_n$. This gives a **Hodge zeta function** $\zeta_L(s)$. Define the **Dirac spectral function** as

$$\zeta_D(s) = \sum_{\lambda > 0} \frac{1}{\lambda^s}$$

A **Dirac tree forest ratio** of a graph could now be defined as

$$\tau_D(G) = \frac{\text{Det}(1 + D)}{\text{Det}(D)}$$

Because non-zero eigenvalues come in pairs $\lambda, -\lambda$, we have

$$\tau_D(G) = \prod_{\lambda>0} \left(1 + \frac{1}{\lambda}\right) \left(1 - \frac{1}{\lambda}\right) = \prod_{\lambda>0} \left(1 - \frac{1}{\lambda^2}\right).$$

If 1 is an eigenvalue of D we would exclude that factor. One could also look at the connection Laplacian zeta function.

4.7. Looking at the zeta function rather than the eigenvalues adds an analytic angle to spectral theory. One can look at a **analytic properties** of the function $\zeta(s)$ and especially at the places, where $\zeta(s)$ is singular or places where $\zeta(s)$ is zero. The above formula shows that the tree-forest ratio measures a sort of growth rate of $\zeta(s)$ along the real axes.

4.8. Unlike trees or forests, the **spectral zeta function** can be considered for manifolds M , even so the interpretation of trees and forests is no more adequate in the continuum. On compact Riemannian manifolds M we have an exterior derivative d and so a Hodge Laplacian $L = D^2 = (d + d^*)^2$. Define

$$\tau(M) = \text{Det}(1 + L) / \text{Det}(L),$$

where both Det is the Ray-Singer determinant defined by a zeta regularization of $L = (d + d^*)^2$. One could also look just at the zeta function defined by the scalar Laplacian. We do not see any use yet for the quantity $\tau(M)$, the point is just that it can be considered.

4.9. Let us look for example at the case $M = \mathbb{T}$, where $D = i \frac{d}{dx}$ which has the eigenfunctions e^{inx} with eigenvalues $-n$. The Dirac zeta function

$$\sum_{n>0} \frac{1}{n^s}.$$

The Laplacian zeta function is $\sum_{n>0} \frac{1}{n^{2s}}$ as the eigenvalues of L are $1/n^2$.

4.10.

Proposition 6. $\tau(\mathbb{T}) = \sinh(\pi)/\pi$.

Proof. We have

$$\tau(M) = \prod_{n>0} \left(1 + \frac{1}{n^2}\right) = \sinh(\pi)/\pi$$

using $\sin(\pi x) = \pi x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2}\right)$ at $x = i$. □

5. THE BARYCENTRAL LIMIT

5.1. The quantity $T(G) = \log(\tau(G))$ is additive with respect to the disjoint union operation $T(G+H) = T(G) + T(H)$. Let $|G|$ denote the number of vertices of G . Define the **tree-forest index**

$$i(G) = \log(\tau(G))/|G|$$

where $|G|$ is the number of vertices. It has a chance to remain finite when taking graph limits. Let G_n denote the n 'th Barycentric refinement of a graph $G = G_0$. Now, the growth of the tree forest ratio is dominated by the growth of the forests. The index $i(G_n)$ is the difference of the Forest index $\log(F(G_n))/|G_n|$ and the tree index.

Theorem 1 (Tree Forest Universality). *$\lim_{n \rightarrow \infty} i(G_n)$ exists for every graph G and only depends on the maximal dimension d of G . For $d = 1$, the limit is $2 \log(\phi)$, where ϕ is the golden ratio.*

Proof. We prove this from the Barycentric central limit theorem which asserts that there is a (only dimension-dependent measure) μ on $[0, \infty)$ such that the spectral measures of $L(G_n)$ (which are point measures for all n) converge weakly to μ . Now, we have $i(G_n) = \int_0^{\infty} \log(1 + 1/x) d\mu(x)$ which is a difference of two **potential values**

$$i(G_n) = \int_0^{\infty} \log(1 + x) d\mu(x) - \int_0^{\infty} \log(x) d\mu(x).$$

The 1-dimensional case is special in that $d\mu$ has compact support on $[0, 4]$ and is the equilibrium measure on that interval. The potential $f(z) = \int \log|a - z| d\mu(z)$ exists for every $a \in \mathbb{Z}$ and is subharmonic and zero exactly on the interval $[0, \pi]$. The limiting index $i(G)$ in 1-dimension is therefore the potential value at $a = -1$.

In the $d = 2$ or higher dimensional case, the measure μ does no more have compact support. This is related to the fact that the maximal vertex degree becomes unbounded under Barycentric refinements. While in one dimension, the measure μ was absolutely continuous with a density concentrated mostly on the boundary $0, 4$ of the support, in higher dimensions, the measure has a tail continuity both at the lower bound 0 as well as at infinity. At zero we need a mild decay to make sure that $\int_0^{\infty} \log(x) d\mu(x)$ is finite. At infinity, we need a stronger decay to make

sure that $\int_0^\infty \log(1+x)d\mu(x)$ exists. But we know also from potential theory of subharmonic functions and in our case from the combinatorial context that $0 \leq \int_0^\infty \log(x) d\mu(x) < \int_0^\infty \log(1+x)d\mu(x)$. It suffices therefore to show that $\int_0^\infty \log(1+x)d\mu(x)$ exists. This amounts to estimate the decay rate of μ at infinity. The fact that there is an exponential decay in dimension 2 or higher in the next section. We are using an estimate $\lambda_k \leq 2d_k$, a Perron-Frobenius result on the limiting dimension distribution in Barycentric limits as well as an estimation of the vertex cardinality points of G_n which depends on the **generation** of a point. A vertex in G_n has generation k if it has become a vertex in G_{n-k} and not before. A vertex in G_n of **generation** zero is a vertex which has in the previous level G_{n-1} not been a vertex. Note that if a simplex is a vertex, it is especially again a vertex in the next generation. What happens is that we have an exponential decay of vertex degrees depending on generation. \square

5.2. A different proof could be done along the lines of the Barycentric spectral limit theorem without using the spectral measure $d\mu$: if G has maximal dimension d then the Barycentric refinement G_1 of a d -simplex in G consists of $(d+1)!$ simplices glued. There is a stratification of growth rates of the different simplices depending on dimension. The number of maximal simplices grows exponentially faster than the number of co-dimension-one simplices, we can asymptotically treat every refined n -simplex as a copy of disjoint $(n+1)!$ simplices plus a gluing correction which as it is lower dimensional will be a definite factor smaller. The limiting tree forest ratio is again Cauchy sequence as we add up a sequence of errors which geometrically decay.

6. TAIL DECAY

6.1. In this section we prove a proposition on the limiting density of states dk in dimension $d \geq 2$.

Proposition 7. *There exists a positive constant c such that $\mu([x, \infty)) \leq e^{-cx}$.*

6.2. In dimension $d = 1$, the density of states dk has compact support $[0, 4]$ and therefore does not need an exponential decay estimate. The lemma of course still applies. Let therefore G be a graph of dimension $d \geq 2$ and let G_n be the Barycentric refinements of G . The vertices of G_n are the simplices (complete subgraphs) of G_{n-1} and two are connected, if one is contained in the other. Every vertex x in G_n can be assigned a **generation** $\gamma(x)$ defined to be the integer k , if it has been a vertex in $G_{n-1}, \dots, G_{n-k+1}$. For example, a vertex has generation 1

TREE FOREST RATIO

if it has been a simplex of positive dimension in G_{n-1} . A point has generation n if has been a vertex in G already. Every vertex in G_n has generation $\gamma(x) \in \{1, 2, \dots, n\}$.

6.3. The f -vector of G is the vector (f_0, f_1, \dots, f_d) , where f_k are the number of k -dimensional simplices in G . There are $\text{gen}_n = f_0(G_0)$ vertices of generation n .

There are $\text{gen}_{n-1} = f_0(G_1) - \text{gen}_n$ vertices of generation $n - 1$.

There are $\text{gen}_{n-2} = f_0(G_2) - \text{gen}_n - \text{gen}_{n-1} = f_0(G_2) - f_0(G_1)$ vertices of generation $n - 2$

There are $\text{gen}_1 = f_0(G_n) - \text{gen}_n - \text{gen}_{n-2} - \dots - \text{gen}_2$ vertices of generation 1.

Since $f_0(G_n)$ grows exponentially at least like $(d + 1)^n$, we also have in G_n at least $C(d + 1)^{n-k}$ vertices of generation k .

The **Barycentric refinement operator** A is a $(d + 1) \times (d + 1)$ upper triangular matrix defined by $f(G_1) = Af(G)$. It is given as $A_{ij} = \text{Stirling}(i, j)i!$, involving the **Stirling numbers of the second kind**. The **Perron-Frobenius eigenvector** of A is the probability eigenvector of A belonging to the largest eigenvalue $(d + 1)!$. For exam-

ple, in the case $d = 2$, when $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 6 \\ 0 & 0 & 6 \end{bmatrix}$, the Perron-Frobenius

eigenvector is $[1/6, 1/2, 1/3]$ to the eigenvalue $(d + 1)! = 6$. This means that for large n , the graph G_n has about 1/6th of the simplices which are points, about a half which are edges and about 1/3rd which are triangles. In the limit $d \rightarrow \infty$, the Perron Frobenius probability measures appear converge weakly to a point measure around 0.7215, a fact which we could not prove yet.

Lemma 1. *There is a $c_1 > 0$ such that for all k , at least $c_1^{n-k}|G_n|$ vertices in G_n are generation k vertices.*

This means that the generation γ as a random variable on the finite set of vertices of G_n asymptotically has an exponential distribution. There very few vertices of large generation and very many of small generation. Now we will see that on large generation vertices, there is an upper bound on the eigenvalues which is exponential too in the generation.

6.4. If $S(x)$ is the **unit sphere** of a vertex x , then the k 'th Barycentric refinement $S_k(x)$ is the unit sphere of x in G_k . Because $|S_k(x)|$ is the **vertex degree** of x , the vertex degree of generation k vertices is at least $|S_k(x)| \sim d!^k$.

Lemma 2. *There exists $c_2 > 0$ such that maximally $c_1^k |G_n|$ vertices have degree larger than c_2^k .*

6.5. It follows from the Gershgorin circle theorem that there less than $c_1^k |G_n|$ vertices for which the largest eigenvalue is larger than $2c_2^k$. Now $c_1 = 1/(d+1)!$ and $c_2 = d!$. Since this implies that the error area decays exponentially for $x \rightarrow 1$, was also have the function itself decay exponentially.

7. QUESTIONS

7.1. Which graphs with n vertices have the maximal tree-forest ratio? The minimum is 1. For small n we see that the linear graph is the maximum. It is pretty safe to conjecture that the linear graph is always the maximum:

7.2. What is the average tree-forest ratio on Erdoes-Renyi spaces? We can also look more generally at the potential average on Erdoes-Renyi spaces which produces a density of states measure for each n and p . And then we can ask for which choice of p_n does the density of states average $dk_{n,p(n)}$ converge weakly in the limit $n \rightarrow \infty$.

7.3. What happens with graph operations like various products or joins? For disjoint unions, we have the product of determinants so that $\tau(G + H) = \tau(G)\tau(H)$. Under graph complements, there is no clear relation yet. Also the Zykov join operation does not produce obvious relations. What happens with the tree-forest ratio when taking Shannon products? We do not see any (at least obvious) relation between the tree number yet.

7.4. For connection Laplacians, we know that the zeta functions multiply. because the connection Laplacian of a product of graphs tensor. $L_{G*H} = L_G \otimes L_H$. We therefore have for the connection Laplacians:

$$\log(\rho(G * H)) = \sum_k (-1)^{k+1} \zeta_G(k) \zeta_H(k) / k .$$

7.5. What is the relation between the Hodge Laplacian and connection Laplacian tree-forest ratio? In order to define a tree-forest ratio from the connection Laplacian itself, we have to look at the spectrum of the square. We can for any positive energized complex look at $\tau_L(G) = \det(L + 1) / \det(L)$ which makes sense as L is always invertible.

7.6. We should reiterate that in the case $d \geq 2$, we have not even a proof yet that there are gaps in the spectrum. This should be easy to do as in the case $d = 2$ we see a prominent gap $[4, 6]$ associated to the integrated density of states $1/2$. While we know now something about the decay of dk at infinity, we do not have an asymptotic near 0 or the end of gaps. As for 0, the Schur estimate does not help as we have always a certain percentage of vertices with minimal vertex degree $(d + 1)!$. That means that we can (using Shur) only prove that there exists a constant c such that $dk([0, x]) \leq cx$ for small x . As we know $U(0) \geq 0$ due to its relation to the growth rate of trees, we know that $U(z)$ is finite in a neighborhood of $z = 0$. It is natural to conjecture that the potential $U(z) < \infty$ for all $z \in \mathbb{Z}$. This would imply that the **logarithmic capacity** $-\int \int \log |z - w| dk(z) dk(w)$ is finite in all dimensions d . We know that the value is 0 in the case $d = 0$, where $U(z) = 0$ on the support $[0, 4]$ of dk . Having finite logarithmic capacity especially implies that the **Lebesgue decomposition** of $dk = dk_{pp} + dk_{sc} + dk_{ac}$ has no pure point part.

7.7. Fourier theory could be an approach to determine the nature of the spectrum $d\mu$. Since the measure μ has exponential decay at infinity, all Fourier coefficients $\hat{d}\mu(t) = \int_0^\infty e^{itx} d\mu(x)$ exist. In order to determine the spectral type, one could also focus on a compact part of the spectrum like for example, $[0, 4]$. Now, we can look at Fourier series of $d\mu$ instead rather than the Fourier transform. For example, by a theorem of Wiener [8], if $\frac{1}{n} \sum k = 0^{n-1} \hat{\mu}(k)^2$ converges to zero for $n \rightarrow \infty$, then $d\mu$ has no point spectrum.

8. CODE

8.1. The following Mathematica code plots the potential function (the analog of the Lyapunov exponent), and the integrated density of states (the analog of the rotation number) as well as the spectrum of the Barycentric measure dk in the planar case. The code is optimized for the two-dimensional case so that we do not have to use general costly clique finding algorithms.

```
T3[s_]:=Module[{v=VertexList[s]}, Union[Partition[Flatten[Union[
  Table[z=EdgeList[Subgraph[s, Complement[VertexList[
    NeighborhoodGraph[s, v[[k]]]], {v[[k]]}]]]; Table[Sort[
    {v[[k]], z[[j, 1]], z[[j, 2]]}], {j, Length[z]}], {k, Length[v]}]]], 3]]];
T2[s_]:=Module[{e=EdgeList[s]},
  Table[Sort[{e[[k, 1]], e[[k, 2]]}], {k, Length[e]}];
T1[s_]:=Module[{v=VertexList[s]}, Table[{v[[k]]}, {k, Length[v]}];
Whitney2D[s_]:=Sort[Map[Sort, Union[T3[s], T2[s], T1[s]]];
GraphFromComplex[G_]:=Module[{n=Length[G], v, e={}}, Do[x=Sort[G[[k]]];
  If[Length[x]==3, e=Union[e, {x->{x[[1]]}, x->{x[[2]]}, x->{x[[3]]},
```

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```
x->Sort[{x[[1]],x[[2]]},x->Sort[{x[[1]],x[[3]]},
x->Sort[{x[[2]],x[[3]]}]];
If[Length[x]==2,e=Union[e,{x->{x[[1]],x->{x[[2]]}}],{k,Length[G]}];
rules=Table[G[[k]]->k,{k,n}]; UndirectedGraph[Graph[e /. rules]];
Barycentric2D[s_-]:=GraphFromComplex[Whitney2D[s]];
Barycentric2D[s_ ,k_-]:=Module[{s1=s},Do[s1=Barycentric2D[s1],{k}];s1];
s=Barycentric2D[CompleteGraph[3],6];
L=Sort[Eigenvalues[1.0*KirchhoffMatrix[s]]];
U[z_-]:=Sum[Log[Abs[z-L[[k]]]],{k,Length[L]}/Length[L];
V[x_-]:=Sum[If[L[[k]]<x,1,0],{k,Length[L]}/Length[L];
f[x_-]:={Red,Point[{x,0}]}];
W=Graphics[Map[f,L],PlotRange->{{-1,10},{-1,7}}];
Show[{W,Plot[{3*U[x],6*V[x]},{x,-2,10},PlotPoints->1000]}
{TreeIndex=U[0],ForestIndex=U[-1],TreeForestIndex=TreeIndex/ForestIndex}
```

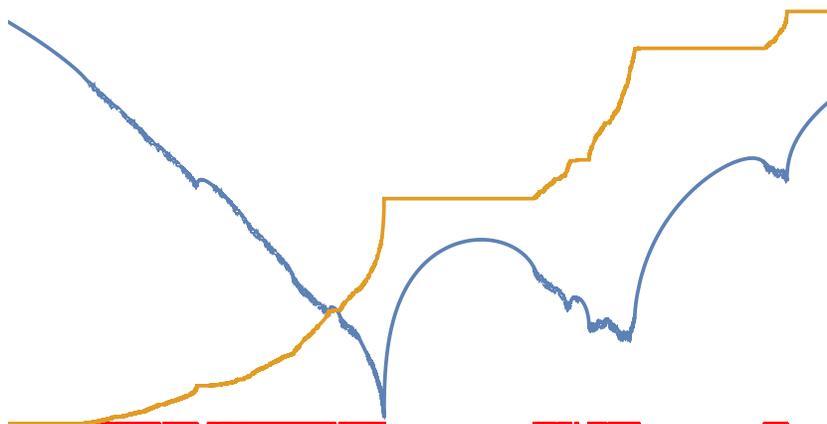


FIGURE 2. The output of the above code shows the potential $\operatorname{Re}(U(x + 0i))$ and the integrated density of states $\operatorname{Im}(U(x + 0i))$ of the complex potential $U(z) = \int_{\mathcal{C}} \log(z-w) dk(w)$ of the Barycentric 2-dimensional density of states dk . We see the 6th Barycentric refinement G_6 for $G = G_0 = K_3$. The prominent gap $[4, 6]$ is very clear numerically, but not theoretically established.

8.2.

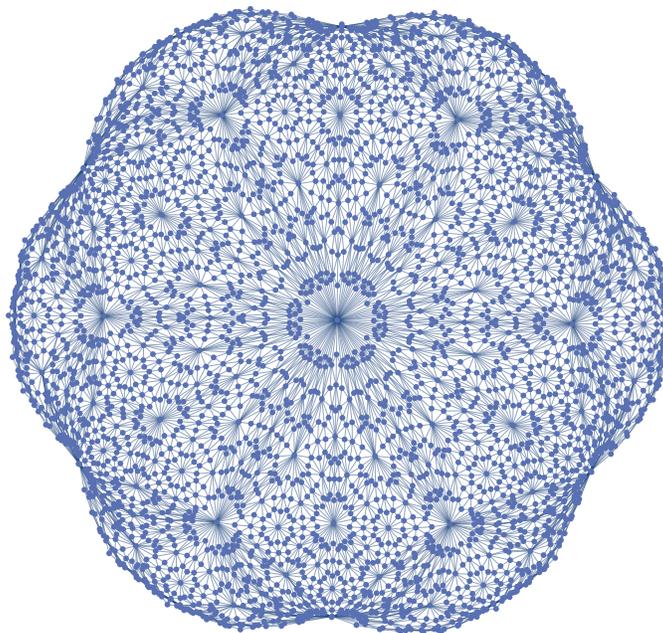


FIGURE 3. The 5'th Barycentric refinement G_5 of $G_0 = K_3$ has 3937 vertices, 11712 edges and 7776 triangles. The next version G_6 used in the above spectral picture already has $23425 = 3937 + 11712 + 7776$ vertices.

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