COMBINATORIAL MANIFOLDS ARE HAMILTONIAN

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ABSTRACT. Extending a theorem of Whitney of 1931 we prove that all connected d-graphs are Hamiltonian for $d \geq 1$. A d-graph is a type of combinatorial manifold which is inductively defined as a finite simple graph for which every unit sphere is a $(d-1)$-sphere. A d-sphere is d-graph such that removing one vertex renders the graph contractible. A graph is contractible if there exists a vertex for which its unit sphere and the graph without that vertex are both contractible. These inductive definitions are primed with the assumptions that the empty graph 0 is the (-1)-sphere and that the one-point graph 1 is the smallest contractible graph. The proof is constructive and shows that unlike for general graphs, the complexity of the construction of Hamiltonian cycles in d-graphs is polynomial in the number of vertices of the graph.

1. INTRODUCTION

1.1. Hassler Whitney [27] proved in 1931: "Given a planar graph composed of elementary triangles in which there are no circuits of 1,2 or 3 edges other than these elementary triangles, there exists a circuit which passes through every vertex of the graph". This theorem implies that "every 4-connected maximally planar graph is Hamiltonian". The later can be restated as "every 2-sphere is Hamiltonian". A 2-sphere is a finite simple graph in which every unit sphere $S(x)$ is a 1-sphere, a circular graph of length 4 or higher and. A 2-ball is a 2-sphere with a removed vertex. The class of 2-spheres is exactly the class of 4-connected maximal planar graphs. Bill Tutte [24] extended Whitney's theorem to all 4-connected planar graphs. An overview over graph theory and history, see [12, 2, 19, 17].

1.2. As Whitney has pointed out already, the interest in triangulated graphs was fueled by the problem of coloring maximal planar graphs as this is the class of planar graphs which are hardest to color among

all planar graphs. Whitney's paper actually was part of the thesis which he wrote under the guidance of George Birkhoff in the area of graph chromatology. The 2-sphere restatement of Whitney's theorem is graph theoretical. The original statement is too, but it only becomes so after rephrasing "planar" within graph theory using the Kuratowski theorem of 1930. What is interesting about the sphere formulation of Whitney's theorem is that it opens the door to generalize coloring and Hamiltonian graph problems to higher dimensions, where it is detached from embedding questions.

1.3. The Whitney result can be extended not only to other dimensions, we can also extend the class by using the language of simplicial complexes. The 1-skeleton complex of a finite abstract simplicial complex is a graph. When is such a graph Hamiltonian? The concept of finite abstract simplicial complex is due to Dehn and Heegard from 1907 [6]. One can restrict the question to shellable complexes, a concept put forward in a combinatorial setting by Hadwiger and Mani from 1972 [11]. The phenomenon of non-shellability of spheres has been discovered already in 1924 in a topological setting.

1.4. The result can be extended to **generalized d-graphs**, finite simple graphs G for which the boundary alone is a $(d-1)$ graph without boundary and for which every inner vertex has a unit sphere which is a $(d-1)$ sphere and such that for every inner vertex x near the boundary there is a triangle (x, a, b), where (a, b) is an edge on the boundary. This goes beyond d-graphs in that we don't need to have interior points for example. Every d-graph with or without boundary is a generalized d-graph.

1.5. A simple example of a generalized d-graph is the d-simplex itself, in which we look at the $d-1$ dimensional skeleton complex. It has no interior point and the boundary is naturally a $(d-1)$-sphere. A wheel graph in which all boundary edges are stellated is **not** a generalized d-graph. While the boundary as a 1-graph, it has an interior point which is not accessible from the boundary. It is not Hamiltonian because the Hamiltonian boundary curve which is forced can not be detoured to the interior. But if one exterior triangle is removed, then the inner point becomes accessible and the graph becomes Hamiltonian. A three dimensional example is the stellated solid octahedron, where the octahedron is the interior unit sphere. But in that three dimensional case, not only the accessibility fails, also some unit spheres are neither 2-spheres, nor 2-balls or 2-simplices.

1.6. How can the Hamiltonian property fail for manifold-like graphs? It can be due to lack of connectivity but this mechanism does not explain non-existence of Hamiltonian cycles of higher dimensional Goldner-Harary graphs. The classical Goldner-Harary graph is a non-Hamiltonian graph which is the 1-skeleton graph of a shellable 3-dimensional complex for which all unit spheres are contractible shellable 2-complexes. After a Barycentric refinement, the Goldner-Harary graph becomes a 3-ball and is Hamiltonian. [Update June 22, 2018: originally, I stated that it is not Hamiltonian, probably tested with the wrong graph at first. But it is a 3-ball. All unit spheres are either 2-spheres or 2-balls.]

1.7. The failure of being Hamiltonian is rare when building up complexes randomly by successive simplex aggregation to the boundary. We observe that one culprit for the failing Hamiltonian property often is the loss of the manifold structure. This happens for the Goldner-Harary graph G, obtained from shellable complexes first in dimension 3. The two dimensional boundary complex has a Barycentric refinement G_1, where unit spheres are not balls or spheres as needed for a manifold with boundary. For the Barycentric refinement G_1 of G the boundary becomes Hamiltonian. Still, there remain defects from being Euclidean: there are still unit spheres which are neither 2-spheres nor generalized 2-balls.

1.8. The question on how to define a d-dimensional "sphere" combinatorially came up already when Ludwig Schläfli generalized the Euler gem formula $v - e + f = 2$ from 2 dimensional to higher dimensional spheres and proving that the Euler characteristic of a d-dimensional sphere is $v_0 - v_1 + v_2 - \cdots \pm v_d = 1 + (-1)^d$ [21]. Without stating explicitly, Schläfli assumed in 1852 that spheres are shellable simplicial complexes. He called a contractible space a "polyschematische Figur" and a discrete sphere a "sphärisches Polyschem". Also Euler implicitly assumed a shell decomposition in two dimensions. However, there are triangulations of Euclidean spheres which are no more shellable, and the first examples appeared already in 1924. Non-shellable spherical simplicial complexes are puzzles, where the individual cells are so entangled within each other that it is not possible to take out one. Related is the existence of 2-dimensional structures like the Bing house or dunce hat which are homotopic to a point but which can not be shrunk to a point. The Bing house for example produces a non-shellable 3-ball.

1.9. In the wake of the foundational crisis in mathematics in the early 20th century, Herman Weyl [25] asked for ways to define spheres within finite mathematics. He refers in his exposé to [6] and mentions that a

combinatorial definition of "Kugelschemata" in dimension 4 and larger
is not yet known. Defining a sphere as a triangulation of a Euclidean
sphere is not possible, as the later uses the notion of Euclidean space
tapping into mathematics which a finitist would not accept. Also the
notions of triangulations can be quite complicated and can contain
higher dimensional simplices than the space itself. Today, one seems to
be less worried about the consistency of the foundations of mathematics
but cares more about realizing mathematical structures faithfully
in a computer without approximations or to see a result in the context
of reverse mathematics trying to localize which axioms or axiom systems
are needed to prove a phenomenon. In any case, combinatorial
mathematics can be seen as a shelter in which one could retreat in the
unlikely event of an emerging inconsistency in the ZFC axiom system.

1.10. Thanks to notions of homotopy put forward by J.H.C. Whitehead [26] which was adapted to the discrete later on, one can define a
d-sphere as a simplicial complex which has locally a unit sphere which
is a $(d-1)$ sphere and such that removing a point renders the complex
contractible. (We use this term here as a synonym with collapsible and
would use "homotopic to the 1 point complex" to describe more general
homotopies). Such notions have emerged in the discrete in the 1990ies
in Fisk's theory [8], in discrete Morse theory of Forman [10] or digital
topology of Evako [13], refined in [4]. The definitions of d-spheres and
d-graphs are done with the goal to be as simple as possible.

1.11. The assumption of having a "contractible graph after puncturing a sphere" restricts the class of spheres slightly because of the existence of non-shellable complexes and the existence of non-contractible
complexes which are only homotopic to a point. It turns out not to
be a substantial problem as the Barycentric refinement of any simplicial complex is always a Whitney complex and the Whitney complex
of a d-sphere is always shellable. The troubles with non-shellability
are washed away with one Barycentric refinement. The troubles with
non-Euclidean properties however (as we see here by linking it to a
Hamiltonian property) can not always removed through Barycentric
refinements. One dimensional star graphs illustrate this. In any case,
the graph theoretical definition of spheres therefore are almost no lack
of generality from a topological point of view and Schläfli's shellability
assumption was not such a big restriction of generality after all when
seen in the Weyl program to finitize Euclidean geometry.

1.12. Graphs not only have the benefit of being intuitively more accessible than simplicial complexes which are sets of sets. Having a simple

and solid graph theoretical definition of what a d-sphere is, one can ask questions which usually are asked only in graph theory. There are two theorems about 2-spheres which are particularly elegant: the first is the 4-color theorem which is equivalent to the statement that *2-spheres have chromatic number* 3 *or* 4, where 3-colorability is equivalent to the property of being Eulerian. Second, there is the already mentioned Whitney theorem on Hamiltonian graphs stating that *2-spheres are Hamiltonian*. The observation that the class of 2-spheres is the same as the class of maximally planar, 4-connected graphs is not difficult to verify [14]. The fact that looking at maximally planar 4-connected graphs covers all of the coloring of planar graphs has been pointed out by Whitney already [27]. It must have been folklore wisdom already then, but not yet formulated graph theoretically before Whitney.

1.13. 1-spheres are cyclic graphs with 4 or more nodes. They have chromatic number 2 or 3 and are Hamiltonian. Having Whitney covering the two dimensional case, it is natural to ask what happens with the coloring statement in dimension 3 and higher. In Fisk theory [8, 9], the coloring problem is reduced to coloring problems in higher dimensions, leading to the conjecture that a d-sphere has chromatic number $d + 1$ or $d + 2$ [14, 15]. Discrete projective planes can have chromatic number 5 and indeed they are not embeddable in an Euclidean 3-space. For tori, the chromatic number is 3,4 or 5 [1].

1.14. The sphere coloring statement bounding the chromatic number of d-spheres between $d + 1$ and $d + 2$ still remains to be proven in dimensions larger than 2 but it looks good: one only has to verify that it is possible to do successive edge refinements in the interior of a $(d+1)$-ball to render it minimally colorable by $d + 2$ colors, coloring so the boundary sphere. The existence result is constructive also in higher dimensions but it will lead in particular to a constructive proof of the 4 color theorem. The second line of question about Hamiltonian cycles is easier and covered in the current article. Unlike in 2 dimensions, where the existence of a Hamiltonian cycle is unrelated to coloring statements in higher dimensions because the coloring argument in two dimensions relies on a Jordan curve statement.

1.15. The main reason why the main result proven here is true is that the boundary of a d-graph with boundary is a union of connected $(d-1)$ graphs which by induction are Hamiltonian. We can use calculus to divide the discrete manifold up and expose the interior to the boundary. In a larger class of "generalized d-graphs", we can make the reduction enhanced by removing "boundary discs" until we have no interior

points near the boundary any more. The Hamiltonian cycle on the boundary can then be extended to the still remaining inner points near the boundary. In some sense, we can extend a Hamiltonian cycle on a $(d-1)$-graph with boundary to a "fattened surface" and achieve with cutting out holes that all the graph becomes a fattened surface. Any pure simplicial d-dimensional complex is Hamiltonian as long one can identify the boundary δG as a $(d-1)$ dimensional complex without boundary and such that δG allows access to the nearby interior points.

1.16. In general, we see that the construction of Hamiltonian cycles can be done quite effectively for d-graphs: take a d-graph with boundary with n vertices, use a function f to chop it into two d-graphs $\{f < 0\}$ and $\{f > 0\}$ with boundary, where each part contains about half the vertices. Now do the Hamiltonian construction in both parts and build a single bridge in the joining part $f = 0$ which does not contain any vertices to get a Hamiltonian cycle in the entire graph. Now repeat the construction on the smaller parts. This informal description needs to be made more precise as the cutting has to be smooth enough. We will bypass this difficulty by cutting away only small holes. This still gives a polynomial complexity in the number of vertices of the graph. Using the already known fact that in the case $d = 2$, the complexity is linear, we get linearity in all dimensions.

1.17. In the proof we cut explicitly near a point in the interior producing a cavity. If the graphs have become so small that it is no more possible to cut a hole into them, we can edge away from the boundary until we have a generalized d-graph without interior. Now we can cover the graph with a Hamiltonian cycle because the Hamiltonian cycle on the "fattened boundary" covers everything. Also the construction of the Hamiltonian cycle on that boundary is constructive. Start with a Hamiltonian cycle on the $(d-1)$ surface given by the theorem for dimension $d-1$. A still constructive "Swiss cheese" approach is then to cut small cavities away to reach the near interior points and then to join the various Hamiltonian cycles on different connected components using bridges.

2. Terminology

2.1. A finite simple graph $G = (V, E)$ is **Hamiltonian** if there exists a simple closed circuit which visits every vertex of G exactly once. A **finite abstract simplicial complex** is a finite set of non-empty sets which is closed under the operation of taking finite non-empty subsets.

The 1-**skeleton graph** of a complex G is the one dimensional sub-complex of all sets in G of cardinality 1 or 2. It naturally is a finite simple graph. A simplicial complex is called **Hamiltonian** if its 1-skeleton graph is Hamiltonian. We can think of the operation "**Skeleton**" either as a map from "simplicial complexes" to "finite simple graphs" or as a projection from arbitrary complexes to 1-dimensional complexes, the association with projection being justified by the identity Skeleton ∘ Skeleton = Skeleton.

2.2. The **Whitney complex** of a finite simple graph (V, E) is the finite abstract simplicial complex G in which the sets are the vertex sets of the complete sub-graphs of (V, E). We often identify a graph with its Whitney complex. Similarly, we associate to a Whitney complex its skeleton graph. The two operations "Whitney" and "Skeleton" are not inverses of each other in general. The Whitney complex of the 1-skeleton graph of a complex G is often different from the complex, but Whitney complexes have the property that they are entirely encoded by the graph alone. The identity Skeleton(Whitney)$((V, E)) = (V, E)$ holds for all finite simple graphs (V, E). Even so in general Whitney(Skeleton)(G) is not equal to G the two constructs are natural operations between "simple graphs" and "simplicial complexes".

2.3. A graph is called **pure** of dimension d, if its Whitney complex is pure: this means that the **facets** (=maximal simplices) in its Whitney complex G all have the same dimension d. A finite simple graph is called **shellable**, if it is pure and if its Whitney complex G is a shellable complex: first formulated in a purely combinatorial setting in [3], an abstract finite simplicial complex is called **shellable** if there exists an ordering $x_1, \ldots, x_n$ of its facets such that for every $k \geq 1$, the intersection H_k of x_k with $\bigcup_{j=1}^{k-1} x_j$ defines a shellable $(d-1)$ complex. By definition, a shellable complex of dimension d produces a d-**connected** graph meaning that the 1-skeleton graph remains connected after removing d vertices or less.

2.4. A finite simple graph G is called a **d-sphere** if it is either the (-1)-sphere $0 = (\emptyset, \emptyset)$ or if every unit sphere $S(x)$ is a $(d-1)$-sphere and the graph $G - x$ without x is contractible for some vertex x. A graph G is **contractible** if it is either the one-point graph $1 = K_1$ or if there exists a vertex x such that $G - x$ and $S(x)$ are both contractible. A d-**ball** is a graph $H = G - x$ which is obtained from a d-sphere G by removing a single vertex x. A d-**graph** is a finite simple graph for which every unit sphere is a $(d-1)$ sphere. If every unit sphere is either

a $(d-1)$-sphere or a $(d-1)$ ball, then we deal with a d-**graph with boundary**.

2.5. We use here a definition of spheres in higher dimensions [14] which relates closely to notions put forward by Evako [7] who also formulated discrete versions of Whitehead homotopy [13]. Both the discrete homotopy as well as the discrete sphere definitions have been refined over time. We use the term **contractible** even so this is often called "collapsible". For us, contractible and collapsible are synonyms. Examples like the dunce hat or Bing house show that they are not the same than **homotopic to the 1-point graph** which is the property that there is a graph H such that H is both contractible and equivalent to G by contraction steps, where a vertex with all connections to $S(x)$ is allowed to be removed if $S(x)$ is contractible.

2.6. The **dual graph** of a d-graph has the maximal d-simplices (=facets) as vertices and connects two different maximal simplices if they intersect in a $(d-1)$ simplex. The dual graph of a d-graph is always zero or one-dimensional. For a connected d-graph of dimension $d \geq 2$ different from a simplex, it is always one-dimensional. The proof: the dual graph does not contain a triangle because if it would, the entire triplet of d-simplices would belong to a larger dimensional simplex.
Examples: 1) the dual graph of a wheel graph with boundary C_n is C_n.
2) the dual graph of the 16 cell, which is a 3-sphere has as a dual graph the **tesseract**.
3) The dual graph of the octahedron graph is the cube graph. Both the cube graph as well as the tessaract are not d-graphs as the unit spheres are not spheres. Also, the dual graph of the cube contains triangles because more than 2 maximal simplices intersect.

2.7. While nice triangulations of Euclidean spheres produce shellable complexes, there exist already for $d = 3$ non-shellable triangulations of spheres [18]. Examples of non-shellable balls are the Furch ball of 1924, the Newman ball of 1926 or the Rudin's ball [20]. Ziegler's ball G with 10 vertices and 21 facets appears to be the smallest known so far. The Barycentric refinement G_1 of any of these examples G however is a shellable complex.

2.8. [Update, June 21, 2018. We originally had a stronger statement here about Strong Hamiltonian claiming that one can even have a Hamiltonia path visit all maximal simplices. This was not proven and assumed to be obvious from the induction. The problem is that after doing the rerouting to the interior of the holes, not all maximal simplices are adjacent any more to a boundary. But we don't need that

strong of a Hamiltonian property. We need only to be able to force the Hamiltonian cycle to go through a particular facet in order to build the bridge.]

A $(d-1)$ simplex is a **boundary face** in a d-graph G if it is contained in only one of the maximal d-simplices. We call a d-graph G without boundary **strongly Hamiltonian**, if given a facet, there exists a Hamiltonian cycle which intersects the facet in an edge.

Here is the argument, which proves the strong Hamiltonian statement: given a facet x. If it is at the boundary, we can use induction already. If it is not at the boundary, cut a few edges so that it is at the boundary. The Hamiltonian cycle is now also a Hamiltonian cycle, when adding back the edges.

2.9. The wheel graphs are 2-balls which are strongly Hamiltonian. There is a Hamiltonian cycle around the boundary which detour once to the center. The **boundary complex** of a d-graph G is the 1-complex of the $(d-1)$ boundary complex of G. In order to have a good induction proof, we need to extend the class of d-graphs and the class of d-graphs with boundary a bit. This is done next.

2.10. A **generalized d-graph** is a graph which either is a d-graph with or without boundary or a graph for which all spheres are $(d-1)$ balls, $(d-1)$ spheres or $(d-1)$ simplices and for which every interior vertex x in distance 1 to the boundary is part of a triangle xab, with an edge (a, b) at the boundary.

The last condition disallows d-subgraphs which have only vertices in common with the boundary. An example is the stellated wheel graph. This condition is only really necessary in the case $d = 2$. In higher dimensions, we get the Hamiltonian property also without it. The reason is that in accessible inner points can still be reached from the outside via a detour, but we don't bother with that for now and leave the condition in all dimensions.

Examples:

1) The diamond graph $C_2(2)$ is an example of a generalized 2-ball because the sphere condition is satisfied and no interior point exist.
2) The stellated square is not a generalized 2-ball. The inner sphere has no common edge with the boundary. After doing edge refinement in the interior of the stellated square, we can get examples of arbitrary

large 2-balls or (after digging holes) get 2-graphs which are isolated 2-subgraphs.

3) The Goldner-Harary graph has no interior but it fails the unit sphere condition. There are unit sphere which are neither $(d-1)$ balls nor $(d-1)$ simplices.

4) The 3-dimensional "Avici" graph $C_3(2)$ obtained by gluing two K_4 along a K_3 is a generalized 3-ball because removing the 2-dimensional interior triangle and the two 3-simplices produces a 2-dimensional complex.

5) For any $d \geq 1$, the d-simplex is a generalized d-ball. After removing the interior d-dimensional facet, it has a Barycentric refinement which is a $(d-1)$ sphere.

6) The **d-dimensional prism** is an example of a generalized d-ball. It has no interior It has a $(d-1)$ dimensional simplex roof and a $(d-1)$ dimensional simplex floor. In d dimensions, it is a union of d simplices of dimension d. It is a generalized ball without interior point. 7) The stellated solid octahedron is not a generalized 3-ball because some unit spheres are not balls, spheres or simplices.

2.11. Let us look a bit more at the threshold, where the Hamiltonian property fails: we can verify directly by induction that a shellable graph of dimension $d \geq 2$ for which the dual graph is a path graph is Hamiltonian. It only amounts to show by induction that the strong Hamiltonian property can get extended when adding a new simplex. The just mentioned Avici graph is an example. More generally, any **cyclic polytope**, where d-simplices are attached to each other in a linear way such that their intersection is a $d-2$ simplex are examples of shellable complexes which are Hamiltonian.

2.12. When trying to extend the result from linear complexes to complexes, where the dual graph is a tree, we in general lose the Hamiltonian property. The Goldner-Harary example is prototypical of this type. The reason is that we might run out of vertices which can use to lift the Hamiltonian cycle to all the newly attached vertices. In the Gardner-Harary graph, this happens, leading to a failure of the Euclidean structure and Hamiltonian graph property, even after Barycentric refinement. We are fine in the case of a d-prism, as there, we have no interior and the boundary sphere possesses a Hamiltonian cycle, because it is a 2-sphere. An other prototype example is the case of an interior sphere which does not meet an other interior unit sphere nor any boundary. The smallest such example is the stellated square or stellated octahedron.

2.13. There are shellable d-complexes for any $d \geq 1$ for which the dual graph contains triangles but these complexes fail the property of being "generalized d-graphs" in one way or the other. They are either not Whitney complexes like for the 1-skeleton complex C_3 of K_3 for which the dual complex is again isomorphic to C_3. An other example in one dimension is a claw graph or the cube graph, where three edges come together. Or then, we have like in windmill graph, three triangles hinged together at a single edge, which is not Hamiltonian.

2.14. Here are the incidence relations: d-spheres $\subset$ d-graphs $\subset$ generalized d graphs $\subset$ Whitney complexes $\subset$ finite abstract simplicial complexes. Similar incidence relations hold in the case of boundary.

3. The theorem

3.1. The main result is about d-graphs, a class of graphs which behave like "combinatorial manifolds". We use the later term only informally, as it used in different ways in the literature, also in non-combinatorial set-ups like in the context of PL manifolds. d-graphs are finite objects, finite simple graphs.

Theorem 1 (Hamiltonian manifold theorem). *Every connected d-graph is strongly Hamiltonian for $d \geq 1$.*

3.2. The proof will use induction with respect to dimension d. We will prove at the same time the analogue statement for d-graphs with boundary. The case $d = 2$ is in some sense the hardest and most subtle case, which is covered by Whitney already (except for the strong Hamiltonian property, which however can be replaced by Hamiltonian connectednedss). Our proof is new and proves the result for slightly more general **generalized d-graphs**. In dimension $d = 2$, it does not cover all of Tutte's theorem but goes beyond in the sense that 2-graphs do not have to be planar.

Theorem 2 (Boundary version). *Every connected d graph with boundary is strongly Hamiltonian for $d > 1$.*

3.3. The boundary version fails in dimension $d = 1$ because connected 1-graphs with boundary are path graphs which are not Hamiltonian. The boundary version fails the strong Hamiltonian property in dimension $d = 2$ because a Hamiltonian cycle would have to stay on the boundary and cover every edge. This already does not work for the wheel graphs.

3.4. The problem to find Hamiltonian cycles in a graph is an NP complete problem. It is known that for 2-spheres, the computation of Hamiltonian cycle is linear [5]. This generalizes. Our proof shows:

Corollary 3. *The construction of Hamiltonian cycles in d-graphs is polynomial in the number of vertices.*

Actually, using the linear complexity result [5] in two dimensions, we get also in general that the construction of Hamiltonian cycles in d-graphs is linear in the number of vertices: there exists a constant C such that a Hamiltonian cycle can be computed in less than Cn computation steps, if n is the number of vertices in the graph G. For recent improvements of the computation of Tutte paths, see [22].

4. PROOF

4.1. The proof goes by induction with respect to dimension $d \geq 1$ and is done for generalized d graphs with boundary for $d \geq 2$. The statement is true for all connected 1-graphs as these are cyclic graphs. Assume the claim has been shown for all $(d-1)$-graphs without boundary. Take a generalized d-graph with boundary, remove the interior parts until every vertex is close to the boundary, then build a Hamiltonian path by taking the one on the boundary and extending it to the vertices close to the boundary.

4.2. To cave out holes, we can use a function f to cut away a small d-ball B around a **strong interior point**, an interior point which has only interior points nearby. In each case, we chose the strong interior point in distance 2 the already established boundary. Given a strong interior point x, we remove that point and add the new sphere boundary $S(x)$ to the boundary. The Hamiltonian path on that boundary $S(x)$ can make a detour to the center x so that the unit ball $B(x)$ is still Hamiltonian. A prismatic bridge, (as explained below) combines this with the rest of the graph.

4.3. After getting rid of all strong interior points, we look by either caving away points such that $G \setminus x$ is still a generalized d-graph. For these points there exists an interior point y near the boundary such that $S(y) \cap \delta G$ consists of one point only. By adding all unit spheres $S(x)$ to the boundary, removing x, we achieve that every interior point which remains is now near the boundary, meaning of distance 1 to the boundary.

(There is a difficulty of having non-accessible interior points which only appears in dimension 2. This is a case already covered in the literature.

In two dimensions, some interior points x can remain which can not be reached from the boundary δG. This happens if the unit sphere of x touches the boundary everywhere in one point y only. We just absorb such point y to a neighboring hole. The hole can grow more. The interior could get a triangle or aggregate more points as long as the subgraph generated by these points remain Hamiltonian. It can be K_3, the diamond graph or any path shellable complex or generalized 2-graph. In the proof we only need one possible extension.)

4.4. Finally, we reach out for the interior points which are in distance 1 to the boundary after all these modifications. These points can be reached by making a small detour from the boundary. To organize this, we drill a small hole into G by removing some edges in the $(d-1)$ dimensional boundary, exposing the interior point to the boundary and making it a boundary point but without changing the property of being a generalized d-graph. In the case $d = 2$, this means removing an edge in the 1-dimensional boundary (and so a triangle). In the case $d = 3$, this means the interior edge and the two triangles of a diamond graph producing a hole with boundary C_4 in the two dimensional boundary (this also removes three tetrahedra). In the case $d = 4$, this means removing the interior edges and the three tetrahedrons of a prism at the boundary, building a three dimensional ball cavity into the boundary which means to remove four 4-simplices and exposes the center point to the boundary.

Here is the explicit construction algorithm of a Hamiltonian path for a connected d-graph G:

- If a boundary δG is available, identity the vertices of this subgraph δG. It can be the empty set.
- Find an interior point x in distance 2 from the boundary δG. Remove x from G and join the unit sphere $S(x)$ to the boundary δG. Find a Hamiltonian path in the unit ball $B(x)$. Build a bridge from $S(x)$ to δG using a prism bridge. Now we have a new generalized d-ball with boundary. Repeat this step 2) until no interior points of distance 2 to the boundary can be found any more.
- If we should end up with an interior point y in distance 1 to the boundary which is "boxed in" in the sense that its unit sphere $S(y)$ has no edge in common with δG, we make the interior of one of the neighboring holes larger to absorb some point $z \in S(y)$, allowing y to be reached. This case only can occur in dimension 2 because for $d \geq 2$, the intersection of

two neighboring spheres in δG has positive dimension and so contains an edge.

- Now we have a graph for which every still remaining interior point can be reached from the boundary. Build Hamiltonian paths in each of the connected components of the boundary and connect them with bridges. This path covers now everything except the interior points in distance 1 from the boundary.
- Find an interior point x in distance 1 from the boundary. Build a detour so have that interior point is included into the Hamiltonian path. Continue with step 5) until no interior points are available any more. Now we have a Hamiltonian path.

5. CALCULUS

5.1. In this section, we describe the cutting procedure using a function. It is a basic property of d-graphs without boundary that the level surface $\{f = c\}$ is always either empty or a $(d-1)$ graph without boundary. Also for a d-graph with boundary, the level surface $\{f = c\}$ is either empty or a $(d-1)$ graph with or without boundary. These things were observed in [16]. It makes use of the definition that the **level set** $\{f = c\}$ is the set of all simplices in G on which f changes sign, then build the graph in which these simplices are the vertices and where two are connected, if one is contained in the other. What is achieved with this definition, that for c different from the range of $f(V)$ on the vertex set V, the level surfaces $\{f = c\}$ are nice "discrete manifolds", hence the name "discrete Sard theorem". We deal here with generalized d-graphs, but apply the cutting procedure only in the case when $\{f = c\}$ is in the interior of a smaller d-graph with boundary. In our proof we will only apply the situation where $f = 1$ on a single vertex x and f is negative on every other vertex and where the ball $B_2(x)$ is a d-graph with boundary (and not only a generalized graph with boundary).

5.2. For a general locally injective function f, while $\{f = c\}$ is nice, the sets $\{f < c\}$ and $\{f > c\}$ do not need any more to be d-graphs with boundary. If f is equal to 1 on a vertex x and negative everywhere else, then $f > 0$ is a single point. for some small but positive ϵ, then $\{f = c\}$ is a $(d-1)$ sphere but $\{f > c\}$ is the 1-point graph K_1. Actually, the vertex set of $\{f < 0\}$ can be an arbitrary subset of the vertex set. We want to use functions to cut a d-graph into smaller pieces, establish the Hamiltonian property in both parts then join the two paths using a single bridge. This "divide and conquer" strategy which provides a computational way to find Hamiltonian paths quickly for d-graphs.

5.3. A level surface $\{f = c\}$ of a function f on the vertex set of a d-graph is called **smooth** if for all vertices x, the induced graphs from $S^-(x) = \{y \in S(x) \;;\mid f(y) < c\}$ and $S^+(x) = \{y \in S(x) \;;\mid f(y) > c\}$ are either $(d-1)$-balls, $(d-1)$ spheres, a k-simplex or empty. A connected 1-graph G is just a cyclic graph C_n for $n \geq 4$. Every locally injective function on a 1-graph is smooth. Almost by definition, we see that a level surface $\{f = c\}$ is smooth if and only if both $\{f > c\}$ and $\{f < c\}$ are d-graphs with boundary or then a k-simplex with $k \leq d$. The same definition can be extended to d-graphs to d-graphs with boundary. The smoothness notion is not really needed in the proof but it can help in process to use smooth surfaces.

6. Cyclic polytopes

6.1. In this section, we cover a class of generalized d-graphs which are Hamiltonian. Before using the Swiss cheese strategy, we used to cut up the graph into lower dimensional layers, eventually reaching cyclic polytopes (stripes). This strategy seems to work also to get a Hamiltonian paths, even more effectively, but we have not shown that we can always cut things up like that. The idea was to cut up a graph G into two d-graphs $G_1 = \{f > 0\}, G_2 = \{f < 0\}$, then find the Hamiltonian paths in each, then glue them together.

6.2. When finding a Hamiltonian path using such subdivision, there will a moment when we can no more divide up a d-graph into smaller d-graphs using smooth cuts. There are arbitrary long 2-balls already which can not be cut into smaller balls. An example is a Birkhoff Diamond: start with a wheel graph W_5 with boundary C_5, then chose a boundary point then take the union with an other wheel graph centered there. This produces a 2-ball with 2 interior points and boundary C_6. It is a 2-ball but it can not be cut into smaller paths which are 2-balls.

6.3. A **cyclic polytope** is an example of a **linear shellable complex**. This means that it is a shellable simplicial complex G of dimension $d \geq 1$ for which the dual graph is a finite path graph. Cyclic polytopes play an important role in combinatorial topology because of McMullen's upper bound theorem telling that these polytopes are the ones which maximize the volume (number of facets) with a fixed number of vertices. We have the following observation:

Lemma 4. *If G is a d-graph which is strongly Hamiltonian, then adding a new vertex over one of its facets is still Hamiltonian. But it might lose the property to go through all the other faces.*

Proof. Just take an edge in each of the faces F_j and "lift" it up to the new vertex x_j. As the intersection of F_j with F_k has no edge, we don't run into the problem of double booking a wire to two extensions.   □

6.4. This immediately implies that we can continue to make extensions on a new set F_j of newly generated faces.

Corollary 5. *Every path shellable complex is Hamiltonian.*

6.5. The reason why not all shellable simplicial complexes are Hamiltonian is that during the extension the strong Hamiltonian property has been lost on other faces. The prototype examples where the Hamiltonian property fails are Goldner-Harary in any dimensions, or stellated cross polytopes, also in any dimension.

7. BRIDGES

7.1. A locally injective function on the vertex set V of a graph $G = (V, E)$ is also called a **coloring**. The discrete Sard theorem [16] assures that the level set $\{f = c\}$ in a d-graph is a $(d-1)$-graph, as long as c is different from the range of f. The level set is defined as the subgraph of the Barycentric refinement of G generated by the set of simplices in G on which f changes sign. If we think of the level surface as water we need to build bridges between the part $\{f < c\}$ and the part $f > c$. In dimension $d = 2$, the level surface is built by the edges and triangles on which f changes sign. Each connectivity component is a circular graph.

7.2. Using functions, we solve the problem of "cutting a d-graph into smaller parts" $\{f < c\}$ and $\{f > c\}$, where $f = c$ does not contain any vertices but is naturally associated to a $d - 1$ complex. As long as c is different from the values $f(V)$, we get two regions A, B for which prismatic bridges exist between the two parts A and B. Such a bridge is given by a $d - 1$ boundary simplex in $f < c$ and a $d - 1$ boundary simplex in $f > c$ bounding a d-ball in dimension $d > 1$. Now, if a Hamiltonian path in A visits the first $d - 1$ face in A and a Hamiltonian path in B visits a d-1 face in B, then we can rewire the prism so that the two Hamiltonian paths merge to a single Hamiltonian path and this path still has the property that it visits every d-simplex as the simplices in the bridge $f = c$ are bound by $(d-1)$-dimensional faces which contain edges.

7.3. Here is the lemma which assures that we can join Hamiltonian paths in $A = \{f < c\}$ and $B = \{f > c\}$. Assume the bridge connects the simplex A' in A with the simplex B' in B.

Lemma 6 (Bridge lemma). *For every edge $e = (a, b)$ in a $(d-1)$-face X in A', there is a $(d-1)$ face Y in B' such that for every edge $f = (c, d)$ in Y, there is a quadrilateral containing a, b, c, d.*

Proof. Given a face X. Look at all edges which go from X to B. This naturally defines a simplicial complex where the edges are the vertices. It has a single maximal $(d-1)$-simplex. This defines an injective map from X to B. The image is a simplex Y in B. Now pick any edge $y = (c, d)$ in Y. We claim that there is a quadrangle containing a, b, c, d.

First of all, the distance between a and c (or d) is smaller or equal than 2. The reason is that c, d are in the union of the unit spheres of a and b. That shows that a is connected either to c or d.

Now, assume a is connected to c. We have to show that b is connected to d. If there were no direct connection from b to d, then because of the maximal distance 2, d has to be connected to c. But then the closure of bcd is there and bd is in.   □

7.4. The set of all d-simplices with edges in X or Y or connections between X and Y is the **prism** generated by the faces X and Y. The edges connecting X and Y actually are the vertices of a $(d-1)$-simplex in the hypersurface $f = c$. The prism itself is not a d-ball but it is a generalized d-ball.

7.5. In the case $d = 2$, for example, the prism is a diamond graph made of 2 triangles and some unit balls are simplices. In the case $d = 3$, the prism is a cylindrical prism made of three tetrahedra, which represent the three edges of a triangle of the level surface $\{f = c\}$.

8. SHELLABILITY

8.1. In this section, we point out that the Whitney complex of a d-sphere G or d-ball is a shellable complex in all dimensions $d \geq 1$. It is a result which is not needed in the Swiss cheese proof of the Hamiltonian manifold theorem. This means that there exists a sequence x_k of d-simplices such that $\{x_j\}_{j=1}^n$ generates the complex G and $G_k = \bigcup_{j=1}^{k-1} X_j \cap X_k$ is a shellable $(d-1)$ complex for every k, if X_k is the complex generated by $\{x_k\}$: it is the smallest set of non-empty sets which contains all x_k and is closed under the operation of taking finite non-empty subsets.

8.2. The usual suspects of non-shellable triangulations of spheres are not Whitney complexes of graphs. The smallest d-ball G due to Ziegler for example produces a 5 dimensional Whitney complex from its 1-dimensional skeleton complex. The Barycentric refinement G_1 of G is then a Whitney complex and indeed it is shellable.
All Barycentric refinements of a simplicial complex whose Euclidean realization gives a d-sphere or d-ball is shellable because a Barycentric refinement is a Whitney complex. But all these non-shellable cases are washed away with Barycentric refinements.

Lemma 7. *The Whitney complex of a d-sphere is a shellable complex.*

Proof. We use induction with respect to dimension d. For a (-1) sphere, which is the empty graph, this is the assumption. Assume the claim holds for $d-1$-spheres. Take a d sphere. By definition, there is a vertex v such that $S(v)$ is a $(d-1)$-sphere and $G-v$ is contractible. As $S(x)$ is shellable, there is a sequence of simplices $y_1, \ldots, y_m$ building up $S(x)$ in such a way that $y_1 \cup \cdots \cup y_{k-1} \cap y_k$ is shellable. Let x_j be the cone extension of y_j with the vertex v. Now, $x_1 \cup \cdots \cup x_{k-1} \cap x_k$ is shellable as the result is just the cone extension with v. $\square$

8.3. Similarly, we have:

Lemma 8. *The Whitney complex of a d-ball is shellable.*

Proof. To verify the statement for d-balls, let us assume, the statement is proven for all $(d-1)$-balls. We make now a second induction with respect to the number n of simplices in the ball. For one simplex, there is nothing to show. Given a d-ball G. By definition of contractibility, there is a vertex x such that both $S(x)$ and $G-x$ are contractible. Now, $H = G - x$ is smaller and so shellable. As $S(x)$ is a smaller dimensional ball, also the unit sphere $S(x)$ is shellable. Let $y_1, \ldots, y_m$ be the simplices in $S(x)$. The complex G is obtained by adding the simplices x_j to $G - x$, where x_j is the cone extension of y_j with v. We can now get G from $G - x$ by successively adding the simplices $x_1, \ldots, x_m$. $\square$

8.4. This of course does not go over to higher genus 2-complexes already. The simplicial complex of a triangulation of a 2-torus for example is not shellable because at some point we have to add a triangle x_k which intersects $\bigcup_{j=1}^{k-1} x_j$ in a 0-dimensional or 1-dimensional non-pure complex violating in both cases the shellability condition.

8.5. Every cone extension $H = G + x$ of a d-sphere is a $(d+1)$-ball: the reason is that this is the removal of a vertex from a suspension of G. The graph H has a boundary $\delta H = G$ which is the unit sphere $S(x)$.

9. QUESTIONS

9.1. A graph is called **Hamiltonian connected** if for every $a \in V, b \in V$, which need not to be distinct, there exists a Hamiltonian path from a to b. Are all d-spheres Hamiltonian connected? This is known for $d = 2$ [23]. We see that we can force a Hamiltonian cycle to go through a particular edge e. The reason is induction and the fact that we can force e to be in a prism, then force the boundary simplices.

9.2. In two dimensions, there are already stronger results like Hamiltonian connectedness which assure that we can force a Hamiltonian path to go through any edge. Is there a simple argument showing, that under the existence of a Hamiltonian path, there is one which intersects every simplex in an edge? There is a simple argument showing that for any 2-sphere and any edge $e = (a, b)$, we can find a Hamiltonian path going trough that edge: remove all edges going from a to some y in $S(a)$ except for two adjacent edges. We have still a Hamiltonian path. It has to go through e now and this path is still a Hamiltonian path for the original sphere.

9.3. We don't know the structure of the set of all Hamiltonian cycles in a d-sphere. Is the class of Hamiltonian cycles of a d-sphere we can define two paths C, D to be connected if there exists a local deformation from one to the other. Local means that we can only change both graphs in a prismatic bridge which connects two $d-1$ simplices. It is natural to ask whether this space of Hamiltonian paths is connected.

9.4. Edge refinements preserve the Hamiltonian property in d-graphs: given an edge (a, b) in d-dimensional complex: The intersection $S(x) \cap S(y)$ is either a $d-2$ sphere or a $d-2$ ball. A Hamiltonian path passing through either a or b, can make a detour of the central point c splitting (a, b) and have the Hamiltonian property for the edge refined complex. Can we find a similar construction for Barycentric refinements? Is there a constructive way for d-graphs to get from a Hamiltonian path of G to a Hamiltonian path in the Barycentric refinement? For general graphs, the Barycentric refinement procedure does not preserve the property of being Hamiltonian.

9.5. Traditionally, coloring questions have been considered in parallel to the property of being Hamiltonian even so there are no general ways to get from a Hamiltonian path a vertex coloring (one can get sometimes get face colorings of planar maps from a Hamiltonian path on the boundary). Anyway, here is again the question for d-graphs: is the chromatic number of a d-sphere always equal to $d+1$ or $d+2$? For $d = 2$ it is the 4-color theorem.

9.6. A generalized **Barnette question** asks whether the dual graph of a d-sphere with chromatic number $d+1$ is Hamiltonian. The condition of having a minimal coloring replaces the Eulerian property in two dimension. For $d = 2$, the problem is the classical Barnette problem which is still open.

10. ILLUSTRATIONS
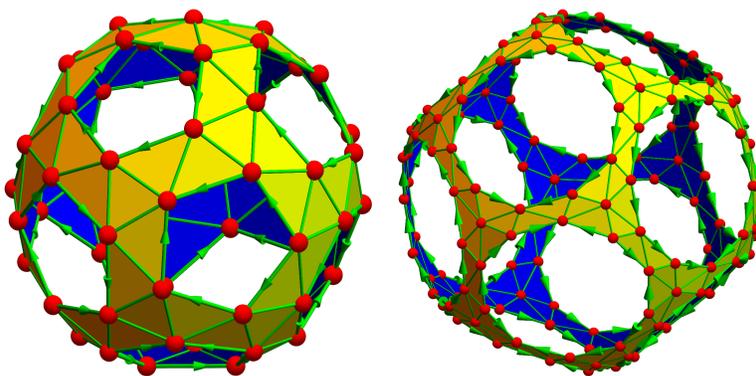
FIGURE 1. The first graph is a generalized d-graph without interior point. The second graph is a generalized d-graph with some interior points. By deleting some edges we can expose them to the boundary.
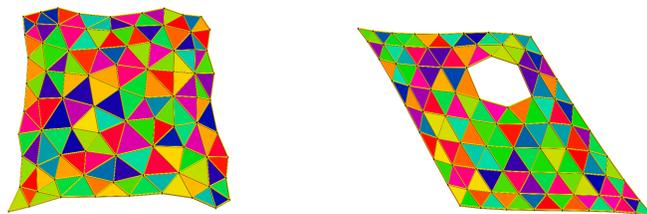
FIGURE 2. We see two generalized 2-graphs with boundary. Removing the extreme triangles which only intersect the rest on an edge produces 2-graphs. These graphs are Hamiltonian.
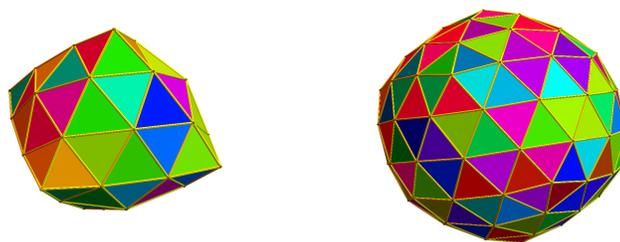
FIGURE 3. Two 2-spheres, tame refinements of the octahedron and the icosahedron. Both are best shown to be Hamiltonian by cutting them up into smaller parts, covering the parts with Hamiltonian paths, then joining them.
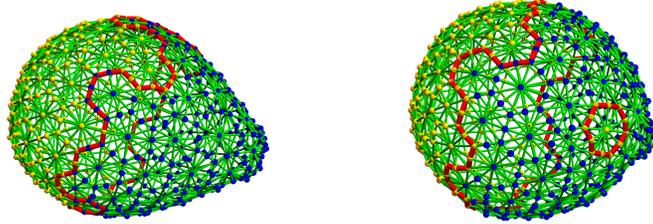
FIGURE 4.   We look at the level curves $\{f = c\}$ of some
functions f on a 2-sphere G. To visualize the graph, we
draw in each case the Barycentric refinement, in which
the curve is an actual curve. In this case there is first
one single curve and then a curve with two components.
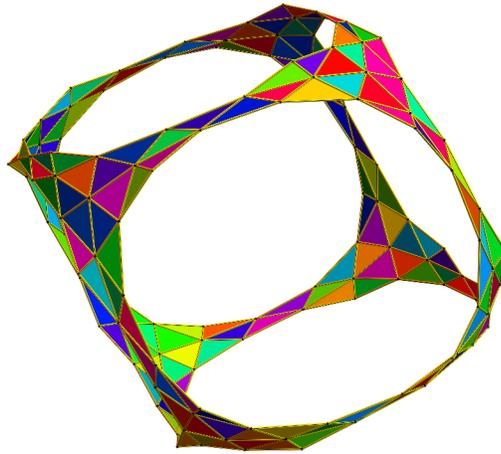
FIGURE 5. The picture shows a generalized 2-graph with boundary. In this case, an edge refinement at every of the 8 corners will produce a generalized 2-graph with boundary.
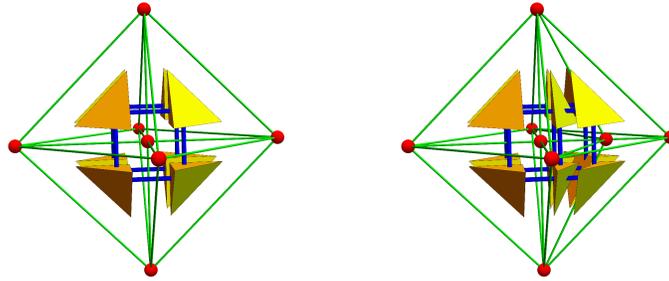
FIGURE 6.   An edge refinement in the case of a 3-ball. It
adds an other interior vertex. Reversing the process is an
edge collapse near the boundary, which is the same than
a homotopy step, where a vertex is removed.The fact that
edge refinements value the Hamiltonian property fueled
the research early on.
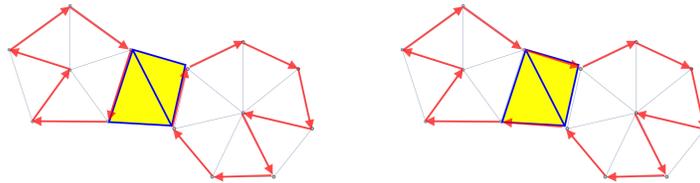
FIGURE 7.   The prism construction can bridge two
$f < c$ with $f > c$. The three edges in the prism rep-
resent three vertices in the level curve $\{f = c\}$. The two
triangles are two edges in that curve.
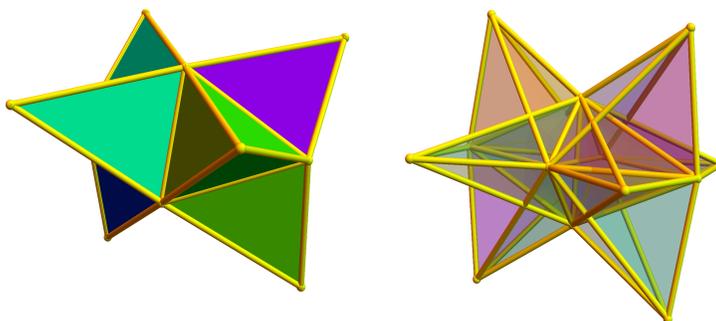
FIGURE 8. The Goldner-Harary graph seen to the left is the 1-skeleton graph of a shellable 3-dimensional simplicial complex whose dual graph is a tree. It is not Hamiltonian. It is an example of an Apollonian network. A 4-dimensional version of the Goldner-Harary graph is seen to the right. We first gluing two 4-simplices together along a 3- dimensional simplex, then stellate each of the eight 3-dimensional boundary faces. The unit spheres in this graph are either Goldner-Harary graphs or tetrahedra. This graph is not Hamiltonian. It is not even a generalized 4-graph.
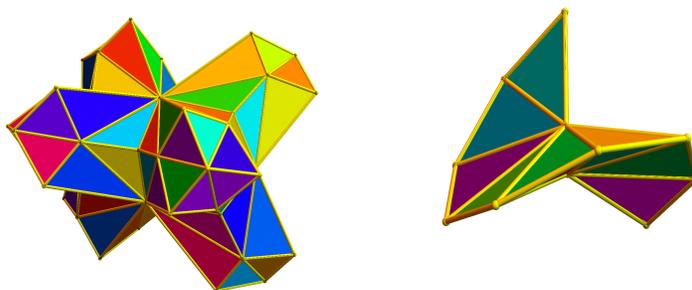
FIGURE 9. The Barycentric refinement of the Goldner-Harary graph G is Hamiltonian It is a 3-ball, all unit balls, nor spheres, nor simplices. To the right, we see one of the unit spheres of a boundary point. It might not appear so, but it is actually a 2-ball. [Update June 22, 2018. This was originally stated wrong. I had made a computation mistake, the Barycentric refinement of Goldner Harary is Hamiltonian and is a 3-ball.]
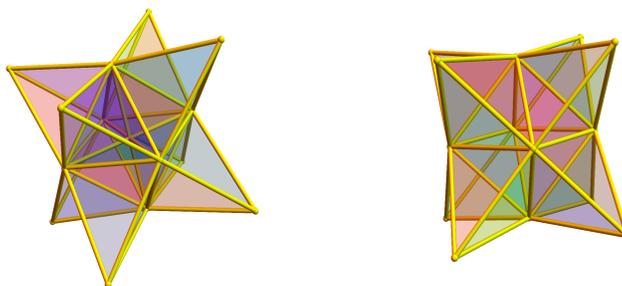
FIGURE 10.   The stellated, filled octahedron, which is
only 3-connected. It is not Hamiltonian and the Hamil-
tonian property fails for similar reasons than for the
Goldner-Harary graph. It is not a generalized 3-graph
because the inner unit sphere (the octahedron) has no
edge in common with the boundary. Also the local Eu-
clidean property fails both in the filled an non-filled case.
This is the deciding factor against the Hamiltonian prop-
erty here as removing only one spike still does not make
it Hamiltonian even so the inner vertex is now accessible
from the boundary.
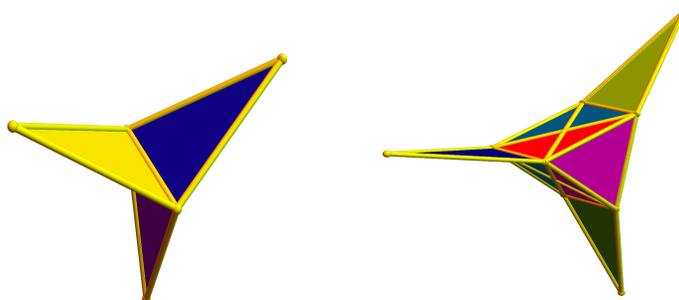
FIGURE 11.   The "windmill graph" is a 2-dimensional
shellable complex which is planar but not Hamiltonian.
Algebraically, it is the join of K_2 with the 3-point graph
$3 = 1 + 1 + 1$. It consists of 3 triangles (blades) con-
nected to a common edge. Having non-spherical unit
spheres is often a decisive factor against the Hamiltonian
property. To the right we see an other non-Hamiltonian
graph, which is a fat windmill.
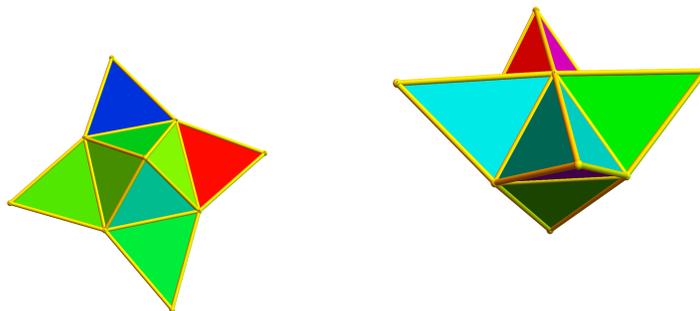
FIGURE 12.   The stellated square to the left is a generalized 2-ball which is not Hamiltonian. It is an example with an isolated 2-ball inside which only touches the boundary in zero dimensional places. When completing it to a sphere, it becomes Hamiltonian even so it is not a 2-sphere (there are unit spheres which are not circular graphs).

FIGURE 13.   The "Avici graph" is a shellable 3-dimensional simplicial complex which is Hamiltonian. It is not a 3-ball. It is a generalized 3-graph. To the right we see the 4 dimensional Avici graph, which is a union of two 4-simplices glued along a 3-simplex. It is a generalized 4-graph and also Hamiltonian.
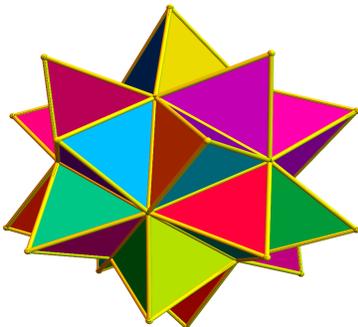
FIGURE 14.   The stellated isosahedron is a shellable 3-dimensional complex which is not Hamiltonian. It is not a generalized 3-ball.
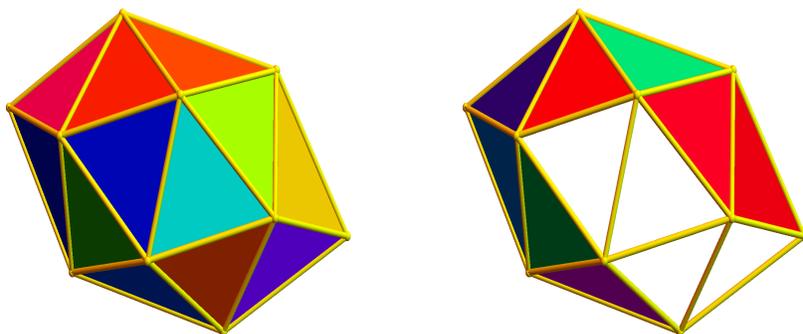
FIGURE 15.   The Birkhoff diamond is a generalized 2-ball. We can remove 4 edges on the boundary to expose the 4 interior points to the boundary producing a stellated diamond. We could also have used a function f and join the Hamiltonian cycles for $f < 0$ (which is a wheel graph) and $f > 0$ which is a diamond graph. Alternatively, we can remove four other edges (seen to the right). A Hamiltonian cycle having edges in all triangles is obtained by going around the boundary of the remaining triangles.
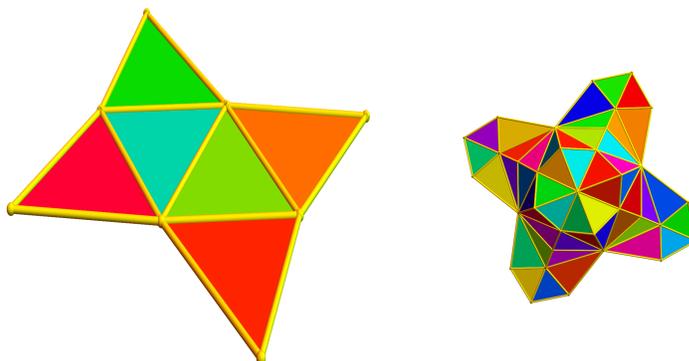
FIGURE 16. The 2-dimensional version of the Goldner-Harary graph is a stellated diamond. It is Hamiltonian. To the right we see the Barycentric refinement G_1 of the stellated square G. While G was not Hamiltonian, the refined G_1 is. The Barycentric refinement of Goldner Harary was not Hamiltonian.
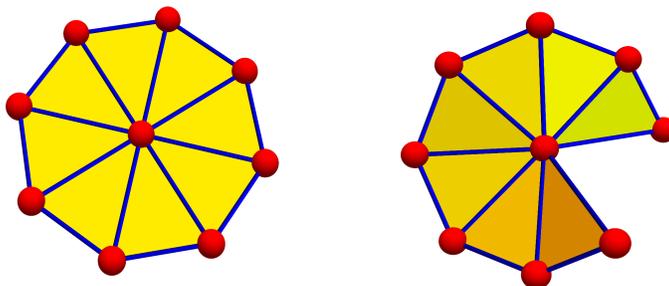
FIGURE 17. A wheel graph is a 2-ball G. It has one interior point. Cutting away one edge at the boundary produces a generalized 2-ball H. It is technically no more a 2-ball because two vertices have unit spheres K_2. After removing the interior edges, it is a 1-sphere which is Hamiltonian. As this boundary has the same vertices than H and G, also G is Hamiltonian. It is only dimension $d = 2$ that we can not insist on the strong Hamiltonian property on the boundary.
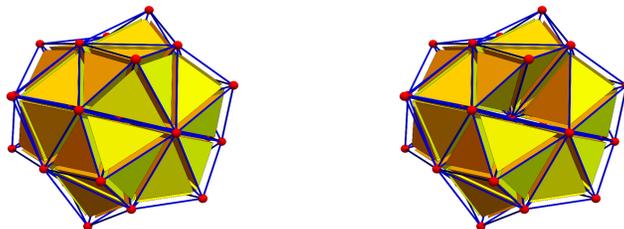
FIGURE 18.   The 3-cube is a 3-ball. It is the graph
product $G_1 \times G_2 \times G_2 = P_2 \times P_2 \times P_2$, where $P_2 = K_2$
is the path graph of length 1 (or complete 1-dimensional
graph). Its vertices are all triples (x_1, x_2, x_3) where x_i are
simplices in G_i and two are connected if one is contained
in the other. To the right we see the generalized 3-ball H
without interior. A prismatic hole (Diamond graph) was
drilled out at the boundary to expose the interior point
to the boundary. After discarding the interior edges we
have a 3-ball without interior. It is Hamiltonian because
it actually has become a 2-sphere. Because H and G
have the same vertices, the Hamiltonian path in H is
also a Hamiltonian path in G.
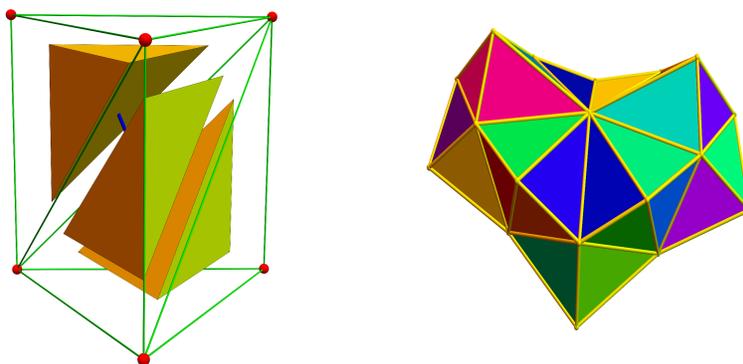
FIGURE 19. The 3-dimensional prism G. It is not a 3-ball because some unit spheres at the boundary are triangles. It plays an important role when briding different 3-graphs $f < 0$ and $f > 0$. To the right we see the Barycentric refinement of the boundary complex. It is a 2-sphere. The vertices of this graph are given by the 2-skeleton complex of G and two vertices are connected if one is contained in the other.
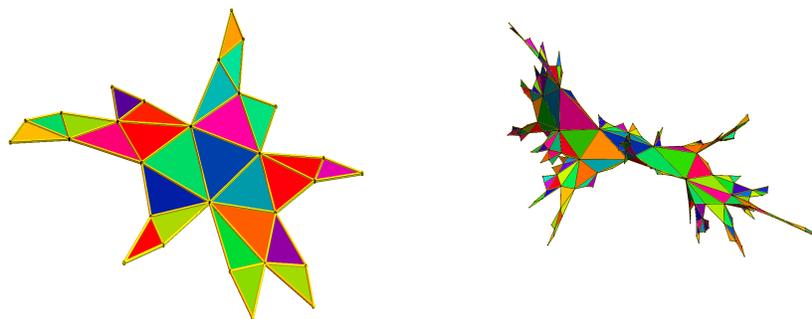
FIGURE 20. Two shellable complexes which are not 2-balls. They are planar but not 4-connected so that Tutte does not apply. They are Hamiltonian. The Hamiltonian path is the boundary curve. We could even add more edges building a 2-ball which still is Hamiltonian.
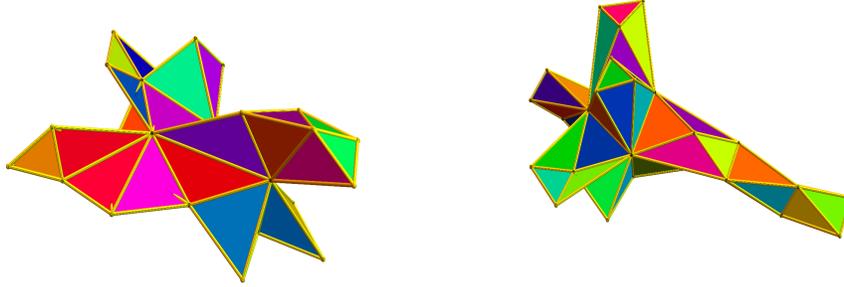
FIGURE 21.   Two shellable 3-complexes. The first one
was obtained by gluing successive Avici graphs to ran-
dom places of the previous case. In the second case, we
glue octahedra together.
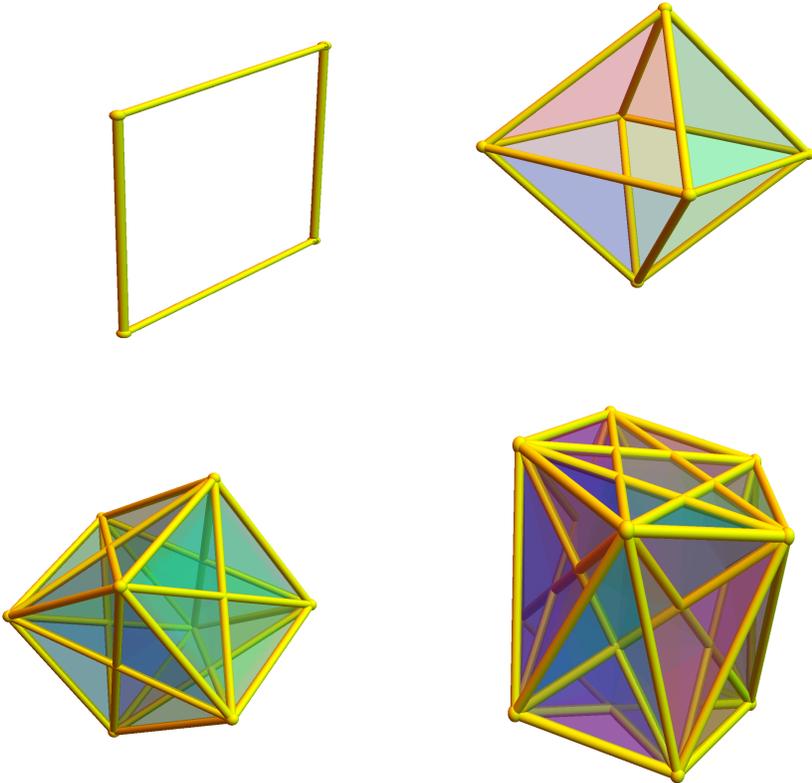
FIGURE 22. The smallest 1-sphere (cyclic graph C_4), then the smallest 2-sphere (octahedron), 3-sphere and 4-sphere. The 3-sphere (16-cell) is the suspension of the 2-sphere, the 4-sphere (32-chamber) is the suspension of the 3-sphere. They are all Hamiltonian as they are all d-graphs.
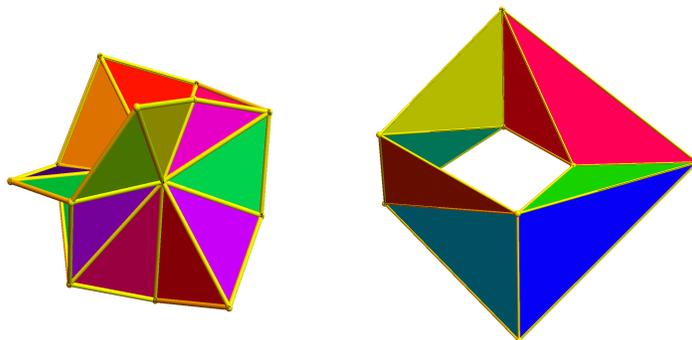
FIGURE 23.   The dunce hat is a non-shellable, non-contractible 2-complex with f-vector $(17, 52, 36)$. It is not a 2-graph as some unit spheres $S(x)$ have Euler characteristic $\chi(S(x)) = -1$. It is Hamiltonian, even so the theorem proven here does not cover it. To the right, we see the Möbius strip, a non-shellable complex. It is a non-orientable generalized 2-graph with boundary. It is Hamiltonian, the boundary is the Hamiltonian path.
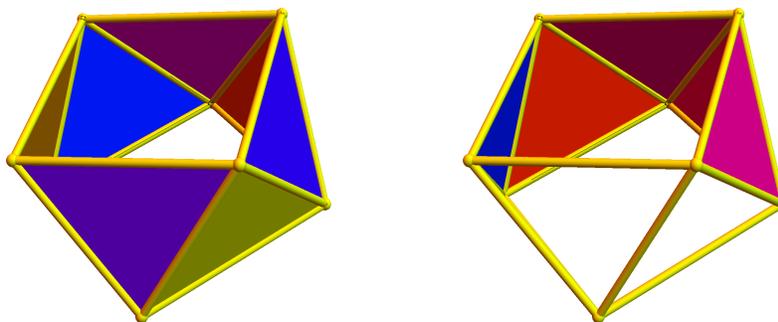
FIGURE 24.   A Hamiltonian path for a cylinder without interior point can be constructed by building a bridge between the two boundary curves. This is equivalent with rewiring using a prism bridge.
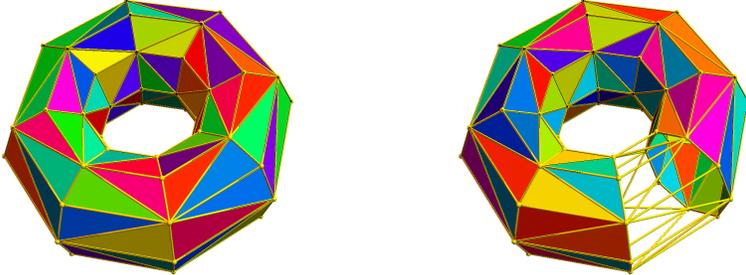
FIGURE 25.   A 2-torus is a 2-graph: every unit sphere is a circular graph. It is not shellable. To find a Hamiltonian path, find one first in an annulus without interior, then use bridges to connect these paths.
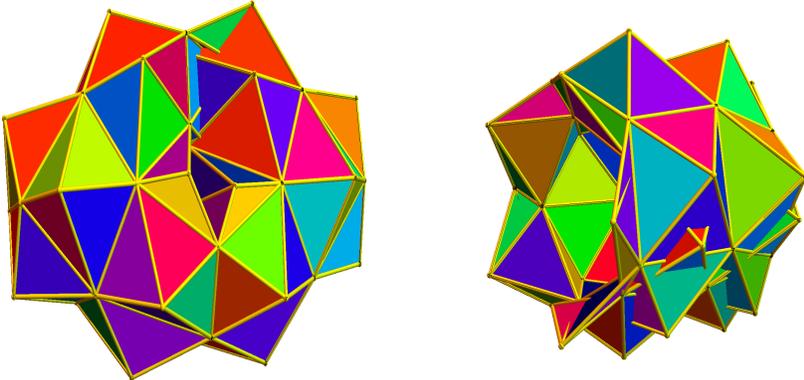

FIGURE 26.   The projective plane and the Klein bottle. Both are 2-graphs and Hamiltonian.
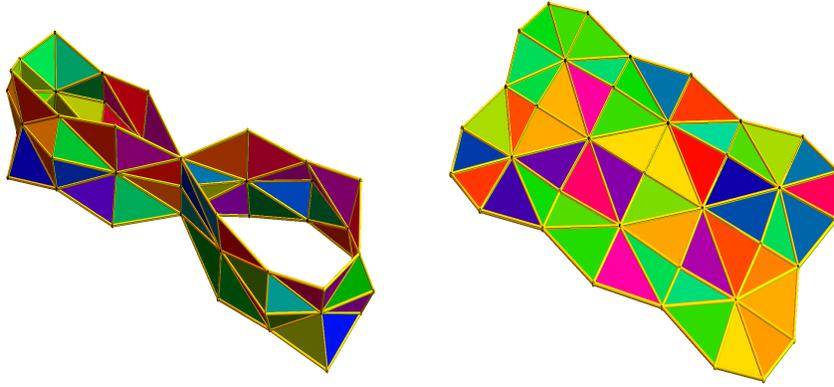
FIGURE 27.   The product of a figure 8 graph and K_2 is
a two dimensional simplicial complex, but not a 2-graph
with boundary. It is not Hamiltonian. To the right, we
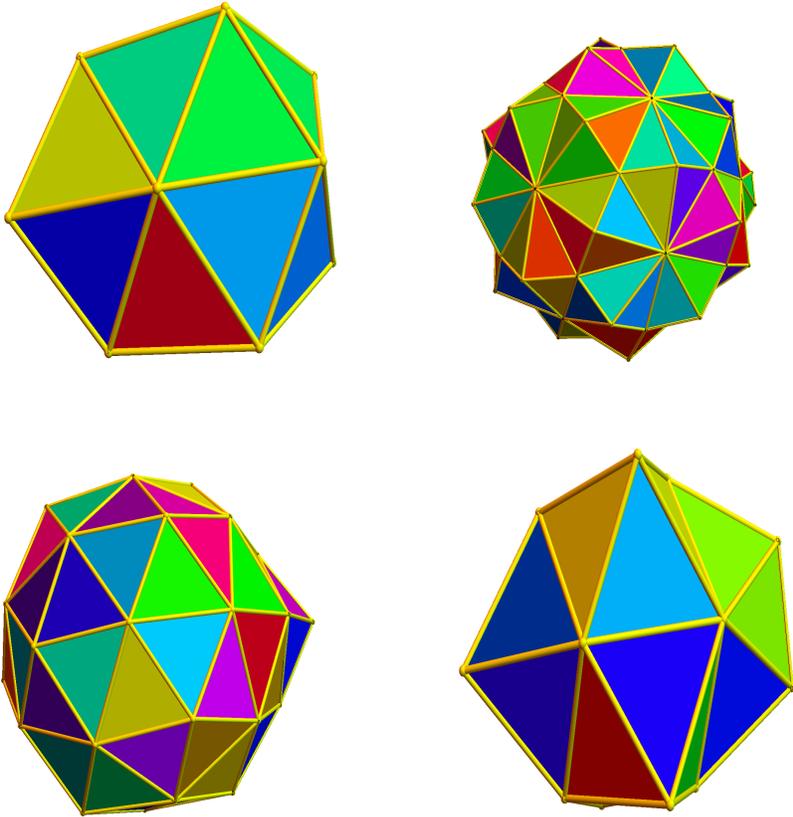see the product of two path graphs which is Hamiltonian.

FIGURE 28. Edge refinements of the icosahedron, the Disdyakis Triacontahedron, the Pentakis Dodecahedron and Tetrakis Hexahedron. These are all 2-spheres. After adding a central point (building the join with a 1-point graph), we get in each case a 3-ball. As the spheres are Hamiltonian, also these unit balls are Hamiltonian (just make a detour through the center once).
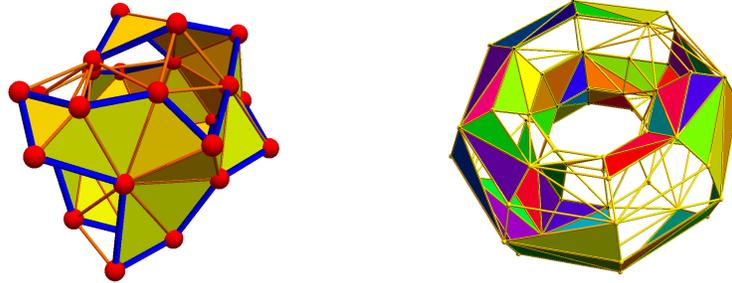
FIGURE 29.   In a 2-graph, we can often find directly a Hamiltonian path by building up a random shellable 2-complex without interior inside G. The cyclic boundary then is a Hamiltonian path. In general this can be used to make the task of covering the rest smaller. This also works in higher dimensions. We see the case of a sphere and a torus.
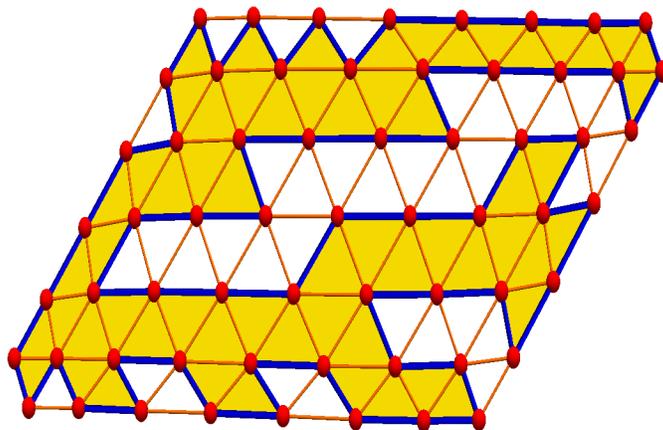
FIGURE 30. A Hamiltonian path found by embedding a shellable simplicial complex without interior points looks often like a Peano curve. It would also have been possible to cut the region up into two, solve the problem in both and then merge them together.
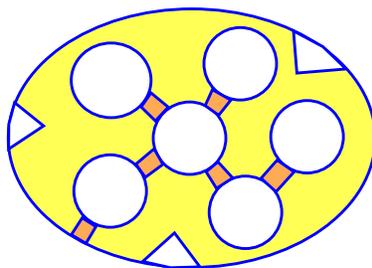

FIGURE 31. The Swiss cheese algorithm to construct a Hamiltonian path is the idea to bild sphere caves around strongly interior points, building bridges between the different boundary components. Then access other interior points from the boundary.
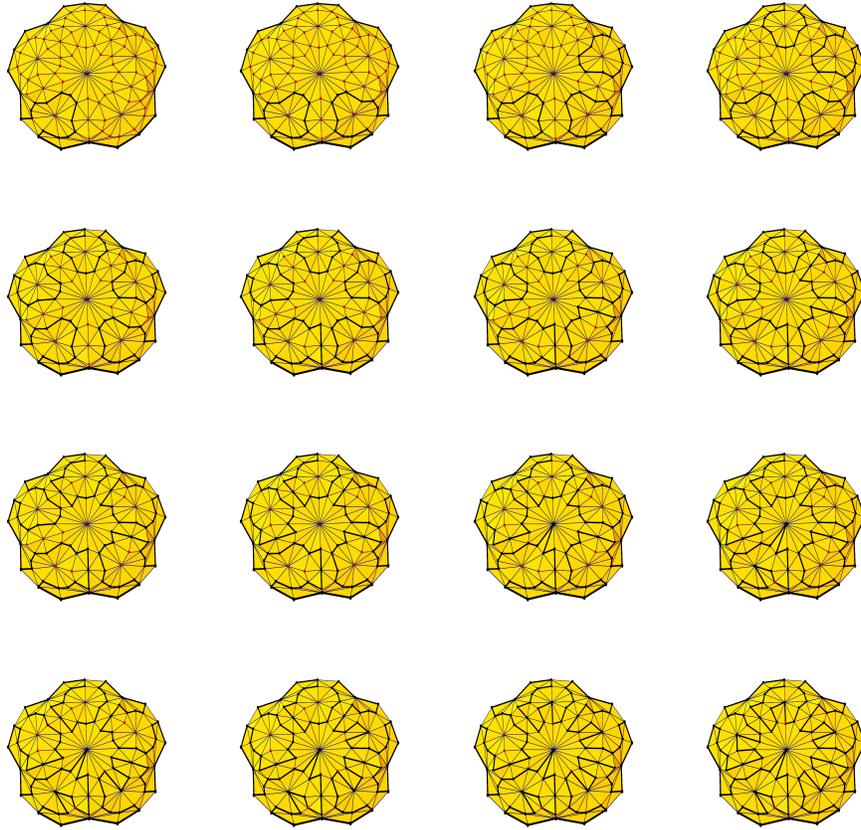
FIGURE 32. The Swiss Cheese strategy applied in an example: start with covering the boundary with a Hamiltonian path. Then cut out some holes and connect them to each other. We chose an example which has the difficulty that after cutting out the hole we have some interior points not reachable. They are included by enlarging the hole slightly. Finally gather the inside the of the holes and make detours to the other remaining interior points. Now we have a Hamiltonian path.

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