

UNIVERSALITY FOR BARYCENTRIC SUBDIVISION

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ABSTRACT. The spectrum of the Laplacian of successive Barycentric subdivisions of a graph converges exponentially fast to a limit which only depends on the clique number of the initial graph and not on the graph itself. Announced in [40]), the proof uses now an explicit linear operator mapping the clique vector of a graph to the clique vector of the Barycentric refinement. The eigenvectors of its transpose produce integral geometric invariants for which Euler characteristic is one example.

1. NOTATIONS

Given a finite simple graph $G = (V, E)$ with vertex set V and edge set E , the **Barycentric refinement** $G_1 = (V_1, E_1)$ is the graph for which V_1 consists of all nonempty complete subgraphs of G and where E_1 consists of all unordered distinct pairs in V_1 , for which one is a subgraph of the other. Denote by G_m the successive Barycentric refinements of G assuming $G_0 = G$. If $\lambda_0 \leq \dots \leq \lambda_n$ are the eigenvalues of the **Kirchhoff Laplacian** $L = B - A$ of G , where B is the diagonal **degree matrix** and A the **adjacency matrix** of G , define the **spectral function** $F_G(x) = \lambda_{[nx]}$, where $[t]$ is the largest integer smaller or equal to t . So, $F_G(0) = 0$ and $F_G(1)$ is the largest eigenvalue of G . The L^1 norm of F satisfies $\|F_G\|_1 = \|\lambda\|_1/|V| = \text{tr}(L)/|V| = 2|E|/|V|$ by the **Euler handshaking lemma** telling that $2|E|$ is the sum of the vertex degrees. In other words, $\|F\|_1 = d(|V| - 1)$, where d is the **graph density** $d = |E|/B(|V|, 2)$ with Binomial coefficient $B(\cdot, \cdot)$. The density d gives the fraction of occupied graphs in the completed graph with vertex set V . If G, H are two sub graphs of some graph with n vertices, define the **graph distance** $d(G, H)$ as the minimal number of edges which need to be modified to get from G to H . If L, K are the Laplacians of G, H , then $\sum_{i,j} |L_{ij} - K_{ij}| \leq 4d(G, H)$ because each edge (i, j) affects the four matrix entries $L_{ij}, L_{ji}, L_{ii}, L_{jj}$ of the Laplacian only: adding or removing that edge changes each of the 4 entries by 1. The **Lidskii-Last inequality** assures $\|\mu - \lambda\|_1 \leq \sum_{i,j=1}^n |A - B|_{ij}$ [59, 45] for any two symmetric $n \times n$ matrices A, B with eigenvalues $\alpha_1 \leq \alpha_2 \leq \dots \leq \alpha_n$ and $\beta_1 \leq \beta_2 \leq \dots \leq \beta_n$. For two subgraphs G, H of a common graph with n vertices, and Laplacians L, H ,

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the inequality gives $\|\lambda - \mu\|_1 \leq 4d(G, H)$ so that $\|F_{G'} - F_{H'}\| \leq 4d(G, H)/n$ if G', H' are the graphs with edge set of G and vertex set of the host graph having $n(G, H)$ vertices. Therefore $\|F_G - F_H\|_1 \leq 4d(G, H)/n(G, H)$ holds, where $n(G, H)$ is the minimum of $|V(G)|$ and $|V(H)|$ assuming both are in a common host graph. The L^1 distance of spectral functions can so estimated by the graph distance. Obviously, if k disjoint copies of the same graph H form a larger graph G , then $F(G) = F(H)$. Given a cover U_j of G , let H be graph generated by the set of vertices which are only in one of the set U_j . Let K be the graph generated by the complement of H . Now, $d(G, H) \leq 4|K|$ and $\sum_j d(U_j, H_j) \leq 4|K|$. If v_k is the number of complete subgraphs K_{k+1} of G , the **clique number** of G is defined to be k if $v_k = 0$ and $v_{k-1} > 0$. A tree for example has clique number 2 as it does not contain triangles. Denote by \mathcal{G}_d the class of graphs with clique number $d + 1$. The class contains K_{d+1} as well as any of its Barycentric refinements. The **clique data** of G is the vector $\vec{v} = (v_0, v_1, \dots)$, where v_k counts the number of K_{k+1} subgraphs of G . We have $v_0 = |V|$, $v_1 = |E|$ and v_2 counts the number of triangles in G . The clique data define the **Euler polynomial** $e_G(x) = \sum_{k=0}^{\infty} v_k x^k$ and the **Euler characteristic** $\chi(G) = e_G(-1)$. The polynomial degree of $e(x)$ is d if $d + 1$ is the clique number. There is a linear **Barycentric operator** A which maps the clique data of G to the clique data of G_1 . It is an upper triangular linear operator on l^2 with diagonal entries $A_{kk} = k!$. It also maps the Euler polynomial of G linearly to the Euler polynomial of G_1 . The **unit sphere** $S(x)$ of a vertex $x \in V$ in G is the graph generated by all vertices connected to x . The **dimension** of G is defined as $\dim(\emptyset) = -1$ and $\dim(G) = 1 + \sum_{v \in V} \dim(S(x))/v_0$, where $S(x)$ is the unit sphere centered at $x \in V$. A graph has **uniform dimension** d if every unit sphere has uniform dimension $d - 1$. The empty graph has uniform dimension -1 . Given a graph G with uniform dimension d , the **interior** is the graph generated by the set of vertices for which every unit sphere $S(x)$ is a $(d - 1)$ -sphere. A **d -sphere** is inductively defined to be a graph of uniform dimension d for which every $S(x)$ is a $(d - 1)$ -sphere and for which removing one vertex renders the graph contractible. This **Evako sphere** definition starts with the assumption that the (-1) -sphere is the empty graph. **Contractibility** for graphs is inductively defined as the property that there exists $x \in V$ for which both $S(x)$ and the graph generated by $V \setminus \{x\}$ are both contractible, starting with the assumption that K_1 is contractible. The **boundary** δG of a graph G is the graph generated by the subset of vertices in G for which the unit sphere $S(x)$ is not a sphere. Also the next definitions are inductive: a **d -graph** is a graph for which every unit sphere is a $(d - 1)$ -sphere; a **d -graph with boundary** is a graph for which every unit sphere is a $(d - 1)$ -sphere or $(d - 1)$ -ball; a **d -ball** is a d -graph with boundary for which the boundary is a $(d - 1)$ -sphere. Let $w_k(G_m)$ denote the number of K_{k+1} subgraphs in the boundary δG_m . For $G = K_{d+1}$, the boundary of G_m is a $(d - 1)$ -sphere for $m \geq 1$ and δG_m

contains the vertices for which the unit sphere in G_m has Euler characteristic 1. Barycentric refinements honor both the class of d -graphs as well as the class of d -graphs with boundary. Starting with $G = K_{d+1}$ which itself is neither a d -graph, nor a d -graph with boundary, the Barycentric refinements G_m are all d -balls for every $m \geq 1$. While the spectral functions F are well suited to describe limits in $L^1([0, 1])$, one can also look at the **integrated density of states** F^{-1} , a monotone $[0, 1]$ -valued function on $[0, \infty)$ which is also called **spectral distribution function** or **von Neumann trace**. It defines the **density of states** $(F^{-1})'$ which is a probability measure on $[0, \infty)$ also called **Plancherel measure**, analogue to the cumulative distribution functions defining the law of the random variable which in absolutely continuous case is the probability density function. Point-wise convergence of F implies point-wise convergence of F^{-1} and so weak-* convergence of the density of states.

2. THE THEOREM

Theorem 1 (Central limit theorem for Barycentric subdivision). *The functions $F_{G_m}(x)$ converge in $L^1([0, 1])$ to a function $F(x)$ which only depends on the clique number of G . The density of states converges to a measure μ which only depends on the clique number.*

We first prove a lemma which is interesting by itself. It allows to compute explicitly the clique vector of the subdivision G_1 from the clique vector of G .

Lemma 2. *There is an upper triangular matrix A such that $\vec{v}(G_1) = A\vec{v}(G)$ for all finite simple graphs G .*

Proof. When subdividing a subgraph K_{k+1} it splits into $A_{kk} = (k + 1)!$ smaller K_{k+1} graphs. This is the diagonal element of A . Additionally, for $m > k$, every K_{m+1} subgraph produces A_{km} subgraphs isomorphic to K_{k+1} , which is the number of interior K_{k+1} subgraphs of the Barycentric subdivision of K_{d+1} . These K_{m+1} subgraphs in the interior correspond to K_{m+2} subgraphs on the boundary. This means that A_{km} is the number of K_{m+2} subgraphs of the boundary of the Barycentric refinement of K_{k+1} . Since this boundary is of smaller dimension, we can use the already computed part of A to determine A_{km} . To construct A , we build up the columns recursively, starting to the left with e_1 . Let $B(n, k) = n!/(k!(n - k)!)$. If n columns of A have been constructed, we apply the upper $n \times n$ part of A to the vector $\vec{v} = [B(n + 1, 1), \dots, B(n + 1, n)]^T$ in order to get the clique data of the $(n - 1)$ -sphere $\delta(K_{n+1})_1$. These numbers encode the number of interior complete subgraphs of the n -ball $(K_{n+1})_1$. If the resulting vector is $[w_1, \dots, w_n]^T$, take $[1, w_1, \dots, w_{n-1}, (n + 1)!, 0, 0, \dots]^T$ as the new column. The Mathematica code below implements this procedure. \square

The **Barycentric operator** A has an inverse A^{-1} which is a bounded **compact operator** on $l^2(\mathbf{N})$. Below we give the bootstrap procedure to

compute A , adding more and more columns, using the already computed A to determine the next column. The eigenvalues λ of A are included in the spectrum $\sigma(A) = \{1!, 2!, 3!, \dots\}$. Any eigenvector f of A^T can lead to **invariants** $X(G) = \langle f, v(G) \rangle$ which correspond to **valuations** in the continuum as $\lambda X(G) = \langle f, v \rangle = \langle A^T f, v \rangle = \langle f, Av \rangle = X(G_1)$. Of particular interest is the **Euler characteristic eigenvector** $f = [1, -1, 1, -1, \dots]^T$ to the eigenvalue $\lambda = 1$ verifying that $\chi(G)$ is an invariant under Barycentric subdivision. An other invariant is the eigenvectors $f = [0, \dots, 0, -2, d+1]^T$ to the eigenvalue $d!$ which measures the “boundary volume” of a d -graph with boundary. For d -graphs, it remains zero under refinement.

Here is the proof of the theorem:

Proof. **(i) For $G \in \mathcal{G}_d$, there are constants $C_d > 0$ such that $v_k(G_0)(d+1)!^m/C_d \leq v_k(G_m) \leq v_k(G_0)C_d((d+1)!)^m$ and $w_k(G_0)(d!)^m/C_d \leq w_k(G_m) \leq w_k(G_0)C_d(d!)^m$ for every $0 \leq k \leq d$.** Proof. This follows from the lemma and **Perron-Frobenius** as the matrix A is explicit. When restricting to \mathcal{G}_d , it is a $(d+1) \times (d+1)$ -matrix with maximal eigenvalue $(d+1)!$, when restricted to G_{d-1} its maximal eigenvalue is $d!$. Diagonalizing $A_d = S_d^{-1}B_dS_d$ we could find explicit bounds C_d .

(ii) For every $G = K_k$, the sequence F_{G_m} converges. Proof: There are $(d+1)!$ subgraphs of $G_{M=1}$ isomorphic to G_m , forming a cover of G_{m+1} . They intersect in a lower dimensional graph. Since the number of vertices of this intersection grows with an upper bound $C_d d!^m$ in each G_m we have $d(G_{m+1}, \bigcup_j G_m^{(j)}) \leq (d+1)!C_d(d!)^m$. This shows $\|F_{G_m} - F_{G_{m+1}}\|_1 \leq C_d(d+1)!(d!)^m/((d+1)!)^m = C_d/(d+1)^{m-1}$. We have a Cauchy sequence in $L^1([0, 1])$ and so a limit in that Banach space.

(iii) Barycentric refinement is a contraction on \mathcal{G}_d in an adapted metric. Proof. Every $G \in \mathcal{G}_d$ can be written as a union of several $H_m \sim K_{d+1}$ subgraphs and a rest graph L in \mathcal{G}_k with $k < d$. Then F_{G_m} and F_{H_m} have the same limit because $F_{H_m} = F_{G_{m-1}}$. The Barycentric evolution of the boundary of refinements of K_{d+1} as well as L grows exponentially smaller. Given two graphs $A, B \in \mathcal{G}_d$, then $d(A_m, B_m) \leq C_d(d!)^m$ so that $\|F_{A_m} - F_{B_m}\|_1 \leq 4C_d/(d+1)^m$. Let m_0 be so large that $c = C_d/(d+1)^{m_0} < 1$. Define a new distance $d'(A, B) = \sum_{k=0}^{m_0-1} d(A_k, B_k)$, so that $d'(A_1, B_1) \leq cd'(A, B)$. Apply the **Banach fixed point theorem**.

(iv) There is uniform convergence of F_{G_m} on compact intervals of $(0, 1)$. Since each F_{G_m} is a monotone function in $L^1([0, 1])$, the exponential convergence on compact subsets of $(0, 1)$ follows from exponential L^1 convergence. (This is a general real analysis fact [46]). In dimension d , one has $\|F_{G_m}\|_1 \rightarrow (d+1)!$ exponentially fast. Indeed, as the number of boundary simplices grows like $(d!)^m$ and the number of

interior simplices grows like $((d + 1)!)^m$, the convergence is of the order $1/(d + 1)^m$. By the **Courant-Fischer mini-max principle** $F_G(1) = \lambda_n = \max(v, Lv)/(v, v) \geq \max(L_{xx}) = \max(\deg(x))$ grows indefinitely, so that the L^∞ convergence can not be extended to $L^\infty[0, 1]$. \square

It follows that the density of states of G converges “in law” to a universal density of states which only depends on the clique number class of G , hence the name “central limit theorem”. The analogy is to think of G or its Laplacian L as the random variable and the spectrum $\sigma(L)$ of the Laplacian L as the analog of the probability density and of the Barycentric refinement operation as the analogue of adding and normalizing two independent identically distributed random variables. For $d = 1$, where the graph is triangle-free and contains at least one edge, we know everything:

Proposition 3. *For $d = 1$, the limiting function is $F_1(x) = 4 \sin^2(\pi x/2)$.*

Proof. As the limiting distribution is universal, we can compute it for $G = C_m$, where $G_m = C_{4 \cdot 2^m}$. As the spectrum of C_n is the set $\{4 \sin^2(\pi k/n) \mid k = 1, \dots, n\}$, the limit is F_1 . \square

For $d = 1$, the limiting spectral function is related to the **Julia set** of the **quadratic map** $z \rightarrow z(4 - z)$ which is conjugated to $z \rightarrow z^2 - 2$ ($c = -2$ at bottom tail of Mandelbrot set) or $z \rightarrow 4z(1 - z)$ which is the **Ulam interval map** conjugated to **tent map**, or $z \rightarrow 4z^2 - 1$ which is a **Chebyshev polynomial**.

3. REMARKS

1) For $d = 2$ already, we expect spectral gaps. A first large one is observed at $x = 1/2$. For G_4 with $G = K_3$, we see a jump at 0.5 of 2.002, Starting with $G = K_3$, the graph G_2 has 25 vertices. Its Laplacian has eigenvalues for which $\lambda_{13} - \lambda_{12} = 2.0647\dots \sim 2$ being already close to the gap. The eigenvalues can be simplified to be roots of polynomials of degree 4 for which explicit radical expressions exist allowing to estimate. We also know $\|F_{G_3} - F_2\|_1 \leq 8 \sum_{k=3}^{\infty} 1/3^k = 8/18 < 1/2$. While we know that $F_{G_m} - F_2$ converges pointwise and uniformly on each compact interval we don't have uniform constant bounds on each interval.

2) Instead of the Laplacian $L = B - A$, we could take the **adjacency matrix** A . We could also take the **multiplicative Laplacian** $L' = AB^{-1}$ or the to L' isospectral selfadjoint **random walk Laplacian** $L'' = B^{1/2}AB^{1/2}$. The corresponding spectral functions always converge exponentially fast, but to different limiting functions. For the adjacency matrix for example, the spectral gaps appear much smaller.

3) The convergence also works for the **Dirac operator** $D = d + d^*$, where d is the **exterior derivative** or the **Hodge Laplacian** $L = D^2$. For the Dirac operator D , for which the density of states is supported on $(-\infty, \infty)$. For the Hodge Laplacian on $[0, \infty)$. The Hodge Laplacian factors into different form sectors L_k and the spectral functions of each **form Laplacian** L_k converge.

The blocks which make up D^2 . The **scalar Laplacian** $L_0 = d^*d$ agrees with the **combinatorial Laplacian** = Kirchhoff Laplacian $L = B - A$ discussed above.

4) Any Barycentric refinement preserves the **Euler characteristic** $\chi(G) = \sum_{k=0} (-1)^k v_k$. We know that G_n is homotopic and even homeomorphic to G . G and H are **Ivashchenko homotopic** [28, 8] if one get H from G by homotopy transformation steps done by adding a vertex, connecting it to a contractible subgraph, or removing one with a contractible sphere. A topology \mathcal{O} on V of G is a **graph topology** if there is a sub-base \mathcal{B} of \mathcal{O} consisting of contractible subgraphs such that the intersection of any $A, B \in \mathcal{B}$ satisfying $\dim(A \cap B) \geq \min(\dim(A), \dim(B))$ is contractible, and every edge is contained in some $B \in \mathcal{B}$. We ask the \mathcal{G} of \mathcal{B} to be homotopic to G , where the **nerve graph** $\mathcal{G} = (\mathcal{B}, \mathcal{E})$ has edges \mathcal{E} consisting of all pairs $(A, B) \in \mathcal{B} \times \mathcal{B}$ for which the dimension assumption is satisfied. A map between two graphs equipped with graph topologies is **continuous**, if it is a graph homomorphism of the nerve graphs such that $\dim(\phi(A)) \leq \dim(A)$ for every $A \in \mathcal{B}$. If ϕ has a continuous inverse it is a **graph homeomorphism**.

5) We know $\dim(G_1) \geq \dim(G)$ with equality for d -graphs, graphs for which unit spheres are spheres. For a visualization of the discrepancy on random Erdős-Renyi graphs, see [43]. We also know that $\dim(G_m)$ converges monotonically to the dimension of the largest complete subgraph because the highest dimensional simplices dominate exponentially.

6) If G is a d -graph, then for $m > 1$, all G_m as well as any finite intersections of unit spheres in G_m are all **Eulerian** so that we can define a **geodesic flow** on G_m or a **billiard** on Barycentric refinements of the ball $(K_k)_m$. For non-Eulerian graphs like an icosahedron, a light propagation on the vertex set is not defined without breaking some symmetry. Also, for any graph and $m > 0$, the **chromatic number** of G_m is the clique number of G . Indeed, for $m \geq 1$ the dimension of the complete subgraph graph x of G_m can be taken as the **color** of the vertex x in G_{m+1} . Since the dimension takes values in $\{0, \dots, d\}$ the chromatic number of G agrees with the clique number $d+1$ of G .

7) If $d_0 \leq \dots \leq d_n$ is the **degree sequence** of G , define the **degree function** $H_G(x) = d_{\lfloor xn \rfloor}$ in the same way as the spectral function. Since the degrees are the diagonal elements of L , the **integrated degree function** $\tilde{H}(x) = \int_0^x H_G(t) dt$ and the **integrated spectral function** $\tilde{F}(x) = \int_0^x F_G(t) dt$ agree at $x = 0$ and $x = 1$. By the **Schur inequality**, we have $\tilde{H}(x) \geq \tilde{F}(x)$. The degree function H_{G_m} also converges.

8) Barycentric refinements are usually defined for realizations of a **simplicial complex** in Euclidean space. The Barycentric subdivision of a graph is not the same than what is called a **simplex graph** [3] as the later contains the empty graph as a node. It also is not the same than the **clique graph** [24] as the later has as vertices the maximal complete subgraphs, connecting

them if they have a non empty intersection. The Barycentric refinement is the **graph product** of G with K_1 [43]. The product $G \times H$ has as vertex set the set of all pairs (x, y) , where x is a complete subgraph x of G and a complete subgraph y of H . Two such vertices (x, y) and (u, v) are connected by an edge, if either $x \subset u$ and $y \subset v$ or $u \subset x$ and $y \subset v$.

9) The limit of Barycentric refinements is “**flat**” in the following sense: the graph **curvature** $K(x) = 1 - V_0(x)/2 + V_1(x)/3 - \dots$ with $\vec{V}(x) = (V_0(x), V_1(x), \dots)$ being the clique data of the unit sphere $S(x)$ is defined for all finite simple graphs [36] satisfying the **Gauss-Bonnet-Chern theorem** adds up to the Euler characteristic of the original graph. Because the total curvature is 1, and as the graph gets larger, the curvature averaged over some subgraph G_{m-k} of G_m goes to zero, while the individual curvatures can grow indefinitely. For $d = 2$ for example, where the curvature is $K(x) = 1 - d(x)/6$, there are vertices with very negative curvature, but averaging this over a smaller patch gives zero. In general, if we look at subgraphs of G which are wheel graphs as 2-dimensional sections, also the **sectional curvatures** become unbounded at some points but averages of sectional curvatures over two dimensional surfaces obtained by Barycentric refinements of a wheel graph go to zero. The limiting holographic object looks like Euclidean space. There is even “**rotational symmetry**”, not everywhere, but with centers at a dense subset of the continuum.

10) The graph theoretical subdivision definition uses the point of view that complete subgraphs of a graph are treated as **points**. When this identification is taken seriously and iterated, we get a **holographic Barycentric refinement sequence** which is already realized in the graph itself. This is familiar when building up number systems: if the set of integers $G = \mathbb{Z}$ is considered to be a graph with adjacent integers connected, the Barycentric refinements G_n contain all **dyadic rationals** $k/2^n$. Modulo 1, this is a **Prüfer group** \mathbb{P}_2 which has as the **Pontryagin dual** the compact topological group \mathbb{D}_2 of dyadic integers, a subring of the **field** \mathbb{Q}_2 of **2-adic numbers**. In the case $d = 1$, there is a limiting random operator on the group of **dyadic integers**.

11) There is an analogy between $(\mathbb{R}, \mathbb{T}, \mathbb{Z}, x \rightarrow x + \alpha \bmod 1, x \rightarrow 2x \bmod 1)$ and $(\mathbb{Q}_2, \mathbb{P}_2, \mathbb{D}_2, x \rightarrow x + 1, \sigma)$, where $x \rightarrow x + 1$ is the addition on the compact topological group. This translation on \mathbb{D}_2 is called **adding machine** [20] and σ is shift which is a return map on half of \mathbb{D}_2 . There is a natural way to get the group \mathbb{D}_2 through **ergodic theory** as it is the unique fixed point of 2:1 **integral extensions** in the class of all dynamical system. It is called also the **von Neumann-Kakutani system** and usually written as an interval map on $[0, 1]$. The ergodic theory of systems with discrete spectrum is completely understood [10, 14] and the von-Neumann-Kakutani system belongs so to a class of systems, where we can solve the **dynamical log-problem**: given any x, y and $\epsilon > -$ find n such that $d(T^n x, y) < \epsilon$ which is essential everywhere in dynamics, from prediction of events up to finding solutions to Diophantine equations. Such systems are typically **uniquely**

ergodic and so by spectral theory naturally conjugated to a group translation on a compact topological group, the dual group of the eigenvalues of the **Koopman operator** on the unit circle in the complex plane. There is an important difference between the real and 2-adic story: while in both cases, the chaotic scaling systems are isomorphic **Bernoulli systems** on compact Abelian topological groups, the group translation on the 2-adic group is naturally unique and **quantized**, while on the circle, there are many group translations $x \rightarrow x + \alpha$. In the **real picture**, there is a continuum of natural translations, in the **dyadic picture**, there is a **smallest translation**. The picture is already naturally quantum. Egyptian dyadic fractions, music notation or the transition from Fourier to wavelet theory can be seen as a move towards **dyadic models**. See [62] for a plethora of other places.

12) For a d -graph G with $d > 1$, the renormalization limit of G is still unidentified. The limiting operator is likely a random operator on a compact topological group. As the group of dyadic integers, it is likely also a **solenoid**, an inverse limit of an inverse system of topological groups. There is a group acting on it producing the operator in a crossed product C^* -algebra. This is at least the picture in one dimensions.

13) As F_{G_m} converges in L^1 , we have convergence of the **integrated density of states** $F_{G_m}^{-1}$ in $L^1[0, \infty)$ and so weak-* convergence of the derivative, the **density of states**, in the Banach space of measures. In the case $d = 1$, the density of states is $f(x) = (x(4-x))^{-1/2}/\pi$ supported on $[0, 4]$. The integrated density of states is $F^{-1}(x) = (2/\pi) \arcsin(\sqrt{x}/2)$. The measure $\mu = f(x)1_{[0,4]}dx$ is the **equilibrium measure** on the Julia set of $T(z) = 4z - z^2$ whose dynamic is conjugated to $z \rightarrow z^2 - 2$ in the Mandelbrot picture or the **Ulam map** $z \rightarrow 4x(1-x)$ for maps on the interval $[0, 1]$. The spectral function F satisfies $T(F(x)) = F(2x)$. The measure μ maximizes metric entropy and equals it to topological entropy $\log(2)$ of T .

14) In the case $d = 2$, the numbers v_k were already known [60, 5] as $v_2(G_m) = 6^m$ and v_0 is the sum of $v_0(m-1), v_1(m-1), v_2(m-1)$ and $v_0 - v_1 + v_2 = 2$ leading to formulas like $v_0(G_m) = 1 - 3(2^{m-1} + 2^m + 2^{m-1}3^{m-1})$, $v_1(G_m) = 3(-2^{m-1} + 2^m + 2^{m-1}3^m)$. As the lemma shows, in general, the clique data of G_1 are a linear image of the clique data of G with a linear map A independent of G . Since A has a compact inverse, we can look at an eigenbasis of A and could write down explicit formulas for the number of vertices of G_n , if the initial clique data of G are known.

15) If H is a subgraph of G , then each refinement H_m is a subgraph of G_m . Also, G_k is a subgraph of G_m if $k \leq m$ and the automorphism group of G_m contains the automorphism group of G . The case $G = K_n$ shows that the **automorphism group** can become larger. In general, $\text{Aut}(G_1) = \text{Aut}(G)$. It is only in rare cases like $G = C_n$ that $\text{Aut}(G_m)$ can grow indefinitely.

16) The matrix A mapping the clique data of G to the clique data of G_1 is a special case of the following: for any graph H , there is a linear map A_H which maps the clique data of any graph G to the clique data of $G \times H$.

This operator A_H depends only on H . While $A = A_{K_1}$ is invertible on each class of finite dimensional graphs. The spectral data $v(G)$ do not determine the graph G as trees already show, the Barycentric refinement operator T is invertible on the image $G(\mathcal{G})$ of the class \mathcal{G} of all finite simple graphs.

17) Barycentric refinements make graphs nicer: the chromatic number is the clique number, the graphs are Eulerian. If we identify graphs with their Barycentric refinements, we have a **holographic picture**. There is a natural geodesic flow on G_1 if G is a d -graph as each every unit sphere has a natural involution, defined inductively by running the geodesic flow on the unit sphere. On a two dimensional sphere for example, the flow is defined because every vertex has even degree. This allows to draw **straight lines** on it and to define an antipodal map. For a 3-sphere, because every unit sphere has such an antipodal map, we can define straight lines there, in turn leading to an antipodal map on such spheres. This can be continued by climbing up the dimensions to get a geodesic flow on the graph G_1 itself.

18) The density of states obtained as a limiting spectral density of finite dimensional situations is central in the theory of **random Schrödinger operators** L [11, 53, 50], where the density of states μ can be defined as the functional $f \rightarrow \text{tr}(f(L))$ as L is an element in a von Neumann algebra with trace, the crossed product of $L^\infty(\Omega)$ with an ergodic group action on the probability space. The **Birkhoff ergodic theorem** has been applied by Pastur to assure in that theory that the density of states of finite dimensional operators defined by orbits starting at some point x converges **almost surely** to the measure μ . The 1-dimensional discrete case $d = 1$, where a single ergodic transformation T on a probability space is given is one of the most studied models, in particular the map $x \rightarrow x + \alpha \bmod 1$ leading to almost periodic matrices like the **almost Mathieu operator**. It is the real analog of the limiting operator which is almost periodic over the dyadic integers. There is no doubt that also in higher dimensions, the limiting model of Barycentric refinement is part of the theory of aperiodic almost periodic media [50, 11, 50, 57, 58, 31].

4. OPEN ENDS

1) Already for $d = 2$, we do not know what the nature of F_d is. One must suspect that there is a renormalization picture, as in one dimensions. It can not be a Julia set of a polynomial, as the spectrum becomes unbounded. We suspect that for $d > 1$, the time of the group action is higher dimensional and that the limiting Laplacian remains almost periodic. Since we can not compute $F_2(G_m)$ yet for large m due to the fast growth of G_m , we see experimentally that the function F_2 on $L^1([0, 1])$ has a **self-similar nature** in the sense that $F_2(6x) \sim \phi_2(F_2(x))$ for some ϕ_2 and $x \in [0, 1/6]$. This self-similar nature is present in $d = 1$ as $F_1(2x) = \phi_1(F_1(x))$, where ϕ_1 is the **quadratic map** $\phi_1(x) = x(4 - x)$. Since $F_2(1)$ is infinite, the map ϕ_2 (if it exists at all) must be an unbounded function and in particular can not

be a polynomial.

2) In order to establish an existence of a **spectral gap** for F_2 , one could first establish an exact finite approximation bound, then estimate the rest. For $d = 2$, the first large gap opens at $x = 1/2$. This could be related to the fact that every step doubles the vertex degrees in two dimensions. Here are the largest spectral jump values for G_5 if $G = K_3$: $\Delta(0.5) = 2.002\dots$, $\Delta(0.835366) = 1.67669\dots$, $\Delta(0.917683) = 2.86249$, $\Delta(0.972561) = 3.89394$, $\Delta(0.98628) = 6.93379$, $\Delta(0.995427) = 7.96794$ and $\Delta(0.998476) = 14.9767$. To investigate gaps, we would need a Lidskii type estimate for the smaller half of the eigenvalues.

3) We see experimentally that for $d = 2$, the sequence F_{G_m} appears monotonically increasing $F_{G_m} \leq F_{G_{m+1}}$ at least for the small m in which we can compute it. In the case $d = 1$, the monotonicity is explicit. It fails for smaller m and $d = 3$ but could hold for any d if m is large enough. If we had more details about the convergence in the middle of the spectrum, we could attack the problem of verifying that spectral gaps exist.

4) Unlike the **Feigenbaum renormalization** in one dimensional dynamics [19, 27, 12], where a hyperbolic attractor with stable and unstable manifold exists, the Barycentric renormalization is a contraction and the convergence proof is more elementary. More so than the **central limit theorem** in probability theory, where the renormalization map is $X \rightarrow \overline{X_1 + X_2}$, where X_i are independent random variables on a probability space Ω with the same distribution than X and $\overline{X} = (X - E[X])/\sigma(X)$. That random variable renormalization is not a uniform contraction in $L^2(\Omega, P)$. For $d = 1$ there is a limiting almost periodic operator where the compact topological **group of dyadic integers** becomes visible in the Schrödinger case. We expect also in higher dimensions, a random limiting operator and that the renormalization map is robust in the sense that one can for example shift the energy E and have a deformed attractor. In the one-dimensional Schroedinger case, where one deals with Jacobi matrices, one gets so to spectra on Julia sets J_E of the quadratic map $T(x) = z^2 + E$ [49, 21, 34, 33, 35]. Also in higher dimensions, we expect that a Schrödinger renormalization picture reveals the underlying topological group if an energy parameter is used to modify the renormalization.

5) In the case $d = 1$, the roots of the **Dirac zeta function** $\zeta(s) = \sum_{\lambda>0} \lambda^{-s}$, defined by the positive eigenvalues λ of the Dirac operator $D = d + d^*$ [38, 47] of G_m , converges to a subset of the line $\text{Re}(s) = 1$. What happens in the case $d = 2$? Already for $d = 1$, the convergence of the roots is slow of the order $\log(\log(v_0(G)))$ as it is initially proportional to m and slowing down exponentially when approaching the line. Any experimental investigations

in $d = 2$ would be difficult as no explicit formulas for the eigenvalues exist.

6) It would be nice to know more about the linear operators A_H which belong to the map $G \rightarrow G \times H$ on \mathcal{G} . Their inverse is compact. In the case $H = K_1$, where we get the Barycentric refinement operator A , the eigenvector of A^T to the eigenvalue 1 gives an invariant for Barycentric refinement: it is the Euler characteristic vector $(1, -1, 1, -1, 1, -1, \dots)$. When restricted to d -graphs, the vector $(0, 0, \dots, 1)$ with 1 at the $d + 1$ 'th entry is an eigenvector to the eigenvalue $(d + 1)!$. It leads to a counting invariant in the limit which is called **volume**. The other eigenvectors lead to similar limits which must be linear combinations of **valuations** in Riemannian geometry [56, 32]. We see that the even **Barycentric invariants** $X_{2k}(G) = \vec{f}_{2k}(A) \cdot \vec{v}(G)$ are constant zero on d -graphs, where $\vec{f}_k(A)$ are the eigenvectors of A^T . It will imply for example that for any compact smooth 4-manifold containing v vertices, e edges, f triangles, g tetrahedra, and h pentatopes satisfies $22e + 40g = 33f + 45h$. In the continuum, the more general operator A_H leads to a linear map on valuations. These maps will help to investigate the connection between the discrete and continuum.

7) In the same way as for Riemannian manifolds, where one studies the eigenfunctions f_k of the Laplacians, the eigenfunctions of the Kirchhoff Laplacians will play an important role for understanding the limiting density of states. The **Chladni figures** are the level sets $f_k = 0$. They can be defined for d -graphs. We can look at $f = 0$. It is by a discrete Sard result [44] a $(d - 1)$ -graph as long as 0 is not a value taken by f . If 0 is a value taken we just can plot $f = \epsilon$ for $0 < \epsilon$ small. The point is that we can so define nice $(d - 1)$ graphs associated to f_k . The topology of these **Chladni graphs** depends very much on the energy as in the continuum and as in the continuum, are not well understood yet.

8) The even eigenvectors \vec{v}_{2k} of A^T appear to have the property that $X_{2k}(G) = \vec{v}_{2k} \cdot \vec{v}(G) = 0$ for any $2d$ -graph. If true, this leads to integral geometric invariants in the limit. For any graph, define $X_{2k}(G) = \lim_{m \rightarrow \infty} \vec{v}_{2k} \vec{v}(G_m) / \lambda_{2k}^{-m}$. The limit is trivial as the right hand side is constant. For d -graphs, we see $X_{2k}(G) = 0$. We have tried random versions of $S^4, S^2 \times S^2, T^4, S^2 \times T^2, S^3 \times T^1, S^6$. The limiting integral theoretic invariants for d -manifolds would be zero for differentiable manifolds. Assume we would find a graph G with uniform dimension d which is not a d -graph but which is homeomorphic to a d -graph H . Then, since Barycentric refinements preserve the homeomorphism relation, the Barycentric limit M of G is a topological manifold homeomorphic to the Barycentric limit N of H . While M, N are topological manifolds which are homeomorphic, they can not be diffeomorphic, as integral geometric integer valued invariants are diffeomorphism invariants. A basis for valuations can be obtained if M is embedded in a projective

n -sphere. We can compute the expectation of the k -volume of a random m -planes with M using a natural probability measure obtained from Haar measure on $SO(n)$ acting on k -planes. This leads to $d + 1$ invariants, for which some linear combinations is the Euler characteristic. The graph theoretical invariants obtained from A could produce invariants allowing to distinguish homeomorphic but not diffeomorphic manifolds.

5. ABOUT THE LITERATURE

Barycentric refinement are of central importance in topology. Dieudonné [16] writes of *"the three essential innovations that launched combinatorial topology: simplicial subdivisions by the barycentric method, the use of dual triangulation and, finally, the use of incidence matrices and of their reduction."* In algebraic topology, it is used for proving the **excision theorem** or the **simplicial approximation theorem** [25, 55]. In topology, Barycentric subdivision is primarily used in an Euclidean setting for subdividing complex polytopes or CW complexes. For abstract simplicial complexes, it is related to flag complexes even so there are various inequivalent definitions of what simplicial subdivision or simplex graph or Barycentric subdivisions are. In graph theory, subdivisions are considered for simplicial complexes which are special **hypergraphs** in topological graph theory [23]. Indeed, most graph theory treats graphs as one-dimensional simplicial complexes, ignoring the **Whitney complex** of all complete subgraphs. Subdivisions classically considered for graphs only agree with the definition used here if the graph has no triangles. Two graphs are **classically homeomorphic**, if they have isomorphic subdivisions ignoring triangles (see e.g. [4, 30, 2]). In the context of **maps** which are finite cell complexes whose topological space is a surface S , Barycentric subdivisions are considered for this cell complex [22] but not for the graph. In the context of convex polytopes, Barycentric subdivision appear for the **order complex** of a polytop which is an abstract simplicial complex [48].

Resources on the spectral theory of graph are in [9, 52, 1, 13, 51, 61]. It parallels to a great deal the corresponding theory for Riemannian manifolds [7, 54, 6].

The definition of d -spheres and homotopy are both due to Evako. Having developed the sphere notion independently in [41, 43] we realized in [42] the earlier definition of Ivashchenko=Evako [29, 28, 17, 18]. The definition of these Evako spheres is based on Ivashchenko homotopy which is homotopy notion inspired by Whitehead [63] but defined for graphs.

The Barycentric refinements for $d = 2$ are studied in [5], where the limit of Barycentric refinement has a dual called the **hexacarpet**. Figure 7) in that

paper shows the eigenvalue counting function in which gaps in the eigenvalues are shown similarly as in the spectral function $F_G(x)$. The work [5] must therefore be credited for the experimental discovery of the gap.

Papers like [15, 26] deal with the geometry of tessellations of a triangle, which defines a random walk leading to a dense subgroup of $SL(2, R)$ which defines a Lyapunov exponent. While the focus of those papers is different, there might be relations.

For illustrations, more motivation and background, see also our first write-up [40].

6. FIGURES AND CODE

Here are the Mathematica routines which produced the graphs:

```

TopDim=2;
Cl[s_., k_.]:=Module[{n,t,m,u,q,V=VertexList[s],W=EdgeList[s],l},
n=Length[V]; m=Length[W]; u=Subsets[V,{k,k}]; q=Length[u]; l={};
W=Table[{W[[j,1]],W[[j,2]]},{j,m}]; If[k==1,l=Table[{V[[j]]},{j,n}],
If[k==2,l=W,Do[t=Subgraph[s,u[[j]]]; If[Length[EdgeList[t]]==
Binomial[k,2],l=Append[l,VertexList[t]},{j,q}]];l];
Ring[s_., a_.]:=Module[{v,n,m,u,X},v=VertexList[s]; n=Length[v];
u=Table[Cl[s,k],{k,TopDim+1}]/.Table[k->a[[k]},{k,n}];m=Length[u];
X=Sum[Sum[Product[u[[k,l,m]],
{m,Length[u[[k,l]]}]],{l,Length[u[[k]]}]],{k,m}];
GR[f_.]:=Module[{s={}},Do[Do[If[Denominator[f[[k]]/f[[1]]]==1 && k!=1,
s=Append[s,k->1]],{k,Length[f]},{l,Length[f]}];
UndirectedGraph[Graph[s]];
GraphProduct[s1_.,s2_.]:=Module[{f,g,i,fc,tc},
fc=FromCharacterCode; tc=ToCharacterCode;
i[l_.,n_.]:=Table[fc[Join[tc[l],IntegerDigits[k]+48]},{k,n}];
f=Ring[s1,i["a"],Length[VertexList[s1]]];
g=Ring[s2,i["b"],Length[VertexList[s2]]]; GR[Expand[f*g]];
NewGraph[s_.]:=GraphProduct[s,CompleteGraph[1]];
Bary[s_.,n_.]:=Last[NestList[NewGraph,s,n]];
Laplace[s_.,n_.]:=Normal[KirchhoffMatrix[Bary[s,n]]];
F[s_.,n_.]:=Module[{u,m},u=Sort[Eigenvalues[1.0*Laplace[s,n]]];
m=Length[u]; Plot[u[[Floor[x*m]]],{x,0,1}];
TopDim=1; Show[Table[F[CycleGraph[4],k],{k,3}],PlotRange->{0,4}]
TopDim=2; Show[Table[F[CompleteGraph[3],k],{k,3}],PlotRange->{0,10}]

```

And here is the recursive computation of the matrix A which gives the clique data $A\vec{v}$ of the Barycentric refinement of G_1 if \vec{v} is the clique vector of G . The computation produces the clique data of the boundary of $(K_n)_1$ which allows to compute the number of interior k -simplices producing the off diagonal matrix elements not in the top row.

```

BarycentricOperator[m_.]:=Module[{}],
b[A_.]:=Module[{n=Length[A],c},c=A.Table[Binomial[n+1,k],{k,n}];
Delete[Prepend[c,1],n+1];
T[A_.]:=Append[Transpose[Append[Transpose[A],b[A]]],
Append[Table[0,{Length[A]}],(Length[A]+1)!];
Last[NestList[T,{1},m]]; BarycentricOperator[7]

```

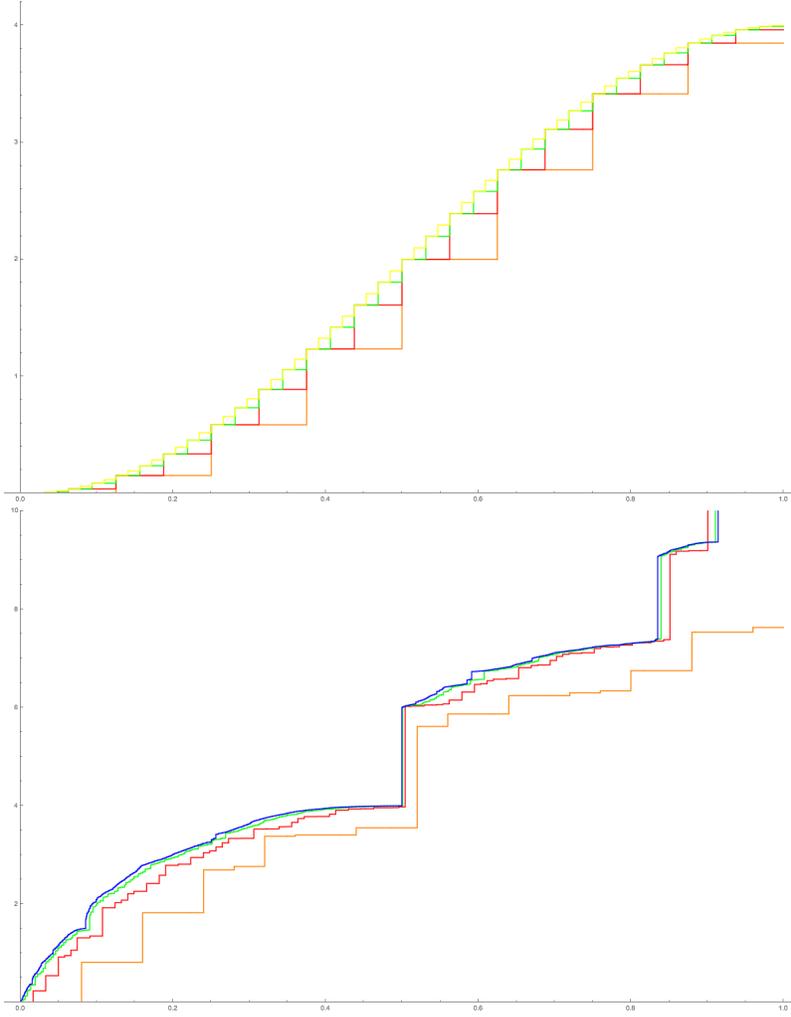


FIGURE 1. The functions F_{G_m} for $G = C_4$ converge to the limiting function $4 \sin^2(\pi x/2)$. The functions F_{G_m} for $G = K_3$ converge to a limiting function which appears to have jumps corresponding to gaps in the spectrum. We have here only established that the universal limit F exists, but already for $d = 2$ do not know about the nature of the limit.

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 6 & 14 & 30 & 62 & 126 & 254 \\ 0 & 0 & 6 & 36 & 150 & 540 & 1806 & 5796 \\ 0 & 0 & 0 & 24 & 240 & 1560 & 8400 & 40824 \\ 0 & 0 & 0 & 0 & 120 & 1800 & 16800 & 126000 \\ 0 & 0 & 0 & 0 & 0 & 720 & 15120 & 191520 \\ 0 & 0 & 0 & 0 & 0 & 0 & 5040 & 141120 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 40320 \end{bmatrix}.$$

The procedure produces finite dimensional versions of this matrix which matter when looking for interesting quantities on \mathcal{G}_d . For $d = 2$ for example, we have

we have $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 6 \\ 0 & 0 & 6 \end{bmatrix}$ whose transpose has the eigenvectors $[1, -1, 1]^T$

(Euler characteristic), the eigenvectors $[0, -2, 3]^T$, a well known invariant for 2-graphs which is proportional to the boundary for 2-graphs with boundary

as well as $[0, 0, 1]^T$ which is area. For $d = 3$, where $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 6 & 14 \\ 0 & 0 & 6 & 36 \\ 0 & 0 & 0 & 24 \end{bmatrix}$,

the eigenvectors besides the Euler characteristic vector $[1, -1, 1, -1]^T$ to the eigenvalue 1 and volume $[0, 0, 0, 1]^T$ to the eigenvalue 24, there is $[0, 22, -33, 40]^T$ with eigenvalue 2 and $[0, 0, 0 - 1, 2]^T$ with eigenvalue 6. The later gives an invariant which is zero for 3-graphs as in general $[0, \dots, 0, -2, d + 1]$ is an invariant for d -graphs. We see experimentally that **for any d -graph, the eigenfunction of A^T to each eigenvalue $(2k)!$ is perpendicular to the clique vector \vec{v}** . For example, there is a discrete $P^2 \times S^2$ with clique vector $\vec{v} = [1908, 26520, 87020, 104010, 41604]$ which is perpendicular to $\vec{v}_4 = [0, -22, 33, -40, 45]$. The invariant does not change under **edge refinement modifications** which are homotopies preserving d -graphs. The invariants $X_{2k}(G) = \vec{v}(G) \cdot \vec{v}_{2k}$ remain zero under such homotopies. They are currently under investigation. It looks promising that integral theoretical methods like [37, 39] for generalized curvatures allow to prove the invariants to be zero. It would be useful also to know whether any two homeomorphic d -graphs have a common refinement when using Barycentric or edge refinements. This looks more accessible than related questions for triangulations as d -graphs can be dealt with recursively and refinements of unit spheres carry to refinements of the entire graph.

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