DENSITY OF WAVE FRONTS

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ABSTRACT. We prove that wave fronts on a flat torus become dense. As a corollary, wave fronts become dense for a square billiard and for the geodesic flow on the flat Klein bottle and for the surface of a three dimensional cube.

1. MOTIVATING QUESTIONS

1.1. Imagine a square room with mirror lining the inner wals. Light emerges from a lamp. The wave front initially forms a perfect circle. After reflecting at the mirror walls, it will bounce back, continuing to be piecewise smooth curve at any time. What happens in the long term with this wave front? Does it fill out the room more and more densely? How long do we have to wait so that every ball of a given fixed radius has and remains to have a non-empty intersection with this curve?

1.2. An earth quake has its epicenter at a point on a closed surface. What is the fate of the resulting tsunami wave front emanating from this point? On a round sphere or round projective plane, this wave will stay circular and oscillate forth and back periodically. This situation is obviously very special on a perfect sphere. It changes if the metric on the sphere is altered in any way. What happens for a general surface like an ellipsoid or a torus, surfaces that lack this Wiederkehr property? What happens with a light front radiating from a point on the surface of a cube?

1.3. As a third example, imagine a duck in a circular pond, making waves. These waves first form circles around the duck, then bounce off the shore and come back. What happens in the long term? If the duck was in the center of the pond, and assuming ideal conditions, the wave front would reach the shore, reflect there and return back to the duck. Assuming no friction or diffusion, the wave front would oscillates back and forth between center and boundary. What would happen if the duck was located at a point off the centerpoint of the pond? Would the wave front produced by the duck become dense on the pond's surface? Intuition says yes because the length of the wave front grows [36]. It gets more complicated as it has to remain wrapped-up in the pond.

1.4. These pictures illustrate some rather general **wave front problems** in **Riemannian geometry**. One can ask for which Riemannian manifolds (with or without boundary or on which polytopes) do the wave front become dense for all initial points, like in the mirrored room example? In which cases does the wave front become dense for some points and not become dense for others like in the duck example case? In which cases does it not become dense for all initial points, like in the tsunami case?

1.5. We explore this theme here in the torus, square and cube surface case, where the metric is flat and where the question can be analyzed with elementary tools only. While the subject could reach far into differential geometry, all we really need here is some single variable calculus.
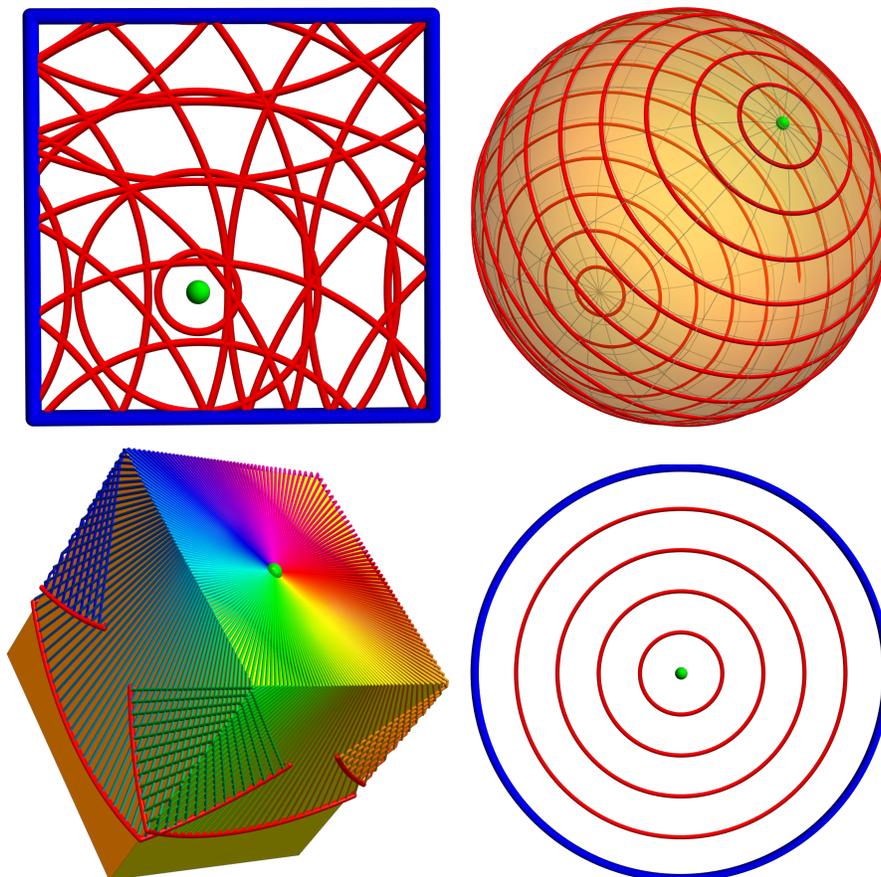

FIGURE 1. What happens with a light wave front in a room of mirrors, a tsunami on a perfect sphere or a cube, and the wave front produced by a duck in a circular pond?

2. BACKGROUND

2.1. We begin with reviewing some elementary notions in differential geometry as covered in standard textbooks like [1, 4, 20, 38]. A good overview of the subject is Chapter 10 in [3]. For manifolds with boundaries, the geodesic dynamics, often referred to as a **billiard**, has become a research topic on its own right [37, 25, 34, 15]. A reference for wave fronts in Riemannian manifolds with boundary is [2]. Motivated by the questions posed in the previous section we work in three different frameworks.

2.2. The first setting involves a compact Riemannian manifold (M, g), where M is a smooth manifold equipped with a positive definite symmetric $(2, 0)$ tensor field g. At each point $P \in M$, the bilinear form $g(P)$ defines an inner product $\langle v, w \rangle_P = g(P)(v, w)$ in the tangent space $T_P M$. This inner product introduces a notion of length $||v||_p = \sqrt{\langle v, v \rangle_p}$, which, in turn, allows us to define the **arc length** of a smooth curve $x : [a, b] \to M$ as $\int_a^b ||\dot{x}(t)|| \, dt$, where $\dot{x} \in T_x M$ is

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the velocity vector. By the fundamental theorem of Riemannian geometry, there is a unique torsion-free connection on (M, g). This covariant derivative is in a local basis $\{e_1, \ldots, e_d\}$ at P determined by the Christoffel symbols $\nabla_{e_i} e_j = \sum_k \Gamma_{ij}^k e_k$. The geodesic flow can then be defined as the solution to the differential equation $\ddot{x}^k + \sum_{i,j=1}^d \Gamma_{ij}^k(x(t)) \dot{x}^i \dot{x}^j = 0$. Given an initial point $P = x(0)$ and an initial velocity $\dot{x}(0)$, there is a unique solution curve $x(t)$ to these equations. If M is compact, it is also a complete metric space and the Hopf-Rynov theorem assures global existence of solutions.

2.3. The second framework extends to compact Riemannian manifold (M, g) with boundary δM. Here, the geodesic flow, referred to as a **billiard**, requires distinguishing between the motion in the interior of M and reflections at the boundary. The boundary $N = \delta M$ is assumed to be a smooth manifold of co-dimension 1 with the metric g inducing a Riemannian metric h on δM. This is achieved by restricting g to the hyper tangent plane $T_P N \subset T_P M$. If a geodesic $x(t)$ starts at the boundary, it can continue as a boundary geodesic or enter the interior of M. A geodesic $x(t)$ starting in the interior hits the boundary at some point $P \in \delta M$, the incoming vector $x'(t^-) \in T_P M$ is reflected at the boundary to produce an out-going vector $x'(t^+)$, analogous to a billiard ball rebounding off a table edge or a light ray reflecting off a mirror. In non-convex scenarios, like the Sinai billiard, where a circular obstacle is placed in a flat Clifford 2-torus, the return map to the boundary may already exhibit non-smooth behavior. After each reflection, the geodesic flow resumes in the interior until the next impact point.

2.4. The third framework considers **piecewise smooth Riemannian polytops**, a concept that was historically challenging to define [33, 29]. This difficulty has led either to restricting the analysis to convex polytopes [21, 41] or then to use a broader definition, where a polytop M is viewed as the geometric realization of a finite abstract simplicial complex G, a finite set of non-empty sets closed under the operation of taking non-empty subsets [19]. A **piecewise smooth Riemannian polytop** is such a polytop that is equipped with a Riemannian metric g that is smooth except on a subset of co-dimension 2. Inductively, this means M is a smooth manifold (with or without boundary) for which the tensor g is defined on an open subset $U = M \setminus K$, where K is a piecewise smooth Riemannian polytope of co-dimension 2. The geodesic flow is defined for geodesic paths avoiding K. One of the simplest examples is the square table, a 2-manifold with circular boundary and a metric that is discontinuous at its four corners. For a 3-dimensional example, consider a billiard in a solid cube. Here, the Riemannian metric tensor g may be discontinuous along the edges and vertices that form a 1-dimensional network.

3. Wave fronts

3.1. Wave fronts can be defined and observed on any **piecewise smooth Riemannian manifold M with or without boundary**. The **wave front** of a point P is defined as the hyper surface $W_t = \exp_P(S_t)$, where $\exp_P : T_P M \to M$ is the **exponential map** and $S_t = \{x \in \mathbb{R}^d, |x| = t\}$ is the sphere of radius t in the tangent space $T_P M \sim \mathbb{R}^d$. The global existence of the exponential map on a closed smooth Riemannian manifold is guaranteed by the Hopf-Rynov theorem. In the case of a manifold with boundary, the geodesic flow dynamics is also known as the **billiard dynamical system**. This is already interesting for a 2-dimensional domain and especially on a convex domains, in which case the system is called the **Birkhoff billiard** [10, 32]. Already in a polygon, like in a square, a path entering one of the corners does

not have a well defined bounce, but the wave front W_t remains defined for all times. Similarly, when looking at a wave front on a surface of a polyhedron like a cube, it is not clear how to continue the geodesic through the corner. We just disregard these trajectories. The wave front W_t is still defined now as piecewise smooth curve. It will even become disconnected after finite t in general. In this article, we limit ourselves to

- 1. Smooth closed manifolds. In this case, W_t always are continuous curves, which are piecewise smooth with singularities at caustic points, points where the Jacobian $d\exp_P$ is singular.
- 2. Convex planar flat billiards. Now, W_t remains a continuous curve. It remains piecewise smooth. Differentiability can fail at caustics or at the boundary.
- 3. Piecewise flat convex polyhedra. In this case, W_t in general can already become disconnected.
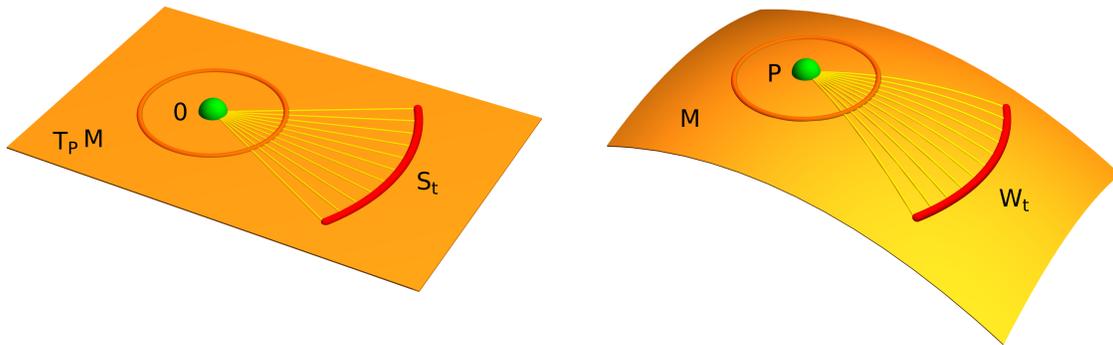

FIGURE 2. In a Riemannian d-manifold M, the wave front $W_t(P)$ of a point P for which a tangent space $T_P M$ exists, is the image $\exp_P(S_t)$ of the geodesic sphere $S_t = \{|x| = t\} \subset T_P M \sim \mathbb{R}^d$ of radius t.

3.2. While for a smooth surface M without boundary or a billiard in a convex region, the wave fronts W_t are piecewise smooth and in particular continuous. This is already different on a closed polyhedral surface. The wave front there can become disconnected after some time. This happens in particular on a cube's surface, specifically the surface bounding the unit cube. Numerical experiments still indicate to us that also for more general non-flat compact surfaces, in general, the wave front becomes dense, meaning that it fills out the surface more and more. An unresolved case is already the dodecahedron.

3.3. Here is the definition of "dense wave front" for a general compact Riemannian manifold M with or without boundary. Given $r > 0$, the **open ball** $B_r(Q)$ of a point $Q \in M$ is the set of points P for which there is a geodesic from P to Q that has geodesic length smaller than r.

Definition 1 (Dense Wave Front). *The wave front $W_t(P)$ becomes dense on M if for every open ball $U = B_r(Q)$ in M with $r > 0$, there is a time $\tau(P, r)$ such that for all $t > \tau(P, r)$, the wave front $W_t(P)$ intersects U in a non-empty set.*

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3.4. Since the $(d-1)$-dimensional volume of the co-dimension-one wave front W_t is expected to grow and the manifold M is finite, we expect it to fill up M more and more. The term "generic" here is used in an informal way, but it could be made precise using the notion of **Baire generic**. The set of Riemannian manifolds of a fixed dimension naturally can be made into a metric space by Nash embedding them into a large Euclidean space E. A metric on Riemannian manifolds can then be defined by taking the smallest possible Hausdorff distance between any embeddings M_1, M_2 in E. Alternatively one could fix a topological manifold M and define a distance between Riemannian metrics on M, using this topology on metrics to define Baire genericity.

Conjecture 1. *For a generic $d \geq 2$-dimensional Riemannian manifold with or without boundary, the wave front $W_t(P)$ becomes dense for all points $P \in M$.*

3.5. For Wiederkehr manifolds M like the round sphere (or in a generalized sense the projective space), the wave front $W_t(P)$ is either a point or a circle for all $t \geq 0$. The story of Wiederkehr manifolds is related to the story of manifolds on which all geodesics are closed [8]. Any round sphere or projective space of constant curvature in any dimension has this property. But this is an exceptional case as already Blaschke realized in 1921 [11]. It is natural to conjecture that for a generic 2-manifold M with or without boundary, every wave front $W_t(P)$ **becomes dense** in the sense of Definition 1.

4. Surfaces

4.1. We confirm here that one can have dense wave fronts even in very simple situations, despite the fact that having a dense wave front displays some sort of mixing property. This can happen even in the case when the geodesic flow is integrable. We look first at the **flat 2-torus** $M = \mathbb{R}^2/\mathbb{Z}^2$. It is a 2-dimensional compact Riemannian manifold with Euclidean flat metric, induced from the plane.

Theorem 1. *On a flat 2-torus $M = \mathbb{R}^2/\mathbb{Z}^2$, all wave fronts $W_t(P)$ become dense.*

4.2. Our proof actually shows that for any arc $A = \{(\cos(t), \sin(t)), t \in [\alpha, \beta]$ of positive length, the wave front $W_t(P, A) = \exp_P(tA)$ becomes dense. We can think of the exponential map restricted to a sector as building a **light cone** of a **headlight**. If $\alpha = \beta$, we have a **laser**, a wave front going in a single direction. In other words this is then a single geodesic ray. While for irrational $\alpha/(2\pi)$ such a **laser beam** $\{W_s(p), s \in [0, t]\}$ becomes dense, the **wave front of this laser** is just the point $W_t(p)$ for all t.

Proof. We look at the wave front on the universal cover $\mathbb{R}^2$ of $\mathbb{T}^2$, where it is a round circle of radius t. We can assume without loss of generality that the point P is at the origin. The wave front can be seen as a union of two half circles which both are graphs. The first is the graph of $f(x) = \sqrt{t^2 - x^2}$. The second is the graph of $g(x) = -\sqrt{t^2 - x^2}$. We forget about the and even on the full graph of f, we concentrate only on a small rectangle R of the circle which is chosen to be so wide that its height is 1. For given $t > 0$, define the region $R = [a, b] \times [f(a), f(b)]$ with $[a, b] = [-2 \cdot \sqrt{t}, -\sqrt{2t}]$. The graph of $W_t(p)$ intersected with R crosses R monotonically from $(a, f(a))$ to $(b, f(b))$. Since the graph of f is concave down, the slope of f is in the interval $I = [\sqrt{2}, 2]/\sqrt{1 - t/4}$ which is for $t > 36/5$ smaller than $3/\sqrt{t}$. We also check using a Taylor expansion of f that that $f(b) - f(a) = 1 + O(1/\sqrt{t})$. This implies that $|f(t+1) - f(t)| \leq 3/\sqrt{t}$ in that interval I. If R is projected to the fundamental domain $[0, 1] \times [0, 1]$, it crisscrosses that

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square and is $3/\sqrt{t}$ dense. We have shown that even a tiny part $W_t(P) \cap R$ of the entire wave front has become $3/\sqrt{t}$ dense. □
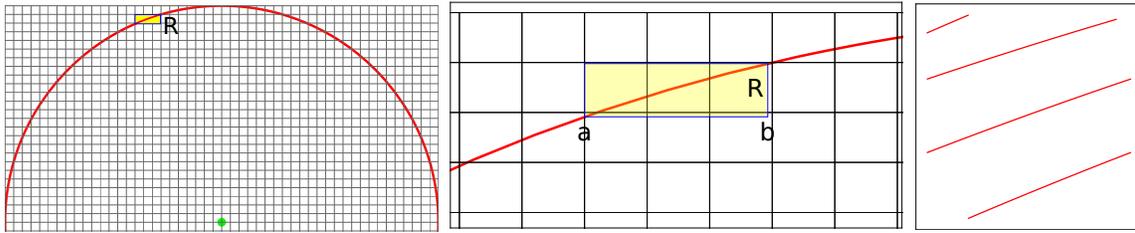

FIGURE 3. To prove the result, we project a small part $R = [a,b] \times [f(a), f(b)]$ to the torus. This restriction to a smaller interval at first seems to make the problem harder. But it allows to isolate part $W_t \cap R$ of the wave front W_t for which W_t is a graph that is asymptotically linear for $t \to \infty$. It is then easier to see that $(W_t \cap R)/\mathbb{Z}^2$ becomes dense.

4.3. The flat torus $S_1 \times S_1$ be made more asymmetric by using $S_r \times S_s$ where S_r, S_s are circles of radius r or s. This could be realized by a conformal distortion of the Riemannian metric. It still is flat. We just need to scale the region and scale the maps accordingly in the proof to see that also on such asymmetric tori, the wave fronts become dense. A similar argument works for the **flat Klein bottle** which has a flat 2-torus as a 2:1 cover. Both the flat Klein bottle as well as the flat 2-torus (called Clifford torus) could be embedded isometrically in $\mathbb{R}^4$ but not in $\mathbb{R}^3$.

Theorem 2. *On a flat Klein bottle M, all wave fronts $W_t(P)$ become dense.*

4.4. We first investigated the torus problem using results from the **Gauss-circle problem**, which tells us how many lattice points we expect in the annulus $A_{t,t+h}$ between W_{t+h} and W_t in the universal cover. Already, Gauss estimated the number of lattice points in this annulus to be close to its area $2\pi th$ and estimated an error $|E(t)| \leq \sqrt{2} \cdot 2\pi t$. It is known, for example, that $E(t) \leq Ct^{2/3}$. [1] This means that the annulus $A_{t,t+h}$ has about $2\pi th + Ct^{2/3}h$ lattice points. For $h = t^{-1/2}$, this area grows like $2\pi\sqrt{t}$ with an error $O(t^{1/6})$. As we expect so many lattice points in $A_{t,h}$ and every of these lattice points is $h = 1/\sqrt{t}$ close to the wave front W_t we see that W_t gets $1/\sqrt{t}$ close to the original point P. Changing the annulus to $A_{t+h,t+2h}$ shows that we have points in distance $[1/\sqrt{t}, 2/\sqrt{t}]$ from P. The **geometry of numbers** approach could even shed better light on **how fast** the wave fronts fill out the torus. The proof using the Gauss-circle problem in higher dimensions would need heavier mathematical machinery like **Ehrhard theory**.

4.5. The simple constructive proof given above was found only later in the summer of 2024. It illustrates that sometimes, it is easier to prove a stronger result: the proof shows that already a small part of the wave front becomes dense. The reader might enjoy to modify the simple single variable calculus argument to show that also on a stretched torus $\mathbb{R}/(\alpha\mathbb{Z}) \times \mathbb{R}/(\beta\mathbb{Z})$, the wave front becomes dense. The argument can also be adapted to wave fronts on higher dimensional flat tori $\mathbb{R}^d/\mathbb{T}^d$. Again, we just have to restrict to a small cuboid domain R, where the wave front can be described in the cover as a graph of a function $f(x_1, \ldots, x_{d-1})$ of $d-1$

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[1]The big open problem in this context is whether $E(t) = O(t^{1/2+\epsilon})$ holds for every $\epsilon > 0$.

variables. There is no doubt that in the case of a non-flat metric, the density of the wave front persists. However this has not been proven; we do not know the shape of the wave front in the universal cover. But it is likely as the wave front gets longer. In the case when the geodesic flow has positive Lyapunov exponents, the wave front length in the universal cover would grow exponentially and make a density result more likely. But we even don't know how to prove this. In the case when the geodesic flow is hyperbolic, like on a compact Riemannian manifold of strictly negative curvature, the density would follow.

Conjecture 2. *For any smooth Riemannian metric on $\mathbb{T}^d$ with $d > 1$, the wave front of any point P becomes dense.*
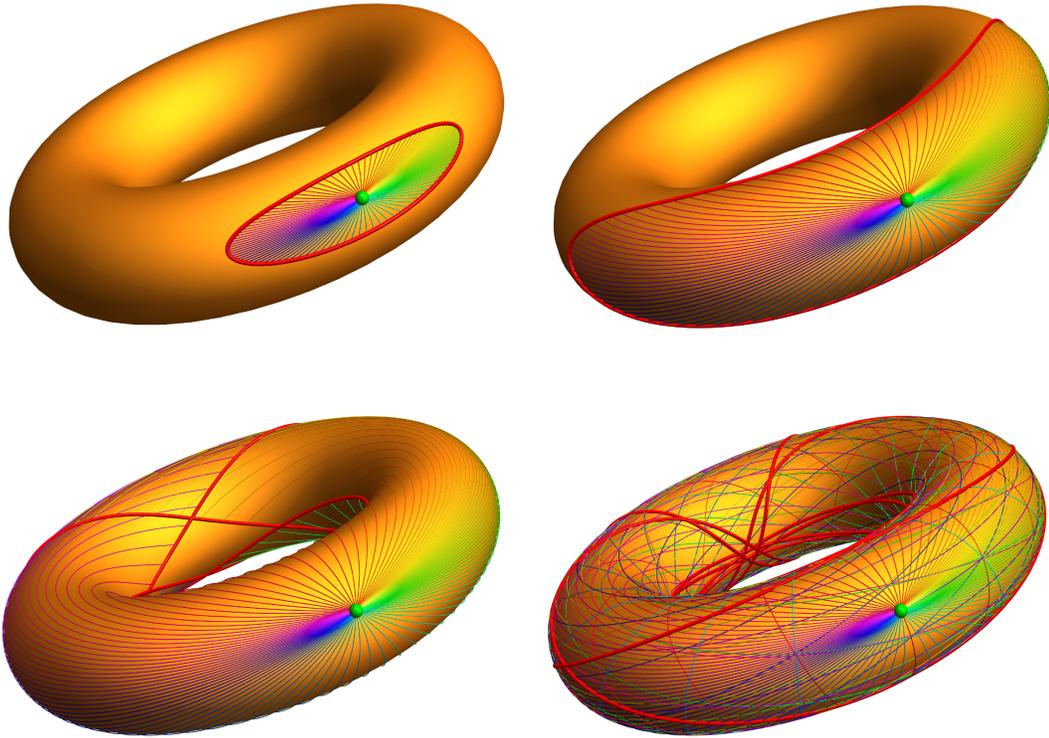

FIGURE 4. Wave fronts W_t on a flat torus. We prove that they become dense.
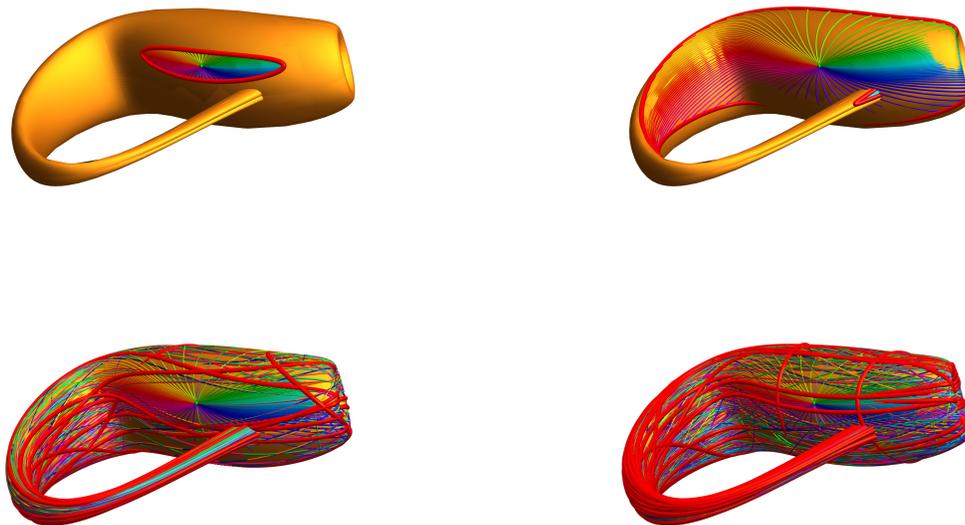
FIGURE 5. Wave fronts W_t on a flat Klein bottle. We prove that they become dense.

5. BILLIARDS

5.1. Billiards in convex tables were used by George Birkhoff [10] (in section 6 of his book) and his postdoc Hillel Poritsky [32] to illustrate principles and theorems in dynamical systems theory, especially in the context of the 3-body problem. Periodic points are critical points of a variational principle. The Birkhoff fixed point theorem even can establish the existence of many periodic points if the billiard map. For modern texts on billiards, see [37, 34]. A plethora of open problems persist: one does not know for example whether every triangular billiard has a periodic orbit. No smooth convex billiard exhibiting chaos (positive Lyapunov exponents on a set of positive measure in the phase space) is known. Candidates are tables like $x^4 + y^4 \leq 1$. Even one of the first problem of Poritski [32], asking whether every integrable smooth convex billiard is an ellipse, remains unsettled in its full form despite partial progress [9].

5.2. There are Wiederkehr situations also for billiards, but it appears to be more restricted. If we start a wave front at the center P of a circular billiard table, then the wave front will periodically return to the point P. But already for the same circle, we expect the wave front $W_t(P)$ to become dense, if the starting point P is not at the center. We are not aware even whether the density of the wave front in the simple duck problem of the circle has been shown. We can not use the method used here for the torus. In general, we expect:

Conjecture 3. *For a generic convex billiard table, all billiard wave fronts $W_t(P)$ become dense.*

5.3. Also convincing is the following statement:

Conjecture 4. *For all convex billiard tables, there is a point P such that the wave front $W_t(P)$ becomes dense.*

5.4. Here is a problem that we feel could be settled with a simple argument, but which also escapes us so far:

Conjecture 5. *For all convex polygon billiards, all the wave fronts $W_t(P)$ become dense.*

5.5. While the **regular equilateral triangle**, the **regular hexagon** or the **regular octagon** can be looked at using a cover, we do not know what happens for a **regular pentagon**. It might even hold for non-convex case as long as the illumination problem is solved and the union of wave fronts covers the entire table up to a finite set of points.

5.6. As a consequence of the torus case, we can state the following result:

Corollary 1. *For the billiard in a square, the wave front of any point P becomes dense.*

Proof. By unwrapping the table we get to a universal cover $\mathbb{R}^2$ of $Q = [0,1] \times [0,1]$. The circle $S_t(P)$ centered at P gets mapped into $K_t(P) = \pi(S_t(P))$, where π is the map which first maps (x,y) to $(x,y) \bmod 2$, then applies the map $(x,y) = (q(x), q(y))$, where q is the tent map $q(x) = x$, if $x \le 1$ and $q(x) = 2 - x$ if $x \in [1,2]$. In short, the piecewise smooth exponential map
$$\pi(x,y) = (1 - |(x \bmod 2) - 1|, 1 - |(x \bmod 2) - 1|)$$
maps the circle $S_t(P)$ to $K_t(P)$. $\qquad\square$

5.7. We were unable to find references which even **ask** such wave density questions. But we suspect that it must have been on the mind of many researchers in the field, as some numerical simulations of wave front beams have been done in [6, 5, 7]. These animations show the beauty and complexity which can happen with wave fronts, even for simple situations like a circle or an ellipse. Related to wave fronts are caustics and the "Jacobi last theorem" type question of how many cusps such a caustic has. See [12].

5.8. Nils Berglund kindly informed us about [36, 18, 17], where the question of the **length of the wave fronts** is addressed. Vicente visualized the problem as "Une goutte d'eau dans un bol", ("a drop of water in a cup"). [36] showed that for the duck problem in the unit disk, the length $|W_t|$ satisfies asymptotically $|W_P(t)| = \arcsin(|P|)t + o(t)$, in agreement with $|W_O(t)| = O(1)$ for the origin O and that the longest wave fronts are produced at boundary of the pond. The length question is not so interesting on the flat torus or rectangle or cube, where $|W_P(t)| = 2\pi t$ for all initial points. One would naively expect at first $|W_P(t)| - |W_Q(t)| = o(t)$ to hold in general for a pair of points. The pond case shows that this is not true. It begs the question whether there is an other 2-manifold with boundary where the growth rate $|W_P(t)|/t$ depends on P.

5.9. One possible attack to the density of wave front problem is to focus on **periodic points** which exist for example in a Birkhoff billiard and which are very much also under investigation in regular polygons [16].
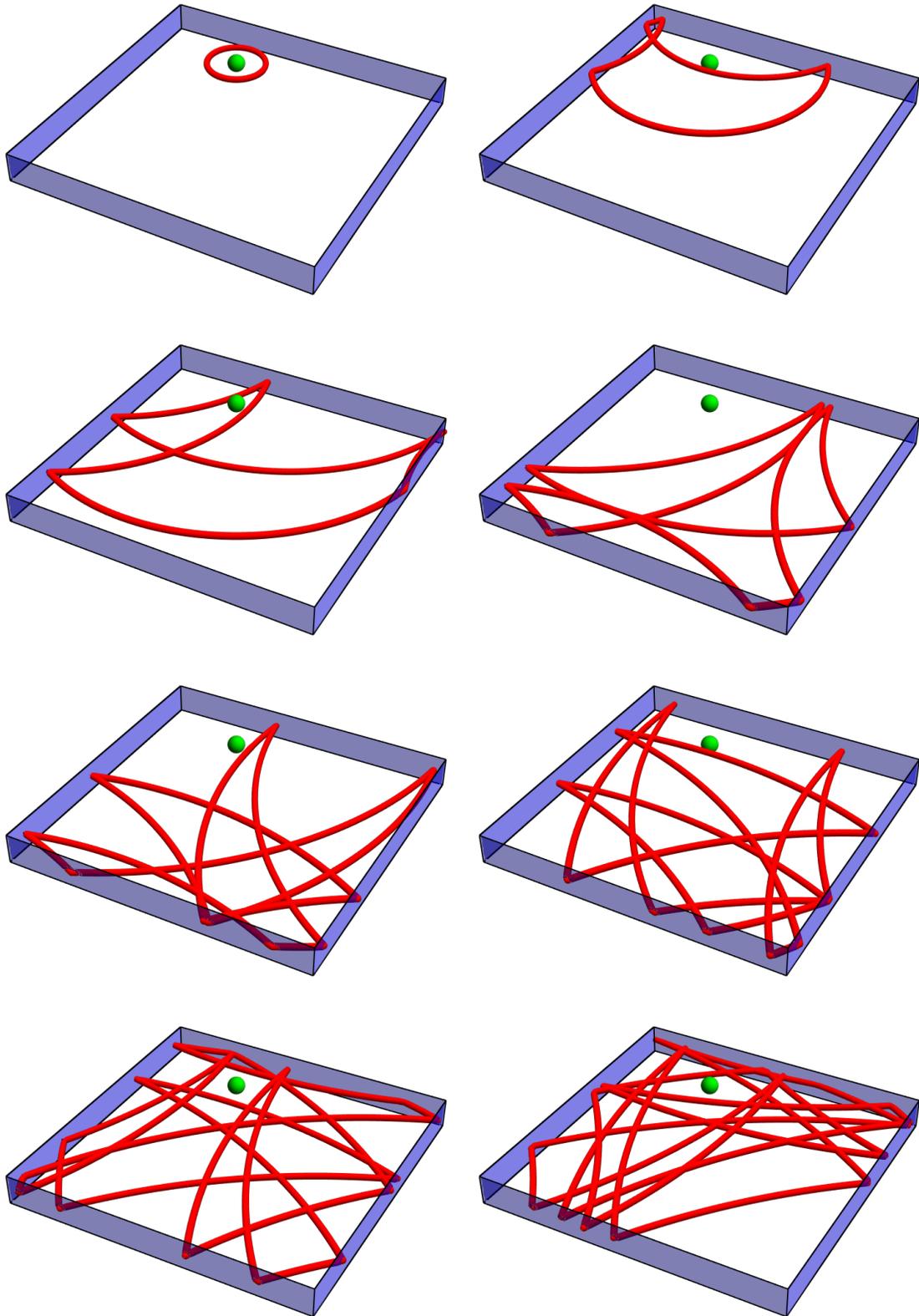
FIGURE 6. Wave fronts W_t in a square billiard. We prove that W_t becomes dense on the table.

6. Polyhedra

6.1. The starting point of this investigation was a numerical simulation of the wave front on the unit cube. The second author has made movies of wave fronts on the cube since 2011 for multi-variable courses to illustrate how quickly simple open questions can appear in mathematics. A wave front on the torus can be seen on [28]. The problem of finding the smallest connection between two points of a cube or to find the path of a billiard shot in a rectangle reaching from a point to one of the corners by reflecting a given number of times on walls can all be solved by unwrapping the situation. This produces nice puzzles in the context of distance questions in multi-variable calculus courses, like finding the shortest connection between two points on a cube or the length of a shortest billiard path in a rectangle that is required to bounce a pre-required number of times on each wall.

6.2. We were fascinated to see that in the cube case, the wave fronts became more and more complicated and that the wave fronts $W_t(P)$ even lost connectivity. Passing through a corner can cut the wave front because the geodesic flow was no longer continuous there. In the universal cover of the cube, once several lattice points were crossed, the wave front already had a chance of been hacked into pieces. Computer experiments indicated that the wave front becomes dense even if we start at a corner. There is no Wiederkehr phenomenon also there. The biggest surprise was maybe that the lack of smoothness of the exponential map could leads to situations on polyhedra, where wave fronts become disconnected and complicated already after a relatively short time.

6.3. The following result again follows from the corresponding result of the torus. The situation is tougher however. In the square case we dealt with a symmetry group that was the Klein group $G = (a, b, a^2 = b^2 = 1, ab = ba)$ with 4 elements, which is a commutative group. In the cube case, we deal with the **cube group** a finite subgroup of $SO(3)$ of order of 24. Every of the 6 faces of the cube comes in 4 possible rotated orders. If we draw a line $\gamma(t)$ connecting point P with a point Q in a square X, then depending on the crossing of the borders we attach a group element to this segment. There is so a rule which assigns to each point Q of the plane an orientation and a point on the surface of the cube. We can model the situation as a 4:1 cover of the open flat manifold in which the 6 corners have been taken away. Imagine that each of the 6 open squares comes in 4 copies. Now these 24 squares can be glued together away from the corners to produce a smooth flat 2 manifold. If we look at the evolution of the wave front $W_t \cap Q$ in a face and denote by W_t^k the by $k\pi/2$ turned front, then we can look at $W_t^0 \cup W_t^1 \cup W_t^2 \cup W_t^3$ over each square. This union is now the union of 4 wave fronts on the torus. There is now a partition of S_t in the universal cover into 4 sets S_t^k which become increasingly dense each such that $\exp_t(S_t^0)$ is the actual wave front $W_t(P)$ on the cube. By symmetry if one of the cases $\exp_t(S_t^k)$ becomes dense, then all 4 do. Since the union does become dense by the theorem on the torus, also $W_t(P)$ does. We have shown:

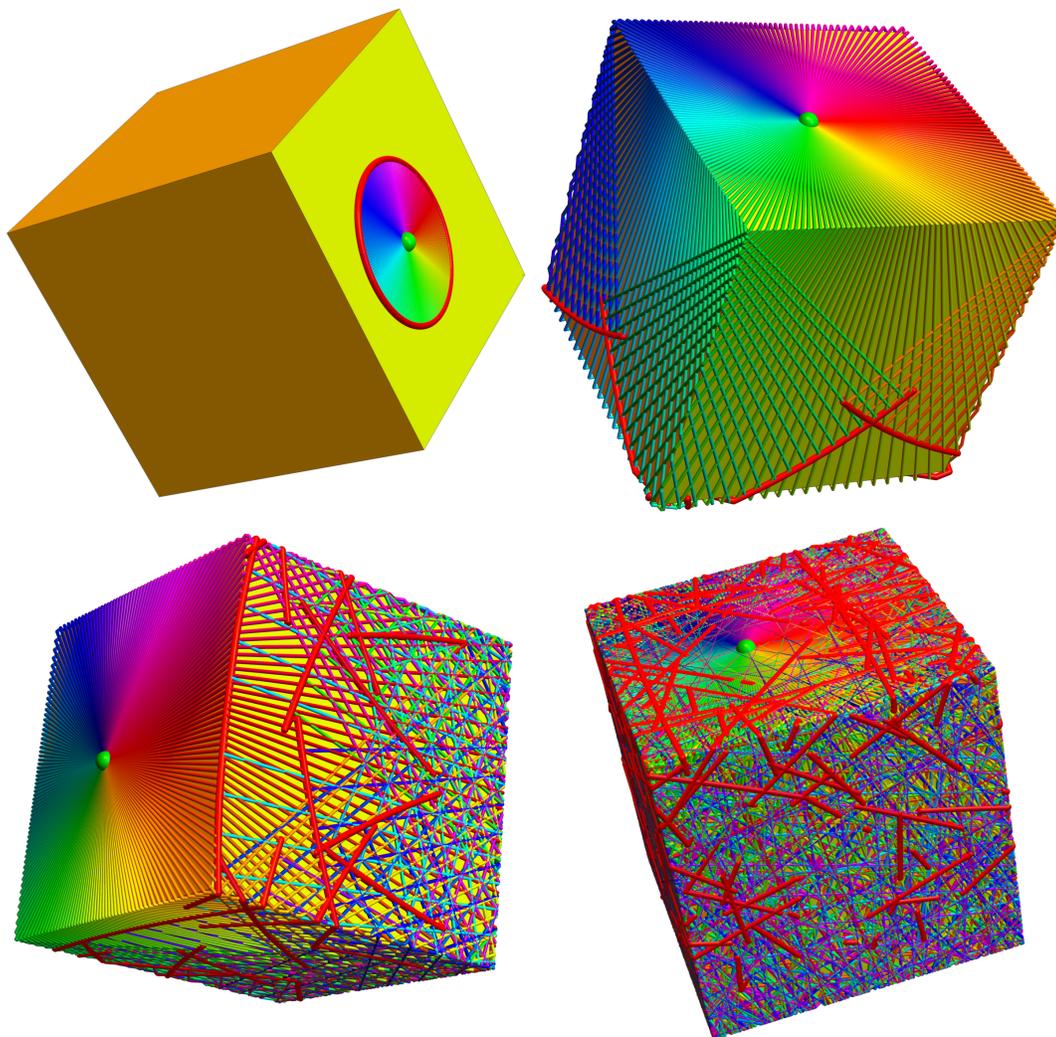**Corollary 2.** *On a cube, all wave fronts become dense.*

FIGURE 7. Wave fronts W_t on a cube. We prove that they become dense.

7. EXISTING LITERATURE

7.1. The study of **wave front propagation** in Riemannian geometry intersects with other areas of mathematics like geometric number theory, partial differential equations and dynamical system theory. The case of a torus also connects with the geometry of numbers or Ehrhart theory. These theories that shed light on the distribution and density of lattice points if the solid is scaled. As an example, we mention Huxley and Nowak's work on primitive lattice points in convex planar domains [23] that provides a detailed analysis of the distribution of these points within a compact convex subset of $\mathbb{R}^2$. There are intricate relationships between the geometry of the domain and the associated error terms. This study offers foundational insights that can be extended to understand how wave fronts might distribute over more complex surfaces. In the geometry of numbers [30], one explores the relationship between geometric structures and lattice point distributions within the context of dynamical systems, particularly focusing on how curvature and boundary shape influence the number of lattice points. Waxman and Yesha's recently looked at lattice points within thin sectors [39]. How does the number of

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integer lattice points within shrinking sectors of a circle grow? Such problems are situated at the intersection of analytic number theory and geometric analysis. The Diophantine properties of the slope of the sector's central line affect the distribution of lattice points. Such studies could help to understand wave front behavior in more constrained geometric settings such as rectangular billiards or on the surface of a cube, where the geometric structure imposes additional complexity.

7.2. The problem of wave fronts is also related to the **illumination problem**: what is the set of points Q such that the wave front $W_t(P)$ never reaches Q? This question was first asked the 1950s. In 1958, Roger Penrose constructed the **Penrose un-illuminable room**. In 1995, the first polygonal unilluminable room was found [35]. In a general polygon, this is not an empty set in general, but it is finite in general [40]. For recent work on translation surfaces see [31]. (Added: January 26th: The Penrose example already shows that there are regions for which entire parts of the room cannot be illuminated implying the wave front can not be dense. Thanks to Sergei Tabachnikov to point this out to us.) On the other hand, having a room that can be illuminated does not prove that fronts become dense. The duck in the center of a circular pond illustrates this.

7.3. An obvious relation of wave front dynamics is with partial differential equations, like wave dynamics. If we have a Riemannian manifold we can look at the wave dynamics on it. If a solution to this partial differential equation starts localized at a point, then the wave front is related to the wave front given by the geodesics. This is related to the **Huygens principle**. Wave dynamics is also closely linked to quantum mechanics because if L is the Laplacian on the manifold and $L = D^2$, then the wave equation $u_{tt} = -Lu$ is by d'Alembert equivalent to two Schrödinger type equations $u_t = -iDu$ and $u_t = +iDu$ for the square root D of the Laplacian. This motivates semi-classical analysis [14]. There are connections between high energy eigenfunctions of the Laplacian and the geodesic or billiard dynamics [24]. We are not aware of any relation between the wave front spread and the eigenfunctions of the Laplacian.

7.4. In Riemannian geometry, a manifold is called a **Wiedersehen manifold** if for every point P there is a **Wiedersehen point** Q such that all geodesics through P pass also through Q. Wilhelm Blaschke conjectured in 1921 [11], that every Wiedersehen manifold is a sphere. Only in 1963, Green proved that a 2-dimensional Wiedersehen manifold (Wiedersehenfläche) must be isometric to a sphere of constant curvature. See [26] Theorem 5.2.5. Note that for a projective plane of constant curvature, the point Q is the same than P so that it is technically not called a Wiedersehen manifold. A natural question is whether it is true that for any non-Wiedersehen manifold (different from a projective space situation), there exists a point P for which the wave front $W_t(P)$ becomes dense and that for any d-manifold with or without boundary that does not allow for a $SO(d-1)$ symmetry, the wave front becomes dense for all points P. An example of a $d = 2$-manifold with boundary is the unit ball which carries a $SO(1)$ rotational symmetry. An other example is an ellipsoid of revolution which again carries a $SO(1)$ rotational symmetry and for which there exist points P which have a Wiedersehen point Q.

7.5. We have already mentioned that in general, we expect wave fronts to become dense. What about specific situations in higher dimensions, like in the case of billiards in a regular polytop? In the case of a polytop that tesselates Euclidean space, we can prove the density by projection. How can one prove for a regular polytop that $W_t(P)$ becomes dense? A good place to start would be a solid regular dodecahedron. The proofs given here do not apply. Still, we feel that

there should be a simple argument proving that wave fronts are dense in **any convex polytop** of dimension 2 or larger.

7.6. Already in the flat case, billiards can show chaotic motion. The wave front length then can grow exponentially in time. We expect the wave front in such a case to be e^{-Ct} dense for some constant C. Can one prove this? A case to consider would be the Bunimovich [13] stadium, which is the boundary of the convex hull of the union of two circles of equal radius which do not have the same center. We would expect there that there is a $c > 0$ such that every ball of radius $B_{e^{-ct}}$ intersects $W_t(P)$ for large enough t. This does not follow from the strong mixing property of the billiard flow. Strong mixing means that for two sets U, V of positive area in the phase space (the area is normalized to 1 in order to have a probability space) that $\lim_{t\to\infty} |T_t(U) \cap V| = |U||V|$. In probability theory, one would rephrase mixing that the random variables $1_{T_t(U)}$ becomes uncorrelated to the random variable 1_V in the limit $t \to \infty$.

7.7. An other case, where we expect that a simple argument could show density of $W_t(P)$ with a rate e^{-Ct} are compact hyperbolic manifolds on which the geodesic flow has long been known to be ergodic [22]. In the hyperbolic case, by a theorem of Hadamard, the wave fronts do not develop singularities. There are metrics on the 2-sphere for which the flow is ergodic. This certainly should also indicate that the wave fronts become dense, but we are not aware of an argument to see this.

7.8. There would be other questions. In the cube case we have seen that the fronts W_t becomes disconnected pretty fast. The total length of the curve of course is $2\pi t$. What is the typical length of a connected component, where we only consider **connected components** $\exp_t(I)$ with I an interval in unit sphere $S \subset TM_P = \mathbb{R}^2$, such that $\exp_s(I)$ is a connected curve for all $s \le t$? These floating pieces get disconnected even more but also start to intersect with other such curves. Is it possible that the set W_t can become connected again for some large t?

METHOD

7.9. All illustrations and experiments were done with the computer algebra system Mathematica. Computer code for the illustrations is available online on the Wolfram blog [27].

DECLARATIONS

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