GAUSS-BONNET FOR MULTI-LINEAR VALUATIONS

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ABSTRACT. We prove Gauss-Bonnet and Poincaré-Hopf formulas for multi-linear valuations on finite simple graphs $G = (V, E)$ and answer affirmatively a conjecture of Grünbaum from 1970 by constructing higher order Dehn-Sommerville valuations which vanish for all d-graphs without boundary. A first example of a higher degree valuations was introduced by Wu in 1959. It is the Wu characteristic $\omega(G) = \sum_{x \cap y \neq \emptyset} \sigma(x)\sigma(y)$ with $\sigma(x) = (-1)^{\dim(x)}$ which sums over all ordered intersecting pairs of complete subgraphs of a finite simple graph G. It more generally defines an intersection number $\omega(A, B) = \sum_{x \cap y \neq \emptyset} \sigma(x)\sigma(y)$, where $x \subset A, y \subset B$ are the simplices in two subgraphs A, B of a given graph. The self intersection number $\omega(G)$ is a higher order Euler characteristic. The later is the linear valuation $\chi(G) = \sum_x \sigma(x)$ which sums over all complete subgraphs of G. We prove that all these characteristics share the multiplicative property of Euler characteristic: for any pair G, H of finite simple graphs, we have $\omega(G \times H) = \omega(G)\omega(H)$ so that all Wu characteristics like Euler characteristic are multiplicative on the Stanley-Reisner ring. The Wu characteristics are invariant under Barycentric refinements and are so combinatorial invariants in the terminology of Bott. By constructing a curvature $K : V \to R$ satisfying Gauss-Bonnet $\omega(G) = \sum_a K(a)$, where a runs over all vertices we prove $\omega(G) = \chi(G) - \chi(\delta(G))$ which holds for any d-graph G with boundary δG. There also prove higher order Poincaré-Hopf formulas: similarly as for Euler characteristic χ and a scalar function f, where the index $i_f(a) = 1 - \chi(S_f^-(a))$ with $S_f^-(a) = \{b \in S(a) \mid f(b) < f(a)\}$ satisfies $\sum_a i_f(a) = \chi(G)$, there is for every multi-linear Wu valuation X and function f an index $i_{X,f}(a)$ such that $\sum_{a \in V} i_{X,f}(a) = X(G)$. For d-graphs G and $X = \omega$ it agrees with the Euler curvature. For the vanishing multi-valuations which were conjectured to exist, like for the quadratic valuation $X(G) = \sum_{i,j} \chi(i)V_{ij}(G)\psi(j) = \langle \chi, V, \psi \rangle$ with $\chi = (1, -1, 1, -1, 1), \psi = (0, -2, 3, -4, 5)$ on 4-graphs, discrete 4 manifolds, where $V_{ij}(G)$ is the f-matrix counting the number of i-simplices in G intersecting with j-simplices in G, the curvature is constant zero. For general graphs and higher multi-linear Dehn-Sommerville relations, the Dehn-Sommerville curvature $K(v)$ at a vertex is a Dehn-Sommerville valuation on the unit sphere $S(v)$. We show $\chi \cdot V(G)\psi = v(G) \cdot \psi$ for any linear valuation ψ of a d-graph G with f-vector $v(G)$. This leads to multi-linear Dehn-Sommerville valuations which vanish on d-graphs.

1. INTRODUCTION

Given a finite simple graph G, a **valuation** is a real-valued map X on the set of subgraphs of G, so that $X(A \cup B) = X(A) + X(B) - X(A \cap B)$ holds for any two subgraphs A, B of G and $X(\emptyset) = 0$. Here $A \cup B = (V \cup W, E \cup F)$ and $A \cap B = (V \cap W, E \cap F)$, if $A = (V, E), B = (W, F)$ are finite simple graphs with

vertex sets V, W and edge sets E, F. With the empty graph $\emptyset$, a graph with no vertices and no edges, the set of all subgraphs is a lattice. If one requires additionally that $X(A) = X(B)$ holds for any two isomorphic subgraphs A, B of G, then X is called an **invariant valuation**. A **quadratic valuation** X is a map which attaches to a pair A, B of subgraphs of G a real number $X(A, B)$ such that for fixed A, the map $B \to X(A, B)$ and for fixed B the map $A \to X(A, B)$ are valuations and such that $X(A, B) = 0$ if $A \cap B = \emptyset$. It is an **intersection number** which extends to a bilinear form on the module of chains defined on the abstract Whitney simplicial complex of the graph G.

More generally, a k-**linear valuation** is a map which attaches to an ordered k-tuple of subgraphs $A_1, \dots, A_k$ of G a real intersection number which is "multi-linear" in the sense that any of the maps $A_j \to X(A_1, \dots, A_j, \dots A_k)$ is a valuation. We also assume they are localized in the sense that $X(A_1, \dots, A_k) \neq 0$ is only possible only if $\bigcap_{j=1}^{k} A_j \neq \emptyset$. For a k-linear valuation X one obtains the self intersection number $X(A) = X(A, A, \dots, A)$. We will see that some multi-linear valuations still honor the product property $X(A \times B) = X(A) \cdot X(B)$, where $A \times B$ is the product graph, the incidence graph of the product $f_A f_B$ of the representations of A, B in the Stanley-Reisner ring. When seen as integer-valued functions on this ring, valuations with this property are multiplicative functions.

Here are footnotes on the choice of the definitions.

(i) The terminology "linear valuation" and "multi-linear valuation" come from fact that the valuation property can be written as $X(A + B) = X(A) + X(B)$ using the symmetric difference $A + B = A \cup B \setminus A \cap B$. But addition throws us out of the category of graphs into the larger class of chains, which form a linear space. Valuations are in this larger ambient class compatible with the algebra. For graphs, we have to to stick to the lattice description when defining the valuation. An example: with the given definitions of union and intersection of graphs, the graphs $A = (V, E) = (\{1, 2\}, \{(1, 2)\})$ and $B = (W, F) = (\{2, 3\}, \{(2, 3)\})$ define $A \cup B = (\{1, 2, 3\}, \{(1, 2), (1, 3)\})$ and $A \cap B = \{\{2\}, \emptyset\})$. The valuation property is satisfied, but with symmetric differences on vertices and edges we get $(V + W, E + F) = \{\{1, 3\}, \{(1, 2), (2, 3)\}\}$ which is only a chain and no more a graph. The absence of a linear algebra structure on the category of graphs is a reason for the complexity for computing valuations (it is an NP complete problem) and the reason to stick with the **Boolean distributive lattice** of subgraphs using union and intersection rather than symmetric difference and intersection which is not defined in the category of graphs. The usual graph complement is not compatible. Also in the continuum, valuations use the lattice structure using that unions and intersections of convex sets rather than allowing complements of convex sets which would allow to build a **measure theory for valuations**. In the discrete, the difficulty is transparent: for a valuation, the complement of the graph $\delta G = a + b$ within $G = a + b + ab$ is only a chain ab which we call the virtual interior. It has the Euler characteristic -1, which is the Wu characteristic of K_2.

(ii) Without the assumption $X(\emptyset) = 0$, we would also count constant functions $X(A) = c$ as valuation, but it is tradition not to include them. It would render the valuation **"affine"** rather than "linear" because linearity requires $X(\emptyset) = 0$ for the empty graph $\emptyset$.

(iii) The invariance property is also a common assumption, both in the continuum and in the discrete. In the continuum case, where valuations are mainly studied on unions of **convex sets** in Euclidean space, the invariance assumption is that the valuation is invariant under Euclidean motion. The reason for assuming this is that the theory is already rich enough with that assumption and that it also also in the continuum leads to a finite dimensional space of valuations. Flat Euclidean space seems confining. This is however not the case as one can for the purpose of Valuations, embed Riemannian manifolds into Euclidean or projective spaces. This induces a

valuation theory on Riemannian manifolds and beyond. More about this in the Hadwiger appendix.

(iv) The assumption $X(A_1, \ldots A_k) = 0$ if $\bigcup_j A = \emptyset$ was done so that the theorems work. Without it, much would fail. The assumption excludes cases like $X(A, B) = Y(A)Z(B)$ for two linear valuations Y, Z. The assumption will imply that the curvatures and indices are localized as is the case in differential geometry. One could look at cubic valuations $X(A, B, C)$ for which X is only zero if the nerve graph defined by A, B, C is not connected. Widening the localization as such would make the curvature functions $K_X(v)$ depend on a disk of radius k, if X is a k-linear valuation. With the assumption, the curvature functions always depend on a disk of radius 2. The assumption of insisting all simplices to intersect simultaneously (as Wu did) implies that cubic and higher order characteristics agree with Euler characteristic in geometric settings. Otherwise, the boundary formula fails and the product property on all graphs fails. We would also expect that getting rid of the localization assumption would for larger k lead to additional "connection terms" due to chains which are not boundaries.
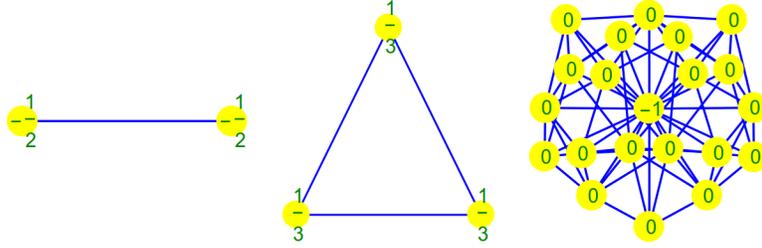

FIGURE 1. We see two graphs $A = K_2, B = K_3$ and its product $G = A \times B$ which has all pairs of simplices $x \subset A, y \subset B$ as vertices and two vertices connected if one is contained in the other. The vertices are labeled by the Wu curvature $K(v)$ which add up to the Wu characteristic ω. The picture illustrates some results of this paper: **1.** There is a curvature $K(v)$ such that its sum is the Wu characteristic ω. **2.** $\omega(G) = \omega(A) \times \omega(B)$. **3.** For a d-graphs G with boundary, $\omega(G) = \chi(G) - \chi(\delta G)$. In this case, G is the product of an interval A with a disc B and so a 3-ball G with a boundary δG which is a 2-sphere. As $\chi(G) = 1, \chi(\delta G) = 2$, we have the boundary formula $\omega(G) = \chi(G) - \chi(\delta G) = -1$ which is true for any $(2k + 1)$-ball.

An example of a non-invariant valuation is the map $A \to X_a(A) = \deg_A(a)$ which assigns to a sub graph A of G the vertex degree of a fixed vertex within A. When summing this valuations $X_a(A)$ over all vertices a, we get by the Euler handshake formula the invariant valuation $\sum_{a \in V} X_a(A) = 2v_1(A)$, counting twice the number of edges in A. The Euler handshake is already a Gauss-Bonnet formula adding up local quantities to get a global quantity. We see it now as adding up local valuations to a global valuation. The prototype of an invariant valuation is the **Euler characteristic** of a graph G. It is

$$\chi(A) = \sum_x (-1)^{\dim(x)} = v_0(A) - v_1(A) + v_2(A) - \ldots ,$$

where x runs over all complete subgraphs of A of non-negative dimension. It is an alternating sum of the components $v_i(A)$ of the f-vector of A. Each $v_i(A)$ is of course an invariant valuation. By Gauss-Bonnet, the Euler characteristic can be written as a sum of generalized local valuations or curvatures $A \to K_A(a) =$

$\sum_{k=0}(-1)^k V_{k-1}(a)/(k+1)$, where $V_k(a)$ counts the number of k-simplices in the unit sphere of A centered at a including $V_{-1}(a) = 1$ counting the empty graph as a -1 dimensional simplex. The map $A \to K_A(a)$ is the **curvature** of A at a. It is generalized as it assigns to the empty graph the constant 1. Any invariant valuation can be written as a sum of such local valuations using the curvature $\sum_{k=0} X(k)V_{k-1}(a)/(k+1)$. This is Gauss-Bonnet for valuations. We rediscovered it in [83] in the case of Euler characteristic but the formula has been discovered and rediscovered before [109, 70, 50]. It appears that we have entered new ground when extending Gauss-Bonnet to general linear valuations and especially to multi-linear valuations. It is important to note that Gauss-Bonnet holds for all finite simple graphs and any multi-linear valuation and no geometric assumptions whatsoever like being a discretization of a manifold are needed. The theorem holds for all networks.

An example of a quadratic valuation is the **Wu characteristic** [149] given by

$$\omega(A) = \sum_{x \cap y \neq \emptyset} (-1)^{\dim(x)+\dim(y)} \ ,$$

where (x, y) runs over all ordered pairs of complete subgraphs of A. It can be written as a quadratic form for the f-**quadratic form** $V_{ij}(A)$ or simply f-**matrix** counting the number of pairs of i-simplices intersecting with j-simplices in A:

$$\omega(A) = \sum_{i,j}(-1)^{i+j}V_{ij}(A) \ .$$

It more generally produces an **intersection number**

$$\omega(A, B) = \sum_{x \cap y \neq \emptyset, x \subset A, y \subset B} (-1)^{\dim(x)+\dim(y)} = \sum_{i,j}(-1)^{i+j}V_{ij}(A, B) \ ,$$

where x runs over all complete subgraphs of the subgraph A and y runs over all complete subgraphs of the subgraph B. In the second equivalent formula for $X(A, B)$, the quadratic form $V_{ij}(A, B)$ counts the number of i-simplices in A and j-simplices in B with $x \cap y \neq \emptyset$. Let $A = \{a\}$ be a one point graph for example, then $B \to \omega(A, B) = \sum_{y \subset B, a \in y}(-1)^{\dim(y)}$ is a linear valuation, but not an invariant valuation as it is local giving nonzero values only near the vertex a.

While the Euler characteristic $\chi = \omega_1$ defines a linear map on the f-**vector** $v = (v_0, v_2, \dots)$ with v_k counting the number of complete subgraphs K_{k+1} of G, the Wu characteristic $\omega = \omega_2$ evaluates a **quadratic f-form** V, where V_{kl} counts the number of complete subgraphs K_{k+1}, K_{l+1} of G which intersect in a non-empty graph. If we write $\chi_1 = (1, -1, 1, -1, \dots, (-1)^d)$ then

$$\chi(G) = \omega_1(G) = \chi_1^T \cdot v, \qquad \omega(G) = \omega_2(G) = \chi_1^T \cdot V\chi_1$$

and $\omega(A, B) = \chi_1^T V(A, B)\chi_1$, where $V_{ij}(A, B)$ counts the number of intersections of i-simplices in A with j-simplices in B. The quadratic form V is not necessarily positive definite but always has a positive maximal eigenvalue by Perron-Frobenius. One can also look at cubic situations like

$$\omega_3(G) = \sum_{x \cap y \cap z \neq \emptyset} (-1)^{\dim(x)+\dim(y)+\dim(z)} \ ,$$

where the behavior on complete graphs is $\omega_3(K_d) = 1$ again for all d and $\omega(G) = \chi(G)$ holds for all d-graphs even if G has a boundary. It can be written using a **cubic f-form** $V_{ijk}(G)$ counting the number of i, j or k-simplices in G which simultaneously intersect: with $\chi_1 = (1, -1, 1, -1, \dots)$,

$$\omega_3(G) = V(G)\chi_1\chi_1\chi_1 = \sum_{ijk} V_{ijk}(G)\chi_1(i)\chi_1(j)\chi_1(k) .$$

The higher order Wu characteristics ω_k are defined similarly using k-linear f-forms $V(G)$ or **f-tensors**. We have $\omega_3(K_d) = \chi(K_d) = 1$ and $\omega_4(K_d) = \omega_2(K_d) = (-1)^d$. All these characteristics turn out to be invariant under Barycentric refinement as well as have the multiplicative property and agree up to a sign with Euler characteristic χ for d-graphs, graphs for which every unit sphere is a $(d-1)$-sphere.

Euler proved $\chi(G) = v - e + f = 2$ with $v = v_0, e = v_1, f = v_2$ for planar graphs, where f counts not only triangles but any region defined by an embedding of G into a 2-sphere. In our terminology, graphs like the dodecahedron or cube graph are 1-dimensional, as they have no triangles. Among the five Platonic solids, only the octahedron and icosahedron are as 2-spheres, the tetrahedron is a 3-dimensional simplex, has Euler characteristic 1 and is contractible. We can reformulate Euler's result then that for a discrete 2-sphere G, the Euler characteristic is 2. We know that χ is a homotopy invariant, that it is the first Dehn-Sommerville invariant [81] and also that it is a Barycentric characteristic number to the eigenvalue 1, obtained by looking at the linear map A on f-vector v of the graph and taking an eigenvector a of A^T and defining $X(G) = a \cdot v$. As χ is the number of even-dimensional simplices minus the number of odd-dimensional simplices, it is the super trace of the identity operator on the $\sum_i v_i(G)$-dimensional Hilbert space of discrete differential forms. It is robust under heat deformation: if d is the exterior derivative, the discrete McKean-Singer formula [113, 88] tells that $\chi(G) = \text{str}(\exp(-tL))$, where $L = (d + d^*)^2$ is the Hodge Laplacian on discrete differential forms and $\text{str}(L) = \sum (-1)^k \text{tr}(L_k)$ if L_k is the part of L acting on k-forms.

By the way, the **total dimension** $X(G) = \sum_i v_i(G)$ of all discrete differential forms is an example of an invariant valuation. It is believed that for d-graphs it is bounded below by $3^d - 1$ by a conjecture of Kalai of 1989 [143] (formulated for polytopes) as one can learn in [14]. This number is also interesting as it is hard to compute for general graphs, that it is the number of vertices of the Barycentric refinement of G or the value $f_G(1, 1, \dots, 1)$ if f_G is the Stanley-Reisner polynomial of G. The smallest d-sphere is believed to be the sharp lower bound like 8 attained for the C_4 graph, 26 attained for the octahedron, 80 for the 3-cross polytop or 242 for the 4-cross polytop; the reason is that $1 + \sum_k v_k x^{k+1} = (1 + 2x)^{d+1}$ for the smallest d-sphere. As the Kalai valuation X has the curvature $K(x) = \sum_{k=0}^{d+1} V_{k-1}(x)/(k+1)$ which is always positive, one could try to use Gauss-Bonnet and induction, as $\sum_{k=0}^{\infty} V_k(x) \geq 3^{d-1}$ implying $\sum_k V_{k-1}(x)/(k+1) \geq 3^{d-1}/(d+1)$. But since the number of vertices of G is only $\geq 2d$ we have $X \geq 3^{d-1}(2d/(d+1)) = 3^d(2/3)(d/(d+1))$. If one could verify the conjecture for all graphs G with $\leq 3d + 3$ vertices, then the induction assumption would give $X \geq (3d + 3)3^{d-1}/(d+1) = 3^d$. The Kalai conjecture is already by definition verifiable in finitely many cases for each d (in principle) but

the just given argument reduces the verification range from 3^d to $3d + d$. The argument still gives inductively $X(G) \geq a^d$ for any $a < 2$ so that $X(G) \geq a^d$ holds for any graph with clique number $d + 1$. But since for $G = K_{d+1}$ we have $X(G) = 2^d$ and $X(H) < X(G)$ if H is a subgraph of G, $X(G) \geq 2^d - 1$ holds already trivially. We believe it should be possible to use Dehn-Sommerville relations to estimate the Kalai curvature better on this space. But this requires to understand quantitatively the projection P onto the Dehn-Sommerville space and then the vector $(1 - P)(1, 1/2, 1/3, \ldots, 1/d)$.

Due to its multi-linear nature, one can not expect much from the Wu characteristic ω at first. Actually, it starts with bad news: ω is not a homotopy invariant, as $\omega(K_1) = 1$ but $\omega(K_2) = -1$, despite that K_2 and K_1 are homotopic. Interestingly, the Wu characteristic ω picks up the dimension of a simplex and $\omega(K_{d+1}) = (-1)^d$ so that one can write more elegantly

$$\omega(G) = \sum_{x \cap y \neq \emptyset} \omega(x)\omega(y) \; .$$

Note that $\omega(x, y) \neq \omega(x)\omega(y)$ as the case $x = y = K_2$ shows where $\omega(x, x) = \omega(x) = -1$, but $\omega(x \times y) = \omega(x)\omega(y)$. As ω is invariant under Barycentric refinement, it has been called **combinatorial invariant** (see section 1.2 of [20]), where polynomial functionals $R_G(z) = \sum_s (-1)^{|s|} z^{b_d(G_s)}$ and $S_G(z) = \sum_s i^{|s| + b_{d-1}(G)} z^{b_d(G_s)}$ were defined, which sum over all possible subsets s of V and where b_k is the k'th Betti number of G_s the graph generated by the set $V \setminus s$. Like Euler characteristics, also the Wu characteristic ω is a functional X on the class of simplicial sub-complexes of G but ω is not a valuation: the kite graph G has $\omega(G) = 1$ but two triangular subgraphs A, B with $\omega(A) = \omega(B) = 1$ but $\omega(A \cap B) = -1$ so that $\omega(A \cup B) = 1$ but and $\omega(A) + \omega(B) - \omega(A \cap B) = 3$. The Wu characteristic is a quadratic valuation as defined above. Just having mentioned polynomial invariants, one could combine all Wu characteristics and define the **Wu function**

$$\omega_G(z) = \sum_k \omega_k(G) z^k \; ,$$

where $\chi = \omega_1$ is the Euler characteristic, $\omega = \omega_2$ the Wu characteristic, ω_3 the cubic Wu characteristic etc. For a graph K_{d+1} or d-ball, we have $\omega(z) = z/(1 - (-1)^d z)$, for a d-graph G, the Wu function is $\omega(z) = \chi(G)z/(1 - z)$. The figure 8 graph G with $\chi(G) = \omega_1(G) = -1$, $\omega(G) = \omega_2(G) = 7$, $\omega_3(G) = 25$, $\omega_4(G) = 79$ is already a case where we don't yet have a closed form for $\omega(z)$.

We will see that any of the ω_k can be computed fast for most graphs as it satisfies a Poincaré-Hopf formula and $\omega(G) = \chi(G) - \chi(\delta(G))$ for d-graphs with boundary. Also $\omega_3(G) = \chi(G)$ etc. What happens is that the index entering in Poincaré-Hopf is a valuation on the unit sphere, allowing to apply Poincaré-Hopf again there etc. So, for most graphs in the Erdös-Renyi space of all graphs with n elements, also the Wu characteristic can be computed quickly. The complexity for computing ω is polynomial in n, if one insists for example that for any finite intersection of unit spheres, maximally 99% of all possible connections between vertices in the unit sphere are present. This especially applies for d-graphs, where any finite intersection of unit spheres is a k-sphere. A consequence of the boundary formula is that for

2-spheres, we still have

$$\omega(G) = v - e + f \ ,$$

where $v = v_0, e = v_1, f = v_2$. The proof of the reduction to Euler characteristic makes use of the Gauss-Bonnet result for multi-linear valuations which in particular holds for the Wu characteristics. It turns out that in the interior of a d-graph, in distance 2 or larger from the boundary, the curvature of the Wu characteristic ω is the same than the curvature of the Euler characteristic χ. For odd dimensional graphs, the curvature of ω lives near the boundary of G, similarly than the curvature for Euler characteristic. For an even dimensional graph with boundary, the curvatures for Euler characteristic or Wu characteristics are in the interior. The proof of the boundary formula $\omega(G) = \chi(G) - \chi(\delta(G))$ reveals what needs to be satisfied: for each unit sphere, we have to be able to use induction. The formula shows that ω measures the Euler characteristic of a "virtual interior" of a graph. This interior is no more a graph but a chain as forming complements throws us out of the category of graphs into a larger category of chains: take the two boundary points of K_2 away, then we end up with an edge which has no vertices attached and which is not a graph any more. It is the virtual interior of K_2 and has Euler characteristic -1 which is the Wu characteristic of K_2. Because of invariance under Barycentric subdivision, the Wu characteristic is defined also in the continuum limit for compact manifolds M with boundary δM, where it satisfies the same boundary formula $\omega(M) = \chi(M) - \chi(\delta M))$. While expressible by Euler characteristic for discrete manifolds, the Wu invariant becomes interesting for **varieties** as we see already in simple examples like the lemniscate

$$(x^2 + y^2)^2 = (x^2 - y^2)$$

which has the Wu characteristic 7 and Euler characteristic -1. Indeed, we can compute the curvature of the Wu-invariant at a singularity of a projective variety by discretization using a graph. Quadratic valuations are useful too as they produce intersection numbers $\omega(A, B)$ for any pairs of subgraphs and so an intersection number of a pair of varieties. Two 1-dimensional circular graphs A, B for example crossing twice have intersection number 2.

We restrict here to the language of graphs. This means that we look only at abstract simplicial complexes which are Whitney complexes of some graph. Like with topologies, measure structures or other constructs placed on a set, one could consider different simplicial complexes on the same graph. The 1-dimensional skeleton-complex consisting of vertices and edges is an example. It is too small for geometry as it treats any graph as a "curve". The neighborhood complex is an other example, but it renders the dimension too large; for a wheel graph with boundary C_n for example, the neighborhood complex would be n-dimensional.. The sweet spot is the Whitney complex defined by all complete subgraphs of G. For the wheel graph W_n for example, it renders it a discrete disc of dimension 2, with Euler characteristic 1, which is contractible and has a 1-dimensional boundary C_n. The geometry, cohomology, homotopy, even spectral theory all behave similarly as in the continuum. The language of graphs is intuitive and not much of generality is lost: while not every abstract finite simplicial complex on G is a Whitney complex of a finite graph - the simplicial complex with algebraic representation $f_G = x + y + z + xy + yz + xz$ is the smallest example which fails to be a Whitney

complex - the Barycentric refinement of an arbitrary finite abstract simplicial complex is always the Whitney complex of a finite simple graph. In the above triangle without 2-simplex, the refinement of the complex is the Whitney complex of C_6. Valuations on graphs satisfy a Gauss-Bonnet formula. For the valuation $v_k(G)$, this is the fundamental theorem of graph theory, a name sometimes applied only for the Euler handshaking lemma which is Gauss-Bonnet for $v_1(G)$. We will see that Gauss-Bonnet generalizes to multi-linear valuations but curvature depend now on the ball of radius 2. The curvature remains local but in some sense has become a second order difference operator similarly as in the continuum, where the curvature tensor uses second derivatives. The language of graphs is equivalent but more intuitive especially when dealing with valuations as a classically, valuation are defined as functionals on simplicial sub complexes of a complex on V so that classically; it is important to realize that a valuation is not a map from subsets of V to $\mathbb{R}$ but a map from sets of subsets of V. It is much easier to work with real-valued maps from the set of subgraphs of G to R. Subgraphs have the intuitive feel of subsets but encode simplicial complexes. The use of valuations as a functional on the set of graphs is language which allows to work with valuations using the intuition we know from measures.

On a graph $G = (V, E)$, the discrete Hadwiger theorem [79] assures that the dimension of the space of valuations on G is the clique number $d + 1$ of G. A basis is given by the functionals $v_i(G)$, counting the number K_{i+1} subgraphs of G. The Euler characteristic $\chi(G) = \sum_k (-1)^k v_k(G)$ can be characterized as the only invariant valuation which stays the same when applying Barycentric subdivision and which assigns the value 1 to all simplices. The multi-linear Dehn-Sommerville invariants we are going to construct which assign the value 0 to d-graphs are not invariant under Barycentric refinements on the class of all graphs, but only vanish on d-graphs, a class, where the Barycentric invariance is clear. The valuation χ can also be characterized as the only invariant valuation which is a homotopy invariant and which assigns 1 to a K_1 subgraph. Similarly, an extension of the discrete Hadwiger result of Klein-Rota shows that the space of invariant quadratic valuations is $(d + 1)(d + 2)/2$-dimensional. A basis is given by the functionals $G \to V_{ij}(G)$ with $j \geq i$, where $V_{ij}(G)$ counts the number of pairs x, y of i-dimensional simplices x and j dimensional simplices y for which $x \cap y \neq \emptyset$. For one dimensional graphs, graphs without triangles for example, the space of linear valuations has dimension 2, and the space of quadratic valuations has dimension 3. Graphs with maximal dimension 1 have a vector space of linear valuations which is two dimensional. It has a basis v_0, v_1 counting the number of vertices and edges, a spanning set for the vector space of quadratic valuations is $V_{00} = v_0$ counting the number of vertices, V_{11} counting the number of pairs (x, y) of edges intersecting and V_{12} counting the number of pairs (x, y), where x is a vertex contains in an edge y. In this case, $V_{12} = 2v_1$ by Euler handshake but that is a relation between linear and quadratic valuations. In an appendix, we review the discrete Hadwiger theorem and prove the extension to quadratic valuations.

One can look at some valuations like Euler characteristic also from an algebraic point of view, as it is possible to write $\chi(G) = -f_G(-1, -1, \ldots, -1)$, where f_G is the element in the Stanley-Reisner ring representing the graph. The later is a

polynomial ring, where each vertex is represented by a variable and every monomial represents a complete subgraph. Each summand in the polynomial is a monomial. For the kite graph G for example obtained by gluing two triangles abc and bcd along an edge bc, we have $f_G(a, b, c, d) = a + b + ab + c + ac + bc + abc + d + ad + cd + acd$ and $\chi(G) = -f_G(-1, -1, -1, -1) = 1$. The Z-module of chains for G is the 11 dimensional set containing elements like $f = 3a - b + 2ab + 4c + ad + 2cd + 7abc - 3acd$ in which the monoids of f_G form the basis elements. Valuations extend naturally to this module. We would for example have $v_2 = -2 - 4 - 1$ and $v_3(f) = 7 - 3$ and $V_{13} = (3 - 1)7 + 3(-3)$. Quadratic valuations are then quadratic forms on this module. We will write the quadratic Wu characteristic algebraically as

$$\omega(G) = (f^2)_G(-1, \ldots, -1) - f_G(-1, \ldots, -1)^2 \,,$$

a representation which will together with Poincaré-Hopf imply that ω is multiplicative when taking Cartesian products of graphs. It reminds of **variance** $\mathrm{E}[X^2] - \mathrm{E}[X]^2$ in probability theory. Similarly,

$$\omega(A, B) = (f_A f_B)(-1, \ldots, -1) - f_A(-1 \ldots, -1) f_B(-1 \ldots, -1)$$

reminds of **covariance** $\mathrm{E}[XY] - \mathrm{E}[X]\mathrm{E}[Y]$ if the evaluation $f \to -f(-1, \ldots, -1)$ is identified with "expectation". In some sense therefore, if Euler characteristic $\chi(G)$ is a topological analog of expectation, the quadratic Wu characteristic $\omega(G)$ is a topological analogue of variance and the product formulas for two graphs an analogue of $\mathrm{E}[X^k Y^k] = \mathrm{E}[X^k]\mathrm{E}[Y^k]$ which all hold for independent random variables as X^k, Y^k are then uncorrelated. For general networks, there is no relation between Euler characteristic and Wu characteristic. It fits the analogy as for general random variables where no relation between expectation and variance exists in general. The fact that for discrete manifolds without boundary, the Euler and Wu characteristics agree, and for discrete manifolds with boundary, the Euler and cubic Wu characteristic agree, comes unexpectedly. We were certainly surprised when discovering this experimentally. Note however that in the entire Erdös-Renyi space of networks with n vertices, the geometric d-graphs form a very thin slice. (While both growth rates are not known, one can expect that the number of non-isomorphic d-graphs with n vertices is bounded above by $\exp(C\sqrt{n})$, an estimate coming from partitioning into connected components, while the number of non-isomorphic graphs with n vertices should have a lower bound of the form $\exp(Cn^2)$ for some positive C, an estimate expected to hold as most pairs of graphs are not isomorphic.)

The reason for the multiplicative property is partly algebraic but there is a topological twist required. This is the Poincaré-Hopf formula which allows to make the connection from algebra to geometry: lets look at Euler characteristic $\chi(G)$ which sums up $I(x) = (-1)^{\dim(x)}$ over all complete subgraphs x of the graph G. Since $I(x)$ is an integer-valued function on the vertex set of the Barycentric refinement graph G_1, one can ask whether it is the **Poincaré-Hopf index** $i_f(x)$ of a scalar function f on the vertex set of G_1. As Poincaré-Hopf tells $\sum_x i_f(x) = \chi(G_1)$ and $\sum_x I(x) = \chi(G)$, it would be nice if there existed a function f for which $i_f(x) = I(x)$. When investigating this experimentally we were surprised to see that this is indeed the case. The scalar function which enumerates the monomials of the Stanley-Reisner polynomial f_G defined by G, where the monomials are ordered according to dimension and lexicographic order, does the job. The proof reveals that the sub graph $S^-(x)$ of G generated

by $\{y \mid f(y) < f(x)\}$ is a $(k-1)$-sphere if the dimension of x is k. This implies $i_f(x) = 1 - \chi(S^-(x)) = 1 - (1 + (-1)^{k+1}) = (-1)^k = I(x)$. This computation works for any finite simple graph G and no geometric assumption on G is necessary. It shows the remarkable fact that the dimension signature $\sigma(x) = (-1)^{\dim(x)}$ on the Barycentric refinement of any finite simple graph is actually a Poincaré-Hopf index of a gradient vector field on the graph. There is no continuum analogue of that; only a shadow of this result can be seen in the proof of the classical Poincaré-Hopf theorem in differential topology [139], where one proofs the theorem first for a particular gradient vector field defined by a triangulation and then proves by a deformation argument that the Poincaré-Hopf sum does not depend on the field.

As the Euler characteristic is the only multiplicative **linear** invariant valuation on the set of graphs, we have to go beyond linear valuations to get more multiplicative invariant functionals of this type. Quadratic valuations are the next natural choice and the Wu characteristic is a natural quadratic valuation as it is invariant under Barycentric refinements, assigns the same value to isomorphic subgraphs and assigns the value 0 to the empty graph and the value 1 to the one point graph K_1. Similarly, the cubic Wu characteristic is a cubic valuation with this property. The Euler characteristic and the Wu characteristics (including cubic and higher order versions) more generally appear to be the only multiplicative invariant valuations among all multi-linear invariant valuations. Grünbaum objected to the claim of Wu that the Wu characteristics are the only combinatorial invariants and pointed out the existence of Dehn-Sommerville invariants. This is a valid objection but **Wu's hypothesis** could still hold when one looks at it as a functionals on **all graphs**. It is well known that the Euler characteristic is the only invariant valuation which assigns the value 1 to K_1 and is a combinatorial invariant [50, 109]. The Wu characteristic appears to be the only quadratic invariant valuation **on the class of all finite simple graphs** which is invariant under Barycentric refinement and assigns the value 1 to K_1. If that is true, we could also say that ω is the only quadratic invariant valuation which is multiplicative. However - and that is the point which Grünbaum made in the case of Euler characteristic - is that in geometric situations like what we call d-graphs, there are other valuations, the Dehn-Sommerville invariants, which are on this class also invariant under Barycentric subdivision because they are zero, even so the values of refinements explode in general when applying refinements in the class of general networks. At present, we must consider it an open problem, whether the Wu invariant is the only quadratic valuation on the class of all graphs, which is invariant under Barycentric refinements and assigns 1 to every K_1 subgraph. Lets call the statement of uniqueness of Wu characteristic the **Wu hypothesis**. To understand it, we have to investigate the behavior of the f-matrix under Barycentric refinement, something we have only just started to look at. The fact that f-vectors transform linearly and looking at the eigenvectors shows immediately that the Euler characteristic is unique. If the transformation on f-matrices $V(G) \to V(G_1)$ were linear, we expect a unique eigenvector to the eigenvalue 1. This would be the Wu eigenvector, leading to the uniqueness of Wu characteristics. If the quadratic case works, then most likely also more general k-linear valuations work and ω_k be unique in the class of k-linear valuations which are invariant under Barycentric refinement as well as assigning 1 to a single vertex.

In a finite simple graph G, a complete subgraph is also called a face or simplex. The set of all subsets of the vertex set of a complete graph can be seen as a simplicial subcomplex which is indecomposable in the sense that it can not be written as a union of two different simplicial subcomplexes of the Whitney complex. Simplices are the "elementary particles" in the Boolean algebra of all simplicial subcomplexes of a graph. The Wu characteristic takes count of "interactions" between these particles. The realization of the signature dimension $I(x) = (-1)^{\dim(x)} = \sigma(x)$ as a Poincaré-Hopf index of a "wave function" f indicate that $I(x)$ is something like a "charge". When deforming the wave function f, these charges change but their total sum does not. There is "charge conservation". For any multi-linear valuation and any scalar function f there is a Poincaré-Hopf index. For a quadratic valuation like the Wu characteristic there is an index $i_f(a, b)$ which vanishes if the distance between x and y is larger than 1. The index function can be seen as an integer-valued function on vertex pairs. The Poincaré-Hopf formula $\omega(G) = \sum_{a,b \in V \times V} i_f(a, b)$ sums over the vertex set and not the set of simplices. It is possible then to push this function from vertex pairs to vertices. We have to stress that graphs include higher dimensional structures without the need to digress to multi-graphs. Much of graph theory literature deals with graphs equipped with the **1-dimensional skeleton simplicial complex** and ignores the two or higher dimensional simplices. The language of graphs alone however is quite powerful to describe a large part of the mightier and fancier language of abstract simplicial complexes and so rather general topology. While the structure of simplicial complexes is more general as there are simplicial complexes which are not Whitney complexes of a graph (like some matroids), refinement rectifies this: the Barycentric refinement of any abstract simplicial complex K is always the Whitney complex of a graph: given an arbitrary simplicial complex K, take the simplices in K as the vertices and connect two if one is contained in the other. The Whitney complex of this graph is then the Barycentric refinement of K. Also for simple polytopes, where now faces are not necessarily triangles, the graph determines the combinatorial structure of the polytope [76]. We don't lose much generality therefore if we stick to the language of graphs, at least if we look for discrete differential geometric structures. The advantage is not only of notational and of didactic advantage - the category of networks can be grasped very early on, as it is familiar from maps and diagrams -, it is also convenient from the computer science point of view as many general-purpose computer algebra languages have the language of graphs hardwired into their language. In an appendix we have given programs which allow to compute all the objects discussed in this article.

The dimension of a simplex K_{d+1} is d. There are various notions of dimensions known for graphs. One is the **maximal dimension** which is defined as d if the clique number of G is $d+1$. In other words, the maximal dimension of G is the maximal dimension which a simplex in G can have. Nice triangulations of d-dimensional manifolds have dimension d but there are triangulations of d-manifolds where G is higher dimensional: take an octahedron for example and attach a new central vertex in each triangle connected to the vertices of the triangle. This is still a triangulation but its dimension is 3 as it contains many tetrahedra. We have defined dimension motivated from Menger-Uhryson as the average of the dimensions of the unit spheres plus 1. The induction assumption is that the empty graph has dimension -1. The original inductive Menger-Uhryson dimension of a graph is 0.

The just defined inductive dimension satisfies all the properties one can wish for and even behaves in many cases like the Hausdorff dimension in the continuum like $\dim(A \times B) \geq \dim(A) + \dim(B)$ in full generality for all finite simple graphs. It is also possible to compute explicitly the average dimension in Erdös-Renyi spaces $G(n, p)$ as it satisfies the recursion $d_{n+1}(p) = 1 + \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} d_k(p)$ with $d_0 = -1$. Each d_n is a polynomial in p of degree $\binom{n}{2}$. See [82].

A d-graph is a finite simple graph for which every unit sphere is a $(d-1)$ graph which is a d-sphere. Being a d-sphere was defined recursively by Evako as the property that every unit sphere is a $(d-1)$-sphere and that removing one vertex renders the graph contractible. We could characterize d-graphs also as graphs for which the Barycentric refinement limit is a smooth, compact d-manifold with boundary. For general graphs or networks, there is a dimension which mathematically very much behaves like the Hausdorff dimension in the continuum: the **inductive dimension** of a graph is defined by setting the dimension of the empty graph to be -1 and in general by adding 1 to the average of the dimensions of the unit spheres of the graph. It is a rational number which similarly as Hausdorff dimension satisfies $\dim(G \times H) \geq \dim(G) + \dim(H)$ for all finite simple graphs G, H where $G \times H$ is the Cartesian product of graphs defined by taking the product in the Stanley-Reisner ring and looking at the graph defined by that algebraic object. We can also computed the expectation of the inductive dimension on Erdös-Renyi spaces of graphs. Now, when looking at valuations, even the presence of a single simplex of dimension d allows us to look at valuations counting in such simplices: counting the largest dimension simplices is the analogue of volume. Having the discarded the 1-dimensional space of constant valuations which assigns to any graph a constant c, we get a $(d+1)$-dimensional space of linear valuations, a $(d+1)(d+2)/2$-dimensional space of quadratic valuations if G has maximal dimension d. This is a generalization of discrete Hadwiger.

The quadratic valuations and intersection numbers we are going to look at, are geometric and do not have much interpretation yet in the case of general networks as they are not homotopy invariants. Here are some attempts for interpretations: in the case of a graph without triangles, there is a physical interpretation in that the Wu characteristic adds up interaction energies between different edges and vertices. Think of the graph as a molecule, the vertices as atom centers and the edges as bonds between atoms, there are positive self-interactions between the positively charged nuclei and positive self-interactions between negatively charged bonds, then there are negative self-interaction energies between the nuclei and bonds. The Wu characteristic $\sum_{x,y} \sigma(x)\sigma(y)$ has now an interpretation as an **interaction energy**. This Hückel type interpretation however fades if triangles are involved. An algebro-geometric perspective comes in by seeing a quadratic valuation $X(A, B)$ as the **intersection number** of pairs (A, B) of subgraphs of a given graph so that they can serve to study intersections in a purely combinatorial way. Two one dimensional graphs intersecting transversely in a point have intersection number 1. A one dimensional graph intersecting transversely with a two-dimensional graph has intersection number -1. Two two dimensional graphs intersecting in a point has again intersection number 1. An other interpretation of the Wu invariant can come by seeing $\text{int}(G) = G - \delta G$ as an interior so that $\chi(G) - \chi(\delta G)$ measures the Euler

characteristic of the interior of G if we think of the interior and boundary being disconnected. Of course it is not possible to define a subgraph of G taking the role of the interior such that $\omega(G)$ is the Euler characteristic of the subgraph. Here is the reason: when looking at star graphs S_n the Euler characteristic of any subgraphs is bounded above by n while the Wu characteristic of a star graph with $n+1$ rays is the value of the Fibonnacci polynomial $n^2 - n - 1$ which grows quadratically with n. Still, in the continuum, some notions along these lines have been developed, like in [140], where a valuation X of the interior is defined as such. The formula $X(M) = \sum_x X(\text{int}(x))$ for a simplicial complex given in Lemma 2 of [140] can be seen an analogue of the formula $\omega(G) = \sum_x \omega(x)$. Whether the picture of seeing the Wu characteristic as the Euler characteristic of some **"virtual interior"** of G, remains to be seen. Anyway, as ω is of kinetic nature as it sums neighboring interactions in a quadratic manner, it kind of measures an interior energy similarly as models in statistical mechanics, the Ising model in particular; only that now the interaction energy is not given by a additionally imposed spin value but geometrically defined by the dimension of the various pieces of space. The interpretation of $\sigma(x) = (-1)^{\dim(x)}$ as a spin value is not so remote as we have identified it as a Poincaré-Hopf index of a gradient vector field.

Finally, one could seriously look at the Wu characteristic as a functional in physics, especially for naive approaches to quantum gravity. The reasons are similar as for Euler characteristic, which in even dimensions like for 4-graphs has the index $i_f(x)$ which is expressible through the Euler characteristic of a 2-graph and so an average over sectional curvatures in a well defined sense so that there is a strong analogy with the Hilbert action in general relativity.

Lets look at a discrete algebro-geometric connection: any quadratic valuation can be seen as a divisor on the **intersection graph** of G, the graph of all complete subgraphs as vertices and where two are connected, if they intersect. A divisor means here an integer-valued map on the vertices of the graph as in Baker-Norine theory. That theory sees graphs as 1-dimensional objects where assigning integer values to vertices is the analogue of what a divisor means in the continuum. The Poincaré-Hopf indices play an important role in that theory. The intersection graph is the graph for which the complete subgraphs are the vertices and two are connected if they intersect. The intersection graph is obtained from the Barycentric refinement by adding more connections. If we "chip-fire" fractions of the divisor to the vertices, we get a rational number at each point which is the curvature. Already the curvature of linear valuations like Euler characteristic can be understood like that: start with the divisor which attaches the value $(-1)^k$ to the k simplices. If we break up this value $(-1)^k$ into $k+1$ pieces and chip fire each part to the vertices, we send $(-1)^k/(k+1)$ to the vertices. Doing that to all gives the Euler curvature value $K(x) = 1 + \sum_{k=1}^{\infty} (-1)^k V_{k-1}(x)/(k+1)$. For the Wu characteristic, things become nonlinear, as the divisor attached to the simplices is no more just a constant but depends on the connections but the proof remains the same.

2. The Wu characteristic

Wenjun Wu introduced in 1959 [149] (possibly already in [148], a reference we could not get hold of yet) the functional

$$\omega(G) = \sum_{x,y} (-1)^{\dim(x)+\dim(y)} \ ,$$

where x, y runs over all pairs of simplices which intersect. We call it the **Wu characteristic**. To get closer to the notation used in models of statistical mechanics like the Ising model, one could define the **signature of a simplex** as $\sigma(x) = (-1)^{\dim(x)}$ and write

$$\omega(G) = \sum_{x \cap y \neq \emptyset} \sigma(x)\sigma(y)$$

which now looks like adding up an **interaction energy**. The invariant was originally formulated by Wu for convex polyhedra but we will look at it in the case of arbitrary graphs equipped with the Whitney complex. It can also be considered for more general simplicial complexes. As explained in the introduction, looking at graphs only, is almost no loss of generality, as the Barycentric refinement of an arbitrary abstract simplicial complex is already the Whitney complex of a finite simple graph.

For example, if $G = K_2$, we have three simplices in G. They are $\{a, b, ab\}$. There are 4 intersections and both give a value -1 and there are three intersections which give value 1. The value is -1. Algebraically, $f_G = a + b + ab$ and $f_G(-1, -1)^2 - (f_G)^2(-1, -1)$ as $f_G^2 = 2ab$ so that $f_G(-1, -1)^2 - (f_G)^2(-1, -1) = -1$. For the kite graph,

$$G = (V, E) = (\{a, b, c, d\}, \{(a, d), (a, b), a, c), (b, c), (c, d)\}) \ ,$$

with "Bosonic simplices" $\{(a, b, c), (a, c, d), (a), (b), (c), (d)\}$ and "Fermionic simplices" $\{(a, d), (a, b), a, c), (b, c), (c, d)\}$, the Wu characteristic is 1. We can see this also by looking at the square free part of $f_G^2 = (a + b + ab + c + ac + bc + abc + d + ad + cd + acd)^2 = 2ab + 2ac + 2bc + 6abc + 2ad + 2bd + 4abd + 2cd + 6acd + 4bcd + 8abcd$.

The Wu characteristic ω is not a linear valuation: the Kite graph with two K_3 subgraphs A, B intersecting in a K_2 shows that $\omega(A \cap B) + \omega(A \cap B) = \omega(A) + \omega(B)$ does not hold as the left hand side is $1 - 1 = 0$, while the right hand side $1 + 1 = 2$. Indeed ω is an example of a multi-linear valuation and is in particular a **quadratic valuation**. It is also not a homotopy invariant, as it is not the same for all complete graphs. It is equal to -1 for odd dimensional simplices and 1 for even dimensional simplices. All complete graphs however are clearly collapsible to a point and so homotopic. Nevertheless, it turns out that the Wu characteristic is multiplicative. We initially also investigated its relation with analytic torsion which is a spectrally defined number for graphs and an other highly dimension and geometry sensitive combinatorial invariant. Like analytic torsion, or Dehn-Sommerville invariants, the Wu characteristic is fragile if we move away from geometric graphs: growing a zero-dimensional dendrite to an odd-dimensional geometric structure for example does not change the homotopy but changes the quadratic valuation. We could build a connected graph with Wu characteristic -1000 for example by growing 501 hairs
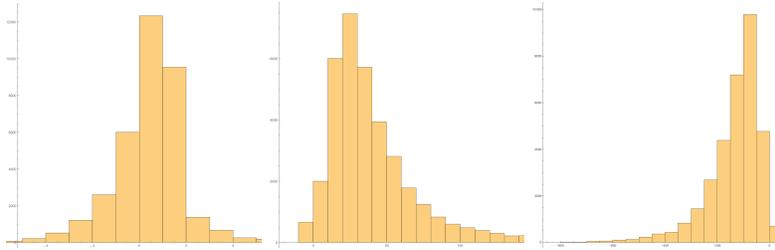
FIGURE 2. We see the distribution of Euler, quadratic and cubic Wu characteristic on a list of 36'000 molecules for which Mathematica has graphs provided. The Wu characteristic ranges from -16 to 1405. The Euler characteristic ranges from -37 to 28 on that list. The cubic Wu characteristic from -815 to 16 with a mean of -133.97. The mean of the Euler characteristic is -0.0368 which is very close to 0, the mean of the Wu characteristic is 42.58. The maximum of ω is attained for an **inulin** molecule with 801 atoms, the minimum of ω_2 which appears here as the maximum of ω is a disconnected graph containing 16 copies of K_2.

to a 2-sphere, where each hair is a one dimensional line graph.

If x is a complete subgraph, then $\omega(x) = (-1)^{\dim}(x) = \sigma(x)$. This will follow from one of the main results Barycentric refinement of a d-simplex x produces a geometric d-ball with boundary for which ω is the difference between the Euler characteristic of the graph minus the Euler characteristic of the boundary. Having the Wu characteristic of a simplex expressed in terms of $\sigma(x)$, we can write

$$\chi(G) = \sum_x \omega(x) ,$$

where x runs over all simplices in G. In some sense, the self-interaction functional ω "explains" the signs in the sum of the Euler characteristic. And also the Wu characteristic ω can now be expressed by itself:

$$\omega(G) = \sum_{x \cap y \neq \emptyset} \omega(x)\omega(y) ,$$

where the sum is again over all ordered pairs of simplices x, y which intersect. In comparison, we have the formula $\sum_{x,y} \omega(x)\omega(y) = \chi(G)^2$, where x, y runs over **all** possible ordered pairs, (pairs which do not necessarily intersect), which follows from $\omega(x \times y) = \omega(x)\omega(y)$ and $\chi(G) = \sum_x \omega(x)$.

Examples.
1) For any cyclic graph C_n with $n \geq 3$, we have $\omega(G) = 0$. For any 2-sphere like the octahedron or icosahedron G, one has $\omega(G) = 2$. For 3-spheres like the 16-cell, the 600 cell or a suspension of a 2-sphere, we have $\omega(G) = 0$. For 4-spheres like a suspension of a 3-sphere or the boundary of $K_2 \times K_2 \times K_2 \times K_2 \times K_2$ we have $\omega(G) = 2$. For a 2-torus graph or discrete Klein bottle, we have $\omega(G) = 0$ again the same than the Euler characteristic. Also for a projective plane, we have Wu characteristic 1.

2) For $G = K_{d+1}$ we have $\omega(G) = (-1)^d$. This remains so after Barycentric subdivision. We see that for a triangulation of a ball, $\omega(G) = (-1)^d$.

3) For a figure 8 graph, $\omega(G) = 7$. For star graph with n rays, we have $\omega(G) = n^2 - 3n + 1$. For a sun graph, we have $\omega(G) = 2n$. For example, for $n = 4$, we get $\omega(G) = 5$. For two 2-spheres touching at a vertex, we have $\omega(G) = 3$.

4) The utility graph G of Euler characteristic $\chi(G) = -3$ has the Wu characteristic $\omega(G) = 15$. The utility graph is the only graph among all connected graphs with 6 vertices for which the Wu characteristic is that high. It is the graph with maximal Wu characteristic in the class of graphs with 6 vertices.

5) For a k-bouquet of 2-spheres glued together at one point, (a wedge sum) the Wu characteristic is $k + 1$.

6) For a k-bouquet of 1-spheres, there are no triangles. The Wu curvature at the central vertex is $d = 2k$ and zero at every other place. The Wu characteristic is $(k - 1)(4k - 1)$.

7) For a sun graph with k rays, the Wu characteristic is $2k$. Such graphs have no triangles. The total curvature contribution of each ray is 2.

8 For a star graph with n rays, the Wu characteristic is $n^2 - 3n + 1$. For example, for $n = 0$, it is 1, for $n = 1$ it is -1 for $n = 9$ it is 55.

9) Adding a one dimensional hair to a 2 sphere reduces the Wu characteristic by 2.

10) The Wu characteristic of the cube graph is 20, the Wu characteristic of the dodecahedron is 50. Both graphs have no triangles and constant vertex degree $d = 3$ so that in both cases, the curvature is constant $5/2$.

11) The Wu characteristic of two crossing circles is 14, the Wu curvature of a crossing being 7 and otherwise being zero.

12) The Wu characteristic of the tesseract is 112. It is a graph without triangles with constant Wu curvature $K = (1 - d/2)(1 - 2d) = 7$, where $d = 4$ is the vertex degree. Since there are 16 vertices, the Wu characteristic is 112. As the Euler curvature is $(1 - d/2) = -1$, the Euler characteristic is -16. Of course, a triangulation of the tesseract, the boundary of $K_2 \times K_2 \times K_2$ is a 3-sphere of Euler characteristic 0 and Wu characteristic also equal to 0.

13) For a suspension of a disjoint union of a circle (which has the Betti vector $(1, 1, 2)$ and so Euler characteristic 2), the Wu characteristic is 2.

14) For the Adenine, Guanine, Cytosine and Thymine graphs, the main bases in DNA and RNA, the Wu characteristics are 15,17,12 and 18. Since the Wu Characteristic measures an interaction between neighboring parts where equal charges repel each other and unequal attract, the interaction energy makes some sense. The bonds are mainly negatively charged, while the atom nuclei are positively charged.
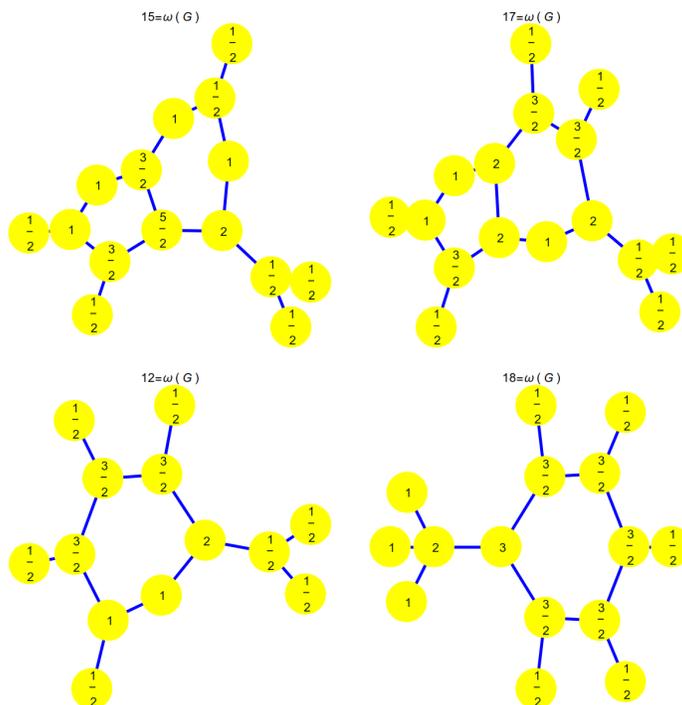
FIGURE 3. The Wu curvatures of Adenine,Guanine, Cytosine and Thymine for which the Wu characteristic are 15,17,12 and 18. For graphs without triangles, the Wu characteristic gives an interaction energy, where bond-bond and vertex-vertex interactions count positive and vertex-bond interactions are negative. As bonds are mostly occupied by electrons and vertices by nuclei, this interpretation Wu functional has some merit. The Wu functional is maybe a too simple functional on molecules to be considered useful as it is even simpler than Hückel theory, which involves the eigenfunctions of the Laplacian. However, it could be important on a more fundamental level when looking at the fabric of space.

3. LINEAR VALUATIONS

The f-**vector** $v(G)$ of a finite simple graph G is defined as

$$v(G) = (v_0, v_1, \ldots, v_d) \ ,$$

where d is the maximal dimension of G. This means that $d+1$ is the clique number and v_d the volume, counting the number of facets, maximal cliques in G. All the entries $v_k(G)$ are invariant valuations. Hadwiger's theorem shows that the list $v_0, \ldots, v_d$ is a basis for the linear space of invariant valuations in G. While one can see the v_k as functionals on graphs, we look at it as a valuation, a functional on the set of subgraphs of G. It naturally defines a functional on the set of simplicial sub complexes of the Whitney complex of G, which is the traditional way to look at valuations. The simplices in G form the analogue of convex sets in integral geometry or geometric probability and subgraphs of G are the analogue of finite union of

convex sets. Euler characteristic $\chi(G) = v_0 - v_1 + v_2 - \dots$ is an important functional. It is a valuation on G, assigning to every subgraph A of G the number $\chi(A)$. Every valuation on G can be assigned a vector ϕ as $X(A) = \phi \cdot v(A)$. For the Euler characteristic, this vector is $\chi_1 = (1, -1, 1, -1 \dots, \pm 1)$. Since we look at multi-linear valuations in a moment, we call classical valuations also linear valuations. A natural basis in the $d + 1$ dimensional vector space of all linear valuations of G are the **Barycentric vectors** $\chi_1, \dots \chi_{d+1}$, the eigenvectors of $A^T(G)$, where A is the **Barycentric refinement operator** which maps the f-vector of G to the f-vector of its Barycentric refinement G_1. The **Barycentric refinement matrix** is explicitly known as

$$A_{ij} = i! S(j, i) \ ,$$

where $S(j, i)$ are the **Stirling numbers** of the second kind. The **Barycentric characteristic numbers** which were algebraically defined like that are natural and especially singles out Euler characteristic. If we would not know about Euler characteristic, we would be forced to consider it now.

Examples.
1) If G has no triangles, then every edge gets mapped into two edges. There are $|V| + |E|$ new vertices in the refinement. The matrix A is

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \ .$$

2) If G is two dimensional without tetrahedra, then every triangle becomes 6 triangles. Every edge becomes doubled and additionally there are 6 new edges for each of the triangles. The number of new vertices is the sum of the number of vertices, edges and triangles. The matrix A is

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 6 \\ 0 & 0 & 6 \end{bmatrix} \ .$$

3) If G is three dimensional without K_5 graphs, then every tetrahedron splits into 24. Every triangle gets split into 6 and then there are 36 new triangles coming from tetrahedra etc. The matrix A is

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 6 & 14 \\ 0 & 0 & 6 & 36 \\ 0 & 0 & 0 & 24 \end{bmatrix} \ .$$

In the case $d = 4$ for example, this matrix is

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 6 & 14 & 30 \\ 0 & 0 & 6 & 36 & 150 \\ 0 & 0 & 0 & 24 & 240 \\ 0 & 0 & 0 & 0 & 120 \end{bmatrix} \ .$$

If χ is an eigenvector of A^T to the eigenvalue λ, then

$$\chi v(G_1) = \chi A v(G) = v(G)^T A^T \chi^T = v(G)^T \lambda \chi^T = \lambda \chi v(G)$$

showing that the valuation scales by a factor λ when applying the Barycentric refinement. Since the matrix A is upper triangular, its eigenvalues $k!$ are all known
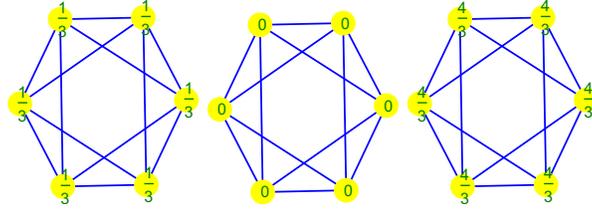
FIGURE 4. The curvatures of the Barycentric numbers in the case $\chi_1 = \chi$, χ_2 which is zero and χ_3 which is the area.

and the eigenvectors χ_k of λ_k form an eigen-basis of the linear space of valuations. In the case $d = 4$ for example, the basis is

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -22 \\ 33 \\ -40 \\ 45 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 19 \\ -38 \\ 55 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ -2 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

The first one is the eigenvector to the eigenvalue 1 leads to Euler characteristic which manifests itself as a **Barycentric characteristic number**. The last one is the **volume**, the number of facets of a sub graph. A statement completely equivalent to the Dehn-Sommerville relations is:

Theorem 1 (Dehn-Sommerville)**.** *If $d + k$ is even, then the Barycentric characteristic numbers satisfy $\chi_k(G) = 0$ for every d-graph.*

This gives more information than the usual Dehn-Sommerville relations as it also proves immediately that the dimension of the Dehn-Sommerville space is $[(d+1)/2]$, where $d + 1$ is the clique number and $[t]$ is the largest integer smaller or equal to t. If the dimension would be larger, then there would be an other invariant which is zero for all geometric graphs. It would also be zero for the cross polytop, where we know the maximality. It also removes any mystery about where these invariants come from or how they can be found. It is linear algebra which forces them on us. We actually discovered this theorem, not realizing first that they are the Dehn-Sommerville relations.

Here is the proof of Theorem (1): It uses Gauss Bonnet and a **suspension decent argument**:

Proof. Let $\mathcal{G}$ be the class of d-graphs for which $\chi_k(G) \neq 0$ or for which there is vertex with nonzero curvature. We show that this class is empty by proving that any graph G in $\mathcal{G}$ for which some unit ball can be extended in G remaining a ball is either a cross-polytope or can be reduced to a smaller example. By definition, for $G \in \mathcal{G}$ there is always a vertex with nonzero curvature. Take a graph G in $\mathcal{G}$ with minimal vertex cardinality in $\mathcal{G}$. Now look at the suspension of the unit sphere $S(a)$. This graph is again in $\mathcal{G}$. It must be G because as a subgraph it has to have less vertices and therefore $\chi_k(G) = 0$ with zero curvature everywhere contradicting the curvature at a to be nonzero. As G agrees with the suspension of $S(a)$, take add an other vertex $b \in S(a)$ to the unit ball $B(a)$ and call it H. If no such other vertex would exist, then G would be a cross polytope. Now the closure of H is a sphere

which is smaller than G and so has everywhere zero curvature. That contradicts that the curvature at a is nonzero. $\qquad\square$

Here is an other local necessary condition for $X(G) = 0$ for a linear valuation.

Lemma 1 (Puiseux type formula). *a) If $X(G) = 0$ for all d-graphs, then for every G and every $v \in V$, we have $2X(B(x)) = X(S(x))$.*
b) Given a d-graph. If $2X(B(x)) = X(S(x))$ for every $x \in V$, then $X(G) = 0$.

Proof. a) Look at the suspension U of $S(x)$ using a second vertex y. Since U is again a d-graph, we have $X(U) = 0$. The valuation condition shows

$$2X(B(x) = X(B(x) + X(B(y)) = X(S(x)) + X(U) = X(S(x)) \ .$$

b) The condition $2X(B(v)) = X(S(v))$ implies that $X(U) = 0$ for any double suspension of $S(x)$. As in the proof above this means $K(x) = 0$ for all x so that $X(G) = 0$. $\qquad\square$

It implies for example that if a graph has all unit spheres of Euler characteristic 2, then $\chi(G) = 0$.

Examples:
1) 2-sphere with a vertex of degree 6 shows that $K(x) = 0$ for a single vertex does not necessarily imply $2X(B(x)) = X(S(x))$. Its only the global condition of zero curvature which implies it.
2) If $k = 1$ and odd d, then $\dim(S(x)) = d - 1$ is even and $\chi(S(x)) = 2$.

In the case $d = 4$, the Dehn-Sommerville space is 2-dimensional and spanned either by the **Barycentric characteristic vectors**

$$\chi_2 = (0, 22, -33, 40, -45) \ , \chi_4 = (0, 0, 0, 2, -5)$$

which are eigenvectors of A_4^T for the Barycentric refinement operator on graphs with clique number $d + 1 = 5$, or then by the **classical Dehn-Sommerville vectors**

$$d_0 = (0, 2, -3, 4, -5) \ , d_2 - (0, 0, 0, 4, -10) \ .$$

In the form defined here, we have $d_{-1} = (-1, 1, -1, 1, -1)$ and $d_3 = (0, 0, 0, 2, -5)$. The valuation χ_4 (which is parallel to d_2) is a trivial boundary invariant, expressing that counting 5 times the number of 4-simplices is 2 times the number of 3-simplices. This is the Euler handshake in the dual graph of G, where the maximal d-simplices are the vertices and two such simplices are connected if they intersect in a $(d-1)$-simplex. Graphs with this property are sometimes called **pseudo manifolds**. The vector $\chi_2 + 2\chi_4 = (0, 22, -33, 44, -55)$ is parallel to the classical Dehn-Sommerville vector $d_0 = (0, 2, -3, 4, -5)$. Let G be the 4-sphere obtained by taking the suspension of the suspension of the octahedron graph. It is a graph G with 10 vertices and 40 edges. Its f-vector is

$$v(G) = (10, 40, 80, 80, 32) \ .$$

When taking the dot product of this with the above Barycentric basis vectors, we get the **Barycentric invariants** $\chi_1(G) = 2, \chi_2(G) = 0, \chi_3(G) = 240, \chi_4(G) = 0, \chi_5(G) = 32$. We see that the Euler characteristic is 2 as it has to be for any 4-sphere. We also see the two Dehn-Sommerville relations and last but not least that the volume is 32. A special case of the above Dehn-Sommerville theorem is that for odd-dimensional d-graphs, the Euler characteristic is zero. Also, in any dimension, the invariant to χ_d, the boundary invariant is always zero. It is a manifestation of the fact that we assumed that the graph G has no boundary. The volume $\chi_{d+1}(G) = (0, 0, \ldots, 0, 1)$ is a valuation which is never zero for a d-graph or more generally for a graph with clique number $d + 1$.

Lets look at an other 4-graph G, the product $H \times H$ of two 2-spheres given as octahedron graphs H. Since the octahedron graph H has the f-vector $v(H) = (6, 12, 8)$, which has $26 = \sum_i v_i$ simplices, the product graph has 676 vertices. Its f-vector is

$$v = (676, 8928, 28992, 34560, 13824) .$$

Again, by taking the dot product of this f-vector with the basis vectors, we get the Barycentric invariants. They are $\chi_1 = 4, \chi_2 = 0, \chi_3 = -2112, \chi_4 = 13824$. The Euler characteristic is 4 as it has to be for the product of two 2-spheres, the two zero values are the Dehn-Sommerville relations and 13824 is the volume. The graph has volume 13824 counting the complete subgraphs K_5.

As a third example, lets look at the suspension G of $S^2 \times S^2$ just constructed before. This is a discrete 5-dimensional graph, a graph with 678 vertices but it is no more a 5-graph, as the there are now by construction two vertices for which the unit sphere is not a sphere S^4. Indeed, the unit sphere is a graph whose topological realization is the standard $S^2 \times S^2$. The f-vector of G is

$$v(G) = (678, 10280, 46848, 92544, 82944, 27648) .$$

The Euler characteristic is now 2 and not zero it would have to be if it were a 5-graph. Interestingly enough, the higher Barycentric invariants are still zero, as

$$(v \cdot \chi_1, v \cdot \chi_2, v \cdot \chi_3, v \cdot \chi_4, v \cdot \chi_5) = (2, -231152, 0, 114432, 0, 27648) .$$

It will be interesting to study for which pseudo d-graphs of this type, higher Dehn-Sommerville relations still hold.

Valuations extend naturally from graphs to chains by linearity. This is the case for linear as well as multi-linear valuations. For a chain $H = \sum_x a_x x$ on a graph with simplices x, define its f-vector $v(H)$ by $v_k(H) = \sum_{\dim(x)=k} a_x$. In particular, if $H = G$, then $v_k(G) = \sum_{\dim(x)=k} 1$. Given a valuation X defined by a vector χ, define $X(G) = \chi \cdot v(G)$.

Given a graph f described in the Stanley-Reisner ring as $f = \sum_i p_i$ with quadratic free monoids p_i in the variables $x_1, \ldots, x_n$ representing the vertex set $V = \{x_1, \ldots, x_n\}$. A chain over G is an element $\sum_i a_i p_i$, where a_i are integers. The set of all chains forms an Abelian group. This and the corresponding construction of homology is one reason why chains were introduced by Poincaré. An other reason for the need of chains is that the boundary of a graph is no more a graph in general, nor are quotients of group actions. As we have noted in [105], for a

group A acting as automorphisms on a graph G, the Riemann-Hurwitz formula $\chi(G) = n\chi(G/A) - \sum_x (e_x - 1)$ holds, where x sums over all simplices in G and $e_x = 1 + \sum_{a \neq 1, a(x) = x} (-1)^{\dim(x)}$ is the **ramification index**. This formula holds also generally on the larger class of chains as it just reduces to the Burnside lemma (which is the special case if G has no edges). In general, one first has to do Barycentric refinements before applying the quotient operation in order to stay within the class of graphs. Still, if G is a d-graph, the quotient G/A is a **discrete orbifold** in general. By the way, the Riemann-Hurwitz idea goes over from Euler characteristic to valuations. One just has to adapt $(-1)^{\dim(x)}$ to $\psi(\dim(x))$ if ψ is the vector defining the valuation $G(V) = v(G) \cdot \psi$. We have not yet investigated Riemann-Hurwitz for k-linear valuations but expect things to work similarly, however to become more interesting.

Here are some examples showing the need to go from graphs to chains: lets take the star graph S_3 for example with $f_G = ab + ac + ad + a + b + c + d$. If the orientation on the simplices is chosen from the way the monomials were written, the boundary δf is $b - a + c - a + d - a = a + c + d - 3a$ which is now only a chain and no more a graph. A second example is to let the group $A = Z_4$ act on $G = C_4$. The quotient G/A is the chain $a + b + 2ab$ which no more a graph. Both the Euler characteristic of G and the quotient are 0, there are no ramification points of the group action. Graphs with multiple connections, multi-graphs or graphs with selfloops must be considered examples of chains.

4. The f-matrix

Given a graph G, define the f-**matrix** or quadratic f-form as

$$V_{ij}(G) = |\{(x,y) \mid x \sim K_{i+1}, y \sim K_{j+1}, x \subset G, y \subset G, x \cap y \neq \emptyset \}| \ .$$

It is a symmetric matrix counting the number of ordered pairs of i-simplices and j-simplices in G which have non-empty intersection. For example, if G is the star graph with 3 spikes, its f-vector is $v(G) = (4, 3)$ as there are 4 vertices and 3 edges. The f-matrix $V(G)$ is $V = \begin{bmatrix} 4 & 6 \\ 6 & 9 \end{bmatrix}$ as there are 4 self-intersections of vertices, $3 + 6 = 9$ intersections of edges and $3 + 3$ intersections of vertices with edges.

A quadratic valuation of a graph G can now be written as

$$X(G) = (V(G)\phi) \cdot \psi = V(G)\phi\psi \ ,$$

where ϕ, ψ are two $(d+1)$-vectors ϕ, ψ, if the clique number of G is $d + 1$. For example, if $\phi = \psi = (1, -1, 1, \dots)$, then X is the Wu characteristic. In the case of the star graph G above, we have

$$[1, -1] \begin{bmatrix} 4 & 6 \\ 6 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 1 \ .$$

The graph G is one of the rare cases, where $V(G)$ has a zero eigenvalue. The Perron-Frobenius eigenvector is $(2, 3)$. As $V(G)$ is symmetric, the eigenvector to 0 is perpendicular: $(-3, 2)$.

Given two subgraphs A, B of G, define the **intersection form**

$$V_{ij}(A, B) = |\{(x,y) \mid x \sim K_{i+1}, y \sim K_{j+1}, x \subset A, y \subset B, x \cap y \neq \emptyset \}|$$

as the number of ordered pairs (x, y), where x is an i-simplex in A and y is a j-simplex in B for which $x \cap y$ is a non-empty graph. A quadratic valuation X can be written using two vectors ϕ, ψ as

$$X(A, B) = \psi \cdot V(A, B)\phi \, .$$

For example, if $A = (a + b + c + ab + ac)$ is a linear subgraph of the above star graph G and $B = (a + c + d + ac + ad)$ is an other linear subgraph of G, then

$$V(A, B) = \begin{bmatrix} 2 & 2 \\ 2 & 1 \end{bmatrix}$$

as there are are two matches aa, cc for vertices, four matches $(ab)(ac),(ab)(ad),(ac)(ac),(ac),(ad)$ for edges and three pairs $a(ac), a(ad), c(ac)$ of vertices in A and edges in B. Now

$$X(A, B) = [1, -1] \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 0 \, .$$

Let G be the "16-cell" again, the regular Platonic 3-sphere which is the suspension of the octahedron. Its f-vector is $(8, 24, 32, 16)$. Its f-matrix is

$$V(G) = \begin{bmatrix} 8 & 48 & 96 & 64 \\ 48 & 264 & 480 & 288 \\ 96 & 480 & 800 & 448 \\ 64 & 288 & 448 & 240 \end{bmatrix} \, .$$

Lets look at the Barycentric eigenspace of the 4-dimensional space of valuations on G:

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 22 \\ -33 \\ 40 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} \, .$$

Lets call them $\{\chi_1, \chi_2, \chi_3, \chi_4\}$. Taking the dot product with the f-vector produces the Barycentric characteristic numbers are $\chi_1(G) = 0, \chi_2(G) = 112, \chi_3(G) = 0, \chi_4(G) = 16$. The Euler characteristic $\chi(G) = \chi_1(G)$ is zero on the graph G as for all 3-graphs. Lets now compute the quadratic Barycentric characteristic matrix. It is defined as

$$\Omega_{ij}(G) = \chi_i \cdot V(G)\chi_j \, .$$

In this example it is given by

$$\Omega(G) = \begin{bmatrix} 0 & 112 & 0 & 16 \\ 112 & 10176 & 224 & 1152 \\ 0 & 224 & -32 & 32 \\ 16 & 1152 & 32 & 240 \end{bmatrix} \, .$$

The first entry $\chi_1 \cdot V(G)\chi_1$ is the Wu characteristic, which is also zero. The first row or column agree with the Barycentric characteristic numbers. We will prove that the first entry is the same and the zero entries in the first row and column are there. These zero entries are the quadratic valuations which were conjectured to be zero by Grünbaum. Establishing the relations

$$\chi_1 V(G)\chi_k = v(G)\chi_k$$

for any k will prove that and so prove the conjecture of Grünbaum positively.

5. Gauss-Bonnet

For a linear valuation X, the curvature

$$K(x) = \sum_{k=0} X(k) \frac{V_{k-1}(x)}{(k+1)}$$

with $V_k(x) = v_k(S(x))$ and $V_{-1}(x) = 1$ satisfies the Gauss-Bonnet formula

$$X(G) = \sum_{x \in V} K(x) = X(G) .$$

Each of the numbers $V_k(x) = v_k(S(x))$ are valuations applied to the unit sphere $S(x)$ counting the number of k-simplices present in $S(x)$. This can be called the "fundamental theorem of graph theory" as for $X(G) = (0, 1, 0, \ldots, 0)$ counting the number of edges the curvature $K(x) = V_0(x)/2$ is half the vertex degree and the Euler handshake lemma is sometimes called as such. The Euler Handshake is maybe the simplest version of a Gauss-Bonnet result for graphs, where the sum of local properties, the degree, adds up to a global property which is twice the length of the graph when seen as a curve.

Lets call $K(x)$ the **Euler curvature** if X is the Euler characteristic. Unlike in the continuum, where curvature is a notion involving second order derivatives, the Euler curvature of a linear valuation is a first order notion. We have experimented with second order curvatures for Euler characteristic in [84] and searched since for conditions in two dimensions for which a second order curvature would work. It turns out that we were too much obsessed with Puiseux formulas in differential geometry and therefore searched in two dimensions for curvatures of the form $K(x) = 2|S_1(x)| - |S_2(x)|$, where $|S_r(x)|$ is the vertex cardinality of the sphere S_r. This search for second order curvatures using Puiseux type discrete formulas was fruitless even in two dimensions. We have now a notion in the form of the Wu curvature which is defined as a second order curvature for general finite simple graphs and which happens to agree with the Euler curvature on d-graphs but manifests as higher order if evaluated on more general spaces. Gauss-Bonnet for linear valuations easily can be proven as follows (see the introduction in [83]). Look first at the curvature on the Barycentric refinement which assigns to a simplex x the value $(-1)^{\dim(x)}$. Now distribute this curvature to vertices by moving to each vertex in x the value $(-1)^{\dim(x)}/(\dim(x) + 1)$. For the valuation $X(G) = v_k(G)$ the same procedure gives the curvature

$$K(x) = V_{k-1}(x) = \frac{1}{k+1}$$

and the theorem:

Theorem 2 (Gauss-Bonnet). *For any linear valuation X, we have*

$$X(G) = \sum_{v \in V} K(v) .$$

We can do the same thing for multi-linear valuations.

Lets define now the curvature for the Wu characteristic. Given a complete subgraph x of G, define

$$V_k(x) = \sum_l (-1)^l v_{kl}(x) \, ,$$

where $v_{kl}(x)$ counts the number of simplices y of G for which $x \cap y \neq \emptyset$. We have now an integer-valued function on the simplices of G which is the sum of the interactions with neighboring simplices including the self interaction. This **simplex curvature** is

$$\kappa(x) = \sum_{k=0} X(k) V_{k-1}(x)$$

By definition, $X(G) = \sum_{x \subset G} \kappa(x)$. If the value $\kappa(x)$ is broken up and distributed equally to the vertices of x, we get a scalar valued function. It is

$$K_X(v) = \sum_{v \in x} \kappa(x)/(\dim(x) + 1) \, ,$$

where the sum is over all simplices x in G which contain v. The same construction works in the quadratic as well as higher degree case.

Theorem 3 (Gauss-Bonnet). *For any multi-linear valuation X, we have*

$$X(G) = \sum_{v \in V} K_X(v) \, .$$

Proof. One can prove it by induction with respect to the degree k and use Gauss-Bonnet for one dimensions. For $k = 1$, we have the case of valuations. To make the induction step reducing it from k to $k-1$, look for the valuation $A \to \omega(A_1, \ldots, A_{k-1}, A)$ and its curvature $k_{A_1, \ldots, A_{k-1}}(v)$ which is a degree $k-1$ valuation. By induction it satisfies Gauss Bonnet $X(A_1, \ldots, A_k) = \sum_w K(w)$ for a curvature $w \to K(w) = K_{A_1, \ldots, A_{k-1}, v}(w)$. This shows $K(A_1, \ldots, A_k) = \sum_{v,w} K(v, w)$ for some curvature depending on two variables. Now move the value of $K(v, w)$ for any $v \neq w$ equally onto the vertex v and w to get a scalar curvature for X. $\square$

The curvature of a quadratic valuation is now a second order difference operator as the geometry of the ball $B_2(x)$ of radius 2 matters. As we assumed k-valuations to be local in the sense that we discard any contributions $X(x_1, \ldots, x_d)$ if their mutual intersection $\bigcap_j x_j$ is empty, the curvature is localized as such. If we require only that the nerve graph of intersections of simplices is connected, then the curvature of a k-linear valuation has longer range too. For a 3-linear valuation for example, we would consider contributions of chains xyz, where x, y, z are edges building a linear graph of length 2. As for now, we don't count such connections in the valuation, the reason being that the theorems would not work. Including long range valuations could be useful when looking at a more exhausting list of invariants. But currently, we want curvature to be local as this is the case in differential geometry.

Examples.
1) If G is a wheel graph W_n with boundary C_n, then the Euler curvature is $1/6$ on the boundary and $1 - n/6$ at the center. For any wheel graph, the Wu curvature is 1 in the interior and 0 on the boundary.
2) If G is a 3-ball like a pyramid construction over an icosahedron, then the Euler

curvature is supported on the boundary. The Wu curvature however is -1 in interior and 0 on the boundary.

Remark.
Also higher degree multi-linear valuations satisfy **Poincaré-Hopf** and index averaging theorems as we will see later. For any function f on the vertex set and any multilinear valuation, we have an **index** $i_f(v)$ on the vertex set V such that $\sum_v i_f(v) = X(G)$. The index of f for the Euler characteristic is defined as

$$i_f(x) = \chi(B_f^-(x)) - \chi(S_f^-(x)) \ ,$$

where $B_f^-(x)$ is the graph generated by $\{y \mid f(y) \leq f(x) \ \}$ and $S_f^-(x)$ is the graph generated by $\{y \mid f(y) < f(x) \ \}$. There is a similar formula for the Wu characteristic. We also can show that the expectation is curvature $\mathrm{E}[i_f] = K$, when integrating over a reasonable space of functions. Unlike for Euler characteristic, the indices can now be nonzero even at places which are usually regular.

6. THE GRÜNBAUM CONJECTURE

Grünbaum [59] conjectured in 1970 that multi-linear Dehn-Sommerville invariants like quadratic valuations

$$X(G) = \sum_{i,j} a_{i,j} V_{ij}(G) \ ,$$

exist which vanish on geometric graphs. Here, $V_{ij}(G)$ is the number of ordered pairs (x, y) of i-simplices and j-simplices which intersect in a nonempty simplex. We answer this positively: for every degree and every classical Dehn-Sommerville invariant, there is a corresponding multi-linear degree d invariant which is zero on d-graphs, as Grünbaum has suspected.

Lets recall the quadratic self-intersection form $V_{ij}(G)$ which counts the number of i-simplices intersecting with the number of j-simplices. Given a graph G with clique number $d + 1$. Define **Dehn-Sommerville space** $\mathcal{D}_d$ is the linear space of 1-dimensional valuations which are spanned by eigenvectors χ_k of the Barycentric operator A^T for which $d + k$ is even. For every X in $\mathcal{D}_d$ and any d-graph G we have $X(G) = 0$. In other words $\chi_k^T v(G) = 0$, if $v(G)$ is the f-vector of G.

We look now at quadratic valuations of the form

$$Y(G) = \sum_{ij} \psi(j)(-1)^i V_{ij}(G) = \psi \cdot V(G)\chi_1$$

and compare this quadratic valuation with the linear valuation

$$X(G) = \sum_i \psi(j)v_j(G) = \psi \cdot \chi_1 \ .$$

Theorem 4. *For any d-graph, the linear valuation $X(G) = v(G)\psi$ evaluates on G to the same value than the quadratic valuation $\chi_1^T V(G)\psi$ Especially, if $X(G) = 0$, then $Y(G) = 0$.*

In particular, this holds for odd-dimensional d-graphs and $\chi = \chi_1$, where we obtain that the Wu invariant $\omega(G) = 0$. We will later prove the stronger claim that $\omega(G) = \chi(G)$ for such graphs, for which the Euler characteristic is zero.

Proof. We use induction with respect to dimension. For $d = 0$ it is trivial since no simplices can interact and $X = Y$ holds trivially. Given the vector $\Psi = (\Psi(0), \ldots, \Psi(d))$, we write for a simplex x $\varphi(x) = \Psi(\dim(x))$ so that

$$X(G) = \sum_x \varphi(x)$$

and

$$Y(G) = \sum_{x \cap y \neq \emptyset} \sigma(x)\varphi(y) \, .$$

Every pair (x, y) of simplices x, y intersect in some simplex z. Partition the sum into subsets, for which the intersections are the simplex z. Then

$$Y(G) = \sum_z \sum_{x,y,x\cap y=z} \sigma(x)\varphi(y) \, ,$$

where the first sum is over all simplices of G. The induction assumption implies

$$\sum_{x,y,x\cap y=\emptyset} \sigma(x)\varphi(y) = \sigma(G)Y(G) = 0$$

We claim that

$$\sum_{x,y,x\cap y=z} \sigma(x)\varphi(y) = \varphi(z)$$

which immediately proves the theorem. To prove this, partition the sum further. Let $z = (z_1, \ldots, z_k)$. Any m-simplex x different from z defines a simplex of dimension $m - k$ in the $(d - k)$-sphere

$$S(z) = S(z_1) \cap S(z_2) \cap \cdots \cap S(z_k) \, .$$

Since $x \cap y = z$, the simplices x', y' defined by $x = z \cup x', y = z \cup z'$ do not intersect. There are four possibilities: either $y' = z' = \emptyset$ or $y' = \emptyset$ or $x' = \emptyset$ or then - and that is the fourth case - that x', y' are both not empty. In the first case we get $\sigma(z)\varphi(z) = \varphi(z)$. In order to show the result we therefore have to show that the sum of the other three cases is zero.
In the second or third case respectively, we have $\sigma(z)\varphi(y)$ or $\sigma(z)\varphi(x)$ and so a contribution $2(-1)^{d+1}\sigma(z)Y(S(z))$ from the second and third case.
In the forth case we have two non-empty non-intersecting simplices in $S(z)$ and a contribution

$$
\begin{aligned}
\sum_{x',y',x'\cap y'=\emptyset} \sigma(x)\varphi(y) &= \sum_{x',y'} \sigma(x)\varphi(y) - Y(S(z)) \\
&= 2\chi(S(z))Y(S(z)) - Y(S(z)) \\
&= 2(\chi(S(z)) - 1)Y(S(z)) \\
&= 2(-1)^d \sigma(z)Y(S(z)) \, .
\end{aligned}
$$

We have used that the unrestricted sum $\sum_{x',y'} \sigma(x)\varphi(y)$ (without intersections) is equal to $(\sum_{x'} \sigma(x))(\sum_{y'} \varphi(y)) = \chi(S(z))Y(S(z))$. $\qquad\square$

Similarly, higher order Dehn-Sommerville valuations are zero. For example, in the cubic case,

$$V(G)(X, X, Z) \, ,$$

where $X(j) = (-1)^j$ and $V(G)(i,j,k)$ counts the ordered lists of simplices (x,y,z) of dimension i,j,k.

Remark.
If A is the Barycentric refinement operator, one could think that since $Av(G)$ is the f-vector of the Barycentric refinement G_1, also $AV(G)A^T$ is the f-matrix of the Barycentric refinement G_1. This is not the case. It would only hold if $V_{ij}(G)$ counted the number of **all** pairs x,y of i-simplices x and j simplices y, without selecting only the pairs which intersect in a nonempty graph.

Examples.
1) Take the 4-sphere G with f-vector $v = (10, 40, 80, 80, 32)$. The Dehn-Sommerville space is 2-dimensional for $d = 4$ and spanned either by the classical Dehn-Sommerville vectors

$$(0,0,0,-2,5), (0,-2,3,-4,5)$$

or then by two eigenvectors of the transpose of the Barycentric refinement operator:

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 6 & 14 & 30 \\ 0 & 0 & 6 & 36 & 150 \\ 0 & 0 & 0 & 24 & 240 \\ 0 & 0 & 0 & 0 & 120 \end{bmatrix}.$$

The quadratic f-**form** encoding the intersection cardinalities is

$$V(G) = \begin{bmatrix} 10 & 80 & 240 & 320 & 160 \\ 80 & 600 & 1680 & 2080 & 960 \\ 240 & 1680 & 4400 & 5120 & 2240 \\ 320 & 2080 & 5120 & 5680 & 2400 \\ 160 & 960 & 2240 & 2400 & 992 \end{bmatrix}.$$

This quadratic form encodes in how many ways a k-simplex intersects with a l-simplex in G. The Barycentric quadratic valuations are

$$\chi_{k,l} = \chi_k^T V(G) \chi_l \ ,$$

where $\chi_1 = (1,-1,1,-1,1), \chi_2 = (0,-22,33,-40,45), \chi_3 = (0,0,19,-38,55)$, $(0,0,0,-2,5)$ and $(0,0,0,0,1)$ are the eigenvectors of A^T. The Barycentric characteristic numbers are

$$\chi = \begin{bmatrix} 2 & 0 & 240 & 0 & 32 \\ 0 & -4560 & 7760 & -800 & 1440 \\ 240 & 7760 & 47440 & 2720 & 5920 \\ 0 & -800 & 2720 & -480 & 160 \\ 32 & 1440 & 5920 & 160 & 992 \end{bmatrix}.$$

The first row are the standard Barycentric characteristic numbers in the eigenbasis of A^T. The two zeros at the place where the classical Dehn-Sommerville space is. But note that we have now evaluated a quadratic invariant. The fact that the values are the same than the values of the valuations is only due to the fact that we deal with geometric graphs, where also Euler characteristic and Wu characteristic agree. The relation

$$\chi_k^T \cdot V(G)\chi_0 = \chi_k^T \cdot v(G)$$

is not true for all networks.

2) Here are all the quadratic invariants evaluated on a Barycentric basis for the octahedron graph G for which

$$V(G) = \begin{bmatrix} 6 & 24 & 24 \\ 24 & 84 & 72 \\ 24 & 72 & 56 \end{bmatrix}$$

and the matrix $\chi_k V \chi_l$ is

$$\begin{bmatrix} 2 & 0 & 8 \\ 0 & -24 & 24 \\ 8 & 24 & 56 \end{bmatrix}.$$

And here are the cubic invariants:

$$\begin{bmatrix} [2,0,8] & [0,-24,24] & [8,24,56] \\ [0,-24,24] & [-24,-120,120] & [24,120,264] \\ [8,24,56] & [24,120,264] & [56,264,344] \end{bmatrix}.$$

3) First the quadratic invariants for the three sphere, the suspension of the octahedron, are

$$\begin{bmatrix} 0 & 112 & 0 & 16 \\ 112 & 10176 & 224 & 1152 \\ 0 & 224 & -32 & 32 \\ 16 & 1152 & 32 & 240 \end{bmatrix}.$$

Here are all the cubic invariants for a three sphere:

$$\begin{bmatrix} [0,112,0,16] & [112,10176,224,1152] & [0,224,-32,32] & [16,1152,32,240] \\ [112,10176,224,1152] & [10176,703264,21216,76480] & [224,21216,-1056,2880] & [1152,76480,2880,14848] \\ [0,224,-32,32] & [224,21216,-1056,2880] & [-32,-1056,-288,0] & [32,2880,0,864] \\ [16,1152,32,240] & [1152,76480,2880,14848] & [32,2880,0,864] & [240,14848,864,2800] \end{bmatrix}.$$

4) Here the quadratic invariants for a discrete 2-dimensional projective plane:

$$\begin{bmatrix} 1 & 0 & 28 \\ 0 & 56 & 224 \\ 28 & 224 & 336 \end{bmatrix}$$

And here the cubic invariants

$$\begin{bmatrix} [1,0,28] & [0,56,224] & [28,224,336] \\ [0,56,224] & [56,798,1638] & [224,1638,2142] \\ [28,224,336] & [224,1638,2142] & [336,2142,2422] \end{bmatrix}.$$

7. Poincaré-Hopf

Lets first look at the Poincaré-Hopf formula for linear valuations. The **unit ball** $B(v)$ at a vertex v is defined as the subgraph of G generated by the set of vertices in distance ≤ 1 from v. The **unit sphere** $S(x) = \{y \in V \mid (x,y) \in E\}$ which is the **boundary** $\delta B(x)$ of the ball. For $f \in \Omega$ and a linear valuation $X \in \mathcal{V}$ define the **index**

$$i_{X,f}(x) = X(B_f^-(x)) - X(S_f^-(x)),$$

where $B^-(x) = S^-(x) \cup \{x\} = \{y \in B(x) \mid f(y) \leq f(x)\}$ and $S^-(x) = \{f(y) < f(x)\}$.

Theorem 5 (Poincaré-Hopf). $\sum_{v \in V} i_{X,f}(v) = X(G).$

Proof. Start with a single vertex v. Now $B^-(v) = \{v\}$ and $S^-(v) = \emptyset$. Now $i_{X,f}(v) = X(\{v\}) - 0$. Now add recursively a new vertex to get a growing set of graphs G_k which covers eventually G. By the properties of valuations, we have $X(G_n) = X(G_{n-1} \cup B^-(v)) = X(G_{n-1}) + X(B^-(v)) - X(S^-(v))$.   □

[Added, Jan 19: there is an even easier proof: just use the function f to chip fire the values of the valuation "down the dimensions" to the vertices. Since this is deterministic, the integer values are preserved. This immediatly verifies that the index value ends up with $X(B_f^-(v)) - X(S_f^-(v))$ as this adds up the values of all simplices containing x, for which f(x) is the maximum. This picture also explains index expectation: when adding up all thes deterministic "flow lines" we get a "diffusion" as the values of the vertices are chip fired in the curvature equally to the vertices. Its amusing to see this in the case of the valuation $v_d(G)$ which is volume: the index $i_f(v)$ is then the number of d-simplices attached to v for which f has the maximum on v. Its trivial that this is still volume since the volume count on each maximal simplex has been shuffled to a single vertex, determined by the function.]

See [85] for our first proof in the case of Euler characteristic, and for [73] for a Morse theoretical inductive proof. We extended the result to valuations in September 2015 while getting interested Barycentric characteristic numbers, a topic which emerged from from the construction of a graph product.

Let $\Omega(G)$ denote the set of **colorings** of G, locally injective function f on $V(G)$. Let P be a **Borel probability measure** on $\Omega(G) = \mathbb{R}^{v_0(G)}$ and let $E[\cdot]$ its **expectation**. Let $c(G)$ be the **chromatic number** of G. Assume either that P is the counting measure on the finite set of colorings of G with $c \geq c(G)$ real colors or that P is a product measure on Ω for which functions $f \to f(y)$ with $y \in V$ are independent identically distributed random variables with continuous probability density function. For all $G \in \mathcal{G}$ and $X \in \mathcal{V}$:

Theorem 6 (Index expectation). *For any finite simple graph G and Euler characteristic X, we have*

$$\mathrm{E}[i_{X,f}(x)] = K_X(x) .$$

Proof. Let V_k denote the number of k-dimensional simplices in $S(x)$ and let V_k^- the number of k-dimensional simplices in $S_f^-(x)$. Given a vertex $x \in V(G)$ and a k-dimensional simplex K_k in $S(x)$, the event

$$A = \{f \mid f(x) > f(y), \forall y \in V(K_k) \}$$

has probability $1/(k+2)$. The reason is that the symmetric group of color permutations acts as measure preserving automorphisms on the probability space of functions, implying that for any f which is in A there are $k+1$ functions which are in the complement so that A has probability $1/(k+2)$. This implies

$$\mathrm{E}[V_k^-(x)] = \frac{V_k(x)}{(k+2)} .$$

The same identity holds for continuous probability spaces. Therefore,

$$
\begin{aligned}
\mathrm{E}[1 - \chi(S^-(x))] &= 1 - \sum_{k=0}^{\infty}(-1)^k \mathrm{E}[V_k^-(x)] = 1 + \sum_{k=1}^{\infty}(-1)^k \mathrm{E}[V_{k-1}^-(x)] \\
&= 1 + \sum_{k=1}^{\infty}(-1)^k \frac{V_{k-1}(x)}{(k+1)} = \sum_{k=0}^{\infty}(-1)^k \frac{V_{k-1}(x)}{(k+1)} = K(x) .
\end{aligned}
$$

$\square$

See [87] for a proof in the case of continuous distributions. In [96] we adapted the result to finite probability spaces after getting interested in graph colorings.

Assume now $X(A, B)$ is a quadratic valuation like the Wu intersection number. For a fixed subgraph B the map $A \to X(A, B)$ is a valuation and has a curvature $K_B(x)$ as well as an index $i_{f,B}(x)$. The Gauss-Bonnet and Poincaré-Hopf results show that they both sum to $X(G, B)$. This especially applies for $B = G$, so that we have

$$
X(G, G) = \sum_x i_{f,G}(x)
$$

Now we use that for any x, $A \to i_{f,A}(x)$ is a valuation and apply Poincaré-Hopf again to see, using the same function f that

$$
X(G, G) = \sum_{x,y} i_f(x, y) .
$$

We have $(v, w) \to i_f(v, w)$ is zero if $d(v, w) > 1$ and that $i_f(v, w) = i_f(w, v)$.

Define the **index a quadratic valuation** X as

$$
i_f(v) = i_f(v, v) + \sum_{w, (v,w) \in E} i_f(v, w) .
$$

More generally, if $X(A_1, \ldots, A_k)$ is a degree k valuation, we inductively define an index $i_f(v_1, \ldots, v_k)$ first which can only be non-zero if $v_1, \ldots, v_k$ are contained in ball of radius 2, then since this is symmetric in permutations of k, it is divisible by k and can be distributed to the k vertices to get an integer value $i_f(v)$ on the vertex set which is the **index** of X.

Theorem 7 (Poincaré-Hopf for quadratic valuations). *If X is a degree k valuation, then*

$$
\sum_v i_f(v) = X(G) .
$$

Proof. Use induction with respect to k. If $k = 1$, it is Theorem (5). Having verified it for degree $k - 1$, use Theorem (5) for the valuation for

$$
A \to X(A_1, \ldots, A_{k-1}, A)
$$

to get a function $i_f(v_1, \ldots, v_k)$ whose total value $\sum_\pi i_f(v_{\pi(1)}, \ldots, v_{\pi(k)})$ when summing over all permutations gives an index value on each simplex which is divisible by k so that one can assign $i_f(v_1, \ldots, v_k)/k$ to each of the adjacent vertices. $\square$

Lets make this more explicit in the Wu case $X = \omega$. For a fixed vertex v in G and a fixed subgraph G, the valuation

$$A \to X_v(A) = i_f(v)(A)$$

is by definition given by

$$\omega(A \cap B_f^-(v)) - \omega(A \cap S_f^-(v)) \ ,$$

where $B_{f,B}^-$ is part of the ball $B_{f,B}(v)$ in G, where f takes values smaller than $f(v)$ and S_f^- is part of the sphere $S_{f,B}(v)$ in B where f takes values smaller than $f(v)$. Now apply this to get

$$i_f(v,w) = \omega(B_f^-(v), B_f^-(w)) - \omega(B_f^-(v), S_f^-(w)) - \omega(S_f^-(v), B_f^-(w)) + \omega(S_f^-(v), S_f^-(w)) \ .$$

In the case $v = w$, it is

$$i_f(v,v) = \omega(B_f^-(v)) - \omega(S_f^-(v)) \ .$$

This splitting up into cases $v = w$ and $v \neq w$ can be done for a general quadratic valuation and leads to:

Theorem 8 (The index). *The Poincaré-Hopf index $i_f(v)$ for a quadratic valuation X is*

$$
\begin{aligned}
i_f(v) \quad = \quad & X(B_f^-(v)) - 2X(S_f^-(v), B_f^-(v)) + X(S_f^-(v)) \\
+ \quad & \sum_w [X(B_f^-(v), B_f^-(w)) - X(B_f^-(v), S_f^-(w)) \\
& -X(S_f^-(v), B_f^-(w)) + X(S_f^-(v), S_f^-(w))] \ .
\end{aligned}
$$

Examples.
1) Let G be the icosahedron graph and X the Wu characteristic valuation. We take the function f enumerating the indices so that the adjacency matrix of G is

$$
\begin{bmatrix}
0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0
\end{bmatrix} .
$$

Now $i_f(v, w)$ is the symmetric matrix

$$
i_f =
\begin{bmatrix}
1 & -1 & -1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1 & 0 & -1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1 & 0 & -1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 0 & 1 & 1
\end{bmatrix} .
$$

When summing all the entries, we get by Poincaré-Hopf the Wu characteristic $\omega(G) = 2$. The scalar function $i_f(v)$ on vertices v is obtained by summing over each row. It gives

$$i_f = (1, 0, 0, 0, 0, 0, 1, -1, -1, 0, 1, 1) .$$

These indices by the way are the same then the Euler index for the Euler characteristic.

2) For the House graph with adjacency matrix

$$
\begin{bmatrix}
0 & 1 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 1 \\
1 & 0 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & 0
\end{bmatrix} ,
$$

vertex degrees $(2, 3, 3, 2, 2)$ and Euler curvatures $(0, -1/5, -1/6, 0, 1/3)$. For the function $f(x_k) = k$, the Wu index matrix is

$$
i_f =
\begin{bmatrix}
1 & -1 & 0 & -1 & 0 \\
-1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
-1 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0
\end{bmatrix} ,
$$

the Wu indices are $(-1, 0, 1, 1, 1)$ adding up to the Wu characteristic 2. The Wu curvatures are $(0, 2/3, 2/3, 0, 2/3)$. We see that some Wu indices $i_f(v, w)$ are nonzero for v, w of distance 2.

We see that like curvature this is a second order notion. As we have in each induction step the expectation being curvature, we have the

Theorem 9 (Index expectation). *For any finite simple graph G and any k-linear valuation X, we have*

$$\mathrm{E}[i_{X,f}(x)] = K_X(x) .$$

Here is an important lemma which is a generalization of the fact that the function $\sigma(x)$ is a Poincaré-Hopf index on G_1.

Lemma 2. *If G and H are two finite simple graphs and f is a function on the vertices of $G \times H$ ordered according to degree and given lexigographic orders of vertices of G and H, then the Poincaré-index of any valuation X assigning the value $1 + (-1)^k$ to a sphere of dimension k is $i_f(x, y) = \sigma(x)\sigma(y) = \sigma(x \times y)$.*

Proof. By definition,
$$i_f(x, y) = 1 - \omega(S_f^-(x, y)) .$$
The set of vertices $w \times z$ in $S_f^-(x, y)$ is the set of subsimplices of $x \times y$. This is the boundary part of the Barycentric refinement of the boundary of $x \times y$. It is therefore a $k = \dim(x) + \dim(y) - 1$ dimensional sphere and has for any of the Wu characteristics, the value $1 + (-1)^k$. This is 2 for even k and 0 for odd k. This means
$$i_f(x, y) = 1 - (1 + (-1)^k) = -(-1)^k = (-1)^{\dim(x)+\dim(y)} = \sigma(x \times y) .$$
$\square$

The Poincaré-Hopf index $i_f(x, y)$ is a function on the product $V \times W$, if V is the vertex set of G and the W the vertex set of H. The fact $\sum_{x,y} \sigma(x)\sigma(y) = \omega(G)$ is the definition of ω. Applying Poincaré-Hopf shows that that this is $\omega(G_1)$, the Wu characteristic of the Barycentric refinement.

Corollary 1. *The Barycentric subdivision G_1 of G satisfies $\omega(G_1) = \omega(G)$ for any Wu characteristic*

Proof. We will see later that all Wu characteristics of a sphere are all the same and agree with the Euler characteristic. $\square$

Poincaré-Hopf generalizes to multi-linear valuations of higher degree: there is a function $i_f(x_1, \ldots, x_k)$ which is symmetric in $x_1, \ldots, x_k)$ and nonzero only if $x_1, \ldots, x_k$ are in a simplex. The values of $i_f(x_1, \ldots, x_k)$ can all be chip-fired to vertices. But now, for quadratic valuations, the index on vertices is a rational number of the form $p/2$. For refined graphs, it is an integer. For cubic valuations, when chip-fired onto vertices, the charges are now fractions of the form $k/3$.

Theorem 10. *The Wu characteristic ω_k satisfies*
$$\omega_k(G) = \sum_{V \times V \cdots \times V} i_f(v_1, \ldots, v_k)$$
for an index function i_f.

8. PRODUCT PROPERTY

The Cartesian product of two graphs G, H is defined as the incidence graph of the product $f_G \cdot f_H$ in the Stanley-Reisner ring. We have introduced this product in [101] for graphs. (We had at that time not been aware of the Stanley ring [141]. There are various products already known in graph theory. The ring theoretically defined product does the right thing on cohomology, with respect to dimension and also with respect to valuations like Euler characteristic. It has been known for simplicial complexes, but was so far unused in graph theory. It has the properties known from the continuum: in full generality, for any finite simple graph, it satisfies the **Kuenneth formula** [101]
$$H^k(G \times H) = \oplus_{i+j=k} H^i(G) \otimes H^j(G) .$$

It also satisfies the **dimension inequality**

$$\dim(G \times H) \geq \dim(G) + \dim(H) .$$

These two results hold for general finite simple graphs which need not necessarily have to be related to any geometric setup. Like cohomology, homotopy or calculus, the product works for any network.

A special case of a graph product is $G \times K_1$ which is the Barycentric refinement. As $\chi(G) = -f_G(-1)$, the Euler characteristic is multiplicative in general on the class of finite simple graphs. One can see the multiplicity of the Euler characteristic also from the general Euler-Poincaré formula equating cohomological and combinatorial Euler characteristic and the fact that the Poincaré polynomial $p_G(x) = \sum_{k=0} \dim(H^k(G))x^k$ satisfies $p_{G \times H}(x) = p_G(x)p_H(y)$ and $\chi(G) = p_G(-1)$ which implies also $\chi(G \times H) = \chi(G)\chi(H)$.

There is a also a direct algebraic way to see that the χ is preserved when taking a Barycentric refinement $G \to G_1$. Given a graph G, the vertices of G_1 are the simplices of G. The function $I(x) = (-1)^{\dim(x)}$ is a $\{-1, 1\}$- valued function on the vertices of G_1. If f_G is the polynomial in the Stanley ring and the monomials are ordered alphabetically, then this defines a function f on the vertex set of G_1. It turns out that $i_f(x) = I(x)$. Proof: $i_f(x) = 1 - \chi(S^-(x))$. If x is zero dimensional, then $S^-(x)$ is empty and $i_f(x) = 1$. If $x = (a, b)$ is one dimensional and represented by ab in the Stanley ring, then $S^-(x) = \{a, b\}$ has two not connected elements and $i_f(x) = -1$. If $x = (a, b, c)$ is two dimensional and represented by abc, then $S^-(x) = \{ab, bc, ac, a, b, c\}$ has a circular graph as $S^-(x)$ and $i_f(x) = 1$. If $x = (a, b, c, d)$ is three dimensional and represented by $abcd$, then $S^-(x) = \{abc, acd, abc, bcd, ab, ac, ad, bc, bd, cd, a, b, c, d\}$ is a two dimensional sphere graph and $\chi(S^-(x)) = 2$ so that $i_f(x) = -1$. Etc. Now, since $\sum I(x)$ is the Euler characteristic of G by definition and $\sum_x i_f(x)$ is the Euler characteristic of G_1 by Poincaré-Hopf applied to the function f, the invariance of the Euler characteristic under Barycentric refinement has become a consequence of the Poincaré-Hopf formula. Also the product formula of Euler characteristic becomes clear as $f_G f_H$ defines the vertex set of the graph $G \times H$ on which the functions $I(x, y) = I_G(x)I_H(y)$ taking values in $\{-1, 1\}$ are the Poincaré-Hopf index of a function and so the Euler characteristic of $G \times H$.

When computing the Wu characteristic of G, it is the sum $\sum_{x,y} \sigma(x)\sigma(y)$ which lives on vertices and edges of G_1. We can look at the 1-dimensional skeleton simplicial complex of G_1 and look at the corresponding element in the Stanley ring on $(G_1)'$. If $I(x, y)$ is the function, then $\sum I$ is the Wu characteristic. Now again, I can be seen as the Poincaré-Hopf index of a function. It has the property that for vertices $S^-(x) = \emptyset$ and for edges either $S^-(x) = \{a, b\}$ or $S^-(x) = \emptyset$.

Lemma 3. *For any finite simple graph G, we have $\chi(G) = -f_G(-1)$ and*

$$\omega(G) = f_G(-1)^2 - (f_G^2)(-1) .$$

Proof. We know that $\omega(G)$ sums over the monomials of $f_G(x) * f_G)(x)$ containing some square and that $f_G^2(x)$ is the part of $f_G(x) * f_G(x)$ containing no squares. So, if we take $f_G(x) * f_G(x)$ and subtract $f_G^2(x)$ we get the sum of the monomials which contain some square. Evaluating at $(-1, -1, \ldots, -1)$ gives the formula. $\qquad\square$

Theorem 11. *If G, H are two arbitrary finite simple graphs, then*

$$\omega(G \times H) = \omega(G) \cdot \omega(H)$$

for any of the Wu characteristics.

Proof. We know that $f_{G \times H} = f_G \cdot f_H$. Now, if we take the part of f_G^2 which contains a square and the part of f_H^2 which contains a square and multiply them, we get the part of $(f_G f_H)^2$ which contains a square in each part of the variables. Any pair of intersecting simplices x, y in $G \times H$ is of the form $x = (x_1, x_2), y = (y_1, y_2)$, where both x_1 and y_1 and x_2 and y_2 intersect.
We have also combinatorially $\omega(G) = \sum_{x,y} \omega(x)\omega(y)$ and $\omega(H) = \sum_{u,v} \omega(u)\omega(y)$ and $\omega(xu) = \omega(x)\omega(u)$ and $\omega(yv) = \omega(y)\omega(v)$ so that The vertices of $G \times H$ are of the form (xu, yv), the Poincaré-Hopf index sum of $G \times H$ is

$$\omega(G) \cdot \omega(H) = \sum_{(x,u,y,v)} \omega(x)\omega(u)\omega(y)\omega(v) \ .$$

This is a function on pairs of vertices in $G \times H$ from which we can get, by summing over one of the variables, a scalar function on the vertices of the $G \times H$. This is a Wu Poincaré-Hopf index for ω and the result is $\omega(G \times H)$. $\square$

Examples.

1) If $G = K_2$, then $f_G(x, y) = x + y + xy$. The usual product in $Z[x, y, z]$ gives $f_G * f_G = x^2 + 2xy + 2x^2y + y^2 + 2xy^2 + x^2y^2$ and the Stanley product gives $f_G f_G = 2xy$. We have $f_G * f_G(-1, -1) = 1$ and $f_G^2(-1, -1) = 2$, and so $\omega(G) = f_G(-1, -1)^2 - (f_G^2)(-1, -1) = -1$.
2) If $G = K_3$, then $f_G(x, y, z) = x + y + xy + z + xz + yz + xyz$ which satisfies $\chi(G) = -f_G(-1, -1, -1)$. The usual product is $f_G * f_G = x^2 + 2xy + 2x^2y + y^2 + 2xy^2 + x^2y^2 + 2xz + 2x^2z + 2yz + 6xyz + 4x^2yz + 2y^2z + 4xy^2z + 2x^2y^2z + z^2 + 2xz^2 + x^2z^2 + 2yz^2 + 4xyz^2 + 2x^2yz^2 + y^2z^2 + 2xy^2z^2 + x^2y^2z^2$ which satisfies $f_G(-1, -1, -1) * f_G(-1, -1, 1) = 1$. In the Stanley ring, the product is $f_G^2 = f_G f_G = 2xy + 2xz + 2yz + 6xyz$ which satisfies $f_G^2(-1, -1, -1) = 0$. We have $\omega(G) = 1 - 0 = 1$.
3) If $G = C_4$, then $f_G = w + x + wx + y + xy + z + wz + yz$ and $\chi(G)(-1, -1, -1, -1) = 0$ and $f_G * f_G = w^2 + 2wx + 2w^2x + x^2 + 2wx^2 + w^2x^2 + 2wy + 2xy + 4wxy + 2x^2y + 2wx^2y + y^2 + 2xy^2 + x^2y^2 + 2wz + 2w^2z + 2xz + 4wxz + 2w^2xz + 2yz + 4wyz + 4xyz + 4wxyz + 2y^2z + 2xy^2z + z^2 + 2wz^2 + w^2z^2 + 2yz^2 + 2wyz^2 + y^2z^2$ and $f_G^2 = 2wx + 2xy + 4wxy + 2wz + 4wxz + 2yz + 4wyz + 4xyz + 4wxyz$. We have $f_G(-1, -1, -1, -1)^2 - (f_G^2)(-1, -1, -1, -1) = 0 - 0 = 0$.

9. Graphs with boundary

A graph G is called a *d*-**graph with boundary** if every unit sphere is either a $(d-1)$-sphere or a $(d-1)$-ball with $(d-2)$ sphere as boundary. The set of vertices, for which $S(x)$ is a sphere forms a subgraph called the **interior** int(G), the set of vertices, for which $S(x)$ is a ball form the **boundary** δG. We ask that the boundary δG is either a $(d-1)$-graph or that it is empty. The class of graphs with boundary plays the role of **compact manifolds with boundary**. The class of *d*-graphs with boundary are invariant under Barycentric refinement because the boundary operation commutes with the process of taking Barycentric refinements.

Lets look first at a Gauss-Bonnet proof of the **Dehn-Sommerville identities**, which tell that $X(G) = 0$ for the Dehn-Sommerville valuations:

$$X_{k,d}(G) = \sum_{j=k}^{d-1} (-1)^{j+d} \binom{j+1}{k+1} v_j(G) + v_k(G) \ .$$

We have changed the signs slightly. The classical way to write the Dehn-Sommerville valuations is

$$\tilde{X}_{k,d}(G) = \sum_{j=k}^{d-1} (-1)^{j-1} \binom{j+1}{k+1} v_j(G) - (-1)^d v_k(G) \ .$$

The reason for our choice of the sign is that with this choice, the valuations are mostly non-negative for any finite simple graph.

The fact that they are zero for d-graphs follows immediately from Gauss-Bonnet and the fact that their curvature is a Dehn-Sommerville valuation of a unit sphere:

Lemma 4 (Curvature of Dehn-Sommerville is Dehn-Sommerville). *Given an arbitrary finite simple graph and let X a Dehn-Sommerville valuation, then its curvature satisfies $K(x) = X_{k-1,d-1}(S(x))$.*

Proof. The curvature of the valuation is a local valuation is a shifted functional $X_{k-1,d-1}/d$. To see this, note the identity

$$X_{k+1,d+1}(l+1)/(l+1) = X(k,d)(l)/(k+2) \ .$$

$\square$

Dehn-Sommerville relations are usually considered only for polytopes or topological manifolds and not for arbitrary finite simple graphs. Interestingly, for most G with clique number d, we see $X_{k,d}(G) \geq 0$ for $k \geq 0$. We see very rare instances of $X_{0,d} < 0$ and have not seen an example with $X_{1,d}(G) < 0$. It is not a surprise as the curvature is a Dehn-Sommerville invariant of a lower dimensional graph, which if rarely negative makes it unlikely that the higher invariant is negative too. The Euler characteristic $X_{-1,d}$ on the other hand is negative pretty frequently. And then there is the classical result of Dehn-Sommerville-Klee on the vanishing of Dehn-Sommerville relations. Note however also here that we never refer to the continuum. The theorem is entirely graph theoretical:

Corollary 2 (Dehn-Sommerville). *If G is a d-graph, then each Dehn Sommerville valuation X_k with $k \geq 0$ satisfies $X(G) = 0$ and the curvature of X is constant zero.*

Proof. We use Gauss-Bonnet and induction with respect to dimension as well as the previous lemma. The curvature is zero because it is a Dehn-Sommervile invariant of a sphere, which is a smaller dimensional graph. $\square$

As before, we write $\sigma(x) = (-1)^{\dim(x)}$ for a complete subgraph x of G.

For a complete subgraph z of G define

$$\omega_z(G) = \sum_{z=x \ capy} \sigma(x)\sigma(y)$$

which sums over all interaction pairs (x, y) of simplices in G which have z as their intersection. Because the intersection of two complete subgraphs x, y is a complete subgraph of G, we have partitioned all intersections and can write

$$\omega(G) = \sum_z \omega_z(G) ,$$

where the summation is over all complete subgraphs z of G.

Given a complete subgraph z of G with vertex set $a_1, \ldots, a_k$, define $S(a_1, \ldots, a_k)$ as the intersection of the spheres $S(a_1) \cap S(a_2) \cdots \cap S(a_k)$. If G is a d-graph, then $S(a_1) \cap S(a_2) \cdots \cap S(a_k)$ is a $(d-k)$-sphere. This can be seen by induction: $S(a_1)$ is a $(d-1)$-sphere by definition. Since $S(a_2) \cap S(a_1)$ is a $(d-2)$-sphere in $S(a_1)$, it is also a $d-2$ sphere in G etc.

Lemma 5. *If G is a d-graph, then $\omega_z(G) = \sigma(z)$.*

Proof. Look at all the interactions in which both x, y contain z properly: this gives $\omega(S_z)\sigma(z)$. Then look at all the interactions in which one is z. This gives $-2\omega(S_z)\sigma(z)$. Then there is the case when both $x = z, y = z$ which gives 1. The sum is either 1 or -1. $\square$

Theorem 12. *Given a d-graph G, then $\omega_k(G) = \chi(G)$ for any Wu characteristics ω_k.*

Proof. We can follow the same proof which worked in the case of a quadratic valuation using the Euler characteristic and a Dehn-Sommerville valuation. Again use induction. What happens still is that $\omega(z) = \sigma(z) = (-1)^{\dim(z)}$ which then implies

$$\omega(G) = \sum_z \sum_{x \cap y = z} \sigma(x)\sigma(y) = \sum_z \omega(z) = \chi(z) .$$

To see this, there are again four cases, $x = y = z$, the cases when one x or y is equal to z or then if both are not equal to z. The contribution of the first is 1, the contribution of the second and third is $2\chi(S(z))$, The contribution of the last $\chi(S(z))^2 - \chi(S(z))$. If $\chi(S(z)) = 2$, we get the sum $1 - \chi(S(z)) = -1$. If $\chi(S(z)) = 0$, the contribution is 1. We see that the $\omega(z) = \sigma(z)$. The multi-linear case follows inductively. $\square$

Theorem 13. *For a d-graph G and a vertex v, then the Wu curvature $K_\omega(v)$ is equal to the Euler curvature $K_\chi(v)$.*

Proof. Fix a vertex v and fix a simplex x of dimension k containing v. It is enough to show that for this x, each contribution

$$\sum_{y \cap x \neq \emptyset} \sigma(x)\sigma(y)$$

to the curvature is zero. We can write this as

$$\sigma(x) \sum_{z \subset x} [\sum_{y \cap x = z} \sigma(y)] .$$

As before, for fixed $z = (a_1, \ldots, a_k)$, we have

$$\sum_{y \cap x = z} \sigma(y) = \sigma(z) .$$

Every y defines a simplex in $S(a_1) \cap \cdots \cap S(a_k) \subset x$ of dimension $\dim(y) - k$. The sum $\sum_{z \subset x} \sigma(x)$ is the Euler characteristic of a contractible set which is 1. This shows that the contribution to the curvature is $\sigma(x)/(dim(x)+1)$ is the same than the contribution for the Euler curvature. $\qquad\square$

Examples.
$d = 1$: there are no other interactions. Curvature is zero.
$d = 2$: every triangle has $d(a) + d(b) + d(c) - 6$ additional triangle-triangle connections and $d(a) + d(b) + d(c) - 6$ additional triangle-edge connections. Every edge has additional $d(a) + d(b) - 2$ edge edge connections and $d(a) + d(b) - 2$ edge-triangle connections.

Theorem 14 (Boundary formula). *The Wu characteristic of a d-graph G with boundary δG satisfies*
$$\omega(G) = \chi(G) - \chi(\delta G))$$
where χ is the Euler characteristic.

Proof. Again, we prove this by induction with respect to dimension d. For $d = 1$, a 1-graph with boundary is a line graph which has $\omega(G) = -1 = \chi(G) - \chi(\delta G)$. Again, write the sum as
$$\sum_{x,y} \sigma(x)\sigma(y) = \sum_z \sum_{x \cap y = z} \sigma(x)\sigma(y) .$$
By induction, For a boundary simplex z, the sum $S(z) = \sum_{x \cap y = z} \sigma(x)\sigma(y)$ is either -1 or 1 depending on the dimension of z. The reason is that this sum $S(z)$ is $\chi(B(z)) - \Omega_z(G)$ which is 1 or -1. But this is the Euler characteristic of a thickened boundary of G which, since homotopic to the boundary has the same Euler characteristic than the boundary. $\qquad\square$

Li Yu [152], it is shown that any real valued (not necessarily linear) invariant of a compact combinatorial manifold with boundary which is invariant under Barycentric subdivision is determined by the two numbers $\chi(G)$ and $\chi(\delta G)$. The Wu invariant is such a real-valued invariant and the boundary formula gives an example for the Yu theorem.

An immediate corollary of this formula is:

Corollary 3. *For even dimensional d-graphs without boundary $\omega(G) = \chi(G)$. For odd dimensional d-graphs without boundary $\omega(G) = 0$.*

Proof. For odd dimensional d-graphs G, we have $\chi(G) = 0$ and for even dimensional d-graphs G the boundary δG is odd dimensional and $\chi(\delta G) = 0$. $\qquad\square$

Remarks.
1) It again follows that the Wu invariant of a d-ball is $(-1)^d$.
2) For a general finite simple graph, let δG denote the **subset** of V where K_ω and K_χ do not agree. If we define $\omega(\delta G)) = \sum_{x \in \delta G} K_\omega(x)$ and $\chi(\delta G)) = \sum_{x \in \delta G} K_\chi(x)$, then the formula $\omega(G) - \omega(\delta G) = \chi(G) - \chi(\delta G)$ would hold. for any finite simple graph.
3) Its follows that the Wu characteristic is a combinatorial invariant for manifolds with boundary.
4) It also follows that the Wu characteristic is a cobordism invariant like Euler

characteristic: if H, K are cobordant using an even dimensional graph, then H, K both have Wu and Euler characteristic zero. If G is odd dimensional, then H, K are odd dimensional without boundary and $\omega(H) = \chi(H) = \chi(K) = \omega(K)$.

It seems that χ and ω are essential prototypes as ω_3 behaves again in the same way than χ:

Theorem 15. *If G is a d-graph with boundary, then*

$$\omega_3(G) = \chi(G) .$$

Remarks. 1) Again, this illustrates the theorem of Yu [152] that any possibly nonlinear combinatorial invariant depends only on the Euler characteristic of G and δG. But as the quadratic Wu characteristic, also the cubic characteristic not only involves the f-vector but also higher f-tensors.
2) Euler characteristic for geometric even dimensional graphs has a nice "Hilbert-action" type interpretation as it is the average over a naturally defined probability space of two dimensional subgraphs and so an average of "scalar curvatures" obtained by averaging all sectional curvatures through a point. [86]. The Wu-invariant makes the Euler characteristic of even dimensional graphs look even more "interaction like". It not only can be seen as a super count of the "indecomposable parts" of space given in terms of simplices; it is also a super count of the interactions between these indecomposable parts. The interactions of equal type (Fermionic-Fermionic pairs or Bosonic-Bosonic pairs) are counted positive, the interactions of opposite type (Bosonic-Fermionic) are counted negative. The result that this number still has a geometric interpretation as is remarkable even in the odd dimensional case, where the Euler characteristic is zero.

10. More examples

Graphs without triangles can be seen as **one dimensional curves**. One can force on any finite simple graph (V, E) a simplicial structure which is one dimensional and ignore the Whitney complex. This is done by taking the 1-dimensional skeleton complex $V \cup E$. Graphs without triangles only have Betti numbers $b_0 = dim(H^0(G))$ counting the number of connectivity components and $b_1 = dim(H^1(G))$ counting the number of generators for the fundamental group. The Euler characteristic of this complex can then be given by Euler-Poincaré in two ways as $v_0 - v_1 = b_0 - b_1$. For example, if G is the cube graph where $v_0 = 8, v_1 = 12$, and $b_0 = 1, b_1 = 5$. As for the Euler curvature, the Wu curvature is local but while Euler curvature depends on the disc of radius 1, the Wu curvature depends on a disk of radius 2.

Lemma 6. *The Wu curvature of a graph G without triangles at a vertex x is $K(x) = 1 - 5d/2 + d^2/2 + \sum_i d_i/2$, where d is the vertex degree at x and d_i are the vertex degrees of vertices neighboring x.*

Proof. The vertices contribute $1 - d$, where 1 is the self interaction and $-d$ the interaction with the d neighboring edges. The edges contribute 1 for the self interaction, -2 for the interaction with the neighboring vertices, and then $d_i - 1$ for the interaction with the neighboring edges. We have so the curvature contribution of the edge (a, b) with vertex degree $d(a) = d, d(b) = d'$:

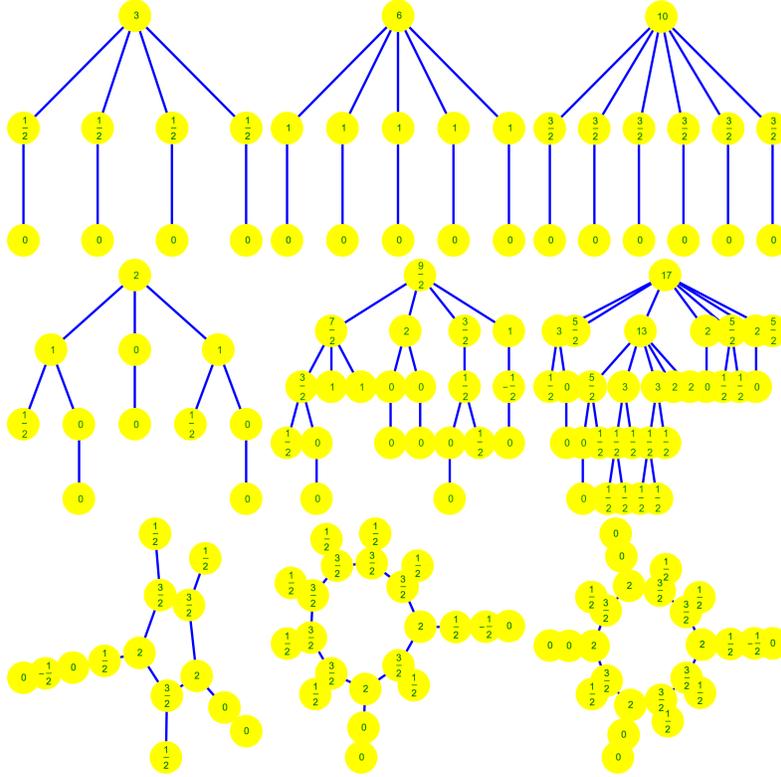$$(1 - 2 + (d - 1) + (d' - 1))/2 = -3/2 + d/2 + d'/2$$

FIGURE 5. Examples of Wu curvatures of graphs without triangles. First we see the Wu curvatures of three star graphs, then we see the Wu curvatures for three random trees and finally for three random sun graphs.

from each edge. The vertex contribution 1 plus the sum over all these edge contributions gives

$$1 + d + \sum_i (-3/2 + d/2 + d_i/2) = 1 - 5d/2 + d^2/2 + \sum_i d_i/2 .$$

$\square$

Remarks.
1) If G is a Barycentric refinement of a graph without triangles, then $K(x) = 1 - 5d/2 + d^2/2 + d$ at every vertex not adjacent to a non flat point. Including the curvatures $(3d - 8)/2$ at the adjacent points to the degree d branch point, we get the total curvature contributing to a branch point

$$K(x) = (1 - d/2)(1 - 2d) .$$

This is the number we should assign to a branch point in the continuum limit. The curvature attached to an intersection of d rays is $(1 - d/2)(1 - 2d)$.
2) Motivated from the structure of **spin networks** in quantum loop gravity, where the edges are equipped with curvature, one could also attach the curvature to the edges and give each edge $d = (a, b)$ a spin value $(d(a) + d(b) - 5)$, where $d(a), d(b)$

are the degrees at the vertices a and b.

While the variational problem of maximizing the Wu characteristic has to be confined to graphs with a fixed number of vertices to be interesting, the global minimum on the class of connected graphs without triangles is known:

Corollary 4. *Among the class of all connected graphs without triangles, the line graphs minimize the Wu characteristic.*

Proof. We see that the only way to get a negative curvature is to have $d = 1$. It follows that the Wu characteristic for a graph without triangles is only negative for a line graph like K_2. It is zero for a union of circular graphs and otherwise positive. $\square$

Examples:
1) For a circle bouquet graph, a collection of k-circles hitting a common point, the Wu characteristic is $4k^2 - 5k + 1$.
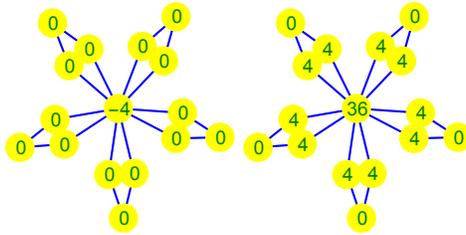
FIGURE 6. The Euler and Wu curvatures of a 1-dimensional sphere bouquet of 5 spheres. The Euler characteristic is -4, the Wu characteristic is 76 which is $4k^2 - 5k + 1$ for $k = 5$.
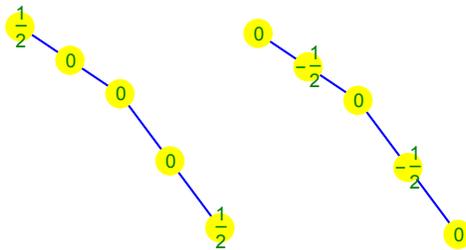

FIGURE 7. The Euler and Wu curvatures of a line graph, a 1-dimensional disk. The curvatures start to be different in distance 1 to the boundary. The Euler characteristic is 1, the Wu characteristic of the disk is -1. The one dimensional line graphs are the only connected graphs without triangle for which the Wu characteristic is negative.

Here are some constructions.

2) adding a circular loop at a two dimensional graph decreases the curvature at the glue point by 1.

3) adding a two dimensional sphere to a two dimensional graph increases the Wu characteristic by 1. This means that it decreases the curvature by 1.

4) Gluing a two and three dimensional sphere along a point produces a graph of Wu characteristic 1.

It follows from $\chi(G) - \chi(\delta G)$ that for 2-graphs with boundary, the Wu curvature is $1 - d(x)/6$ in distance 2 to the boundary. At the boundary, the Wu curvature is 0.
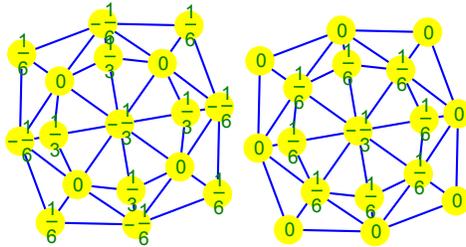
FIGURE 8. The Euler curvatures and Wu curvatures of a 2-disk. Both, Euler characteristic and Wu characteristic are 1. The later follows because the boundary, a circular graph, has zero Euler characteristic.

For 3-graphs with boundary, the curvatures in the interior is $K_1(x) = (1 - V_1(x)/2 + V_2(x)/3 - V_3(x)/4)$. Also the Dehn-Sommeville invariant and $K_{d=1}(x) = (2V_2 - 3V_3)$ is zero in the interior. If we take $2K_1(x)$ as the curvature on a ball, then this gives the right curvature for a ball and for a torus.

More examples

Lemma 7. *If G, H are two d-graphs and x is a vertex in G and y a vertex in H, then the connected sum $G \cup_{xy} H$ of G and H connected along $S(x)$ and $S(y)$ satisfies $\omega(G \cup_{xy} H) = \omega(G) + \omega(H) - \omega(S)$, where S is a d-sphere.*

11. INTERSECTION NUMBERS

Given a graph G and two subgraph graphs K, H of G, we can look at

$$\omega(H, K) = \sum_{x,y} (-1)^{\dim(x) + \dim(y)}$$

where x is a subsimplex of K and y is a subsimplex of H and the sum is over all pairs for which the intersection is not empty. When seen like this, the Wu characteristic is $\omega(G) = \omega(G, G)$. But we can look at the number $\omega(H, K)$ as an **intersection number**.

Examples:
1) if K, H are two simple curves on a graph intersecting simply in a point, then $\omega(K, H) = 1$. Here is the computation: assume xyz is the first arc and abc the second. The interactions are *by* counting 1 and xyb, yzb, aby, bcy counting -1 and

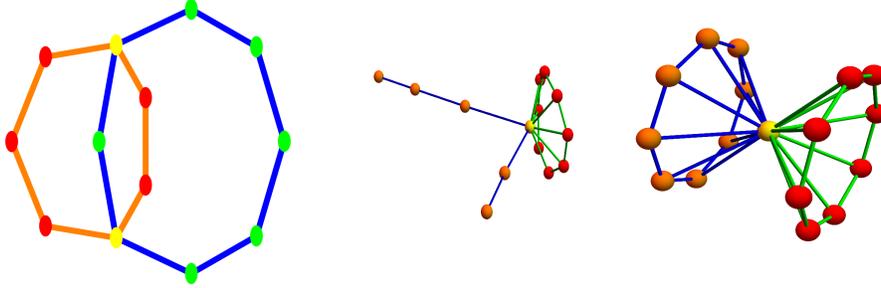then $xyab, yzab, yzab, yzbc$ counting 1. The total sum is 1.

FIGURE 9. Intersection examples: for two one dimensional graphs
A, B intersecting in n points, the intersection number is $\omega(A, B) = n$. For a 1-dimensional graph intersecting with a two dimensional
surface (here a disc), the intersection number is -1. For two two
dimensional surfaces intersecting in a point, the intersection num-
ber is 1.

Lets look at the cubic valuation $\omega_3(G) = \sum_{x,y,z} \sigma(x)\sigma(y)\sigma(z)$ summing over all
triples x, y, z of simplices which have a nonempty common intersection.

Examples.
1) In the case of $G = K_3$ with $f_G = x + y + z + xy + yz + xz + xyz$, there are
already 175 possible ordered triples of non-intersecting complete sub graphs like

$$\{x, xy, xyz\}, \{x, x, x\}, \{x, x, y\}, \{xyz, xy, xy\} .$$

2) The behavior of cubic valuations on complete graphs K_k is the same than for χ.
We have $\omega_3(K_d) = 1$.

Theorem 16. *For d-graphs with boundary and any* **even** $k > 1$:

$$\omega_k(G) = \chi(G) - \chi(\delta G)$$

For any **odd** k *we have*

$$\omega_k(G) = \chi(G)$$

Proof. The proof bootstraps and uses that for a unit sphere $\omega_2(S(x)) = \chi(S(x))$.
□

In other words, we know the Wu polynomial for d graphs.
The examples of star illustrates how branch points produces quadratic growth in
the number of branches. These numbers are of algebra-geometric nature as they go
over to the continuum. Lets explain a bit more algebro-geometric relation.

12. QUESTIONS

A) Is the Wu characteristic the only multiplicative quadratic valuation on graphs
which assigns the value 1 to points? Such a result was suggested by Wu in the
50ies and Grünbaum cautioned in the 70ies to consider Dehn-Sommerville cases.
We know that the Euler characteristic is the only linear valuation on the category

of finite simple graphs which assigns the value 1 to points. This fact has been proven a couple of times and can be seen from the Barycentric refinement operator A, which maps $v(G)$ to $v(G_1)$ as it has only one eigenvalue 1 and the eigenvector of A^T is $(1, -1, 1, -1, \dots)$ leading to Euler characteristic. While we understand the behavior of the f-vector under Barycentric refinement, we don't understand yet the behavior of the f-matrix V under Barycentric refinement but we have not looked yet really. In the case of Barycentric refinements, we have searched for the law first by data fitting, proved it and then found it in the literature. Any possible law $V(G) \to V(G_1)$ should be detectable like that if it exists.

B) Assume we know all the multi-linear valuations of a graph. How much of the graph G is determined by these numbers? Are there non-graph isomorphic graphs with the property that all k-linear valuations are the same? Yes, if we restrict to the local valuations, we can already find trees which are not isomorphic but have all valuations the same. More difficult is to answer the question modulo homeomorphism equivalence [97] or even homotopy. Can we get a complete set of invariants by relaxing the local property? As the story of invariants for geometric objects has shown repetitively, it would be surprising to have an exhaustive set of generalized valuations which are invariants. The complexity of the graph isomorphism problem is not settled but it is not excluded yet that a polynomial number of valuations together with maybe other numbers like spectra suffice to determine the isomorphism type. History has shown that these are tough questions and that even the search for counter examples can be computationally hard. Here is a quick experiment: on all connected graphs of 4 and 5 vertices, linear and quadratic valuations determine the isomorphism class uniquely (there are 6 in the case of 4 vertices and 21 in the case of 5 vertices). But for 6 vertices already, there are 112 isomorphism classes of connected graphs but only 101 different linear and quadratic valuation patterns so that some graphs have the same pattern. Including the cubic valuations already allows to distinguish 108 isomorphism classes still missing 4. Including quartic valuations still does not resolve more. Indeed, the following two graphs G, H with Stanley-Reiner ring elements $f_G = a + b + c + d + e + ab + bc + cd + de + df$ and $f_H = a + b + c + d + e + ab + bc + cd + de + cf$ have the same f-forms for all degrees. We have $v(G) = v(H) = (-1, 5)$ and $V(G) = V(H) = \begin{bmatrix} 1 & 5 \\ 5 & 15 \end{bmatrix}$ and the f-cubic form is $V_3(G) = V_3(H) = \begin{bmatrix} \{5, 11\} & \{11, 21\} \\ \{11, 21\} & \{21, 41\} \end{bmatrix}$. By the way, the two graphs have different Kirchhoff spectra.

C) Is there any significance of the f-**spectrum**, the spectrum of the quadratic f-form V defined by G? This spectrum is of course the same for isomorphic graph. Are there isospectral graphs in the sense that they are not isomorphic but have the same f-spectrum? Is there a significance to the number of negative eigenvalues of V? Is there a significance of the Perron-Frobenius eigenvector which always exist as V is a positive matrix if the graph is connected? What happens with the spectrum when applying Barycentric refinements repetitively? We have seen that the spectrum of the Laplacian etc converges universally [103, 98]. The number of negative eigenvalues of the f-matrix varies from graph to graph. The star graph S_3

with 4 vertices has the f-matrix $V(G) = \begin{bmatrix} 4 & 6 \\ 6 & 9 \end{bmatrix}$ with eigenvalues 13 and 0. For
all other star graphs $V(G)$ is positive definite. For circular graphs C_n with $n \geq 4$,
there is always one negative and positive eigenvalue. For complete graph K_k, the
number of negative eigenvalues of $V(G)$ is $[k/2]$. We see that in general the number
of negative eigenvalues in the f-spectrum can change if G undergoes a Barycentric
refinement. For the f-vector already, the inverse problem of characterizing the pos-
sible f-vectors is interesting. In the quadratic case we don't know anything about
the possible f-matrices.

D) For each valuation functional one can look at its variational problem: it is
the problem to maximize or minimize among all graphs with n vertices. When
restricting to d-graphs of a fixed number of vertices, maximizing volume is related
to the upper bound conjecture. This shows that the problem of maximizing a
general linear or quadratic valuation can be hard even in very intuitive cases. As
for Euler characteristic, one can look at the expectation of the functional on Erdös-
Renyi graphs. For any n, we get so a function on p. We can ask to maximize or
minimize this. For $n = 6$, the utility graph is the only maximum and the maximal
value of the Wu characteristic is 15. We have still to find a connected graph G for
which $\omega(G) < -v_0$. Do such examples exist? Attaching hairs to 2-spheres shows
that we can for any $C < 1$ find graphs G of n vertices for which $\omega(G) < -C\sqrt{n}$.
Also for any $C < 1$, there are graphs of order n with $\omega(G) \geq Cv_0^2$. Is it true that
for any connected graph G with n vertices, the bound

$$-\sqrt{n} \leq \omega(G) \leq n^2$$

holds for Wu characteristic? Monte Carlo experiments with smaller random graphs
suggest such an estimate to hold, but this can be misleading and be a case for
the law of small numbers. For Euler characteristic, where we know the expecta-
tion exactly on the Erdös-Renyi space $E(n,p)$ explicitly [82]. The expectation is
$\mathrm{E}_{n,p}[\chi] = \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} p^{\binom{k}{2}}$ showing that there is $C > 0$ and arbitrary large
graphs with n vertices for which $\chi(G) \geq \exp(Cn)$ despite the fact that we can not
explicitly construct such examples. This might also happen for the Wu character-
istic.

E) Valuations can be studied from a statistical mechanics point of view. Define for
every graph and a fixed function J on edges of the intersection graph of simplices
and fixed 0-form h on the simplices the interaction energy

$$H(G) = -\sum_{(x,y)} J(x,y)\sigma(x)\sigma(y) - \sum_{x} h(x)\sigma(x) .$$

For a particular graph theoretical approach, see [16]. A simple case to look at
$H(G) = \omega(G)$, where $J = 1$ and $h = 0$. As notation from the Ising model is close,
the spin values $\sigma(x) = \omega(x)$ of a simplex x is geometrically defined and (x,y) are
all pairs of simplices which intersect. As custom, one can then define a probability
distribution function $e^{-\beta H(G)}/Z_\beta$ on the Erdös-Rényi space of all graphs on a fixed
vertex set, where $Z_\beta = \sum_G e^{-\beta H(G)}$ is the partition function. For any functional f
on graphs, there is a mean value $\sum_G f(G)P_\beta(G)$ which could be studied in the limit
$n \to \infty$. One can also consider models where a host graph E is fixed and exhausts
it using natural sequences E_n of graphs. But unlike the Ising model, where the

underlying graph is fixed and the spin configurations σ are changed, the geometry alone determines the Wu energy. This renders the story different. If we look at random Erdös-Rényi models, then we can look at the average Wu characteristic. Unlike in the case of Euler characteristic, we were able to give an explicit formula for the expectation value $\mathrm{E}_{p,n}[\chi]$ we don't have an expectation for the Wu characteristic yet.

F) The Euler characteristic satisfies in full generality the Euler-Poincaré formula equating cohomological and combinatorial versions of the Euler characteristic

$$b_0 - b_1 + b_2 - b_3 \cdots = v_0 - v_1 + v_2 - \ldots .$$

As noticed in [90], it was Benno Eckmann [38] appeared have been the first to point out this connection in a purely discrete setting without digressing to the continuum. An intriguing question is whether there is a cohomology defined for general finite simple graphs which produces an Euler-Poincaré formula for the quadratic Wu characteristic. It would have to be a sort of discrete intersection cohomology, where a general finite simple graph now plays the role of a perverse sheaf. This is not so remote, as finite simple graphs are an Abelian category which contain discretizations of perverse sheaves. If a cohomology should exist which produces the Euler-Poincaré formula for the Wu characteristic, then one can expect that its exterior derivative d defines an operator $D = d + d^*$ whose Laplacian $L = D^2$ produces a McKean-Singer formula equating the Wu characteristic with the super trace of the heat kernel $\exp(-tL)$.

G) For d-graphs with boundary, the formula $\omega(G) = \chi(G) - \chi(\delta G)$ computes the Euler characteristic of the "virtual interior" chain of the graph. This also works for complete graphs K_{d+1} for which the Barycentric refinement is a d-ball. For the triangle $G = K_3 = xyz + xy + yz + xz + x + y + z$ for example, $\omega(G)$ computes the Euler characteristic of the chain xyz. Unlike graphs, chains form a Boolean algebra and valuations are linear and nice. The category of graphs is somehow like a manifold in the linear space of chains as adding two graphs throws us out of the graph category. Lets write $\oplus$ for the Boolean addition on the Boolean algebra $2^{(2^V)}$ of chains. The addition is the symmetric difference on each simplex level. The example $(xy + x + y) \oplus (yz + y + z) = xy + yz + x + z$ shows that the sum is no more a graph. The product $(xy + x + y) \star (yz + y + z) = y$ which is the intersection on each dimension however is a graph. The example $f = xyz + xy + yz + xz + x + y + z = (xy + yz + yx + x + y + z) \oplus xyz = g + h$ shows how to break up a triangle into two chains. The first is the boundary chain, the second is the interior chain. But this decomposition is only possible in the ambient Boolean ring and not in the category of graphs. As in the Stanley-Reisner picture (even so the multiplication in that ring is different), the Euler characteristic the triangle as $-f(-1,-1,-1) = 1$, the Euler characteristic of the boundary is $-g(-1,-1,-1) = 0$, and the interior is $-h(-1,-1,-1) = 1$. We see that the Wu characteristic of K_n is the Euler characteristic of the interior of K_n and the same holds for a d-ball. In [97] we defined a notion of homeomorphism for graphs which treats graphs as higher dimensional structures which is compatible with homotopy, cohomology and dimension by design. The topology of a graph is given by a base of subgraphs whose nerve is homotopic to the graph. We have seen that such a topology always works and is given by the star graph. This picture can now be

generalized a bit by allowing the base to consist of virtual open balls.

H) This question is still under investigation: what happens with the f-matrix $V(G)$ of a graph G if it undergoes Barycentric refinement $G \to G_1$? As the f-vector $v(G)$ is transformed with a universal linear transformation, one can expect something similar to happen for the f-matrix, or more generally for the intersection matrix $V(A, B)$ for two intersecting graphs. If $G = C_n$, then $V(G) = \begin{bmatrix} n & 2n \\ 2n & 3n \end{bmatrix}$. For graphs without triangles, the f-matrix is a symmetric 2×2 matrix and there are three independent components. We can get the Barycentric refinement using linear algebra: as for a cycle graph C_4 we have $(V_{11}, V_{12}, V_{22}) = (4, 8, 12) \to (8, 16, 24)$, K_2, we get $(2, 2, 21) \to (3, 4, 4)$ and for the figure 8 graph, we have $(7, 16, 32) \to (15, 32, 56)$. The transformation matrix can be found easily as

$$A = \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 0 & 2 & 0 \\ 0 & \frac{3}{2} & 1 \end{bmatrix} .$$

It has eigenavalues $2, 1, 1$. At the moment, it looks as if this does not work any more for 2-dimensional graphs with clique number 3, where we look for a 6×6 transformation matrix A. For K_3 for example, we have $(3, 6, 3, 9, 3, 1)$ mapping to $(7, 24, 18, 78, 54, 36)$ reflecting the fact that the f-matrix of K_3 is $\begin{bmatrix} 3 & 6 & 3 \\ 6 & 9 & 3 \\ 3 & 3 & 1 \end{bmatrix}$ and the f-matrix of its Barycentric refinement (a wheel graph with C_6 boundary) is $\begin{bmatrix} 7 & 24 & 18 \\ 24 & 78 & 54 \\ 18 & 54 & 36 \end{bmatrix}$. When trying to find a linear transformation relating the two f-matrices using a set of 6 independent graphs, it does not work any more in general. There could be a possible on some subclasses of graphs, as there appear linear relations necessary. Also a nonlinear transformation rule $V(G) \to V(G_1)$ can not yet be ruled out.

13. CONCLUSION

According to [17], there are two basic counting principles which underlie most of arithmetic. It is that for two disjoint finite sets, $|A \cup B| = |A| + |B|$ and $|A \times B| = |A| \times |B|$. This fundamental structure is also present for graphs and there is one quantity $|A|$ which in particular satisfies these two properties: it is the Euler characteristic. In this light, the Wu characteristics and its higher order versions which were discussed here, are also fundamental as they satisfy both counting principles.

A quadratic valuation attaches to a pair of simplicial complexes a number $\omega(A, B)$ such that $A \to \omega(A, B)$ and $B \to \omega(A, B)$ are both valuations. Similarly, as valuations are linear in the f-vector v where v_i is the number of simplices of dimension i, a quadratic valuation is a linear function on f-quadratic forms V, where V_{ij} is the number of simplices which are intersections of i and j dimensional simplices. The function $G \to \omega(G, G)$ in particular defines a functional on simplicial complexes which measures a self interaction energy. It does so especially for finite simple graphs equipped with the Whitney complex. An example of a quadratic valuation

is the Wu characteristic $\omega(G)$ [149] which sums over $\omega(x \times y)$ where (x, y) is an ordered pair of complete subgraph of G which have non-empty intersection. It is more subtle than Euler characteristic as unlike the later, the higher order Wu characteristics are not a homotopy invariant, and so is more subtle like the Bott invariant defined in or Reidemeister (analytic) torsion. We prove that ω is invariant under edge or Barycentric refinements, that it behaves additively with respect to connected sums and multiplicatively with respect to Cartesian products. We prove a Gauss-Bonnet formula for multi-linear valuations and use this to show that for a geometric graph G with boundary δG, all the higher order Wu characteristic satisfy $\omega(G) = \chi(G) - \chi(\delta G)$, where χ is the Euler characteristic. Simple examples like discrete curves show that the curvature is now nonlinear in the vertex degree. The setup allows to construct higher order Dehn-Sommerville relations, answering so positively a conjecture of Grünbaum [59] from 1970. The simplest example is the vanishing of the quadratic valuation $X_{1,3}(M) = \sum_{k,l} \chi_1(k)\chi_3(l)V_{kl}(G) = \chi_1^T V \chi_3$ with $\chi_1 = (1, -1, 1, -1, 1), \chi_3 = (0, -22, 33, -40, 45)$, where the quadratic f-form $V_{kl}(G)$ counts the number of intersecting K_k and K_l subgraphs in a triangulation of a 4-manifold M. More generally, we show that if Y is a Dehn-Sommerville invariant then the quadratic valuation $X(G) = \sum_{k,l} Y(l)(-1)^k V_{kl}(G)$ is zero on d-graphs. The lack of homotopy invariance is not a surprise, as most linear valuations already are not homotopy invariant. It is still not known whether the quadratic Wu invariant is the only quadratic invariant valuation on general graphs which is invariant under Barycentric refinement. For discrete curves, graphs without triangles for example, the curvature for Euler characteristic is $1 - d(x)/2$ while the curvature of the Wu characteristic is $K(x) = 1 - 5d/2 + d^2/2 + \sum_i d_i/2$, where d is the vertex degree at x and d_i the neighboring vertex degrees. Quadratic valuations also produce intersection numbers and are interesting for algebraic sets, where they produces combinatorial invariants for varieties which go beyond Euler characteristic as the quadratic Dehn-Sommerville curvatures at isolated singularities report on their internal structure.
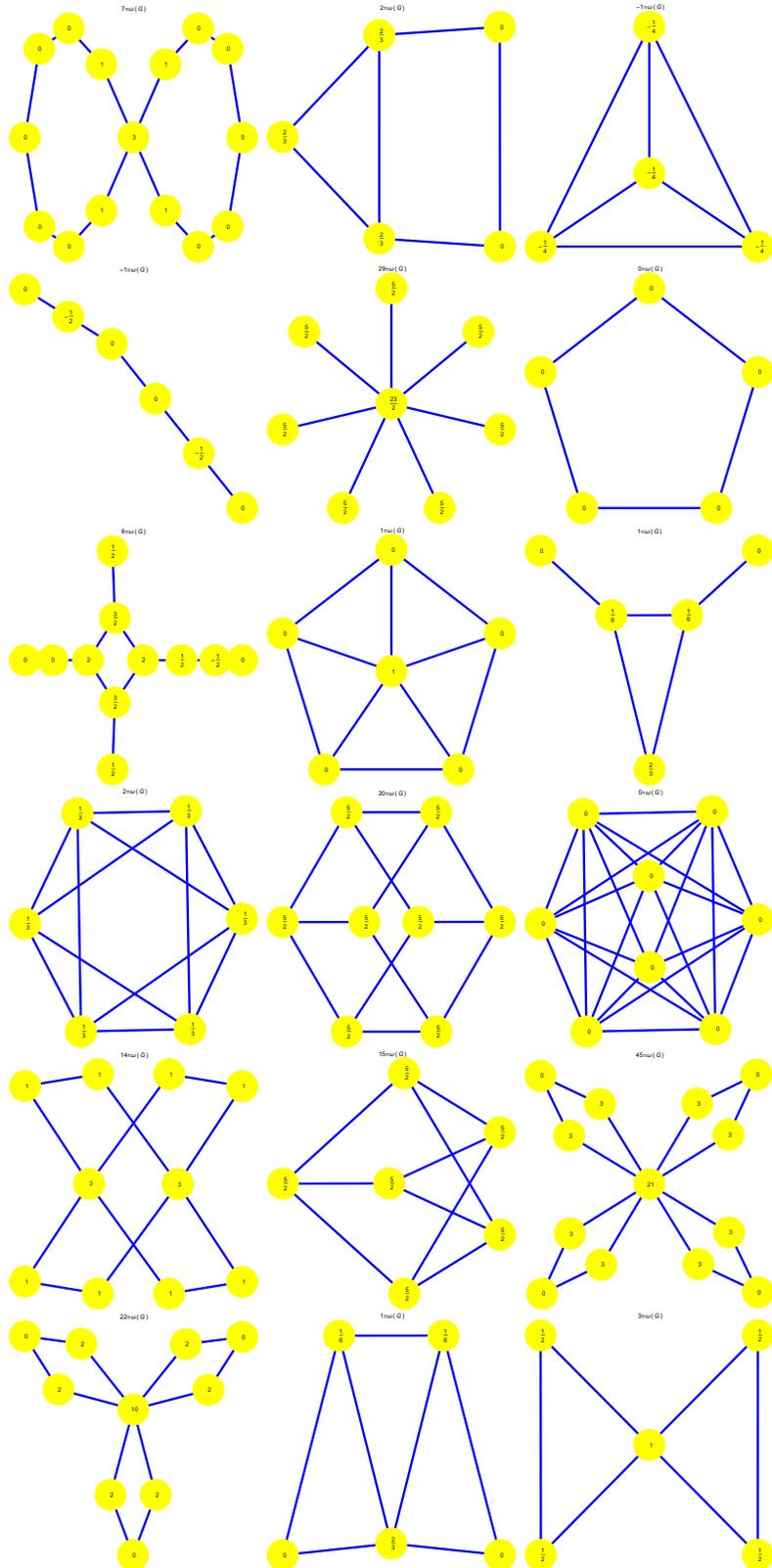
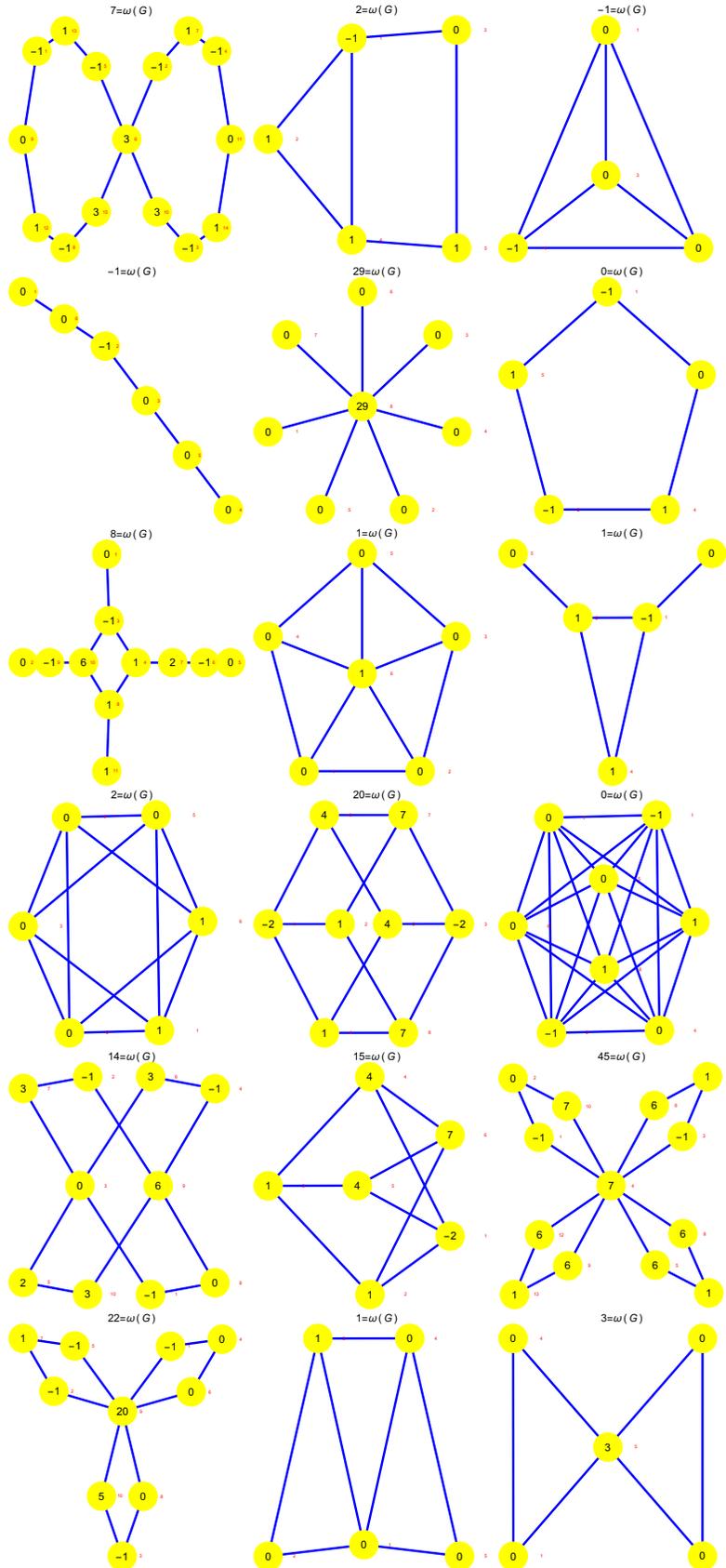FIGURE 10. Examples of graphs with quadratic Wu curvatures.

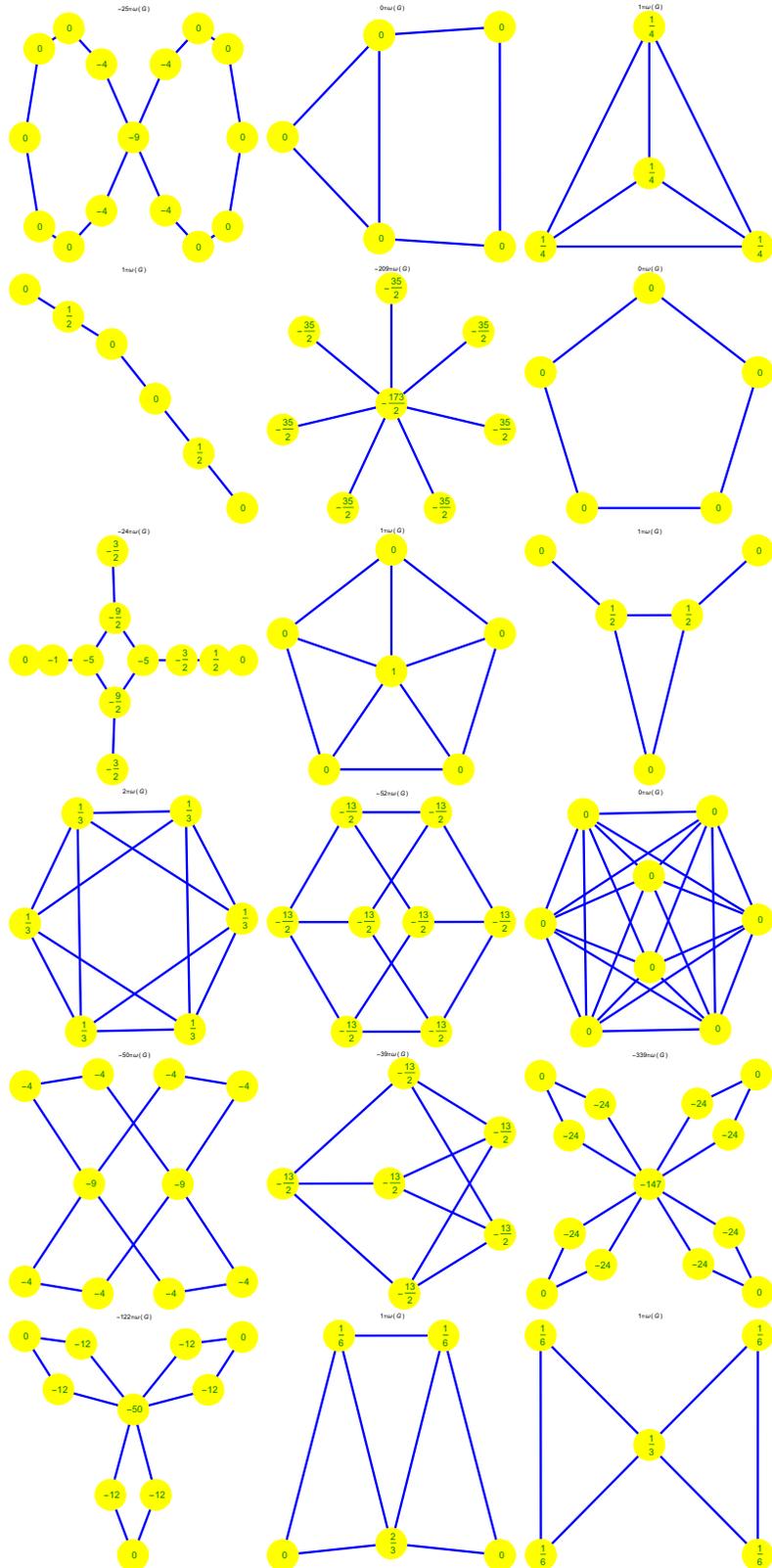FIGURE 11. Examples of graphs with quadratic Wu indices.

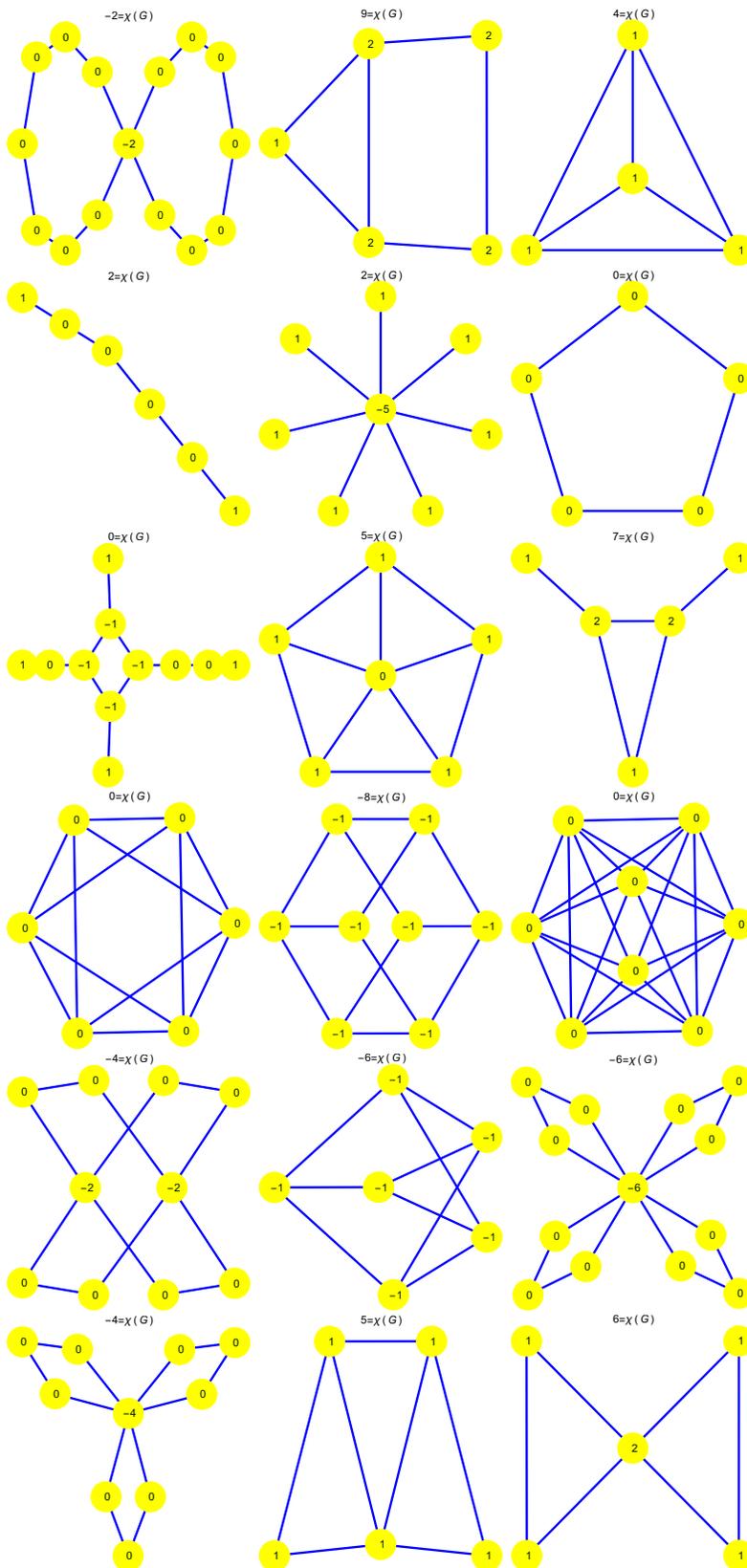FIGURE 12. Examples of graphs with cubic Wu curvatures.

FIGURE 13. Examples of graphs with linear Dehn-Sommerville curvatures for X_0. We use notation, where X_{-1} is the Euler characteristic. The curvatures are zero on the 1-graph C_5 and the 2-graph given as the 2-sphere, the octahedron.
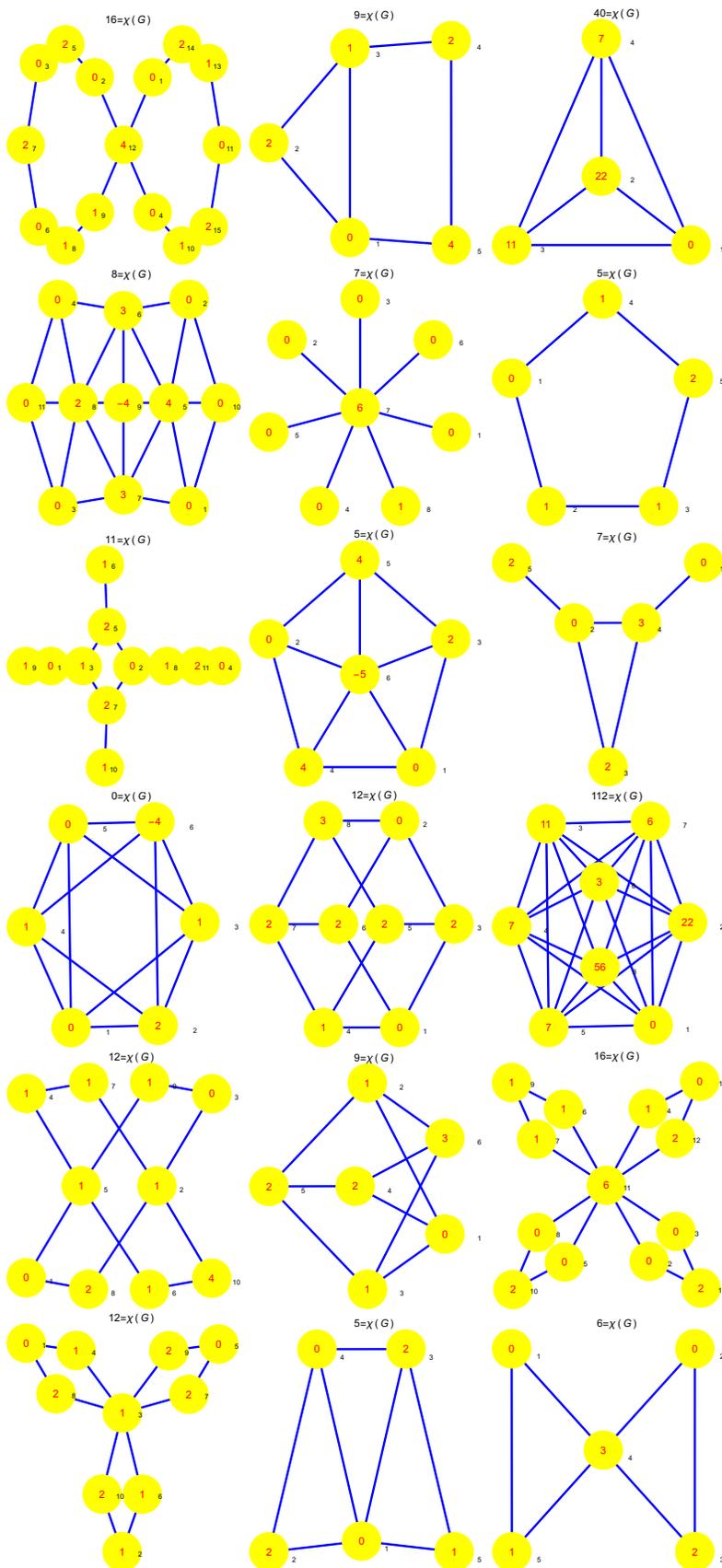
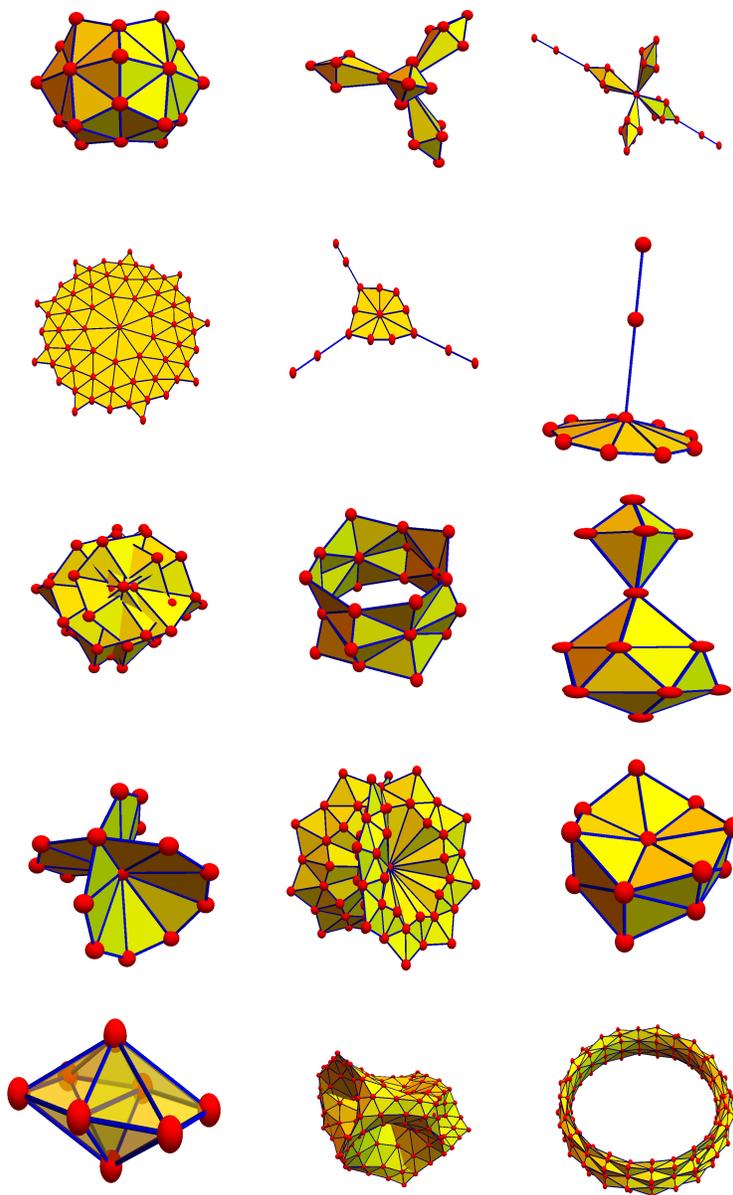FIGURE 14. [Added Jan 19:] Indices of the second Barycentric valuation.

FIGURE 15. **1.** A 3-ball $\omega(G) = -1$. **2.** Bouquet of 2 spheres $\omega(G) = 5$. **3.** Dito with hairs $\omega(G) = 1$. 4. A disk with $\omega(G) = 1$ **5.** A disk with hairs with $\omega(G) = 1$. 6. A flower with $\omega(G) = -1$ **7.** A cylinder $\omega(G) = 0$, **8.** An 2 sphere bouquet $\omega(G) = 3$, **9.** A sphere shell $\omega(G) = -2$. **10.** A pyramid over figure 8, $\omega(G) = -3$, **11.** A figure 8 suspension $\omega(G) = 1$, **12.** A suspension of a Moebius strip $\omega(G) = 2$ **13.** A fat figure eight suspension $\omega(G) = 3$ **14.** A dunce hat is homotopic to K_1 but not contractible. $\omega(G) = 1$. **15.** A torus with $\omega(G) = 0$.

Appendix: Code

More detailed code and examples can be obtained by downloading the LaTeX source file of this preprint on the ArXiv, then copy paste the Mathematica part from the file. We compressed the code as much as possible as shortness of code benefits both communication and verification. The Wolfram language is suitable to serve as pseudo code. Even if the language might evolve in the future, the following code should remain readable. Since Mathematica has not yet built in a procedure to list of all complete subgraphs in a graph, we do that first by hand. One can also use the iGraph library which uses compiled code and is faster. We don't use it by default as this is a complex third party library. But the general problem is NP complete, so that one can not expect a very efficient algorithm.

```
CliqueNumber[s_]:=Length[First[FindClique[s]]];
ListOfCliques[s_,k_]:=Module[{n,t,m,u,r,V,W,U,l={},L},L=Length;
  VL=VertexList;EL=EdgeList;V=VL[s];W=EL[s]; m=L[W];  n=L[V];
  r=Subsets[V,{k,k}];U=Table[{W[[j,1]],W[[j,2]]},{j,L[W]}];
  If[k==1,l=V,If[k==2,l=U,Do[t=Subgraph[s,r[[j]]];
  If[L[EL[t]]==k(k-1)/2,l=Append[l,VL[t]]],{j,L[r]}]]];l];
Whitney[s_]:=Module[{F,a,u,v,d,V,LC,L=Length},V=VertexList[s];
  d=If[L[V]==0,-1,CliqueNumber[s]];LC=ListOfCliques;
  If[d>=0,a[x_]:=Table[{x[[k]]},{k,L[x]}]];
  F[t_,l_]:=If[l==1,a[LC[t,1]],If[l==0,{},LC[t,l]]];
  u=Delete[Union[Table[F[s,l],{l,0,d}]],1];  v={};
  Do[Do[v=Append[v,u[[m,l]]],{l,L[u[[m]]]}],{m,L[u]},v={}];v];
(* Import["IGraphM`"]; Whitney=IGCliques;    *)
```

Here is example code to compute the Euler characteristic χ Euler curvature entering Gauss Bonnet and Euler indices $i_f(x)$ entering Poincaré-Hopf:

```
Euler[s_]:=Module[{c=Whitney[s]},
  If[c=={},0,Sum[(-1)^(Length[c[[k]]]+1),{k,Length[c]}]]];
EulerC[s_,v_]:=Module[{a,n},
  a=Whitney[VertexDelete[NeighborhoodGraph[s,v],v]];
  1+Sum[n=Length[a[[k]]];(-1)^n/(n+1),{k,Length[a]}]];
EulerCurvatures[s_]:=Module[{V=VertexList[s]},
  Table[EulerC[s,V[[k]]],{k,Length[V]}]];
EulerIndex[f_,s_,v_]:=Module[{a={},t,V},
  V=VertexList[NeighborhoodGraph[s,v]];
  Do[If[f[[V[[k]]]]<f[[v]],a=Append[a,V[[k]]]],{k,Length[V]}];
  t=Subgraph[s,a];1-Euler[t]];
EulerIndices[f_,s_]:=Module[{V=VertexList[s]},
  Table[EulerIndex[f,s,V[[k]]],{k,Length[V]}]];
s=RandomGraph[{40,100}];
{Euler[s],Total[EulerIndices[Range[40],s]]}
```

The following procedures compute the Wu characteristic and the cubic Wu Characteristic as well as the intersection number $\omega(G,H)$

```
Wu[s_]:=Module[{c=Whitney[s],v=0,k},If[c=={},1,
  Do[Do[If[Length[Intersection[c[[k]],c[[l]]]]>0,
  v+=(-1)^(Length[c[[k]]]+Length[c[[l]]])];,
  {k,Length[c]},{l,Length[c]}];v]];
Wu3[s_]:=Module[{c=Whitney[s],v=0,k,l,m},If[c=={},v=1,
  Do[Do[Do[If[Length[Intersection[c[[k]],c[[l]],c[[m]]]]>0,
  v+=(-1)^(Length[c[[k]]]+Length[c[[l]]]+Length[c[[m]]]+1);,
  {k,Length[c]},{l,Length[c]},{m,Length[c]}];v]];
Wu[s1_,s2_]:=Module[{c1=Whitney[s1],c2=Whitney[s2],v=0},
  Do[Do[If[Length[Intersection[c1[[k]],c2[[l]]]]>0,
  v+=(-1)^(Length[c1[[k]]]+Length[c2[[l]]])],
  {k,Length[c1]},{l,Length[c2]}];v];

Wu[RandomGraph[{40,100}]]
```

Here are procedures to test the Gauss-Bonnet theorem for the Wu characteristic as well as the Poincaré-Hopf theorem for the Wu characteristic, two results proven in this paper:

```
WuC[s_,v_]:=Module[{a=Whitney[NeighborhoodGraph[s,v,2]]},
  Sum[Sum[If[Length[Intersection[a[[k]],a[[l]]]]>0 &&
```

```
  MemberQ[a[[l]],v],(-1)^(Length[a[[l]]]+Length[a[[k]]]),0],
  {k,Length[a]}]/(Length[a[[l]]]),{l,Length[a]}]];
WuCurvatures[s_]:=Module[{V=VertexList[s]},
  Table[WuC[s,V[[k]]],{k,Length[V]}]];
UnitSphere[s_,a_]:=Module[{b},b=NeighborhoodGraph[s,a];
  If[Length[VertexList[b]]<2,Graph[{}],VertexDelete[b,a]]];
WuIndex[f_,s_,a_,b_]:=Module[{vl,sp,sq,v,w,sa,sb,ba,bb,P,p,S,A,L,V,k},
  P=Position;  p[t_,u_]:=P[t,u][[1,1]];S=Subgraph;A=Append;
  L=Length;V=VertexList;sa={};sb={};
  vl=V[s];sp=UnitSphere[s,a];v=V[sp];sq=UnitSphere[s,b];  w=V[sq];
  Do[If[f[[p[vl,v[[k]]]]]<f[[p[vl,a]]],sa=A[sa,v[[k]]]],{k,L[v]}];
  Do[If[f[[p[vl,w[[k]]]]]<f[[p[vl,b]]],sb=A[sb,w[[k]]]],{k,L[w]}];
  ba=A[sa,a];bb=A[sb,b];Wu[S[s,bb],S[s,ba]]-Wu[S[s,bb],S[s,sa]]-
                       Wu[S[s,sb],S[s,ba]]+Wu[S[s,sb],S[s,sa]]];
WuI[f_,s_]:=Module[{V=VertexList[s],L=Length,k,l},
  Table[WuIndex[f,s,V[[k]],V[[l]]],{k,L[V]},{l,L[V]}]];
WuIndices[f_,s_]:=Total[WuI[f,s]];


s=RandomGraph[{10,30}];{Wu[s],Total[WuCurvatures[s]]}
s=RandomGraph[{10,30}];{Wu[s],Total[WuIndices[Range[10],s]]}
```

Finally, here is an implementation of the graph product and Barycentric refinement. Unlike in the [101], where code for the procedure was algebraically given using the Stanley-Reisner ring, we perform the product here directly. We also have added an example line computing the Euler and Wu characteristics for two graphs and its product.

```
GraphProduct[s1_,s2_]:=Module[{a,b,S,L=Length,A=Append,v,e,p,q},
a=Whitney[s1];b=Whitney[s2];S=SubsetQ;R=Sort;p=Range[L[a]];
q=Range[L[b]];v={};Do[v=A[v,{k,i}],{k,L[p]},{i,L[q]}];e={};Do[
If[(S[a[[k]],a[[l]]]&&S[b[[i]],b[[j]]]&&{k,i}!={l,j}) ||
   (S[a[[l]],a[[k]]]&&S[b[[j]],b[[i]]]&&{k,i}!={l,j}),
 e=A[e,R[{k,i}->{l,j}]]],{k,L[p]},{l,L[p]},{i,L[q]},{j,L[q]}];
v=Union[v];  e=Union[e];  UndirectedGraph[Graph[v,e]]];
Barycentric[s_]:=GraphProduct[s,CompleteGraph[1]];


s1=StarGraph[4];  s2=CycleGraph[4];  s=GraphProduct[s1,s2];
Print[Wu[s1],"__times___", Wu[s2],"__is_equal_to:_",Wu[s]];
Print[Euler[s1],"__times___",Euler[s2],"__is_equal_to",Euler[s]]
```

Here is the inductive dimension implementation for graphs. It is discussed in detail in [82]. It can also be used to illustrated the proven $\dim(A \times B) \geq \dim(A) + \dim(B)$ formula holding for any pair of graphs.

```
Dimension[s_]:=Module[{v,VL=VertexList,u,n,m,e,L},
  L=Length;  v=VL[s];  n=L[v];e=EdgeList[s];  m=L[e];
  If[n==0,u={},u=Table[UnitSphere[s,v[[k]]],{k,n}]];
  If[n==0,-1,If[m==0,0,
  Sum[If[L[VL[u[[k]]]]==0,0,1]+Dimension[u[[k]]],{k,n}]/n]]];

{Dimension[Graph[{},{}]],Dimension[WheelGraph[5]]}
```

Here are procedures to compute the f-vector or f-matrix or f-tensor in the cubic case of a graph:

```
Fvector[s_]:=Delete[BinCounts[Map[Length,Whitney[s]]],1];
Fmatrix[s_]:=Module[{c=Whitney[s],v,m,n,V},
  v=Delete[BinCounts[Map[Length,c]],1];d=Length[v];n=Length[c];
  V=Table[0,{d},{d}];  Do[If[Length[Intersection[c[[k]],c[[l]]]]>0,
  V[[Length[c[[k]]],Length[c[[l]]]]]++],{k,n},{l,n}];V];
Fmatrix3[s_]:=Module[{c=Whitney[s],v,d,n,V},
  v=Delete[BinCounts[Map[Length,c]],1];d=Length[v];n=Length[c];
  V=Table[0,{d},{d},{d}];  Do[Do[Do[If[Length[Intersection[c[[k]],c[[l]],c[[m]]]]>0,
  V[[Length[c[[k]]],Length[c[[l]]],Length[c[[m]]]]]++],{k,n}],{l,n}],{m,n}];V];
```

Finally, lets look at some linear and multi-linear Dehn-Sommerville invariants. To test the produced we build a three or four dimensional sphere:

```
BarycentricOperator[n_]:=Table[StirlingS2[j,i]i!,{i,n+1},{j,n+1}];
Bvector[k_,d_]:=Module[{A,v},A=BarycentricOperator[d-1];
  v=Reverse[Eigenvectors[Transpose[A]]];v[[k]]];
Dvector[k_,d_]:=Module[{f},f[i_]:=Table[If[i==j-1,1,0],{j,d}];
  Sum[(-1)^(j) Binomial[j+1,k+1]f[j],{j,k,d-1}]+(-1)^d f[k]];
```

```
Binvariants[s_]:=Module[{v=Fvector[s],n},
  n=Length[v]; Table[Bvector[k,n].v,{k,n}]];
Dinvariants[s_]:=Module[{v=Fvector[s],n},
  n=Length[v]; Table[Dvector[k,n].v,{k,n}]];
Binvariants2[s_]:=Module[{V=Fmatrix[s],n},
  n=Length[V]; Table[Bvector[k,n].(V.Bvector[l,n]),{k,n},{l,n}]];
Dinvariants2[s_]:=Module[{V=Fmatrix[s],n},
  n=Length[V]; Table[Dvector[k,n].(V.Dvector[l,n]),{k,n},{l,n}]];
Binvariants3[s_]:=Module[{V=Fmatrix3[s],n},n=Length[V];
  Table[((V.Bvector[l,n]).Bvector[k,n]).Bvector[m,n],{k,n},{l,n},{m,n}]];
Dinvariants3[s_]:=Module[{V=Fmatrix3[s],n},n=Length[V];
  Table[((V.Dvector[l,n]).Dvector[k,n]).Dvector[m,n],{k,n},{l,n},{m,n}]];

Suspension[s_]:=Module[{v,e,a,b,r,n},
  v=VertexList[s]; e=EdgeRules[s]; n=Length[v];
  a=n+1; b=n+2; r=Table[v[[k]]->k,{k,Length[v]}]; v=v /. r; e=e /. r;
  Do[e=Append[e,v[[k]]->a]; e=Append[e,v[[k]]->b],{k,n}];
  UndirectedGraph[Graph[e]]];
twosphere=Suspension[CycleGraph[4]];
threesphere=Suspension[twosphere];
foursphere=Suspension[threesphere]
Binvariants2[threesphere]
Binvariants2[foursphere]
```

Like the category of sets, the category of graphs has addition and multiplication given by the symmetric difference and multiplication given by the intersection. But note that the ring product in the Stanley-Reiner ring uses a different multiplication which corresponds to a direct product of sets. Here is how to work with the basic Boolean algebra of graphs in the Wolfram language. Note that the built-in "GraphIntersection" procedure does not take the intersection of the vertex sets. The vertex list of $A \cap B$ with $V(A) = \{1, 2, 3, 4, 5, 6\}$ and $V(B) = \{1, 2, 3, 4\}$ is $\{1, 2, 3, 4, 5, 6\}$ when using "GraphIntersection". We therefore implement the procedure "GProduct", which replaces that procedure and produces the true graph intersection which has a vertex set $\{1, 2, 3, 4\}$. Note that if $G = (V, E)$ and $H = (W, F)$, then $G \cdot H$ is not $G \cap H$. The graph product of the two graphs $G = a + b + ab$ and $b + c + bc$ is 0 while the intersection is b. The sum of the two graphs is $a + b + c + ab + bc$. On the vertex level this is not the symmetric difference $\{a, c\}$ of the vertex sets $\{a, b\}, \{b, c\}$. Mathematica gives the zero dimensional three point graph $a + b + c$ as the graph intersection. Both the intersection as well as the Mathematica graph intersection do not honor distributivity. If modifying the product, we can salvage distributivity, but we still have situations where A has a different graph B with $A + B = A + A = 0$.

```
SetProduct[x_,y_]:=Intersection[x,y];
SetAddition[x_,y_]:=Union[Complement[x,y],Complement[y,x]];
GIntersection[A_,B_]:=UndirectedGraph[Graph[
  SetProduct[VertexList[A],VertexList[B]],
  SetProduct[EdgeList[A],EdgeList[B]]]];
GProduct[A_,B_]:=UndirectedGraph[Graph[
  SetProduct[EdgeList[A],EdgeList[B]]]];
GAddition[A_,B_]:=UndirectedGraph[Graph[
  SetAddition[VertexList[A],VertexList[B]],
  SetAddition[EdgeList[A],EdgeList[B]]]];
EmptyGraph = Graph[{},{}];

A = RandomGraph[{55,15}]; B=RandomGraph[{16,48}]; U=RandomGraph[{107,1000}];
P = GProduct[GAddition[A,B],U]; Q = GAddition[GProduct[A,U],GProduct[B,U]];
Print["Does_distributativity_hold?_",IsomorphicGraphQ[P,Q] ]
P = GProduct[GProduct[A,B],U]; Q = GProduct[A,GProduct[B,U]];
Print["Does_associativity_hold?_",IsomorphicGraphQ[P,Q] ]
P = GAddition[GAddition[A,B],U]; Q = GAddition[A,GAddition[B,U]];
Print["Does_associativity_hold?_",IsomorphicGraphQ[P,Q] ]
P = GAddition[A,EmptyGraph]; Q=A;
Print["Is_the_empty_graph_the_zero_element?_",IsomorphicGraphQ[P,Q] ]
A=Graph[{1->2}]; B=Graph[{2->3}]; EmptyGraphQ[GProduct[A,B]]
```

Appendix: Discrete Hadwiger

The following result of the discrete Hadwiger theorem parallels the statement for simplicial complexes proven in [79] (Theorem 3.2.4). The only change is vocabulary as it is formulated for subgraphs rather than general sub simplicial complexes. But the graph case is not new because the set of subgraphs of a graph $G = (V, E)$ defines a lattice $(L, \emptyset, \cap, \cup)$, where $\emptyset$ is the empty graph and a linear valuation is a valuation on that lattice. The set of complete subgraphs in G is then a generating set of this lattice and $\emptyset$ is the **bottom element** in this distributive lattice.

Theorem 17 (Discrete Hadwiger). *The vector space of invariant valuations on a finite simple graph G is equal to the clique number $d + 1$ of G. A basis is given by the valuations $v_k(A)$ counting the number of K_{k+1} subgraphs in A and where $k = 1, \ldots, d + 1$.*

Proof. We have to show that (i) the functionals v_k are linearly independent and (ii) that every valuation X a linear combination of the functionals v_k.
(i) Assume that we can find a vector $\vec{a} = (a_0, \ldots, a_d)$ such that

$$\sum_i a_i v_i(H) = 0$$

for all subgraphs H of G. By taking complete subgraphs K_k we see that $\vec{a}$ is perpendicular to the vectors $\vec{v}(K_k) = (\binom{k+1}{1}, \binom{k+1}{2}, \ldots, \binom{k+1}{k+1})$. Since all these $d + 1$ vectors are linearly independent for $k = 0, \ldots, d$, the vector $\vec{a}$ must be zero.
Barycentric refinement gives an algebraic proof of linear independence as the eigen-functions of A^T, where A is the **Barycentric refinement operator** are linearly independent as the $(d + 1) \times (d + 1)$ matrix A has $d + 1$ different eigenvalues $k!$ for $k = 1, \ldots, d + 1$.

(ii) Given an arbitrary invariant valuation X on G. By the invariance property in the definition, there is for every $k = 0, \ldots, d$ a real number b_k such that $X(x) = b_k$ for all complete subgraphs x of dimension k. Also $X(\emptyset) = 0$ by definition. Define now $Y(A) = \sum_{k=0}^{d} b_k v_k(A)$. Since every subgraph A of G can be written as a union of complete subgraphs and both $X(\emptyset) = Y(\emptyset) = 0$, we can use the inclusion-exclusion property to see inductively that $X(A) = Y(A)$. $\square$

This can be generalized to higher dimensions. Lets just discuss the quadratic case:

Theorem 18 (Hadwiger for quadratic valuations). *The vector space of quadratic valuations has dimension $(d + 1)(d + 2)/2$.*

Proof. The proof is very similar. One has to get an independent set of $(d + 1)(d + 2)/2$ valuations and then show that every valuation can be written as a linear combination of this set. Lets look at the valuation $V_{ij}(G)$ measuring the number of i-simplices intersecting with j-simplices in G. This number grows like $(i+1)!(j+1)!$ under Barycentric subdivision. These are growth rates which are independent. To show that these number span, note that every quadratic valuation can be written as $X(A) = \sum_{i,j} V_{ij}(G)\phi_i\psi_j$ with $i \leq j$. $\square$

For example, for $d = 0$, the set of quadratic valuations is 1, it is the valuation which counts the discrete set of points. In the one dimensional case $d = 1$, there the valuation space is 3 dimensional, we look at the number of vertices, the number

of pairs of edges intersecting and the number of vertex-edge pairs, which is twice the number of edges.

In the continuum, one restricts the theory of valuations to convex sets or **polyconvex sets**, which is a term introduced in [79] and means a finite union of convex sets in R^n. Also polyconvex sets form a lattice. Things are related of course as any finite simple graph has a geometric realization in which it is a polyconvex set. The fact that in graph theory, the values $v_k(G)$ form a basis of all valuations can also be seen as a manifestation of the Groemer extension theorem from a generating set to the full lattice.

The structure of valuations on discrete spaces is much richer than the structure of measures. Note that on a finite set V, there is only a 1-dimensional space of measures if we ask the measure to be invariant. It is a multiple of the counting measure, which is the Haar measure with respect to the symmetric group acting on V. So, thinking about a valuation as a measure is somewhat misleading. On a finite set, there is a natural σ algebra, the set of all subsets and the structure of all signed measures on this algebra is a $|V|$ dimensional space given by all $\{p_1, \ldots, p_n\}$, a trivial manifestation of the Riesz representation theorem applied to linear functionals on the set of continuous functions $C(V) = \mathrm{R}^n$ which happens to adopt even a Hilbert space structure in the finite case. The fact that set of measures invariant under permutations is 1-dimensional settles the theory of invariant measures on a finite set. On the other hand, if the discrete set is quipped with a graph structure and the corresponding simplicial complex, the theory of valuations is more interesting, as the Hadwiger theorem illustrates. The theory of measures does not look at the internal structure of the sets which are measured, unlike the theory of valuations which look inside. In the continuum, it is a bit harder to explore this internal structure as one needs tomographic methods, but the language of probability theory like Crofton or kinematic formulas allow to deal with it. As we have hoped to demonstrate in this article, some interesting mathematics in discrete differential geometry like Gauss-Bonnet, or Poincaré-Hopf can be seen naturally as results on valuations. We also hoped to show that the language of graphs works well also to discover new results. Working with subgraphs of a graph is a bit like working with subsets but hides the difficulty that the theory of valuations really does more: it deals with the distributive lattice of simplicial subcomplexes of a given simplicial complex rather than the lattice of subsets of a given set. But working with sets of subsets as the basic structure is harder to think and write about. The language of graphs is more intuitive similarly as the concept of metric topologies is easier to deal with than arbitrary topologies.

The analogy between discrete Hadwiger and continuous Hadwiger is so close that one might wonder whether it is possible to treat them in a unified manner. This is indeed possible as emerged while working with Barry Tng [146]. Let us sketch the connection: a measure μ on the set Ω of linear functions on R^d defines a length $|\gamma|$ on smooth parametrized curves $\gamma : [0, 1] \to R^d$ by the **Crofton formula**: take smooth random linear function f_k with equidistant level surfaces and produce the random variable $X_k(\gamma) \in N$ counting the number of intersections with the curve. The law of large numbers shows that the expectation of this random variable can be explored by

a Monte Carlo computation. It defines a length functional on curves, the property that it is additive is related to the additivity of probability. We have now only a **semi-metric** $d(A, B) = \inf_\gamma |\gamma|$ on R^d but the Kolmogorov quotient is now a metric space. The Kolmogorov quotient just takes equivalence classes of points for which the semimetric is zero. For the semi metric $d((x_1, y_1), (x_2, y_2)) = x_1 - x_2$ in the plane for exmple, the Kolmogorov quotient is the real line. If μ is a measure which is invariant under translations and rotations, then it recovers the usual metric on R^d up to a scaling factor. This is the Crofton formula which for polygons reduces to the Buffon needle computation. If μ is a finite point measure, then the Kolmogorov quotient is a finite graph. This setup is rather general and most conveniently described in the projective situation, where translation is part of the projective group. Any **compact Riemannian manifold** can be treated like that: Nash embed it into some projective space P^d, look at the Haar measure on all linear functions on P^d invariant under projective transformations. Then, as the arc length on curves in P^d is the same than arc length on M, we can look at the measure μ on the set Ω of Morse functions on M given by the push forward of the Haar measure on linear functions to the space of Morse functions on M. This defines now a probability space on the class of Morse functions on M which allows to recover the Riemannian metric **integral-geometrically** within the manifold M. The point is however that by choosing a different measure μ on Morse functions, we get different metric spaces. In particular, if we take a discrete finite point measure, we get a discrete space. Going back to the case R^d, the probability space Ω is a set of linear functions, we have convex sets obtained by inequalities $|f_i - k| \le c_i$. Why does the dimension of valuations in R^d agree with the dimension of valuations on graphs? For any finite measures μ_m we have a $(d+1)$-dimensional set of valuations by discrete Hadwiger. Now approximate the original Haar measure with finite measures leading to d-dimensional graphs by doing Barycentric subdivisions, we can get to the Haar measure. Since valuations go over to the limit and the dimension of the set of valuations is upper semi continuous, the discrete approximation argument shows that we have at least a $(d+1)$-dimensional set of valuations in the continuum. We can also go backwards and approximate a given discrete finite measure by absolutely continuous measures. Assume that the vector space of valuations is k-dimensional with $k > d + 1$. We would get k different valuations in the discrete limit which is not the case. On simplices we can compute the k'th valuation by adding up the k-dimensional measures of k-dimensional subsimplices. On convex subsets K, we can make a triangulation and sufficiently many Barycentric subdivisions in Euclidean space allowing to compute the valuation numerically by adding up the valuations on subsimplices. Smooth manifolds allowing an approximation by polytopes, the valuation is still defined as a limit. The setup is much more than just a unification the theory of valuations on Euclidean space or graphs. We can see that for any measure μ, which even **might be singular continuous**, we get to geometric spaces which have a Hadwiger theorem. These spaces can be objects with fractal dimension. They can be seen as approximations of graphs or then as limiting cases of smooth manifolds. The more general setup can use to make sense of curvature also on more general spaces as we can define curvature integral geometrically. We can use the function f to define an index $i_f(x)$ at a point and use the measure μ to average this index to get a curvature function $K(x)$. While we have not gone into this general framework here and stayed strictly within graph theory, we hope that

the prospect of a much more general geometry which includes both graph theory as well as Riemannian geometry, makes the graph theoretical setup more relevant.

Appendix: About the literature

The results are formulated in the language of graph theory [18, 9, 10] which itself has various topological graph theory [58] or algebraic graph theory [15]. There is overlap with work on polytopes [153, 60], simplicial complexes [141, 142, 72] or combinatorial topology like [61, 138]. See [37] for history. Various flavors of discrete topologies have emerged: digital topology [108, 62, 41], discrete calculus [56], Fisk theory [2, 45, 43, 44, 46] to which we got in the context of graph colorings [96, 99] leading to the notion of spheres which appeared in [40] which is based on homotopy [69, 24], based on notions put forward in [75, 150], networks [121, 120, 147, 28, 121, 147, 68], physics [35, 129, 31, 53, 117, 134], computational geometry [36, 12, 34, 123, 151] discrete Morse theory [47, 48, 49, 51], eying classical Morse theory [114], discrete differential geometry in relation to classical differential geometry [13, 54, 74]. We got to into the subject through [84] and generalized it to [83] and summarized in [89] after [85]. The general Gauss-Bonnet-Chern result appeared in [83] but was predated in [109]. It seems Gauss-Bonnet for graphs has been rediscovered a couple of times like [70, 50]. We noticed the first older appearance [70] in [100] and found [109, 50] while working on the present topic. Various lower dimensional versions of curvature have appeared [57, 126, 127, 64, 119, 145]. The first works on Gauss-Bonnet in arbitrary dimensions include [66, 4, 42, 3, 25, 26]. For modern proofs, see [32, 132]. The story of Euler characteristic is told in [52, 130, 111]. A historical paper is [39]. For uniqueness of Euler characteristic as a linear valuation see [80, 133, 70, 50, 131, 152, 110]. For Hadwiger's theorem of 1957, see [79, 78]. The first works on Poincaré-Hopf were [124, 66, 116]. For more history, see [139, 115, 19, 55, 65]. Poincaré-Hopf indices are central in discrete approaches to Riemann-Roch [6, 5]. The index expectation result is [87]. The closest related work is Banchoff [8, 7]. For integral geometry and geometric probability, see [79, 135]. It is a popular topic for REU research projects like [22]. The index formula for Euler characteristic appeared in [86] which proves a special case of the result here along the same lines. See also [92] for the recursion. The Sard approach in [102] simplified this. That paper gives a discrete version of [136]. We got into the Barycentric invariants through [98, 103] after introducing a graph product [101] which was useful in [100], a paper exploring topology of graphs [97, 73, 90]. Originally we were interested in the spectral theory of graphs [27, 128, 1, 33, 125, 144] which parallels the continuum [23, 132, 13]. The linear algebra part of networks was explored in [88] which is a discrete version of [113]. See also [91, 95, 94, 93, 104, 106] and [67, 112] for discrete combinatorial Laplacians. For the Dehn-Sommerville relations, see [81, 122, 118, 110, 21, 63, 77]. In [122, 77] appear Dehn-Sommerville-Klee equations for discrete for manifolds with boundary. In [110], it was noted that the Euler characteristic is the only invariant, using the operator A. The combinatorics of the Barycentric operator was studied in[11] in the case $d = 2$. The explicit formula using Stirling numbers appeared in [21]. For Polytopes [36, 130, 137, 29, 60, 71, 107, 30]. The theory of valuations on distributive lattices has been pioneered by Klee [80] and Rota [133] who proved that there is a unique valuation such that $X(x) = 1$ for any join-irreducible element. This is the Euler characteristic. The example of the lattice of subgraphs of a graph fits within this framework.

APPENDIX: SUMMARIES

The next two pages are from October 2, 2015. The new results are then added where indicated. Originally we used the central manifolds to argue the vanishing of the Barycentric invariant numbers. There were too statements which overreached in the October 2 summary: the symmetric index $j_G(x)$ is not always constant zero for Dehn-Sommerville valuations. Also, the index expectation does not generalize without modifications from Euler characteristic to general valuations. The index formula still produces probabilistic statements about random geometric subgraphs of spheres, but these results still need to be harvested.

If $\mathcal{G}$ is the category of **finite simple graphs** $G = (V, E)$, the linear space $\mathcal{V}$ of **valuations** on $\mathcal{G}$ has a basis given by the f-**numbers** $v_k(G)$ counting complete subgraphs K_{k+1} in G. The **barycentric refinement** G_1 of $G \in \mathcal{G}$ is the graph with K_l subgraphs as vertex set where new vertices $a \neq b$ are connected if $a \subset b$ or $b \subset a$. Under refinement, the clique data transform as $\vec{v} \to A\vec{v}$ with the upper triangular matrix $A_{ij} = i!S(j, i)$ with **Stirling numbers** $S(j, i)$. The eigenvectors χ_k of A^T with eigenvalues $k!$ form an other basis in $\mathcal{V}$. The χ_k are normalized so that the first nonzero entry is > 0 and all entries are in $\mathbb{Z}$ with no common prime factor. χ_1 is the **Euler characteristic** $\sum_{k=0}^{\infty} (-1)^k v_k$, the homotopy and so cohomology invariant on $\mathcal{G}$. Half of the χ_k will be zero **Dehn-Sommerville-Klee invariants** like half the **Betti numbers** are redundant under **Poincaré duality**. On the set $\mathcal{G}_d \subset \mathcal{G}$ with clique number d, the valuations $\mathcal{V}$ have dimension $d + 1$ by **discrete Hadwiger**. A basis is the eigensystem $\vec{\chi}$ of the $(d + 1) \times (d + 1)$ submatrix matrix A_d^T of A^T. The functional $\chi_{d+1}(G)$ is **volume**, counting the **facets** of G. For $x \in V$, define $V_{-1}(x) = 1$ and $V_k(x)$ as the number of complete subgraphs K_{k+1} of the unit sphere $S(x)$, the graph generated by the neighbors of x. The **fundamental theorem** of graph theory is the formula $\sum_{x \in V} V_{k-1}(x) = (k + 1)v_k(G)$. For $k = 1$, it is the **Euler's handshake**. For a valuation $X(G) = \sum_{l=0}^{\infty} a(l)v_l(G)$, define **curvature** $K_X(x) = \sum_{l=0}^{\infty} a(l)V_{l-1}(x)/(l + 1)$. Generalizing the fundamental theorem:

Theorem (Gauss-Bonnet). $\sum_{x \in V} K_X(x) = X(G)$.

Example. For an icosahedron with $\vec{v} = (12, 30, 20)$ and $\vec{v}(S(x)) = (5, 5)$, we have $a_1 = (1, -1, 1)$, $K_1(x) = 1 - 5/2 + 5/3 = 1/6$, $\chi_1 = 2$, $a_2 = (0, 2, -3)$, $K_2(x) = 10/2 - 15/3 = 0$, $\chi_2 = 0$, $a_3 = (0, 0, 1)$, $K_3(x) = 5/3$, $\chi_3 = 20$.

Let $\Omega(G)$ be the set of **colorings** of G, locally injective function f on $V(G)$. The **unit ball** $B(x)$ at x is the graph generated by the union of $\{x\}$ and the **unit sphere** $S(x) = \{y \in V \mid (x, y) \in E\}$ which is the **boundary** $\delta B(x)$. For $f \in \Omega$ and $X \in \mathcal{V}$ define the **index** $i_{X,f}(x) = X(B^-(x)) - X(S^-(x))$, where $B^-(x) = S^-(x) \cup \{x\} = \{y \in B(x) \mid f(y) \leq f(x)\}$ and $S^-(x) = \{f(y) < f(x)\}$. It is local and a **divisor**. Inductive attaching vertices gives:

Theorem (Poincaré-Hopf). $\sum_{x \in V} i_{X,f}(x) = X(G)$.

Let P be a **Borel probability measure** on $\Omega(G) = \mathbb{R}^{v_0(G)}$ and $E[\cdot]$ its **expectation**. Let $c(G)$ be the **chromatic number** of G. Assume either that P is the counting measure on the finite set of colorings of G with $c \geq c(G)$ real colors or that P is a product measure on Ω for which functions $f \to f(y)$ with $y \in V$ are

independent identically distributed random variables with continuous probability density function. For all $G \in \mathcal{G}$ and $X \in \mathcal{V}$:

Theorem (**Banchoff index expectation**). *For Euler characteristic* $E[i_{X,f}(x)] = K_X(x)$.

The **empty graph** $\emptyset$ is a (-1)-graph and (-1)-sphere. Inductively, a d-**graph** is a $G \in \mathcal{G}$ for which the unit spheres are $(d-1)$-spheres. An **Evako** d-**sphere** is a d-graph which when punctured becomes contractible. Inductively, G is **contractible** if there exists $x \in V(G)$ such that both $S(x)$ and the graph without x are contractible. The graph K_1 is contractible. Given $f \in \Omega$ and $c \notin f(V)$, define the graph $\{f = c\}$ in the refinement of G consisting of vertices, where $f - c$ changes sign. In that case, at every vertex x, there is a $(d-2)$-graph $S_f(x)$ defined as the **level surface** $\{f(y) = f(x)\}$ in $S(x)$. The next **Sard** result belongs to **discrete multivariable calculus**:

Theorem (**Implicit function theorem**). *For a d-graph and $f \in \Omega$ and $c \notin f(V)$, the hyper surface $\{f = c\}$ is a $(d-1)$-graph.*

The **symmetric index** of f at x is defined as $2j_{X,f}(x) = i_{X,f}(x) + i_{X,-f}(x)$

Theorem (Index formula). *For $G \in \mathcal{G}$ and Euler characteristic, then*
$$2j_f(x)(1 - \chi(S(x))/2) - X(R_f(x))/2$$

Poincaré-Hopf allows fast recursive computation of X for most G in $\mathcal{G}_d$ quantified using **Erdös-Renyi** measures. For d-graphs, the symmetric index is is $-\chi(R_f(x))/2$ if d is odd and $1 - \chi(R_f(x))/2$ if d is even. If $k + d$ is even, we have $\chi_k(B(x)) = \chi_k(S(x))$, as curvature is supported on δG. Furthermore, we have $\chi_k(\{x\}) = 0$ if $k > 1$.

Theorem (**Dehn-Sommerville-Klee**). *For a d-graph and even $d + k$, the functions K_k are supported on δG. If $\delta G = \emptyset$, then $\chi_k(G) = 0$.*

χ_k with even $k+d$ span classical invariants. Zero curvature $K_k(x) = 0$ for all $x \in V$ also follows $\chi_k = 0$ from Gauss-Bonnet and suspension. Curvature functionals are linear combination of Barycentric functionals for $d - 1$. The classical DS-invariants in dimension d can be derived from Gauss-Bonnet and the fact that curvatures are DS-invariants in dimension d-1.

Illustration: $\chi_2(G) = 0$ on 4-graphs shows that a 4-graph **triangulation** G of a compact 4-manifold with v vertices, e edges, f triangles, t tetrahedra, and p pentatopes satisfies $22e + 40t = 33f + 45p$. Examples: for the 4-crosspolytop G, a 4-sphere with $\vec{v} = (10, 40, 80, 80, 32)$, we get $\vec{\chi}(G) = (2, 0, 240, 0, 32)$. For a discrete $G = S^2 \times T^2$, a product graph constructed using the **Stanley-Reisner ring**, with $\vec{v} = (1664, 23424, 77056, 92160, 36864)$, we get $\vec{\chi}(G) = (0, 0, -10496, 0, 36864)$. For a discrete $G = P^2 \times S^2$ with $\vec{v} = (1898, 26424, 86736, 103680, 41472)$ we get $\vec{\chi}(G) = (2, 0, -10896, 0, 41472)$.

This ends the summary from October 2. Here is is a summary of what has been found since written in a compressed form.

A k-**linear** valuation is a real valued map X on ordered k tuples of subgraphs such that each $A \to X(A_1, \ldots, A, \ldots, A_k)$ is a linear valuation, a map from the set of

subgraphs satisfying $X(A \cup B) + X(A \cap B) = X(A) + X(B)$ and $X(\emptyset) = 0$. Given two valuations X_1, X_2, the quadratic valuation $X_1(A)X_2(B)$ is an example. We assume them to be **localized** in the sense $X(A, B) = 0$ if $A \cap B = \emptyset$. A k-linear valuation X defines a **nonlinear valuation** $X(A) = X(A, A, \ldots, A)$. Of special interest are quadratic valuations $X(A, B)$ which can be seen as intersection numbers. Every linear valuation which is invariant in the sense that $X(A) = X(B)$ for isomorphic graphs defines a linear map on f-vectors $v(A) = (v_0, \ldots, v_k)$. The linear map is represented by a $(d+1)$-vector like $\chi = (1, -1, \ldots, \pm 1)$. We have then $X(A) = \chi \cdot v(A)$. If $V(A, B)$ is the **quadratic f-form**, where $V_{ij}(A, B)$ counts how many x simplces in A intersect in a non-empty set with a y simplex in B. We especially have the f-matrix $V_{ij}(A)$ which encodes the intersections of simplices in A. A quadratic valuation can now be given by two $(d+1)$ vectors χ, ψ: one has $X(A, B) = \phi \cdot V(A, B)\psi$ or $X(A) = \phi \cdot V(A)\psi$. An example is the **Wu characteristic** $\omega(A) = \chi V(A)\chi$. Similarly, one can define k-linear valuations and have the **higher Wu characteristic** ω_k. By Hadwiger, the space of linear valuations is $d+1$ dimensiona, the space of quadratic valuations is $(d+1)(d+2)/2$ dimensional.

Given two finite simple graphs A, B, define a new graph $A \times B$ as follows: assume the vertex sets of A, B are disjoint. The vertex set of $A \times B$ consists of all pairs (x, y) with x being a simplex in A and y being a simplex in B. Two such elements are connected by an edge, if one is contained in the other. For a finite simple graph G, the product $G \times K_1$ is called the **Barycentric refinement** of G. It has the set of simplices of G as vertices and two simplices connected if one is contained in the other. The addition of two graphs $A + B$ is defined as the disjoint union of the two graphs. Note that $A \times B$ is only associative, when the product is describe algebraically as the product in the Stanley-Reisner ring. The following four theorems hold in the class of all finite simple graphs:

Theorem (Gauss-Bonnet). *Every k-linear valuation has a curvature K defined on vertices of G such that $X(A) = \sum_v K(v)$.*

Lets elaborate a bit in the case of quadratic valuations: By definition $X(A) = \sum_{x,y \subset A} a(x, y)$, where $a(x, y)$ only depend on the dimensions of x and y. There are two vectors such that using the quadratic f-form $V(G)$, we have $X(A) = \phi \cdot V(G)\psi$. To get the curvature, write $X(A) = \sum_x \kappa(x)$ with $\kappa(x) = \sum_y a(x, y)$. Now distribute the value of $\kappa(x)$ equally to all of the $\dim(x) + 1$ vertices of x. This gives the curvature function $K(v)$ on vertices.

Theorem (Poincaré-Hopf). *For every k-linear valuation and any locally injective scalar function f on the vertices of G, there is an index $i_f(x)$ such that $X(A) = \sum_v i_f(v)$.*

Given a valuation X, its Poincaré-Hopf index is defined as

$$i_f(v) = \omega(B_f^-(v)) - \omega(S_f^-(v)) \, ,$$

where $B_f^-(x)$ is the graph generated by $\{y \mid f(y) \leq f(x)\}$ and $S_f^-(x)$ is the graph generated by $\{y \mid f(y) < f(x)\}$. For the same type of probability measures as in the linear case, we have:

Theorem (Index expectation). *The expectation of $i_f(v)$ over the probability space of functions is the curvature $K(v)$.*

As for linear valuations, a deformation of the probability measure on functions (like for example given by a wave evolution) changes the curvature but keeps Guass-Bonnet intact. The deformation of the probability measure allows for other type of curvatures.

Theorem (**Topological Invariants**). *Every Wu characteristic is invariant under Barycentric refinements.*

Since $\omega(G) = \sum_x \omega(x)$, we can see ω as a sum of values of a function on the vertex set of the Barycentric refinement G_1. This function is an index for a natural ordering of the vertices of G_1.

Theorem (**Multiplicative function**). *Every Wu characteristic is multiplicative* $\omega_k(A \times B) = \omega_k(A)\omega_k(B)$.

To illustrate the quadratic case, first show $\omega(x \times y) = \omega(x)\omega(y)$ for simplices then write $\omega(G) = \sum_{x,y} \omega(x)\omega(y)$, finally uses the Barycentric invariance.

The following two theorems hold only for d-graphs or d-graphs with boundary:

Theorem (**Boundary formula**). *For every d-graph G with boundary δG, we have* $\omega(G) = \chi(G) - \chi(\delta G)$.

For d-graphs without boundary this is shown by verifying that the Wu curvature and Euler curvatures agree. What happens at the boundary is that for simplices hitting the boundary there is an additional contribution one or minus one. The corresponding sum is the Euler characteristic of a thickend boundary which is homotopic to the actual boundary.

Theorem (**Grünbaum question**). *For every d-graph G without boundary and every linear valuation $X(A) = v(A) \ldots \psi$, the quadratic valuation $Y(A) = \chi_1 V(A)\psi$ satisfies $X(G) = Y(G)$.*

This could be generalized to k-linear cases. Like that $X(A)$ and the cubic valuation $Y(A) = V(A)\chi_1\chi_1\psi$ agree up to a sign on the d graph G. The proof goes by writing the valuation as a sum over pairs of intersecting simplices and then partition according to which simplex z they intersect. When looking at a sum for fixed z, then the part of the interacting simplices is zero. This uses that any lower dimensional spheres have the correct Euler characteristic.

Theorem (**Dehn Sommerville space**). *The Dehn-Sommerville space of k-valuations has the same dimension as the Dehn-Sommerville space of linear valuations $[(d + 1)/2]$.*

The fact that the $[(d + 1)/2]$ Dehn-Sommerville valuations are linearly independent follows from the fact that there is a basis given by eigenvectors to different eigenvalues. An alternative basis are the classical Dehn-Sommerville valuations

$$X_{k,d}(v) = \sum_{j=k}^{d-1} (-1)^{j+d} \binom{j+1}{k+1} v_j(G) + v_k(G) \ .$$

There can not be a larger dimensional space of valuations since a simple example of cross polytopes obtained by doing multiple suspension of a circular graph C_4 shows that only $[(d + 1)/2]$ can be zero in general. For some graphs of course, the space

of valuations which vanish can be larger. An example is the 2-torus $G = C_4 \times C_4$ for which the f-matrix is

$$V = \left[\begin{array}{ccc} 0 & 0 & 128 \\ 0 & 640 & 1408 \\ 128 & 1408 & 1920 \end{array} \right] .$$

The Dehn-Sommerville space of the 2-torus is 2-dimensional. For $d = 2$ dimensional graphis we have only a $[3/2] = 1$-dimensional Dehn-Sommerville space. By the way, with $\chi = (1, -1, 1)$, then $V\chi = (128, 768, 640)$ and the Wu invariant is $\chi \cdot (V\chi) = 0$ as the Wu characteristic is the Euler characteristic, which is 0 for the 2-torus.

Finally, Gauss-Bonnet for linear valuations immediately shows that

Theorem. Flatness of D-S *For any d-graph and any Dehn-Sommerville valuation X, we have $X(G) = 0$ and G is flat in the sense that all curvatures are constant zero.*

The proof uses that the curvature at a vertex v is a Dehn-Sommerville valuation of the unit sphere $S(v)$ of the graph at v.

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