

THE COHOMOLOGY FOR WU CHARACTERISTICS

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ABSTRACT. While the Euler characteristic $\chi(G) = \omega_1(G) = \sum_x \omega(x)$ super counts simplices, the Wu characteristics $\omega_k(G) = \sum_{x_1 \sim x_2 \dots \sim x_k} \omega(x_1) \cdots \omega(x_k)$ super counts simultaneously pairwise interacting k -tuples of simplices in a finite abstract simplicial complex G . More generally, one can define the k -intersection number $\omega_k(G_1, \dots, G_k)$, which is the same sum but where $x_i \in G_i$. For every $k \geq 1$ we define a cohomology $H_k^p(G_1, \dots, G_k)$ compatible with ω_k . It is invariant under Barycentric subdivision. This interaction cohomology allows to distinguish spaces which simplicial cohomology can not: for $k = 2$, it can identify algebraically the Möbius strip and the cylinder. The vector spaces $H_k^p(G)$ are defined by explicit exterior derivatives d_k which generalize the incidence matrices for simplicial cohomology. The cohomology satisfies the Kuenneth formula: for every k , the Poincaré polynomials $p_k(t)$ are ring homomorphisms from the strong ring to the ring of polynomials in t . The case $k = 1$ for Euler characteristic is the familiar simplicial cohomology $H_1^p(G) = H^p(G)$. On any interaction level k , there is now a Dirac operator $D = d_k + d_k^*$. The block diagonal Laplacian $L = D^2$ leads to the generalized Hodge correspondence $b_p(G) = \dim(H_k^p(G)) = \dim(\ker(L_p))$ and Euler-Poincaré $\omega_k(G) = \sum_p (-1)^p \dim(H_k^p(G))$ for Wu characteristic and more generally $\omega_k(G_1, \dots, G_k) = \sum_p (-1)^p \dim(H_k^p(G_1, \dots, G_k))$. Also, like for traditional simplicial cohomology, an isospectral Lax deformation $\tilde{D} = [B(D), D]$, with $B(t) = d(t) - d^*(t) - i(b(t) + \bar{b}(t))$, $D(t) = d(t) + d(t)^* + b(t)$ can deform the exterior derivative d which belongs to the interaction cohomology. Also the Brouwer-Lefschetz fixed point theorem generalizes to all Wu characteristics: given an endomorphism T of G , the super trace of its induced map on k 'th cohomology defines a Lefschetz number $L_k(T)$. The Brouwer index $i_{T,k}(x_1, \dots, x_k) = \prod_{j=1}^k \omega(x_j) \text{sign}(T|x_j)$ attached to simplex tuple which is invariant under T leads to the formula $L_k(T) = \sum_{T(x)=x} i_{T,k}(x)$. For $T = Id$, the Lefschetz number $L_k(Id)$ is equal to the k 'th Wu characteristic $\omega_k(G)$ of the graph G and the Lefschetz formula reduces to the Euler-Poincaré formula for Wu characteristic. Also this generalizes to the case, where automorphisms T_k act on G_k : there is a Lefschetz number $L_k(T_1, \dots, T_k)$ and indices $\prod_{j=1}^k \omega(x_j) \text{sign}(T_j|x_j)$. For $k = 1$, it is the known Lefschetz formula for Euler characteristic. While Gauss-Bonnet for ω_k can be seen as a particular case of a discrete Atiyah-Singer result, the Lefschetz formula is an particular case of a discrete Atiyah-Bott result. But unlike for Euler characteristic $k = 1$, the elliptic differential complexes for $k > 1$ are not yet associated to any constructions in the continuum.

1. INTRODUCTION

1.1. An abstract finite simplicial complex G is a finite set of non-empty subsets that is closed under the process of taking non-empty subsets. It defines a new complex G_1 , where G_1 is the set of subsets of the power set 2^G which are pairwise contained in each other. This is called the Barycentric refinement of G . Quantities which do not change under Barycentric refinements are **combinatorial invariants** [2]. Examples of such quantities are the **Euler characteristic**, the **Wu characteristics** [35, 21] or **Green function values** [26]. An important class of complexes are Whitney complexes of a finite simple graph in which G consists of all non-empty subsets of the vertex set, in which all elements are connected to each other. A Barycentric refinement G_1 of a complex G is always a Whitney complex. The graph has the sets of G as vertices and connects two if one is contained in the other.

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1.2. While Euler characteristic $\chi(G) = \sum_x \omega(x)$ super counts the self interactions of simplices within the geometry, the Wu characteristic ω_k super counts the intersections of k different simplices

$$\omega_k(G) = \sum_{x_1 \sim \dots \sim x_k} \omega(x_1) \cdots \omega(x_k),$$

where $x_1 \sim \dots \sim x_k$ means that $(x_1, x_2, \dots, x_k) \in G^k = G \times G \times \dots \times G$ all pairwise intersect. While originally introduced for polytopes by Wu in 1959, we use it for simplicial complexes and in particular for Whitney complexes of graphs. Especially important is the quadratic case

$$\sum_{x \sim y} \omega(x)\omega(y)$$

which produces an Ising type nearest neighborhood functional if we think of even or odd dimensional simplices as spins of a two-state statistical mechanics system. We call the cohomology for Wu characteristic an **interaction cohomology** because the name "intersection cohomology" is used in the continuum for a cohomology theory by Goresky-MacPherson.

1.3. As in [21], where we proved Gauss-Bonnet, Poincaré-Hopf, and product formulas for Wu characteristics in the case of Whitney complexes, we can work in the slightly more general framework of finite abstract simplicial complexes. If we look at set of subsets of $G \times G \cdots \times G$ of a simplicial complex G such that all k elements pairwise intersect, we obtain a k -intersection complex. For $k = 1$ the complex is G itself, the connection-oblivious complex. For $k = 2$ the intersection relations give the interaction graph G' which was studied in [22, 28, 27, 23, 25, 26]. It contains the Barycentric refinement graph G_1 . For every $k > 1$ we get an example of an "ordinary cohomology" generalizing the simplicial cohomology.

1.4. A major reason to study the Wu characteristic, is that each ω_k is multiplicative with respect to the graph product [19] $G, H \rightarrow (G \times H)_1$, in which the pairs (x, y) are vertices with $x \in G$ and $y \in H$ and where two vertices are connected if one is contained in the other. It is already multiplicative with respect to the Cartesian product $G \times H$, which unlike $(G \times H)_1$ is not a simplicial complex any more. The multiplicative property makes the ω_k functionals a natural counting tool, en par with Euler characteristic $\chi(G) = \omega_1(G)$, which is the special case $k = 1$. The corresponding cohomologies belong to a generalized calculus which does not yet seem to exist in the continuum. The multiplicative property of the functionals is especially natural when looking at the Cartesian product $G \times H$ in the strong ring generated by simplicial complexes in which the simplicial complexes are the multiplicative primes and where the objects are signed to have an additive group. The subring of zero-dimensional complexes can be identified with the ring of integers.

1.5. Traditionally, graphs are seen as one-dimensional simplicial complexes. A more generous point of view is to see a graph $G = (V, E)$ as a structure on which simplicial complexes can live, similarly as topologies, sheaves or σ -algebras are placed on sets. The 1-dimensional complex $V \cup E$ is only one of the possible simplicial complexes. An other example is the graphic matroid defined by trees in the graph. But the most important complex imposed on a graph is the **Whitney complex**. It is defined by the set of complete subgraphs of G . Without specifying further, we always associate a graph equipped with its Whitney complex.

1.6. The class of all finite simple graphs $G = (V, E)$ is a distributive Boolean lattice with union $(V_1 \cup V_2, E_1 \cup E_2)$ and intersection $(V_1 \cap V_2, E_1 \cap E_2)$. The same is true for finite abstract simplicial complexes, where also the union and intersection remain both simplicial complexes. But unlike the Boolean lattice of sets, the Boolean lattices of graphs or complexes are each not a ring: the set theoretical addition $G \Delta H = G \cup H \setminus G \cap H$ is in general only in the larger category of chains. But if we take the disjoint union of graphs as "addition" and the Cartesian multiplication of sets as multiplication, we get a ring [27].

1.7. All of the Wu characteristics ω_k are functions on this ring. They are of fundamental importance because they are not only additive $\omega_k(A \dot{\cup} B) = \omega_k(A) + \omega_k(B)$ but also multiplicative $\omega_k(A \times B) = \omega_k(A)\omega_k(B)$. In other words, all the Wu characteristic numbers ω_k and especially the Euler characteristics $\chi = \omega_1$ are **ring homomorphisms** from the strong ring to the integers. This algebraic property already makes the Wu characteristics important quantities to study.

1.8. Euler characteristic is distinguished in having additionally the property of being invariant under homotopy transformations. It is unique among linear valuations with this property. The lack of homotopy invariance for the Wu characteristic numbers for $k \geq 2$ is not a handicap, in contrary: it allows for finer invariants: while for all **trees** for example, the Euler characteristic is 1, the Wu characteristic $\omega(G) = \omega_2(G)$ is more interesting: it is $\omega(G) = 5 - 4n + M(G)$, where $n = \sum_{v \in V} 1$ is the vertex cardinality and M is the **Zagreb index** $M(G) = \sum_v \deg(v)^2$. This **tree formula** relating the Wu characteristic with the Zagreb index has been pointed out to us by Tamas Reti.

1.9. There are other natural operations on simplicial complexes. One is the **Zykov addition** which is in the continuum known as the **join**. Take the disjoint union G, H of the graphs and additionally connect any pair of vertices from different graphs. The generating function $f_G(x) = 1 + \sum_{k=0} v_k x^{k+1}$ satisfies $f_{G+H}(x) = f_G(x)f_H(x)$ and $\chi(G) = 1 - f_G(-1)$ from which one can deduce that $i(G) = 1 - \chi(G)$ is multiplicative: $i(G+H) = i(G)i(H)$. Evaluating f at $x = 1$ instead shows that the total number of simplices $f_G(x) - 1 = \sum_k v_k$ is multiplicative. We don't use the join much here.

1.10. Looking at finite abstract simplicial complexes modulo Barycentric refinements already captures some kind topology. But Barycentric refinements are still rather rigid and do not include all notions of topology in which one wants to have a sense of rubber geometry. One especially wants to have dimension preserved. We introduced therefore a notion of homeomorphism in [17]. It uses both homotopy and dimension. The Barycentric refinement of a graph G is homeomorphic to G in that notion. This notion of homeomorphism could be ported over to simplicial complexes because the Barycentric refinement of an abstract simplicial complex is a Whitney complex of a graph. From a topological perspective there is no loss of generality by looking at graphs rather than simplicial complexes. We believe that the interaction cohomology considered here is the same for homeomorphic graphs but this is not yet proven.

1.11. As topological spaces come in various forms, notably manifolds or varieties, one can also define classes of simplicial complexes or classes of graphs. In the continuum, one has historically started to investigate first very regular structures like Euclidean spaces, surfaces, then introduced manifolds, varieties, schemes etc. In the discrete, simplicial complexes can be seen as general structure and subclasses play the role of manifolds, or varieties. In this respect, Euler characteristic has a differential topological flair, while higher Wu characteristics are more differential topological because second order difference operations appear in the curvatures.

1.12. Complexes for which all unit spheres are $(d-1)$ spheres are discrete d -manifolds. The definitions are inductive: the empty complex 0 is the (-1) sphere, a d -complex is a complex for which every unit sphere $S(v)$ is a $(d-1)$ -sphere. A d -sphere is a d -graph which after removing one vertex becomes contractible. A graph $G = (V, E)$ is contractible, if there is a vertex v such that both $S(v)$ and the graph generated by $V \setminus \{v\}$ is contractible. We have seen that for d -graphs, all Wu characteristics are the same. Wu characteristic is interesting for discrete varieties like the figure 8 graph. Discrete varieties of dimension d are complexes for which every unit sphere is a $(d-1)$ -complex. There can now be singularities, as the unit spheres are not spheres any more. In the figure 8 graph for example, the unit sphere of the origin is 4, the 0-dimensional graph with 4 points. (When writing 4, we identify zero dimensional complexes with \mathbb{N} .)

2.1. Attaching an **orientation** to a simplicial complex G means choosing a base permutation on each element of G . The choice of orientation is mostly irrelevant and corresponds to a choice of basis. Calculus on complexes deals with functions F on G which change sign if the orientation of a simplex is changed. Functions on the set of k -dimensional simplices are the **k -forms**. It produces **simplicial cohomology** with the exterior derivative defined as $dF(x) = F(\delta x)$, where δx is the ordered boundary complex of a simplex (which is not a simplicial complex in general). While $G = \delta K_3 = \{\{1, 2\}, \{2, 3\}, \{3, 1\}, \{1\}, \{2\}, \{3\}\}$ is a 1-sphere (because $G_1 = C_6$ being a circular graph with 6 vertices), the boundary of a star graph is not even a simplicial complex any more. Summing a k -forms over a collection A of k -simplices gives **integration** $\int_A F dx$.

2.2. Differentiation and integration are related by **Stokes theorem** $\int_G dF dx = \int_{\delta G} F$ which follows by extending integration linearly from simplices to simplicial complexes and chain. The boundary of a simplicial complex is not a simplicial complex any more (for the star complex $G = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{1\}, \{2\}, \{3\}, \{4\}\}$ for example, $\delta G = \{-3\{1\}, \{2\}, \{3\}, \{4\}\}$ is only a chain). It is only for orientable discrete manifolds with boundary that we can achieve δG to be a simplicial complex and actually a closed discrete manifold without boundary, leading to the classical theorems in calculus.

2.3. In calculus we are used to split up the exterior derivatives $d_p : \Lambda^p \rightarrow \Lambda^{p+1}$. It is more convenient to just stick them together to one single derivative $d : \Lambda \rightarrow \Lambda$. For a simplicial complex G with n elements, the exterior derivative matrix d on the union of all forms together is given by a $n \times n$ matrix. The corresponding **Dirac operator** $D = d + d^*$ defines the **Hodge Laplacian** $L = D^2$ which is block diagonal, each block L_k acting as a symmetric Laplacian on k -forms.

2.4. The operator D not only simplifies the notation; it is also useful: the solution to the wave equation $u_{tt} = Lu$ on the graph for example is given by an explicit **d'Alembert solution formula** $u(t) = \cos(Dt)u(0) + \sin(Dt)D^{-1}u'(0)$, where D^{-1} is the pseudo inverse of D . Classically, in the case $D = -i\partial_x$, this is the **Taylor formula** $u(t) = e^{iDt}u(0) = \exp(\partial_x t)u$. In the case of a line graph, this is the **Newton-Gregory formula** (= discrete Taylor formula) $f(x+t) = \sum_k f^k(x)t^k/k!$, where t^k are deformed polynomials. On a general graph, it is a Feynman-Kac path integral, which is in the discrete a finite weighted sum over all possible paths of length t . (See [16] for an exhibit).

2.5. A general solution of the **Schrödinger evolution** $e^{Dt}\psi(0)$ with complex valued initial condition $\psi(0) = u(0) + D^{-1}u'(0)$ is identical to the real wave evolution which so because in an abstract way Lorentz invariant in space-time (where time is continuous). Also the heat evolution $e^{-tL}u(0)$ is of practical use, as it produces a numerical method to find harmonic forms representing cohomology classes. The reason is that under the evolution, all the eigenmodes of the positive eigenvalues get damped exponentially fast, so that the attractor on each space of k -forms is the space of harmonic k -vectors.

2.6. The first block of L , the matrix L_0 acting on scalar functions is known under the name **Kirchhoff matrix**. The eigenspace $E_k(0)$ of the eigenvalue 0 of L_k defines the linear space of harmonic forms whose dimension agrees with the dimension of the cohomology group $H^k(G)$. By McKean-Singer, for any t , the super trace of e^{-tL} is the Euler characteristic $\chi(G)$ which is $\chi_1 \cdot v(G)$, where $\chi_1 = (1, -1, 1, \dots)$ and the **f -vector** $v(G) = (v_0(G), v_1(G), \dots)$ which encodes the number $v_k(G)$ of complete subgraphs of dimension k . In general, $b_k(G) = \dim(H^k(G))$ is equal to the number of zero eigenvalues of L_k . This is part of discrete Hodge theory. We also know that the collection of the nonzero eigenvalues of all Fermionic Laplacians L_{2k+1} is equal to the collection of the nonzero eigenvalues of all Bosonic Laplacians L_{2k} . This **super symmetry** leads to the **McKean-Singer formula**

$$\chi(G) = \text{str} \exp(-tL) .$$

For $t = 0$, this is the combinatorial definition $\chi(G) = \text{str}(1)$ of the Euler characteristic. For $t \rightarrow \infty$, it gives the super sum of the Betti numbers. If $G = (V, E)$ is a finite simple graph, let $v = (v_0, \dots, v_d)$ denote the **f -vector** of G , where v_i is the number of i -dimensional subgraphs of G .

2.7. If $v = (v_0, v_1, \dots, v_r)$ is the f -vector of G , then $n = v_0 + \dots + v_r = v(G) \cdot (1, \dots, 1)$ simplices in G . The number $\chi(G) = v(G) \cdot (1, -1, \dots, \pm 1)$ is the **Euler characteristic** of G . We know $\chi(G \times H) = \chi(G)\chi(H)$ and also $\chi(G) = \chi(G_1)$, where G_1 is the Barycentric refinement. The graph G_1 is a subgraph of the **connection graph**, where two simplices are connected if they intersect. Let $V(G)$ be the **f -matrix** of G counting in the entry V_{ij} how many i -simplices intersect with j -simplices. More generally, we have a symmetric V_{i_1, \dots, i_k} which tells, how many i_1 simplices, i_2 simplices to i_k simplices intersect. For any k , the Euler characteristic satisfies the **Euler-Poincaré formula** $\chi_k(G) = b_0(G) - b_1(G) + b_2(G) \dots$, where $b_j(G) = \dim(H^j(G))$ is the k 'th **Betti number** of G [4]. We will below generalize this Euler-Poincaré formula from $\chi(G) = \omega_1(G)$ to all the Wu characteristics $\omega_k(G)$.

3. INTERACTION COHOMOLOGY

3.1. In this section we define a cohomology compatible with the Wu characteristic

$$\omega(G) = \sum_{x \sim y} \omega(x)\omega(y) ,$$

a sum taken over all intersecting simplices in G , where $\omega(x) = (-1)^{\dim(x)}$. Wu characteristic is a quadratic version of the Euler characteristic $\chi(G) = \sum_x \omega(x)$. Using $\chi_1 = (1, -1, 1, \dots)$ and the f -vector $v(G)$ and f -matrix $V(G)$ we can write $\chi(G) = \chi_1 \cdot v(G)$ and $\omega(G) = \chi_1 \cdot (V(G) \cdot \chi_1)$. Cubic and higher order Wu characteristics are defined in a similar way. All these characteristic numbers are multi-linear valuations in the sense of [21]. One can also generalize the setup and define

$$\omega(G, H) = \sum_{x \in G, y \in H, x \sim y} \omega(x)\omega(y) .$$

It is a **Wu intersection number**. This can be especially interesting if H is a geometric object embedded in G .

3.2. Example: Let G be the complete graph $G = K_n$. By subtracting the number of all non-intersecting pairs of simplices from all pairs of simplices, we get the f -matrix $V_{ij} = \binom{n}{i} \binom{n}{j} - \frac{n!}{i!j!(n-i-j)!}$ which gives $\omega(K_{d+1}) = \sum_{i,j} V_{ij} (-1)^i (-1)^j = (-1)^d$. If f_G is the Stanley-Reisner polynomial of G , then $\omega(G) = f_G^2(-1, \dots, -1) - f_G^2(-1, \dots, -1)$. We can therefore write

$$\chi(G) = \sum_x \omega(x), \omega(G) = \sum_{x \sim y} \omega(x)\omega(y) .$$

The higher order versions $\omega_k(G) = \sum_{x_1 \sim \dots \sim x_k} \prod_{j=1}^k \omega_j$ are computed in the same way. We have $\omega_k(K_n) = \chi(K_n)$ for odd n and $\omega_k(K_n) = 1$ for even n .

3.3. When looking at valuations on **oriented simplices**, we require $F(A) = -F(\bar{A})$, where \bar{A} denotes the simplex with opposite orientation. We think of these valuations as **discrete differential forms**. Multi-linear versions of such valuations play the role of quadratic or higher order differential forms. Given a k -linear form F , it leads to the **exterior derivative**

$$dF(x_1, \dots, x_k) = \sum_j (-1)^{\sum_{i < j} \dim(x_i)} F(A_1, \dots, dA_j, \dots, A_k) ,$$

where dA is the boundary chain of G . In the case $k = 1$, this is the usual exterior derivative

$$dF(x) = F(dx) .$$

For $k = 2$, it is

$$dF(x, y) = F(dx, y) + (-1)^{\dim(x)} F(x, dy) .$$

Because $ddA = 0$, the operation d can serve as an exterior derivative: we get a **cohomology**

$$H^p(G) = \ker(d_p)/\text{im}(d_{p-1}) .$$

We call it the **intersection cohomology** or the **quadratic intersection cohomology** or simply the **quadratic simplicial cohomology**. It deals with interacting k -tuples of simplices. All these cohomologies $H_k^p(G_1, \dots, G_k)$ generalize simplicial cohomology, which is the case $k = 1$.

3.4. Define $b_{k,p}(G)$ to be the dimension of the p 'th cohomology group $H_k^p(G)$. It is the p 'th **Betti number on the level k** . More generally, if G_1, \dots, G_k are simplicial complexes, define $b_p(G_1, \dots, G_k)$ to be the dimension of the cohomology group $H^p(G_1, \dots, G_k)$.

3.5. Functions on simplex k -tuples of pairwise intersecting simplices satisfying

$$\sum_{i=1}^k \dim(x_i) = p$$

are called **p -forms f** in the k -interaction cohomology. As for simplicial cohomology, if $df = 0$, we have **cocycles** and if $f = dg$ we have **coboundaries**.

Examples.

1) If G is the octahedron graph, then, for $k = 1$ there are only 0-forms, 1-forms and 2-forms. The Betti vector for simplicial cohomology is $b(G) = (1, 0, 1)$ which super sums to $\chi(G) = 2$. For $k = 2$, there are only non-trivial cohomology classes for 2-forms, 3-forms as well as 4-forms. The Betti vector for quadratic interaction cohomology is $b_2(G) = (0, 0, 1, 0, 1)$. The same Betti vectors appear for the icosahedron or Barycentric refinements of these 2-spheres. For the Barycentric refinement of the icosahedron, the Dirac operator is already a 9602×9602 matrix.

2) The following example shows how can use cohomology to distinguish topologically not-equivalent complexes, which have equivalent simplicial cohomology. The pair has been found by computer search. Let G be the Whitney complex of of the graph with edges $E = \{ (1, 2), (1, 3), (1, 6), (1, 7), (1, 8), (1, 9), (2, 3), (2, 6), (2, 8), (2, 9), (3, 7), (3, 9), (4, 5), (4, 6), (4, 8), (4, 9), (5, 6), (7, 8) \}$. Let H be the Whitney complex with edges $E = \{ (1, 2), (1, 3), (1, 4), (1, 5), (2, 4), (2, 5), (2, 6), (2, 8), (2, 9), (3, 4), (3, 5), (3, 9), (4, 6), (4, 8), (6, 8), (6, 9), (7, 8), (7, 9) \}$, the f-vector is $(9, 18, 9, 1)$ and the f-matrix is $[[9, 36, 27, 4], [36, 136, 96, 13], [27, 96, 65, 8], [4, 13, 8, 1]]$. The Betti vector is the same $(1, 2, 0, 0)$ so linear simplicial cohomology of G and H are the same. The quadratic Wu cohomology of G and H however differ: it is $b_2(G) = (0, 0, 6, 3, 0, 0, 0)$ and $b_2(H) = (0, 0, 4, 1, 0, 0, 0)$ for H .

3) This is an interesting prototype computation which we have written down in 2016 (see the appendix). For the **cylinder** G given by the Whitney complex of the graph with edge set $E = \{ (1, 2), (2, 3), (3, 4), (4, 1), (5, 6), (6, 7), (7, 8), (8, 5), (1, 5), (5, 2), (2, 6), (6, 3), (3, 7), (7, 4), (4, 8), (8, 1) \}$ and the **Möbius strip** given as the Whitney complex of the graph with edge set $E = \{ (1, 2), (2, 3), (3, 4), (4, 5), (5, 6), (6, 7), (7, 1), (1, 4), (2, 5), (3, 6), (4, 7), (5, 1), (6, 2), (7, 3) \}$ we have identical simplicial cohomology but different quadratic cohomology. The cylinder has the Betti vector $(0, 0, 1, 1, 0)$, the Möbius strip has the trivial quadratic Betti vector $(0, 0, 0, 0, 0)$. In the appendix computation, all matrices are explicitly written down. An other appendix gives the code.

3.6. In the same way as in the case $k = 1$, we have a cohomological reformulation of Wu-characteristic. This Euler-Poincaré formula holds whenever we have a cohomology given by a concrete exterior derivative [4, 5, 10]:

Theorem 1 (Euler-Poincaré). *For any $k \geq 1$, we have $\omega_k(G) = \sum_{p=0}^{\infty} (-1)^p \dim(H_k^p(G))$.*

Proof. Here is the direct linear algebra proof: Assume the vector space C_m of m -forms on G has dimension v_m . Denote by $Z_m = \ker(d)$ the kernel of d of dimension z_m . Let $R_m = \text{ran}(d)$ the range of d of dimension r_m . From the rank-nullity theorem $\dim(\ker(d_m)) + \dim(\text{ran}(d_m)) = v_m$, we get $z_m = v_m - r_m$. From the definition of the cohomology groups $H^m(G) = Z_m(G)/R_{m-1}(G)$ of dimension b_m we get $b_m = z_m - r_{m-1}$. Adding these two equations gives

$$v_m - b_m = r_{m-1} + r_m .$$

Summing over m (using $r_{-1} = 0, r_m = 0$ for $m > m_0$)

$$\sum_{m=0}^{\infty} (-1)^m (v_m - b_m) = \sum_{m=0}^{\infty} (-1)^m (r_{m-1} + r_m) = 0$$

which implies $\sum_{m=0}^{\infty} (-1)^m v_m = \sum_{m=0}^{\infty} (-1)^m b_m$. □

3.7. As we will see below, there is a more elegant deformation proof. It uses the fact that D produces a symmetry between non-zero eigenvalues of D^2 restricted to even forms and non-zero eigenvalues of $L = D^2$ restricted to odd forms. This implies $\text{str}(L^n) = 0$ for all $n > 0$ and so the Mc-Kean-Singer relations $\text{str}(\exp(-tL)) = 0$. For $t = 0$, this is the definition of ω_k . In the limit $t \rightarrow \infty$ we get to the projection onto Harmonic forms, where

$$\lim_{t \rightarrow \infty} \text{str}(\exp(-itL)) = \sum_{p=0}^{\infty} (-1)^p \dim(H_k^p(G)) .$$

4. EXAMPLE COMPUTATIONS

4.1. For $G = \{\{1, 2\}, \{2, 3\}, \{1\}, \{2\}, \{3\}\}$, there are 15 interacting pairs $x \sim y$ of simplices:

$$\left\{ \begin{array}{cc} \{3\} & \{3\} \\ \{2\} & \{2\} \\ \{1\} & \{1\} \\ \{3\} & \{2, 3\} \\ \{2\} & \{2, 3\} \\ \{2\} & \{1, 2\} \\ \{1\} & \{1, 2\} \\ \{2, 3\} & \{3\} \\ \{2, 3\} & \{2\} \\ \{1, 2\} & \{2\} \\ \{1, 2\} & \{1\} \\ \{2, 3\} & \{2, 3\} \\ \{2, 3\} & \{1, 2\} \\ \{1, 2\} & \{2, 3\} \\ \{1, 2\} & \{1, 2\} \end{array} \right\} .$$

Laplacian to this pair (G, H) is

$$L = \begin{bmatrix} 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 2 \end{bmatrix}.$$

Its null-space of L is spanned by a single vector $(0, 0, 0, 0, 0, -1, 1, -1, 1)$ so that the Betti vector is $(0, 0, 1)$.

4.3. If G is a 3-sphere, obtained by suspending the octahedron, then $\chi(G) = \omega(G) = 0$ and $b_{k=1}(G) = (1, 0, 0, 1)$ and $b_{k=2}(G) = (0, 0, 0, 1, 0, 0, 1)$. Also for a 4-sphere, like a suspension of a 3-sphere G , the simplicial cohomology has just shifted up $b_{k=2}(G) = (0, 0, 0, 0, 1, 0, 0, 0, 1)$.

4.4. Here is a table of some linear, quadratic and cubic cohomology computations:

$\chi = \omega_1$	Betti	$\omega = \omega_2$	Betti	ω_3	Betti	Complex
1	(1)	1	(1)	1	(1)	K_1 (point)
1	(1,0)	-1	(0,1,0)	1	(0,0,1,0)	K_2 (1-ball)
1	(1,0,0)	1	(0,0,1,0,0)	1	(0,0,0,0,1,0,0)	K_3 (triangle)
1	(1,0,0,0)	-1	(0,0,0,1,0,0,0)	1	(0,0,0,0,0,0,1,0,0,0)	K_4 (tetrahedron)
0	(1,1)	0	(0,1,1)	0	(0,0,1,1)	C_4 (circle)
2	(1,0,1)	2	(0,0,1,0,1)	2	(0,0,0,0,1,0,1)	Octahedron
2	(1,0,1)	2	(0,0,1,0,1)	2	(0,0,0,0,1,0,1)	Icosahedron
0	(1,0,0,1)	0	(0,0,0,1,0,0,1)	0	(0,0,0,0,0,0,1,0,0,1)	3-sphere
2	(1,0,0,0,1)	2	(0,0,0,0,1,0,0,0,1)	2	(0,0,0,0,0,0,0,0,1,0,0,0,1)	4-sphere
1	(1,0,0)	1	(0,0,1,0,0)	1	(0,0,0,0,1,0,0)	2-ball
1	(1,0)	1	(0,0,1)	-5	(0,0,0,5)	3-star
1	(1,0)	5	(0,0,5)	-23	(0,0,0,23)	4-star
1	(1,0)	11	(0,0,11)	-59	(0,0,0,59)	5-star
1	(1,0)	1	(0,0,0,0,1)	25	(0,0,0,0,0,0,25)	3-star x 3-star
-1	(1,2)	7	(0,0,7)	-25	(0,0,0,25)	Figure 8
-2	(1,3)	22	(0,0,22)	-122	(0,0,0,122)	3 bouquet
-3	(1,4)	45	(0,0,35)	-339	(0,0,0,339)	4 bouquet
-4	(1,5)	76	(0,0,76)	-724	(0,0,0,724)	5 bouquet
1	(1,0)	3	(0,0,3,0,0)	-5	(0,0,0,6,1,0,0)	Rabbit
0	(1,1)	2	(0,0,2,0,0)	0	(0,0,0,1,1,0,0)	House
-4	(1,5)	20	(0,0,20)	-52	(0,0,0,52)	Cube
-16	(14,30)	112	(0,0,112)	-400	(0,0,0,400)	Tesseract
0	(1,1,0)	0	(0,0,0,0,0)	0	(0,0,0,0,1,1,0)	Moebius strip
0	(1,1,0)	0	(0,0,1,1,0)	0	(0,0,0,0,1,1,0)	Cylinder
1	(1,0,0)	1	(0,0,0,0,1)	1	(0,0,0,0,0,0,1)	Projective plane
0	(1,1,0)	0	(0,0,0,1,1)	0	(0,0,0,0,0,1,1)	Klein bottle
1	(1,0,0)	1	(0,0,0,1,2)	1	(0,0,0,0,0,1,2)	Dunce hat

4.5. Let us look at more examples of quadratic interaction cohomology $H^p(G, H)$ with two different complexes G, H . The cohomology groups $H^p(G, H)$ are trivial vector spaces if $G \cap H = \emptyset$, as there are then no interaction. The Moebius strip is a rare example of trivial interaction cohomology $H^p(G, G)$.

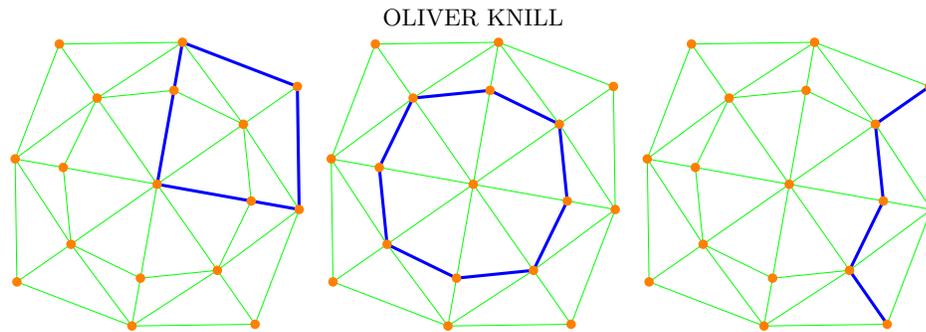


FIGURE 1. The background complex G is a Barycentric refinement of the wheel graph. The $k = 2$ cohomology has the Betti vector $b(G) = (0, 0, 1, 0, 0)$. To the very left and very right we see unit sphere subgraphs H which touch the boundary. The Betti vector for the cohomology $H(G, H)$ is $b(G, H) = (0, 0, 1, 0)$ which matches with $\omega(G, H) = 1$. In the middle we see a circular subgraph H which does not hit the boundary. In that case the cohomology is $b(G, H) = (0, 0, 1, 1)$ leading to the Wu intersection number $\omega(G, H) = 0$.

$\omega = \omega_2$	Betti vector	Complex G	Complex H
-1	(0,1,0)	interval	interval
-1	(0,1)	interval	point inside
0	(0,0)	interval	boundary point
0	(0,1,1)	circle	circle
-1	(0,1)	circle	point
-2	(0,2)	circle	two points
-1	(0,1,0)	circle	sub interval
11	(0,0,11)	star(5)	star(5)
4	(0,4)	star(5)	central point
0	(0,0)	star(5)	boundary point
-1	(0,1,0)	star(5)	radius interval
0	(0,0,1,1)	2D disk	Circle in interior
1	(0,0,1,0)	2D disk	Circle touching boundary
1	(0,0,1,0)	2D disk	2D disc
1	(0,0,1)	2D disk	Point inside
0	(0,0,0)	2D disk	Point at boundary

4.6. We can have seen examples, where H was a subcomplex of G . It is also possible that G, H just intersect. In that case, only a neighborhood of the intersection set matters. We have computed Wu characteristic of nicely intersecting one dimensional graphs. To analyze this, we only need to understand a single intersection cross, where the cohomology is $b(G, H) = (0, 0, 1)$. Now, if G is a circle and H is a circle and the two circles intersect nicely, we have $b(G, H) = (0, 0, 2)$ as there are two intersection points.

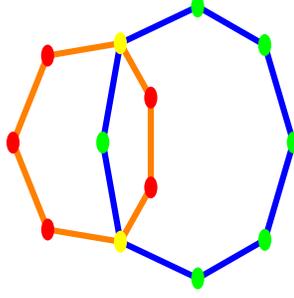


FIGURE 2. The intersection of two circles G, H has the Betti vector $b(G, H) = (0, 0, 2)$ and $\omega(G, H) = 2$.

5. HODGE RELATIONS

5.1. The Euler-Poincaré relations equating combinatorial and cohomological expressions for $H_k^p(G)$ can be proven dynamically using the heat flow $\exp(-tL)$. To do so, it is important to build a basis of the cohomology using harmonic forms. For $t = 0$, we have by definition the Euler characteristic and for $t \rightarrow \infty$, the super trace of the identity on harmonic forms. These Hodge relations play there an important role. It is good to revisit them in the more general frame work of interaction cohomology. The proofs are the same than for $k = 1$ however.

5.2. The Hodge theorem states that $H^p(G_1, \dots, G_k)$ is isomorphic to the space $E_0^p(G_1, \dots, G_k)$ of harmonic p -forms.

- Lemma 1.** a) L commutes with d and d^* ,
 b) $Lf = 0$ is equivalent to $df = 0$ and $d^*f = 0$
 c) $f = dg$ and $h = d^*k$ are orthogonal.
 d) $\text{im}(d) \cup \text{im}(d^*)$ span $\text{im}(L)$.

Proof. a) follows from $d^2 = (d^*)^2 = 0$ so that $d^*L = Ld^* = d^*dd^*$ and $dL = Ld = dd^*d$.
 b) if $Lf = 0$ then $0 = \langle f, Lf \rangle = \langle d^*f, df \rangle + \langle df, df \rangle$ shows that both $d^*f = 0$ and $df = 0$. The other direction is clear.
 c) $\langle f, h \rangle = \langle dg, d^*k \rangle = \langle d dg, k \rangle = \langle g, d^*d^*k \rangle = 0$.
 d) Write $g = Lf = d(d^*f) + d^*(df)$. □

This shows that any p -form can be written as a sum of an exact, a co-exact and a harmonic p -form:

Lemma 2 (Hodge decomposition). *There is an orthogonal decomposition $\Omega = \text{im}(L) + \ker(L) = \text{im}(d) + \text{im}(d^*) + \ker(L)$. Any p -form g can be written as $g = df + d^*h + k$, where k is harmonic.*

Proof. The matrix $L : \Omega(G_1, \dots, G_k) \rightarrow \Omega(G_1, \dots, G_k)$ is symmetric so that the image and kernel are perpendicular. By the lemma, the image of L splits into two orthogonal components $\text{im}(d)$ and $\text{im}(d^*)$. □

Theorem 2 (Hodge-Weyl). *The dimension of the vector space of harmonic p -forms is equal to $b_p(G_1, \dots, G_k)$. Every cohomology class has a unique harmonic representative.*

Proof. If $Lf = 0$, then $df = 0$ and so $Lf = d^*df = 0$. Given a closed n -form g then $dg = 0$ and the Hodge decomposition shows $g = df + k$ so that g differs from a harmonic form only by df and so that this harmonic form is in the same cohomology class than f . □

The **harmonic k -form** represent therefore cohomology classes for the cohomologies $H^p(G_1, \dots, G_k)$.

6. LEFSCHETZ FIXED POINT FORMULA

6.1. In the case of Euler characteristic **Lefschetz number** of an automorphism T of a complex G is the super trace of the induced linear map on simplicial cohomology. If T is the identity, then this is the definition of the cohomological Euler characteristic. We want to show here that the Lefschetz fixed point formula relates more generally the Lefschetz number with the fixed simplices

of T . If T is the identity, then every simplex is fixed and the Lefschetz fixed point formula is the Euler-Poincaré formula. Since we have a cohomology for Wu characteristic, we can look at the super trace of the map induced on the cohomology groups and get so a new Lefschetz number.

6.2. We also have fixed pairs of simplices (x, y) and an **2-interaction index**

$$i_T(x, y) = \omega(x)\omega(y)\text{sign}(T|x)\text{sign}(T|y) .$$

Now, the analogue quadratic theorem is that $L_2(T) = \sum_{x \sim y \in G, T(x)=x, T(y)=y} i_{x,y}(T)$, where the sum is over all pairs of simplices, where one is fixed by T . This is the Lefschetz fixed point formula for Wu characteristic. In the case of the identity, every pair of intersecting simplices is a fixed point and the Lefschetz fixed point formula reduces to the Euler-Poincaré formula for Wu characteristic. For example, for an orientation reversing involution on the circular graph $G = C_n$ with $n \geq 4$. The space $H^{0,1}(G)$ is 6-dimensional as is $H^{1,1}(G)$.

6.3. An **automorphism** T of G is a permutation on the finite set V of zero-dimensional elements in G such that T induces a permutation on G . More generally, one could look also at endomorphisms, maps which map simplices to simplices and preserve the order, but since we are in a finite frame work, we can always restrict to the attractor $\bigcap_n T^n(G)$ on which T then induces an automorphism. It is therefore almost no restriction of generality to look at automorphisms.

6.4. Let $\text{Aut}(G)$ be the group of automorphisms of G . Any $T \in \text{Aut}(G)$ induces a linear map on each interaction cohomology group $H_k^p(G)$. To realize that the matrix belonging to the transformation take an orthonormal basis for the kernel of L_k and define $A_{ij} = f_i(T)f_j$. If $L|H(G)$ denotes the induced map on cohomology, define the **Lefschetz number** $\chi_T(G) = \text{str}(L|H(G))$. In the special case, when T is the identity $T = 1$, the Lefschetz number is the Wu characteristic. In the case $k = 2$, a pair of intersecting simplices (x, y) is called a **fixed point** of T , if $T(x) = x, T(y) = y$. More generally, we call a k -tuple (x_1, \dots, x_k) of pairwise intersecting simplices to be a fixed point of T if $T(x_j) = x_j$. Define the **index** of a fixed point as

$$i_T(x_1, \dots, x_k) = \prod_{j=1}^k (-1)^{\dim(x_j)} \text{sign}(T|x_j) .$$

6.5. In the case $k = 1$, the index is

$$i_T(x) = (-1)^{\dim(x)} \text{sign}(T|x) .$$

In the case $k = 2$, the index is

$$i_T(x, y) = (-1)^{\dim(x)+\dim(y)} \text{sign}(T|x)\text{sign}(T|y) .$$

Here is the Brouwer-Lefschetz-Hopf fixed point formula generalizing [10].

Theorem 3 (Brouwer-Lefschetz fixed point theorem). $\chi_T(G) = \sum_{T(x)=x} i_T(x)$

The heat proof works as before. For $t = 0$ we get the right hand side, for $t \rightarrow \infty$ we get the left hand side.

Proof. Combine the unitary transformation $U : f \rightarrow f(T)$ with the heat flow $V(t) = \exp(-tL)$ to get the time-dependent operator $W(t) = UV(t)$ on the linear space of discrete differential forms. For $t = 0$, the super trace of W_t is $\sum_{x \sim y} i_T(x, y)$. McKean-Singer shows that the super trace of any positive power L^n of L is always zero. This is the **super symmetry** meaning that the union of the positive eigenvalues on the even dimensional part of the interaction Laplacian L is the same than the union of the positive eigenvalues on the odd dimensional part of L . For $t \rightarrow \infty$, the heat flow kills the eigenspace to the positive eigenvalues and pushes the operator $W(t)$ towards a projection onto the kernel. The projection operator P onto harmonic interaction differential forms is the attractor. Since by Hodge theory the kernel of the Laplacian $L_{\text{sub}_i m_i / \text{sub}_i}$ on m -interaction forms parametrized representatives of the m -th cohomology group, the transformation $W(\infty)$ is nothing else than the induced linear transformation on cohomology. Its super trace is by definition the Lefschetz number. The super symmetry relation $\text{str}(L^n)$ for $n > 0$ assures

that the super trace of $\exp(-tL)$ does not depend on t . The heat flow (mentioned for example in [24]) washes away the positive spectrum and leaves only the bare harmonic part responsible for the cohomology description. For $T = 1$, the heat flow interpolates between the combinatorial definition of the Wu characteristic and the cohomological definition of the Wu characteristic, similarly as in the case of the Euler characteristic. \square

6.6. Also, as in the case $k = 1$, we can average the Lefschetz number over a subgroup A of the automorphism group of G to get $\omega_k(G/A)$, where G/A is the quotient chain. If we look at the Barycentric refinement, we can achieve that the quotient chain is again a simplicial complex. These are generalizations of Riemann-Hurwitz type results.

7. BARYCENTRIC INVARIANCE

7.1. Almost by definition, the Wu characteristic $\omega(G)$ is multiplicative

$$\omega(G \times H) = \omega(G)\omega(H)$$

because

$$\sum_{(x,y) \sim (u,v)} \omega(x)\omega(y)\omega(u)\omega(v) = \left(\sum_{x \sim u} \omega(x)\omega(u)\right) \left(\sum_{y \sim v} \omega(y)\omega(v)\right).$$

The same product formula holds for the other cases $\omega_k(G \times H) = \omega_k(G)\omega_k(H)$. This relation has used the Cartesian product $G \times H$ of simplicial complexes which is not a simplicial complex any more. In [19] we have looked at the product $(G \times H)_1$, the Barycentric refinement of $G \times H$ so that we would remain in the class of simplicial complexes. But because calculus can be done equally well on the ring itself, there is no need any more to insist on having simplicial complexes. Indeed, in the strong ring, the simplicial complexes are the multiplicative primes.

Proposition 1. *All the Wu characteristics ω_k are ring homomorphisms from the strong ring to the integers.*

Proof. The additivity $\omega_k(G + H) = \omega_k(G + H)$ for the disjoint union is clear as all ω_k are multilinear valuations and $\omega_k(0) = 0$. The compatibility with multiplication follows from $\omega_k(1) = 1$ for all k and

$$\begin{aligned} \omega_k(G \times H) &= \sum_{(x_1, \dots, x_k) \sim (y_1, \dots, y_k)} \omega(x_1) \cdots \omega(x_k) \omega(y_1) \cdots \omega(y_k) \\ &= \left(\sum_{(x_1, \dots, x_k)} \omega(x_1) \cdots \omega(x_k)\right) \left(\sum_{(y_1, \dots, y_k)} \omega(y_1) \cdots \omega(y_k)\right) \\ &= \omega_k(G)\omega_k(H) \end{aligned}$$

\square

7.2. Given a simplicial complex G defined over a set V of “vertices”, zero dimensional sets in G , the complex G_1 is defined over the set $V_1 = G$. We call a vertex x in G_1 a **p -vertex**, if it had been a p -dimensional simplex in G .

Theorem 4 (Barycentric Invariance). $H_k^p(G) = H_k^p(G_1)$.

Proof. There is an explicit map ϕ from the space of forms on G to the space of forms in G_1 . Lets look at it first in the case $k = 1$.

Given a p -simplex (x_1, \dots, x_p) in G_1 , there is maximally one of the vertices where F is non-zero. Define $F(x_1, \dots, x_p) = F(x_k)/p!$ to be this value. On the other hand, given a form F on G_1 , we get a form on G . Given a simplex x in G it contains $p!$ simplices of the same dimension in G_1 . Now just sum up the F values over these simplices. As ϕ and its inverse commute with the exterior derivative d in the sense $d(G_1)\phi = \phi d(G)$, we have also $L(G_1)\phi = \phi L(G)$. Harmonic forms are mapped into Harmonic forms.

More generally, the Barycentric invariance works for $H^p(G_1, \dots, G_k)$, which is isomorphic to $H^p((G_1)_1, \dots, (G_k)_1)$, where $(G_j)_1$ is the Barycentric refinement of G_j . The map ϕ from forms in

$\Omega(G_1, \dots, G_k)$ to forms in $\Omega((G_1)_1, \dots, (G_k)_1)$ just distributes the value $F(x_1, \dots, x_k)$ on pairwise intersecting simplices x_1, \dots, x_k equally to all k -tuples of pairwise intersecting simplices y_1, \dots, y_k with y_j being simplices of the same dimension than x_j within x_j . The inverse of ϕ sums all values up. These averaging procedures commute with d . \square

7.3. We don't know yet whether there is a more robust statement. We would think for example that if H_1, H_2 are two complexes, where one is a Barycentric refinement of the other and both are embedded in the same way in a larger graph G , then $H^p(G, H_1) = H^p(G, H_2)$. Experiments indicated that this is true. We also have hopes that if H_1, H_2 are two different knots embedded into a three dimensional discrete sphere G , then we could distinguish the embedding by computing $H^p(G, H_1)$ and $H^p(G, H_2)$. We also need to look at the cohomology of the complement $H^p(G, G \setminus H_1)$ and $H^p(G, G \setminus H_2)$. Classically, one has used simplicial cohomology $H^p(G \setminus H_1), H^p(G \setminus H_2)$ already.

8. KUENNETH FORMULA

8.1. The Barycentric refinement invariance shows that that the cohomology of $(G \times H)_1$ is the same than the cohomology of $G \times H$. So, we could for $k = 1$ refer to [19] to get

Theorem 5 (Kueneth). $H_k^p(G \times H) = \oplus_{i+j=p} H_k^i(G) \oplus H_k^j(H)$.

8.2. The proof follows from Hodge and the proposition stated below. Instead of generalizing [19] from $k = 1$ to larger k , it is better to prove the relation directly first for the Cartesian product (which is not a simplicial complex), and then invoke the Barycentric refinement, to recover [19]. Now this is just Hodge. We have to show that for every k , every harmonic p -form f can be written as a pair (g, h) , where g is an i -form and h is a j form.

8.3. Künneth could not be more direct:

Proposition 2. *A basis for $H^p(G, H)$ is given by L -harmonic p -forms v . A basis for $L(G, H)$ -Harmonic p -forms is given by all the vectors $v \oplus w$, where v is $L(G)$ -harmonic k -form and w is $L(H)$ -harmonic m -form so that $k + m = p$.*

8.4. In [19] the construction of a chain homotopy was more convoluted as we dealt with the Barycentric refinements (as at time, we felt we needed to work with simplicial complexes). Separating the two things and establishing the Barycentric invariance separately is more elegant.

8.5. For a given k and ring element G , define the **Poincaré polynomial**

$$p_{G,k}(t) = \sum_{l=0} b_{k,l}(G)t^l .$$

If $k = 1$, then $p_G(t) = p_{G,1}(t)$ is the usual Poincaré polynomial.

Corollary 1. *The map $G \rightarrow p_{G,k}(t)$ is a ring homomorphism from the strong connection ring to $\mathbb{Z}[t]$.*

Proof. If d_i are the exterior derivatives on G_i , we can write them as partial exterior derivatives on the product space. We get from $df(x, y) = d_1f(x, y) + (-1)^{\dim(x)}d_2(f(x, y))$

$$d^*df = d_1^*d_1f + (-1)^{\dim(x)}d_1^*d_2f + (-1)^{\dim(x)}d_2^*d_1f + d_2^*d_2f ,$$

$$dd^*f = d_1d_1^*f + (-1)^{\dim(x)}d_1d_2^*f + (-1)^{\dim(x)}d_2d_1^*f + d_2d_2^*f .$$

Therefore $Hf = H_1f + H_2f + (-1)^{\dim(x)}(d_1^*d_2 + d_1d_2^* + d_2^*d_1 + d_2d_1^*)f(x, y)$. Since Hodge gives an orthogonal decomposition

$$\text{im}(d_i), \text{im}(d_i^*), \ker(H_i) = \ker(d_i) \cap \ker(d_i^*) ,$$

there is a basis in which $H(v, w) = (H(G_1)(v), H(G_2)(w))$. Every kernel element can be written as (v, w) , where v is in the kernel of H_1 and w is in the kernel of H_2 . \square

8.6. In the special case $k = 1$, the Kuenneth formula would follow also from [19] because the product is $(G \times H)_1$ and the cohomology of the Barycentric refinement is the same. It follows that the **Euler-Poincaré formula** holds in general for elements G in the ring: the cohomological Euler characteristic $\sum_{k=0}^{\infty} b_k(G)(-1)^k$ is equal to the combinatorial Euler characteristic $\sum_{k=0}^{\infty} v_k(G)(-1)^k$, where (v_0, v_1, \dots) is the f -vector of G . Note that the Cartesian product $G \times H$ of two signed simplicial complexes is still a signed discrete CW complex so that one can define $v_k(G \times H)$ as the number of k -dimensional cells in $G \times H$.

9. EULER POLYNOMIAL

9.1. Every ring element G in the strong ring has a **f -vector** $(v_0, v_1, v_2, \dots, v_r)$. Unlike for a simplicial complex, the entries are now general integers and not non-negative integers. This defines the **Euler polynomial**

$$e_G(t) = \sum_{l=0}^{\infty} v_l(G)t^l .$$

The Euler-Poincaré relations can be stated as $e_G(-1) = p_G(-1)$. This is a relation which holds for any ring element.

9.2. The Poincaré polynomial relates with the Euler polynomial, which also has a nice algebraic description:

Proposition 3 (Euler polynomial). *The map $G \rightarrow e_G(t)$ is a ring homomorphism from the strong connection ring to $\mathbb{Z}[t]$.*

Proof. When doing a disjoint union, the cardinalities add. For the Cartesian product, we take the set theoretical Cartesian product $G \times H$ of sets of sets. Now, if x has dimension p and y has dimension q , then $x \times y$ has dimension $p + q$. \square

9.3. There are higher dimensional versions of the Euler polynomial. Let $V = V_{k,l}$ denote the **f -vector** of G , where $V_{k,l}$ are the number of pairs (x, y) of simplices which intersect and $\dim(x) = k$ and $\dim(y) = l$. We can now define multi-variate Euler polynomials like the quadratic Euler polynomial

$$e_{G,2}(s, t) = \sum_{i,j} V_{i,j} s^i t^j ,$$

which is a generating function for the f -matrix. The definition of the Wu characteristic gives $\omega_2(G) = e_{G,2}(-1, -1)$.

9.4. Examples.

1) For $G = K_2$, we have $e_G(t) = 2 + t$ and $e_{G,2}(t, s) = 2 + 2t + 2s + ts$ which leads to $\chi(G) = e_G(-1) = 1$ and $\omega(G) = e_{G,2}(-1, -1) = -1$.

2) For $G = K_3$, we have $e_G(t) = 3 + 3t + t^2$ and $e_{G,2}(t, s) = s^2 t^2 + 3s^2 t + 3s^2 + 3st^2 + 9st + 6s + 3t^2 + 6t + 3$ leading to $\omega(G) = e_{G,3}(-1, -1) = 1$.

9.5. Similarly, one can define **f -tensors** for any k . They encode the cardinalities of interacting k -tuples of simplices. The **multivariate Euler polynomial** is then

$$e_{G,k}(t_1, \dots, t_k) = \sum_{j_1, j_2, \dots, j_k} V_{j_1, \dots, j_k} t_1^{j_1} \cdots t_k^{j_k} .$$

For example, for $G = K_2$, we have $e_G(t_1, t_2, t_3) = 2 + 2t_1 + 2t_2 + 2t_3 + 2t_1 t_2 + 2t_1 t_3 + 2t_2 t_3 + t_1 t_2 t_3$ which gives $\omega_3(G) = e_G(-1, -1, -1) = 1$.

Proposition 4 (Euler polynomial). *The map $G \rightarrow e_{G,k}(t_1, \dots, t_k)$ is a ring homomorphism from the strong ring to $\mathbb{Z}[t_1, \dots, t_k]$.*

Proof. Again, the additive part is clear. For the multiplication, if $(x_1 \times \cdots \times x_k)$ intersects $(y_1 \times \cdots \times y_k)$, this happens if each x_j intersects each y_j . The intersection of simplices x_1, x_2, \dots, x_k of dimension r_1, \dots, r_k is represented algebraically by $t_1^{r_1} t_2^{r_2} \cdots t_k^{r_k}$. The intersection of simplices y_1, y_2, \dots, y_k of dimension s_1, \dots, s_k is represented algebraically by $t_1^{s_1} t_2^{s_2} \cdots t_k^{s_k}$. If we take the product, we get $t_1^{r_1+s_1} t_2^{r_2+s_2} \cdots t_k^{r_k+s_k}$ which represents the intersection of simplices $(x_1 \times y_1), (x_2 \times y_2), \dots, (x_k \times y_k)$ of dimension $r_1 + s_1, \dots, r_k + s_k$. \square

9.6. Having a multivariate ring homomorphism for the Euler polynomial suggests going back to the cohomology and redefining the cohomology so that the Poincaré polynomial becomes a multivariate polynomial too. This is indeed possible because the exterior derivative $d = d_1 + \cdots + d_k$ is a sum of **partial exterior derivatives** which each satisfy $d_j^2 = 0$. So, we get a **cohomology Betti tensor** b which is in the case $k = 2$ a **cohomology Betti matrix**. One could define htna a **multivariate Poincaré polynomial** $p_{G,k}(t_1, \dots, t_k)$ and the map $G \rightarrow p_{G,k}(t_1, \dots, t_k)$ is a ring homomorphism from the strong ring to $\mathbb{Z}[t_1, \dots, t_k]$. One can get the Poincaré polynomial (as we have defined it) by setting $p_{G,k}(t, t, \dots, t)$. This is how we have initially defined the cohomology (see the appendix). Euler-Poincaré $p_{G,k}(-1, -1, \dots, -1) = e_{G,k}(-1, -1, \dots, -1)$ still holds, but there is a problem: the Betti matrix is not invariant under Barycentric refinements, similarly as the f -matrix is not invariant under Barycentric refinement. A sensible cohomology should be invariant under Barycentric refinement.

10. MISCELLANEOUS

10.1. Any exterior derivative can be deformed via a Lax deformation [13, 12]. There is a version which keeps the exterior derivative d real

$$\dot{D} = [B(D), D], B(t) = d(t) - d^*(t), D(t) = d(t) + d(t)^* + b(t).$$

Then there is a version which allows the operators to become complex:

$$\dot{D} = [B(D), D], B(t) = d(t) - d^*(t) - ib(t), D(t) = d(t) + d(t)^* + b(t).$$

This works in the same way also for the exterior derivative d of interaction cohomology. In the first case, we have a scattering situation, in the second we get asymptotically to a linear wave equation.

10.2. Having deformations in the division algebras \mathbb{R} and \mathbb{C} , one can ask whether it is also possible in the third and last associative division algebra, the quaternions \mathbb{H} . Given a real exterior derivative we can form $D = d + d^*$ and deform it with $D' = [B, D]$, where $B = d - d^* + I(b + b^*)$, where $I = oi + pj + qk$ is a space quaternion. Given by three real commuting complexes D_1, D_2, D_3 , we can start with the initial condition $D = iD_1 + jD_2 + kD_3$. We initially have the Pythagorean relation $L = D^2 = -L_1^2 - L_2^2 - L_3^2$. Quaternions are best implemented as complex 2×2 matrices using Pauli matrices σ_i . The real number 1 is represented by the identity matrix, the standard imaginary square root of -1 is $i\sigma_3$, the quaternion j is $i\sigma_2$ and the quaternion k is $i\sigma_1$. In other words, the explicit translation from a quaternion $a + ib + jc + kd$ to a complex 2×2 matrix $\begin{bmatrix} a + ib & c + id \\ -c + id & a - ib \end{bmatrix}$. The complex parameter I is now just an arbitrary element in the Lie algebra of $SU(2)$.

10.3. Example. Take the smallest simplicial complex $G = \{\{1\}, \{2\}, \{1, 2\}\}$ for which $D = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$. To evolve in the quaternion domain, take three copies and glue them together

to get the new quaternion valued Dirac operator:

$$D = \begin{bmatrix} 0 & 0 & 0 & 0 & -i & -1-i \\ 0 & 0 & 0 & 0 & 1-i & i \\ 0 & 0 & 0 & 0 & i & 1+i \\ 0 & 0 & 0 & 0 & -1+i & -i \\ -i & -1-i & i & 1+i & 0 & 0 \\ 1-i & i & -1+i & -i & 0 & 0 \end{bmatrix}.$$

Its Laplacian $L = D^2$ is just the original Hodge Laplacian, where each entry 1 is replaced with a 2×2 identity matrix. Here is the deformed operator at time $t = 0.1$ in the case $I = i$ (the quaternion i):

$$D(t) = \begin{bmatrix} 0 + 0.36i & 0.38 + 0.33i & 0 - 0.36i & -0.38 - 0.33i & -0.19 - 0.85i & -0.86 - 0.86i \\ -0.38 + 0.33i & 0 - 0.36i & 0.38 - 0.33i & 0 + 0.36i & 0.86 - 0.86i & -0.19 + 0.85i \\ 0 - 0.36i & -0.38 - 0.33i & 0 + 0.36i & 0.3 + 0.33i & 0.19 + 0.85i & 0.86 + 0.86i \\ 0.38 - 0.33i & 0 + 0.36i & -0.38 + 0.33i & 0 - 0.36i & -0.86 + 0.86i & 0.19 - 0.85i \\ 0.19 - 0.85i & -0.86 - 0.86i & -0.19 + 0.85i & 0.86 + 0.86i & 0 - 0.72i & -0.67 - 0.77i \\ 0.86 - 0.86i & 0.19 + 0.85i & -0.86 + 0.86i & -0.19 - 0.85i & 0.67 - 0.77i & 0 + 0.72i \end{bmatrix}.$$

10.4. Let G be a finite abstract simplicial complex with Barycentric refinement G_1 and let n be the number of simplices of G . The **connection graph** G' of G is defined as the graph with vertex set G , and where two vertices are connected, if they intersect. The graph G_1 is a subgraph of G' . Already the case of a cyclic graph shows that G' has in general a higher dimension than G . Define the **connection matrix** L by $L(x, y) = 1$ if $x \cap y \neq \emptyset$ and $L(x, y) = 0$ else.

10.5. The diagonal entries of L correspond to self interactions, the side diagonal elements are interactions between simplices. With $S = \text{Diag}((-1)^{\dim(x)})$, define $J = S^{-1}IS$, where I is the matrix containing only 1. The matrix satisfies $J_{ij} = (-1)^{i+j}$. It is a checkerboard matrix. Define the **Wu matrix** WJ . We are interested in the matrix W because it gives the Wu characteristic: the Wu characteristic $\omega(G)$ of a finite simple graph is equal to $\text{tr}(WJ)$.

10.6. A square matrix is called **unimodular** if its determinant is either -1 or 1 . This implies that all entries of the inverse are integer-valued. While the Fredholm determinant of a general graph can be pretty arbitrary, the Fredholm determinant of a connection graph is always plus or minus 1 implying that L^{-1} is always an integer matrix. Define the **Fredholm characteristic** $\psi(G) = \det(L)$ and the **Fermi characteristic** $\phi(G) = \prod_x \omega(x)$, where the sum is over all simplices in G and $\omega(x) = (-1)^{\dim(x)}$.

10.7. For any finite simple graph, the Fredholm matrix L of its connection matrix is unimodular: $\psi(G) = \phi(G)$. (We were writing this paragraph of the current paper, when the unimodularity discovery happend, delaying this by 2 years).

10.8. Finally, we notice that the graph G'_1 and G_1 are homotopic. It is not true without a refinement step. The octahedron graph G has a connection graph G' which is not homotopic to G : the Euler characteristic of the connection graph is 0. Its f -vector is $(26, 180, 556, 918, 900, 560, 224, 54, 6)$. Its Betti vector is $(1, 0, 0, 1, 0, 0, 0, 0, 0)$. It is topologically a 3-sphere, a Hopf fibration of the 2-sphere as the graph has developed a nontrivial three dimensional cohomology. It is not contractible. If one removes all zero dimensional vertices, we still have the same cohomology but a smaller graph with f -vector $v(G) = (20, 132, 388, 582, 480, 224, 56, 6)$.

10.9. The Fredholm determinant of a Barycentric refinement G_1 of a graph G is always 1. There is an explicit linear map $f(G) \rightarrow Af = f(G_1)$ which gives the f -vector of the Barycentric refinement G_1 from G . The image always has an even number of odd-dimensional simplices.

10.10. There are various other connection matrices or connection tensors one could look at. For any k one can look at the tensor $L(x_1, \dots, x_k)$ which is 1 if the simplices x_1, \dots, x_k pairwise intersect and 0 else. But it is the quadratic case which obviously is the most important one as we then have a matrix which has a determinant and eigenvalues. An other structure we have looked at but not found useful yet is to look at the set of intersecting simplices (x, y) . If there are n such pairs, look at the $n \times n$ matrix which is $L((x, y), (u, v)) = 1$ if (x, y) and (u, v) intersect somewhere and 0 else. One can similarly define such intersection matrices for $k > 2$ also. So far, we have not yet discovered a nice algebraic fact like the unimodularity of the connection matrix.

10.11. In any case, whatever Laplacian is taken, we can look at it in the Barycentric limit [18, 20].

11. HISTORICAL REMARKS

11.1. About simplicial complexes.

11.2. Abstract simplicial complexes were first defined in 1907 by Dehn and Heegaard [3] and used further by Tietze, Brouwer, Steinitz, Veblen, Whitney, Weyl and Kneser. In [1], Alexandroff calls a simplicial complex an **unrestricted skeleton complex**, whereas any finite sets is a **skeleton complex**. Still mostly used in Euclidean settings like [7], modern topology textbooks like [33, 32] or [31] use the abstract version. A generalization of simplicial complexes are simplicial sets.

11.3. In [32], the join of two simplicial complexes is defined. The definition for graphs goes back to Zykov [36]. In [8, 34], the empty set, the "void" is included in a complex. This leads to reduced f -vectors (v_{-1}, v_0, \dots) with $v_{-1} = 1$ and **reduced Euler characteristic** $\chi(G) - 1$ known in enumerative combinatorics. The number $1 - \chi(G)$ is in one-dimensions the **genus**. It is multiplicative when taking joins of simplicial complexes.

11.4. About Wu characteristic. Wu defines the characteristic numbers through the property that none of the simplices intersect. Lets call this $\tilde{\omega}(G)$. For $k = 2$, one can write this as $\chi(G)^2 - \tilde{\omega}(G)$ so that we have nothing new when defining this. For the higher order versions however, the relations are no more so direct. We should also note that Wu, as custom for combinatorial topology in the 20'th century, dealt with polytopes. But since there are various definitions of polytopes [6], we don't use that language.

12. QUESTIONS

12.1. Since we do not have homotopy invariance but invariance under Barycentric refinement, an important question is whether the cohomology is a topological invariant in the sense of [17]. Under which conditions is it true that if H_1, H_2 are homeomorphic and $G \setminus H_1$ and $G \setminus H_2$ are topologically equivalent, then then $H^p(G, H_1) = H^p(G, H_2)$? Is it true that $H^p(H_1) = H^p(H_2)$? We believe that both statements are true. The intuition comes from the believe that there is a continuum limit, where we can do deformations. One attempt to construct a continuum cohomology is to look at the product complex of the usual de-Rham complex on the product $M \times M$ of a manifold M , then take a limit of equivalence relations where a pair (f, g) of k and l forms with $k + l = p$ is restricted to the diagonal. If the Hodge story goes over, then the homotopy invariance follows from elliptic regularity assuring that there is a positive distance between the zero eigenvalues and the next larger eigenvalue. A small deformation of the geometry changes the eigenvalues continuously so that the kernel of the Hodge operators on p -forms stay robust. This strategy could work while staying within the discrete: when doing Barycentric refinements, we can make the spectral effect of a homotopy deformation small.

12.2. Also not explored is what the Betti numbers of a d -complex with or without boundary are. We have already a boundary formula $\omega_2(G) = \omega(G) = \chi(G) - \chi(\delta G)$ for Wu characteristic. Are there formulas for the Betti numbers in that case which relate the simplicial Betti numbers of G and δG with the Betti numbers of the $k = 2$ cohomology.

12.3. We have seen that for the quadratic cohomology, there are examples of complexes like the Möbius strip which have trivial cohomology in the sense that the Betti vector is 0. This is equivalent to the fact that the Hodge Laplacian is invertible. This obviously does not happen for simplicial cohomology but the Möbius strip is an example for interaction cohomology. One can ask which spaces have this property. Such a space necessarily has to have zero Wu characteristic.

12.4. We have tried to compute the cohomology of the Poincaré Homology sphere which is generated by $\{\{1, 2, 4, 9\}, \{1, 2, 4, 15\}, \{1, 2, 6, 14\}, \{1, 2, 6, 15\}, \{1, 2, 9, 14\}, \{1, 3, 4, 12\}, \{1, 3, 4, 15\}, \{1, 3, 7, 10\}, \{1, 3, 7, 12\}, \{1, 3, 10, 15\}, \{1, 4, 9, 12\}, \{1, 5, 6, 13\}, \{1, 5, 6, 14\}, \{1, 5, 8, 11\}, \{1, 5, 8, 13\}, \{1, 5, 11, 14\}, \{1, 6, 13, 15\}, \{1, 7, 8, 10\}, \{1, 7, 8, 11\}, \{1, 7, 11, 12\}, \{1, 8, 10, 13\}, \{1, 9, 11, 12\}, \{1, 9, 11, 14\}, \{1, 10, 13, 15\}, \{2, 3, 5, 10\}, \{2, 3, 5, 11\}, \{2, 3, 7, 10\}, \{2, 3, 7, 13\}, \{2, 3, 11, 13\}, \{2, 4, 9, 13\}, \{2, 4, 11, 13\}, \{2, 4, 11, 15\}, \{2, 5, 8, 11\}, \{2, 5, 8, 12\}, \{2, 5, 10, 12\}, \{2, 6, 10, 12\}, \{2, 6, 10, 14\}, \{2, 6, 12, 15\}, \{2, 7, 9, 13\}, \{2, 7, 9, 14\}, \{2, 7, 10, 14\}, \{2, 8, 11, 15\}, \{2, 8, 12, 15\}, \{3, 4, 5, 14\}, \{3, 4, 5, 15\}, \{3, 4, 12, 14\}, \{3, 5, 10, 15\}, \{3, 5, 11, 14\}, \{3, 7, 12, 13\}, \{3, 11, 13, 14\}, \{3, 12, 13, 14\}, \{4, 5, 6, 7\}, \{4, 5, 6, 14\}, \{4, 5, 7, 15\}, \{4, 6, 7, 11\}, \{4, 6, 10, 11\}, \{4, 6, 10, 14\}, \{4, 7, 11, 15\}, \{4, 8, 9, 12\}, \{4, 8, 9, 13\}, \{4, 8, 10, 13\}, \{4, 8, 10, 14\}, \{4, 8, 12, 14\}, \{4, 10, 11, 13\}, \{5, 6, 7, 13\}, \{5, 7, 9, 13\}, \{5, 7, 9, 15\}, \{5, 8, 9, 12\}, \{5, 8, 9, 13\}, \{5, 9, 10, 12\}, \{5, 9, 10, 15\}, \{6, 7, 11, 12\}, \{6, 7, 12, 13\}, \{6, 10, 11, 12\}, \{6, 12, 13, 15\}, \{7, 8, 10, 14\}, \{7, 8, 11, 15\}, \{7, 8, 14, 15\}, \{7, 9, 14, 15\}, \{8, 12, 14, 15\}, \{9, 10, 11, 12\}, \{9, 10, 11, 16\}, \{9, 10, 15, 16\}, \{9, 11, 14, 16\}, \{9, 14, 15, 16\}, \{10, 11, 13, 16\}, \{10, 13, 15, 16\}, \{11, 13, 14, 16\}, \{12, 13, 14, 15\}, \{13, 14, 15, 16\}\}$. The corresponding Laplacian is a 70616×70616 matrix. Unfortunately, our computing resources were not able yet to find the kernel of its blocks yet. It would be nice to compare the interaction cohomology of the homology sphere with the interaction cohomology of the 3-sphere.

The Mathematica procedure "Coho2" computes in less than 12 lines a basis for all the cohomology groups $H^p(G, H)$ for a pair of simplicial complexes G, H . We display then the Betti numbers $b_p(G, H) = \dim(H^p(G, H))$. As in the text, the case $H_2^p(G)$ is the same than $H^p(G, G)$.

```

Coho2[G_., H_.]:=Module[{ , n=Length[G]; m=Length[H]; len[x_]:=Total[Map[Length, x]]; U={};
Do[If[Length[Intersection[G[[i]], H[[j]]]]>0, U=Append[U, {G[[i]], H[[j]]}], {i, n}, {j, m}];
U=Sort[U, len[#1]<len[#2] & ]; u=Length[U]; l=Map[len, U]; w=Union[l];
b=Prepend[Table[Max[Flatten[Position[l, w[[k]]]], {k, Length[w]}], 0]; h=Length[b]-1;
deriv1[{x_., y_}]:=Table[{Sort[Delete[x, k]], y}, {k, Length[x]}];
deriv2[{x_., y_}]:=Table[{x, Sort[Delete[y, k]]}, {k, Length[y]}];
d1=Table[0, {u}, {u}]; Do[v=deriv1[U[[m]]]; If[Length[v]>0,
Do[r=Position[U, v[[k]]]; If[r!={}, d1[[m, r[[1, 1]]]]=(-1)^k, {k, Length[v]}], {m, u}];
d2=Table[0, {u}, {u}]; Do[v=deriv2[U[[m]]]; If[Length[v]>0,
Do[r=Position[U, v[[k]]]; If[r!={}, d2[[m, r[[1, 1]]]]=(-1)^(Length[U[[m, 1]]]+k),
{k, Length[v]}], {m, u}]; d=d1+d2; Dirac=d+Transpose[d]; L=Dirac.Dirac; Map[NullSpace,
Table[Table[L[[b[[k]]+i, b[[k]]+j]], {i, b[[k+1]]-b[[k]]}, {j, b[[k+1]]-b[[k]]}], {k, h}]];
Betti2[G_., H_.]:=Map[Length, Coho2[G, H]]; Coho2[G_]:=Coho2[G, G]; Betti2[G_]:=Betti2[G, G];
Generate[A_.]:=Sort[Delete[Union[Sort[Flatten[Map[Subsets, A], 1]], 1]], 1]];

moebius=Generate[{{1, 2, 5}, {1, 4, 5}, {1, 4, 7}, {2, 3, 6}, {2, 5, 6}, {3, 6, 7}, {4, 3, 7}}];
Print["Moebius_Strip: ", Betti2[moebius]];
cylinder=Generate[{{1, 2, 5}, {1, 4, 8}, {1, 5, 8}, {2, 3, 6}, {2, 5, 6}, {3, 4, 7}, {3, 6, 7}, {4, 7, 8}}];
Print["Cylinder: ", Betti2[cylinder]];
circle=Generate[{{1, 2}, {2, 3}, {3, 4}, {4, 5}, {5, 1}}];
Print["Circle: ", Betti2[ circle ]];
figureeight=Generate[{{1, 2}, {1, 4}, {2, 3}, {2, 5}, {2, 7}, {3, 4}, {5, 6}, {6, 7}}];
Print["Figure_Eight: ", Betti2[figureeight]];

point=Generate[{{1}}];
Print["1-simplex: ", Betti2[point]];
segment=Generate[{{1, 2}}];
Print["2-simplex: ", Betti2[segment]];
triangle=Generate[{{1, 2, 3}}];
Print["3-simplex: ", Betti2[triangle]];
tetrahedron=Generate[{{1, 2, 3, 4}}];
Print["4-simplex: ", Betti2[tetrahedron]];

Print["Point_in_interval: ", Betti2[{{1, 2}, {2, 3}, {1}, {2}, {3}}, {{2}}]];
Print["Point_on_boundary: ", Betti2[{{1, 2}, {2, 3}, {1}, {2}, {3}}, {{1}}]];

house=Generate[{{1, 2}, {2, 3}, {3, 4}, {4, 1}, {2, 3, 5}}];
Print["House_graph: ", Betti2[house]];

octahedron=Generate[{{1, 2, 3}, {1, 2, 4}, {1, 3, 5}, {1, 4, 5}, {2, 3, 6}, {2, 4, 6}, {3, 5, 6}, {4, 5, 6}}];
Print["Octahedron: ", Betti2[octahedron]];
equator=Generate[{{2, 3}, {3, 5}, {5, 4}, {4, 2}}];
Print["Circle_in_octahedron: ", Betti2[octahedron, equator]];

kleinbottle=Generate[{{1, 2, 3}, {1, 2, 6}, {1, 3, 5}, {1, 4, 7}, {1, 4, 8}, {1, 5, 7},
{1, 6, 8}, {2, 3, 7}, {2, 4, 6}, {2, 4, 8}, {2, 5, 7}, {2, 5, 8}, {3, 4, 6}, {3, 4, 7}, {3, 5, 6}, {5, 6, 8}}];
Print["Klein_bottle: ", Betti2[kleinbottle]];

projectiveplane=Generate[{{1, 2, 5}, {1, 2, 9}, {1, 4, 5}, {1, 4, 7}, {1, 8, 7}, {1, 8, 9}, {2, 3, 6},
{2, 3, 10}, {2, 5, 6}, {2, 9, 10}, {3, 6, 7}, {3, 10, 11}, {4, 3, 7}, {4, 3, 11}, {4, 5, 12},
{4, 11, 12}, {5, 6, 13}, {5, 12, 13}, {6, 7, 14}, {6, 13, 14}, {8, 7, 14}, {8, 9, 15},
{8, 14, 15}, {9, 10, 15}, {10, 11, 15}, {11, 12, 15}, {12, 13, 15}, {13, 14, 15}}];
Print["Projective_Plane: ", Betti2[projectiveplane]];

Print["The_next_computation_deals_with_4160x4160_matrices:_patience_please_"];
threesphere=Generate[{{1, 3, 5, 7}, {1, 3, 5, 8}, {1, 3, 6, 7}, {1, 3, 6, 8},
{1, 4, 5, 7}, {1, 4, 5, 8}, {1, 4, 6, 7}, {1, 4, 6, 8}, {3, 5, 7, 2}, {3, 5, 8, 2},
{3, 6, 7, 2}, {3, 6, 8, 2}, {4, 5, 7, 2}, {4, 5, 8, 2}, {4, 6, 7, 2}, {4, 6, 8, 2}}];
Print["3-sphere: ", Betti2[threesphere]];

RandomSimplComplex[n_., m_.]:=Module[{A={}, X=Range[n, k], Do[k:=1+Random[Integer, n-1];
A=Append[A, Union[RandomChoice[X, k]], {m}]; Generate[A]];
Print["Random_Complex: ", Betti2[RandomSimplComplex[6, 10]]]

```

This was a Handout to "Wu Characteristic", a Harvard math table talk given on March 8, 2016 [30]. It was formulated in the language of graphs. In that talk, the exterior derivative was not yet finalized. It gave a cohomology which was not yet invariant under Barycentric refinements.

Definitions. Let $G = (V, E)$ be a **finite simple graph** with vertex set V and edge set E . The **f-vector** $v(G) = (v_0(G), v_1(G), \dots, v_d(G))$ contains as coordinates the number $v_k(G)$ of complete subgraphs K_{k+1} of G . These subgraphs are also called **k-simplices** or **cliques**. The **f-matrix** $V(G)$ has the entries $V_{ij}(G)$ counting the number of intersecting pairs (x, y) , where x is a i -simplex and y is a j -simplex in G . The **Euler characteristic** of G is $\chi(G) = \sum_i (-1)^i v_i(G)$. The **Wu characteristic** of G is $\omega(G) = \sum_{i,j} (-1)^{i+j} V_{ij}(G)$. If A, B are two graphs, the **graph product** is a new graph which has as vertex set the set of pairs (x, y) , where x is a simplex in A and y is a simplex in B . Two such pairs $(x, y), (a, b)$ are connected in the product if either $x \subset a, y \subset b$ or $a \subset x, b \subset y$. The product of G with K_1 is called the **Barycentric refinement** G_1 of G . Its vertices are the simplices of G . Two simplices are connected in G_1 if one is contained in the other. If W is a subset of V , it **generates the graph** (W, F) , where F is the subset of $(a, b) \in E$ for which $a, b \in W$. The **unit sphere** $S(v)$ of a vertex v is the subgraph generated by the vertices connected to v . A function $f : V \rightarrow R$ satisfying $f(a) \neq f(b)$ for $(a, b) \in E$ is called a **coloring**. The minimal range of a coloring is the **chromatic number** of G . Define the **Euler curvature** $\kappa(v) = \sum_{k=0}^{\infty} (-1)^k v_{k-1}(S(v)) / (k+1)$ and the **Poincaré-Hopf index** $i_f(v) = 1 - \chi(S_f^-(v))$, where $S_f^-(x)$ is the graph generated by $S_f^-(v) = \{w \in S(v) \mid f(w) < f(v)\}$. Fix an orientation on **facets** on G simplices of maximal dimension. Let $\Omega^{-1} = \{0\}, \Omega^k(G)$ be the set of functions $f(x_0, \dots, x_k)$ from the set of k -simplices of G to R which are anti-symmetric. The **exterior derivatives** $d_k f(x_0, \dots, x_k) = \sum_{j=0}^k (-1)^j f(x_0, \dots, \hat{x}_j, \dots, x_k)$ defines linear transformations. The orientation fixes **incidence matrices**. They can be collected together to a large $n \times n$ matrix d with $n = \sum_{i=0}^d v_i$. The matrix d is called the **exterior derivative**. Since $d_{k+1} d_k = 0$, the image of d_k is contained in the kernel of d_{k+1} . The vector space $H^k(G) = \ker(d_k) / \text{im}(d_{k-1})$ is the **k'th cohomology group** of G . Its dimension $b_k(G)$ is called the **k'th Betti number**. The symmetric matrix $D = d + d^*$ is the **Dirac operator** of G . Its square $L = D^2$ is the **Laplacian** of G . It decomposes into blocks L_k the **Laplacian on k-forms**. The **super trace** of L is $\text{str}(L) = \sum_k (-1)^k \text{tr}(L_k)$. A **graph automorphism** T is a permutation of V such that if $(a, b) \in E$, then $(T(a), T(b)) \in E$. The set of graph automorphisms forms a **automorphism group** $\text{Aut}(G) = \text{Aut}(G_1)$. If A is a subgroup of $\text{Aut}(G)$, we can form G_k/A which is for $k \geq 2$ again a graph. Think of G as a **branched cover** of G/A . For a simplex x , define its **ramification index** $e_x = 1 - \sum_{1 \neq T \in A, T(x)=x} (-1)^{\dim(x)}$. Any $T \in \text{Aut}(G)$ induces a linear map T_k on $H^k(G)$. The **super trace** $\chi_T(G) = \sum_k (-1)^k \text{tr}(T_k)$ is the **Lefschetz number** of T . For $T = 1$ we have $\chi_{\text{Id}}(G) = \chi(G)$. For a simplex $x = T(x)$, define the **Brouwer index** $i_T(x) = (-1)^{\dim(x)} \text{sign}(T|x)$ [10]. An integer-valued function X in Ω is called a **divisor**. The **Euler characteristic** of X is $\chi(X) = \chi(G) + \sum_v X(v)$. If $v_2(G) = 0$, it is $1 - b_1 + \text{deg}(X)$, where $b_1 = g$ is the **genus**. If f is a divisor, then $(f) = Lf$ is called a **principal divisor**. Two divisors are **linearly equivalent** if $X - Y = (f)$ for some f . A divisor X is **essential** if $X(v) \geq 0$ for all $v \in V$. The **linear system** $|X|$ of X is $\{f \mid X + (f) \text{ is essential}\}$. The **dimension** $l(X)$ of X is -1 if $|X| = \emptyset$ and else $\max k \geq 0$ so that for all $m \leq k$ and all $\chi(Y) = m$ the divisor $X - Y$ is essential. The **canonical divisor** K is for graph without triangles defined as $K(v) = -2\kappa(v)$, where $\kappa(v)$ is the curvature. The **dimension** of a graph is -1 for the empty graph and inductively the average of the dimensions of the unit spheres plus 1. For the rabbit graph G , the f-vector is $v(G) = [5, 5, 1]$, the f-matrix is $V(G) = [[5, 10, 3], [10, 21, 5], [3, 5, 1]]$. Its dimension is $\dim(G) = 7/5$, the chromatic number is 3, the Euler Betti numbers are $b(G) = (1, 0)$ which super sums to Euler characteristic $\chi(G) = \omega_1(G) = \sum_i (-1)^i v_i(G) = \sum_i (-1)^i b_i(G) = 1$. The Wu Betti numbers are $b(G) = (0, 2, 6, 1, 0)$. The super sum is the Wu characteristic $\omega_2(G) = \sum_{i,j} (-1)^{i+j} V(G)_{ij} = 3$. The cubic Wu Betti numbers are $(0, 2, 16, 34, 16, 1, 0)$ which super sums to cubic Wu $\omega_3(G) = -5$.

Theorems

Here are adaptations of theorems for Euler characteristic to graph theory. Take any graph G . The left and right hand side are always **concrete integers**. The theorem tells that they are equal and show: “**The Euler characteristic is the most important quantity in all of mathematics**”.

$$\text{Gauss-Bonnet: } \chi(G) = \sum_{v \in V} \kappa(v)$$

$$\text{Poincaré-Hopf: } \chi(G) = \sum_{v \in V} i_f(v)$$

$$\text{Euler-Poincaré: } \chi(G) = \sum_i (-1)^i b_i(G)$$

$$\text{McKean-Singer: } \chi(G) = \text{str}(e^{-tL})$$

$$\text{Steinitz-DeRham: } \chi(G \times H) = \chi(G)\chi(H)$$

$$\text{Brouwer-Lefschetz: } \chi_T(G) = \sum_{x=T(x)} i_T(x)$$

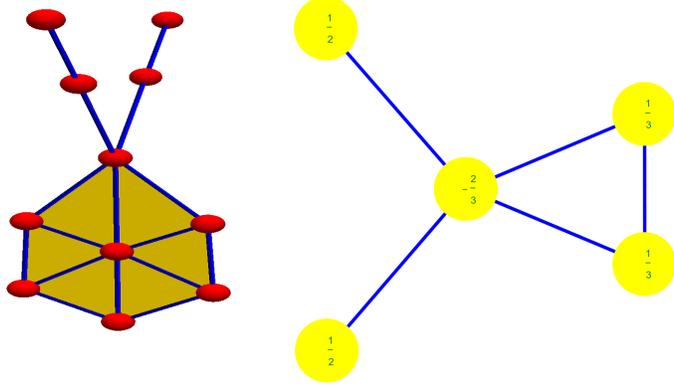
$$\text{Riemann-Hurwitz: } \chi(G) = |A| \chi(G/A) - \sum_x (e_x - 1)$$

$$\text{Riemann-Roch: } \chi(X) = l(X) - l(K - X)$$

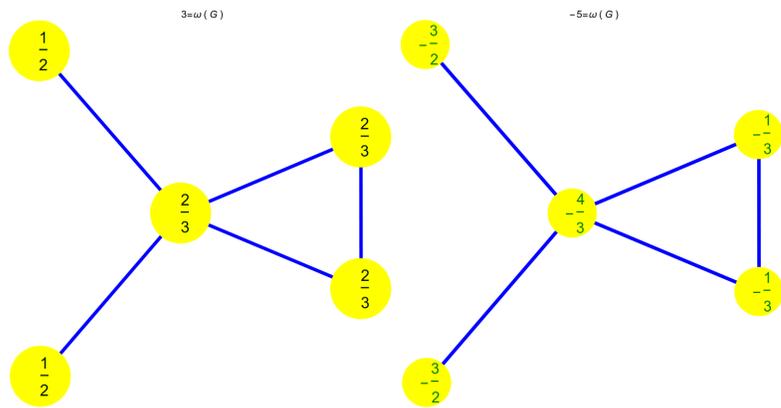
All theorems except Riemann-Roch hold for general finite simple graphs. The complexity of the proofs are lower than in the continuum. In continuum geometry, lower dimensional parts of space are accessed using “sheaves” or “integral geometry” or “tensor calculus”. For manifolds, a functional analytic framework like elliptic regularity is needed to define the **heat flow** e^{-tL} . We currently work on versions of these theorems to all the **Wu characteristics** $\omega_k(G) = \sum_{x_1 \sim \dots \sim x_k} \omega(x_1)\omega(x_2) \cdots \omega(x_k)$, where the sum is over all ordered k tuples of simplices in G which intersect. Besides proofs, we built also computer implementation for all notions allowing to compute things in concrete situations. There are Wu versions of curvature, Poincaré-Hopf index, Brouwer-Lefschetz index and an interaction cohomology. All theorems generalize. Only for Riemann-Roch, we did not complete the adaptation of **Baker-Norine theory** yet, but it looks very good (take Wu curvature for canonical divisor and use Wu-Poincaré-Hopf indices). What is the significance of Wu characteristic? We don't know yet. The fact that important theorems generalize, generates optimism that it can be significant in physics as an **interaction functional** for which extremal graphs have interesting properties. **Tamas Reti** noticed already that for a triangle free graph with n vertices and m edges, the Wu characteristic is $\omega(G) = n - 5m + M(G)$, where $M(G) = \sum_v \deg(v)^2$ is the **first Zagreb index** of G . For Euler characteristic, we have guidance from the continuum. This is no more the case for Wu characteristics. Nothing similar appears in the continuum. Related might be **intersection theory**, as Wu characteristic defines an **intersection number** $\omega(A, B) = \sum_{x \sim y, x \subset A, y \subset B} \omega(x)\omega(y)$ for two subgraphs A, B of G . To generalize the parts using cohomology, we needed an adaptation of simplicial cohomology to a cohomology of interacting simplices. We call it **interaction cohomology**: one first defines discrete **quadratic differential forms** $F(x, y)$ and then an exterior derivative $dF(x, y) = F(dx, y)$. As $d^2 = 0$, one gets cohomologies the usual way.

The Rabbit

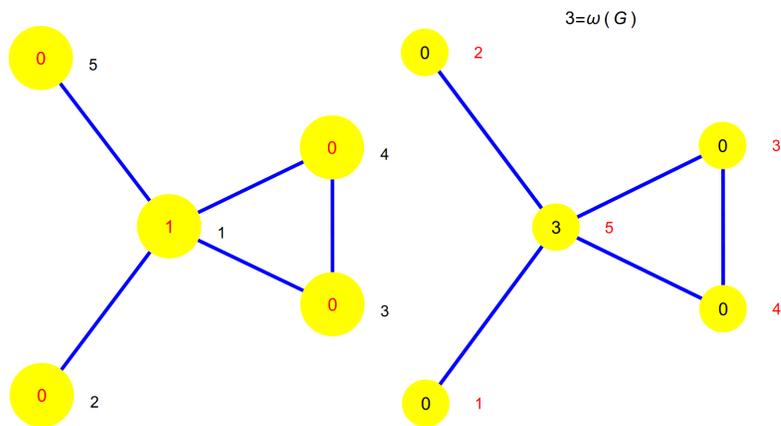
$$1 = \chi(G) = \sum_{k=1}^{|V|} K[k]$$



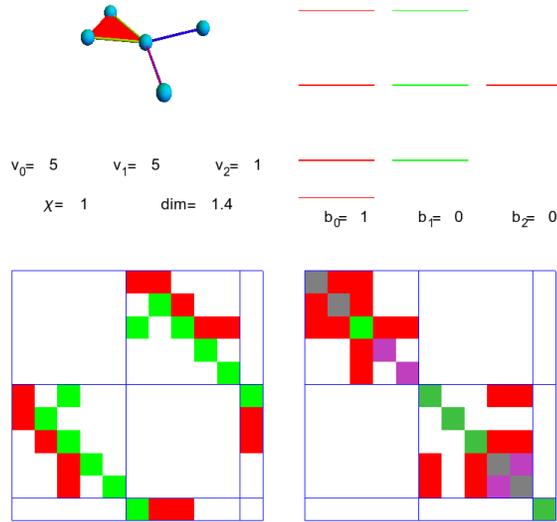
1. Barycentric refinement of the rabbit graph R . 2. Euler curvatures.



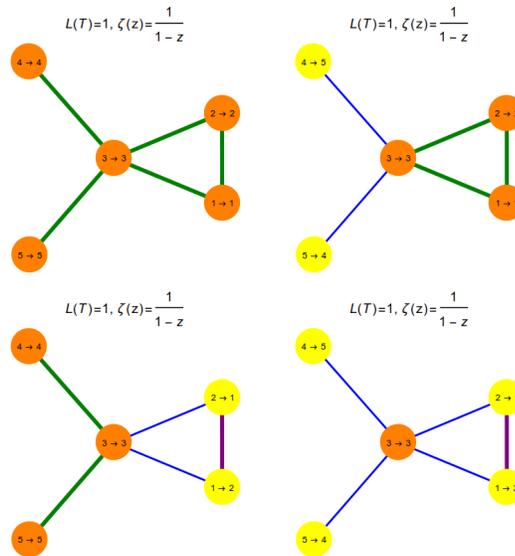
3. Wu curvatures adding up to ω . 4. Cubic Wu curvatures adding up to ω_3 .



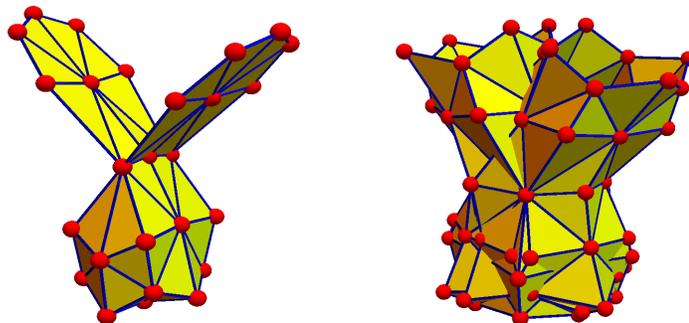
5. Poincaré-Hopf indices. 6. Wu-Poincaré-Hopf indices



7. Spectrum $\sigma(L_0) = \{5, 3, 1, 1, 0\}, \sigma(L_1) = \{5, 3, 3, 1, 1\}, \sigma(L_2) = \{3\}$, Betti numbers= $\dim(\ker(L_k))$, Dirac operator D and Laplacian L of R with 3 blocks. **Super symmetry: Bosonic spectrum** $\{5, 3, 1, 1\} \cup \{3\}$ agrees with **Fermionic spectrum** $\{5, 3, 3, 1, 1\}$.



8. Lefschetz numbers of the 4 automorphisms in $\text{Aut}(R) = Z_2 \times Z_2$.



9. **3D rabbit** $R \times K_2$ with $\omega = -3, \dim = 2.59433 \geq \dim(R) + 1 = 2.4$.

10. **curled rabbit** $G = R \times C_4$ with $\chi(G) = \omega(G) = 0, b_0 = 1, b_1 = 1$.

Calculus

Fix an orientation on simplices. The exterior derivatives are the **gradient**

$$\text{grad} = d_0 = \begin{bmatrix} -1 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \end{bmatrix},$$

the **curl**

$$\text{curl} = d_1 = [1 \quad -1 \quad -1 \quad 0 \quad 0]$$

which satisfies $\text{curl}(\text{grad}) = 0$ and the **divergence**

$$\text{div} = d_0^* = \begin{bmatrix} -1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 1 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The **scalar Laplacian** or **Kirchhoff matrix** $L_0 = d_0^*d_0 = \text{div}(\text{grad})$ is

$$L_0 = \begin{bmatrix} 2 & -1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ -1 & -1 & 4 & -1 & -1 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \end{bmatrix}.$$

It does not depend on the chosen orientation on simplices and is equal to $B - A$, where B is the diagonal matrix containing the **vertex degrees** and A is the **adjacency matrix** of G . Has eigenvalues $\sigma(L_0) = \{5, 3, 1, 1, 0\}$ with one-dimensional kernel spanned by $[1, 1, 1, 1, 1]^T$. Also the **1-form Laplacian** $L_1 = d_1^*d_1 + d_0d_0^*$ does not depend on the orientation of the simplices:

$$L_1 = \begin{bmatrix} 3 & 0 & 0 & -1 & -1 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 3 & -1 & -1 \\ -1 & 0 & -1 & 2 & 1 \\ -1 & 0 & -1 & 1 & 2 \end{bmatrix}$$

with eigenvalues $\sigma(L_1) = \{5, 3, 3, 1, 1\}$ with zero dimensional kernel. This reflects that the graph is **simply connected**.

The 2-form Laplacian $d_1d_1^*$

$$L_2 = [3]$$

has a single eigenvalue $\{3\}$.

Dirac operator $D = d + d^*$ is

$$D = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 0 & 0 & 0 \end{bmatrix}.$$

The form Laplacian or **Laplace Beltrami operator** $L = D^2 = (d + d^*)^* = dd^* + d^*d$ is

$$L = \begin{bmatrix} 2 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 4 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

Partial difference equations

Any continuum PDE can be considered on graphs. They work especially on the rabbit. The Laplacian $L = (d + d^*)^2$ can be either for the exterior derivative d for forms or quadratic forms. Let λ_k denote the **eigenvalues** of the form Laplacian L :

The Wave equation

$$u_{tt} = Lu$$

has **d'Alembert solution** $u(t) = \cos(Dt)u(0) + \sin(Dt)D^{-1}u'(0)$. Using $u(0) = \sum_n u_n f_n$ and $v(0) = u'(0) = \sum_n v_n f_n$ with **eigenvectors** f_n of L , one has the solution

$$u(t) = \sum_n \cos(\sqrt{\lambda_n}t)u_n f_n + \sin(\sqrt{\lambda_n}t)v_n f_n / \sqrt{\lambda_n}.$$

The Heat equation

$$u_t = Lu$$

has the solution $u(t) = e^{-Lt}u(0)$ or $\sum_n e^{-\lambda_n t}u_n f_n$, where $u(0) = \sum_n u_n f_n$ is the eigenvector expansion of $u(0)$ with respect to eigenvectors f_n of L .

The Poisson equation

$$Lu = v$$

has the solution $u = L^{-1}v$, where L^{-1} is the pseudo inverse of L . We can also assume v to be perpendicular to the kernel of L .

The Laplace equation

$$Lu = 0$$

has on the scalar sector Ω^0 only the locally constant functions as solutions. In general, these are **harmonic functions**.

Maxwell equations

$$dF = 0, d^*F = j$$

for a 2-form F called **electromagnetic field**. In the case of the rabbit, F is a number attached to the triangle. j , the **current** is a function assigned to edges. If $F = dA$ with a **vector potential** A . If A is Coulomb gauged so that $d^*A = 0$, then we get the Poisson equation for one forms

$$L_1 A = j.$$

This solves the problem to find the electromagnetic field $F = dA$ from the current j . In the rabbit case, since it is simply connected, there is no kernel of L_1 and we can just invert the matrix.

The electromagnetic field $F = dA$ defined on the rabbit is a number attached to the triangle:

Let there be light!

Simplicial Cohomology

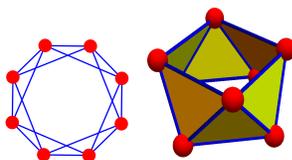
Simplicial cohomology is defined by an **exterior derivative** $dF(x) = F(dx)$ on **valuation forms** $F(x)$ on subgraphs x of a **finite simple graph** G , where dx is the **boundary chain** of a simplex x . Evaluation $F(A)$ is **integration** and $dF(A) = F(dA)$ is **Stokes**. Since $d^2 = 0$, the kernel of $d_p : \Omega^p \rightarrow \Omega^{p+1}$ contains the image of d_{p-1} . The vector space $H^p(G) = \ker(d_p)/\text{im}(d_{p-1})$ is the p 'th **simplicial cohomology** of G . The **Betti numbers** $b_p(G) = \dim(H^p(G))$ define $\sum_p (-1)^p b_p$ which is **Euler characteristic** $\chi(G) = \sum_x (-1)^{\dim(x)}$, summing over all complete subgraphs x of G . If T is an automorphism of G , the **Lefschetz number**, the super trace $\chi_T(G)$ of the induced map U_T on $H^p(G)$ is equal to the sum $\sum_{T(x)=x} i_T(x)$, where $i_T(x) = (-1)^{\dim(x)} \text{sign}(T|x)$ is the **Brouwer index**. This is the **Lefschetz fixed point theorem**. The **Poincaré polynomial** $p_G(x) = \sum_{k=0} \dim(H^k(G))x^k$ satisfies $p_{G \times H}(x) = p_G(x)p_H(x)$ and $\chi(G) = p_G(-1)$ so that $\chi(G \times H) = \chi(G) \cdot \chi(H)$. For $T = Id$, the Lefschetz formula is **Euler-Poincaré**. With the **Dirac operator** $D = d + d^*$ and **Laplacian** $L = D^2$, discrete **Hodge** tells that $b_p(G)$ is the nullity of L restricted to p -forms. By **McKean Singer super symmetry**, the positive Laplace spectrum on even-forms is the positive Laplace spectrum on odd-forms. The super trace $\text{str}(L^k)$ is therefore zero for $k > 0$ and $l(t) = \text{str}(\exp(-tL)U_T)$ with **Koopman operator** $U_T f = f(T)$ is t -invariant. This heat flow argument proves Lefschetz because $l(0) = \text{str}(U_T)$ is $\sum_{T(x)=x} i_T(x)$ and $\lim_{t \rightarrow \infty} l(t) = \chi_T(G)$ by Hodge.

Interaction Cohomology

Super counting ordered pairs of intersecting simplices (x, y) gives the **Wu characteristic** $\omega(G) = \sum_{x \sim y} (-1)^{\dim(x) + \dim(y)}$. Like Euler characteristic it is **multiplicative** $\omega(G \times H) = \omega(G)\omega(H)$ and satisfies **Gauss-Bonnet** and **Poincaré-Hopf**. **Quadratic interaction cohomology** is defined by the **exterior derivative** $dF(x, y) = F(dx, y) + (-1)^{\dim(x)} F(x, dy)$ on functions F on ordered pairs (x, y) of intersecting simplices in G . If b_p are the Betti numbers of these interaction cohomology groups, then $\omega(G) = \sum_p (-1)^p b_p$ and the **Lefschetz formula** $\chi_T(G) = \sum_{(x,y)=(T(x),T(y))} i_T(x, y)$ holds where $\chi_T(G)$ is the **Lefschetz number**, the super trace of U_T on cohomology and $i_T(x, y) = (-1)^{\dim(x) + \dim(y)} \text{sign}(T|x)\text{sign}(T|y)$ is the **Brouwer index** introduced in the discrete in [10]. The heat proof works too. The **interaction Poincaré polynomial** $p_G(x) = \sum_{k=0} \dim(H^k(G))x^k$ again satisfies $p_{G \times H}(x) = p_G(x)p_H(x)$.

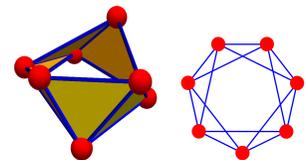
The Cylinder and the Möbius strip

The cylinder G and Möbius strip H are homotopic but not homeomorphic. As simplicial cohomology is a homotopy invariant, it can not distinguish H and G and $p_G(x) = p_H(x)$. But interaction cohomology can see it. The interaction Poincaré polynomials of G and H are $p_G(x) = x^2 + x^3$ and $p_H(x) = 0$. Like **Stiefel-Whitney classes**, interaction cohomology can distinguish the graphs. While Stiefel-Whitney is defined for vector bundles, interaction cohomologies are defined for all finite simple graphs. As it is invariant under Barycentric refinement $G \rightarrow G_1 = G \times K_1$, the cohomology works for continuum geometries like manifolds or varieties.



The cylinder G is an orientable graph with $\delta(G) = C_4 \cup C_4$.

The Möbius strip H is non-orientable with $\delta(H) = C_7$.



Classical Calculus

Calculus on graphs either deals with **valuations** or **form valuations**. Of particular interest are **invariant linear valuations**, maps X on non-oriented subgraphs A of G satisfying the **valuation property** $X(A \cup B) + X(A \cap B) = X(A) + X(B)$ and $X(\emptyset) = 0$ and $X(A) = X(B)$ if A and B are isomorphic subgraphs. We don't assume invariance in general. By **discrete Hadwiger**, the vector space of invariant linear valuations has dimension $d + 1$, where $d + 1$ is the **clique number**, the cardinality of the vertex set a complete subgraph of G can have. Linear invariant valuations are of the form $\sum_{k=0}^d X_k v_k(G)$, where $v(G) = (v_0, v_1, \dots, v_d)$ is the f -vector of G . Examples of invariant valuations are $\chi(G)$ or $v_k(G)$ giving the number of k -simplices in G . An example of a non-invariant valuation is $\deg_a(A)$ giving the number of edges in A hitting a vertex a . To define **form valuations** which implements a discrete "integration of differential forms", one has to orient first simplices in G . No compatibility is required so that any graph admits a choice of orientation. The later is irrelevant for interesting quantities like cohomology. A **form valuation** X is a function on oriented subgraphs of G such that the valuation property holds and $X(A) = -X(\bar{A})$ if \bar{A} is the graph A in which all orientations are reversed. Form valuations in particular change sign if a simplex changes orientation and when supported on p -simplices play the role of p -forms. The defining identity $dF(A) = F(dA)$ is already Stokes theorem. If A is a discrete p surface made up of $(p + 1)$ -simplices with cancelling orientation so that dA is a p -graph, then this discretizes the continuum Stokes theorem but the result holds for all subgraphs of any graph if one extends F to **chains** over G . For example, if $f_G = abc + ab + bc + ca + a + b + c$ is the algebraic description of a triangle then $f_{dG} = bc - ac + ab + b - a + c - b + a - c = ab + bc - ac$ is only a chain. With the orientation $f_G = abc + ba + bc + ca + a + b + c$ we would have got additionally the terms $2a - 2b$. The vector space $\Omega^p(G)$ of **all form valuations** on G has dimension $v_p(G)$ as we just have to give a value to each p -simplex to define F .

We use a cylinder G , with f -vector $(8, 16, 8)$ which super sums to the Euler characteristic $\chi(G) = 0$. An orientation on facets is fixed by giving graph algebraically like with $f_G = a + b + ab + c + bc + d + ad + cd + e + ae + be + abe + f + bf + cf + bcf + ef + bef + g + cg + dg + cdg + fg + cfg + h + ah + dh + adh + eh + aeh + gh + dgh$. The automorphism group $A = D_8$ of G has 16 elements. The Lefschetz numbers of the transformations are $(0, 2, 2, 0, 2, 0, 0, 2, 0, 2, 2, 0, 2, 0, 0, 2)$. The average Lefschetz number is $\chi(G/A) = 1$. The **gradient** d_0 and **curl** d_1 are

$$d_0 = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}, d_1 = \begin{bmatrix} -1 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & -1 \end{bmatrix}$$

The Laplacian $L = (d + d^*)^2$ has blocks $L_0 = d_0^* d_0$, which is the Kirchhoff matrix. The form Laplacian $L_1 = d_0 d_0^* + d_1^* d_1$ is a matrix on 1-forms. Together with the form Laplacian $L_2 = d_1 d_1^*$,

subgraphs x, y by the valuation properties. While in linear calculus, integration is the evaluation of the valuation on subgraphs, in quadratic interaction calculus, we evaluate=integrate on pairs of subgraphs. The reason for the name “interaction cohomology” is because for F to be non-zero, the subgraphs x, y are required to be **interacting** meaning having a non-empty intersection. (The word “Intersection” would also work but the name “Intersection cohomology” has been taken already in the continuum). We can think about $F(A, B)$ as a type of **intersection number** of two oriented interacting subgraphs A, B of the graph G and as $F(A, A)$ as the **self intersection number** $F(A, A)$. An other case is the intersection of the diagonal $A = \bigcup_x \{(x, x)\}$ and graph $B = \bigcup_x \{(x, T(x))\}$ in the product $G \times G$ of an automorphism T of G . The form version of the Wu intersection number $\omega(A, B)$ is then the Lefschetz number $\chi_T(G)$. A particularly important example of a self-intersection number is the Wu characteristic $\omega(G) = \omega(G, G)$ which is quadratic valuation, not a form valuation. Our motivation to define interaction cohomology was to get a Euler-Poincaré formula for Wu characteristic. It turns out that Euler-Poincaré automatically and always generalizes to the Lefschetz fixed point theorem, as the heat flow argument has shown; Euler-Poincaré is always just the case in which T is the identity automorphism.

Lets look now at **quadratic interaction cohomology** for the cylinder graph G and the Möbius graph H . We believe this case demonstrates well already how quadratic interaction cohomology allows in an algebraic way to distinguish graphs which traditional cohomology can not. The f -matrices of the graphs are

$$V(G) = \begin{bmatrix} 8 & 32 & 24 \\ 32 & 112 & 72 \\ 24 & 72 & 40 \end{bmatrix}, V(H) = \begin{bmatrix} 7 & 28 & 21 \\ 28 & 98 & 63 \\ 21 & 63 & 35 \end{bmatrix}, V(\tilde{H}) = \begin{bmatrix} 8 & 32 & 24 \\ 32 & 114 & 74 \\ 24 & 74 & 42 \end{bmatrix}.$$

The f -matrix is the matrix V for which V_{ij} counts the number of pairs (x, y) , where x is an i -simplex, y is a j -simplex and x, y intersect. The Laplacian for quadratic intersection cohomology of the cylinder is a 416×416 matrix because there are a total of $416 = \sum_{i,j} V_{ij}$ pairs of simplices which do intersect in the graph G . For the Möbius strip, it is a 364×364 matrix which splits into 5 blocks L_0, L_1, L_2, L_3, L_4 . Block L_p corresponds to the interaction pairs (x, y) for which $\dim(x) + \dim(y)$ is equal to p . The scalar interaction Laplacian L_0 for the cylinder is the diagonal matrix $\text{Diag}(8, 8, 8, 8, 8, 8, 8)$. For the Möbius strip H , it is the matrix $L_0(H) = \text{Diag}(8, 8, 8, 8, 8, 8)$. The diagonal entries of $L_0(H)$ depend only on the vertex degrees. In general, for any graph, if all vertex degrees are positive, then the scalar interaction Laplacian L_0 of the graph is invertible. For the cylinder, the block L_1 is an invertible 64×64 matrix because there are $V_{12} + V_{21} = 64$ vertex-edge or edge-vertex interactions. Its determinant is $2^{58} 3^8 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17^8 103^2 373^2 2089^2$. For the Möbius strip, the block L_1 is a 56×56 matrix with determinant $2^{46} 3^{75} 7^3 17^7 42924041^2$.

For the cylinder graph G , the block $L_2(G)$ is a 160×160 matrix, which has a 1-dimensional kernel spanned by the vector $[0, -5, -5, 10, 10, 2, 2, 5, 0, -5, 2, 10, 10, 2, 5, 0, -5, -2, -2, -10, -10, 5, 5, 0, 2, 10, 10, 2, -10, 2, 0, 9, -2, 9, 10, -10, -2, -9, 0, -9, -10, 2, -2, -10, 9, 0, 9, -10, -2, -10, -2, -9, 0, -2, -9, -10, 2, 10, 10, 2, 0, 5, 5, -2, -10, 9, 0, 9, -10, -2, 2, -10, -9, 0, -2, -9, -10, 2, 10, -5, 10, 2, 0, 5, -2, 10, -9, 0, -9, 10, -2, 10, -2, 9, 9, 0, 2, -10, -10, -2, -5, -10, -2, 0, 5, 2, 10, -5, 2, 10, -5, 0, -7, -7, -8, -7, -7, -8, 8, 7, 7, -7, -7, -8, 8, 7, 7, 7, 7, 8, -8, -7, -7, 8, 7, 7, -7, 7, -8, -7, -7, 8, -7, -7, 8, -7, 7, 8, -8, 7, -7, -8, 7, 7, -8, 7, 7, 8, -7, 7]$. This vector is associated to edge-edge and triangle-vertex interactions. So far, harmonic forms always had physical relevance. We don't know what it is for this interaction calculus.

On the other hand, for the Möbius strip H , the interaction form Laplacian $L_2(H)$ is a 140×140 matrix which is invertible! Its determinant is $2^{42} 3^{145} 5^3 7^6 11 \cdot 17^{14} 11087^2 212633^2 42924041^4$.

The quadratic form Laplacian L_3 which describes the edge-triangle interactions. For the cylinder G , the interaction form Laplacian $L_3(G)$ is a 144×144 matrix. It has a 1-dimensional kernel spanned by the vector $[4, 3, -6, -3, 6, 3, 4, -6, 3, -6, -3, -3, 4, -6, 6, 3, 4, -6, -3, -6, 2, -3, -3, 2, -2, -3, 2, -3, 3, 2, -2, -3, -2, 3, -3, 2, 6, 6, -4, -3, 3, 3, 2, -2, -3, -2, 3, 3, 2, 6, -3, 6, -4, -3, -3, 2, -2, 3, 3,$

-2, 3, -2, -6, 3, 6, -4, 3, 6, -3, -6, 3, -4, -4, -3, 3, -2, 2, -3, -6, 3, 6, -3, -4, -3, 3, -2, 2, -6, -3, -6, 6, 6, 3, -2, 2, -3, 4, 3, -3, -3, 3, -4, 3, -2, 2, -6, -3, -6, 6, 6, 3, -2, 3, 2, -3, 4, 3, 3, -4, 3, 3, -3, -2, 2, -6, 6, -6, 6, -2, 3, -3, 2, -3, 4, -3, -6, 6, 3, -2, 3, -3, 2, -3, 4] .

The surprise, which led us write this down quickly is that for the Möbius strip H , the interaction form Laplacian $L_3(H)$ is a 126×126 matrix which is **invertible** and has determinant $2^{28}3^75^37^311^217^711087^4212633^442924041^2$. Since L_3 is invertible too, there is also no cohomological restrictions on the $p = 3$ level for the Möbius strip. The quadratic form Laplacian is invertible. This is in contrast to the cylinder, where cohomological constraints exist on this level. The interaction cohomology can detect topological features, which simplicial cohomology can not.

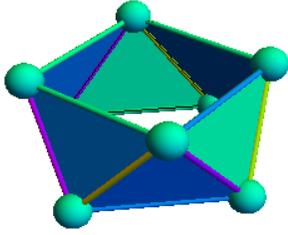
The fact that determinants of L_3, L_2 have common prime factors is not an accident and can be explained by **super symmetry**, as the union of nonzero eigenvalues of L_0, L_2, L_4 are the same than the union of the nonzero eigenvalues of L_1, L_3 .

Finally, we look at the L_4 block, which describes triangle-triangle interactions. For the cylinder G , this interaction Laplacian is a 40×40 matrix which has the determinant $2^{26}3^35 \cdot 11^323^229^271^2241^2$. For the Möbius strip, it is a 35×35 matrix which has determinant $2^{11}5 \cdot 11 \cdot 11087^2212633^2$.

The derivative d_0 for the Moebius strip H is the 56×7 matrix

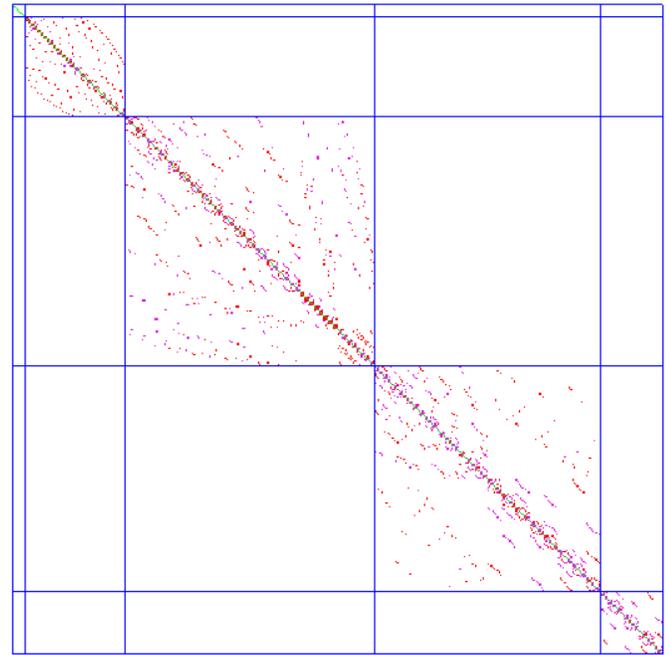
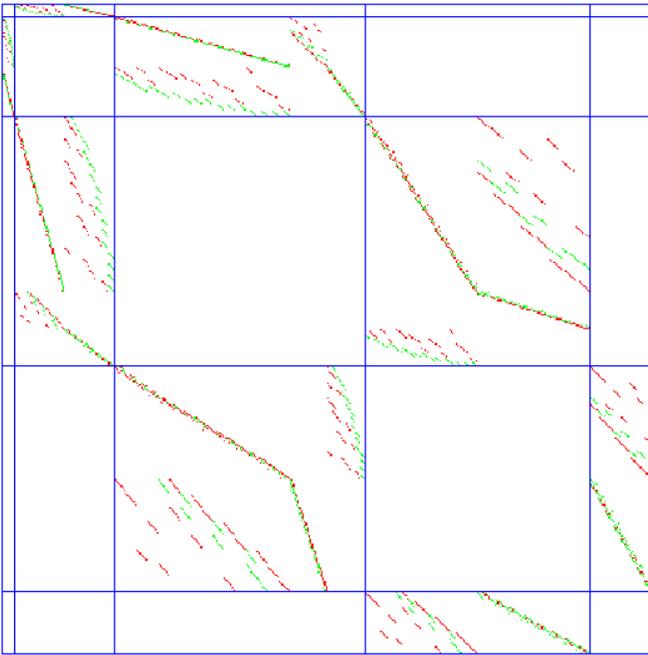
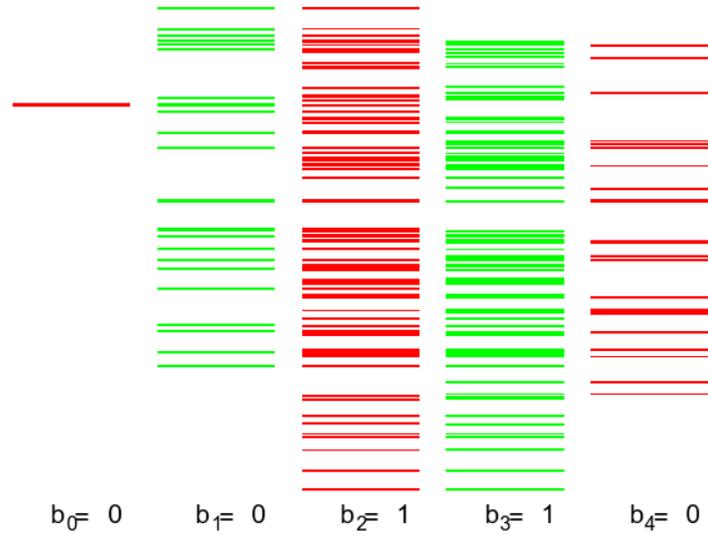
$$d_0 = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}.$$

It takes as arguments a list of function values on pairs of self-interacting vertices (x, x) and gives back a function on the list of pairs of interacting vertex-edge or edge-vertex pairs (x, y) . The exterior derivative d_1 for H is a 140×56 matrix. We use now the notation $i = -1$ for formatting reasons:

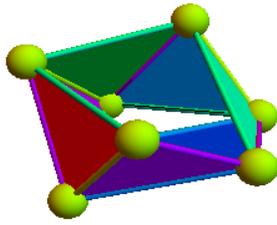


$$v_0 = 8 \quad v_1 = 64 \quad v_2 = 160 \quad v_3 = 144 \quad v_4 = 40$$

$$\omega = 0 \quad \dim = 2.$$

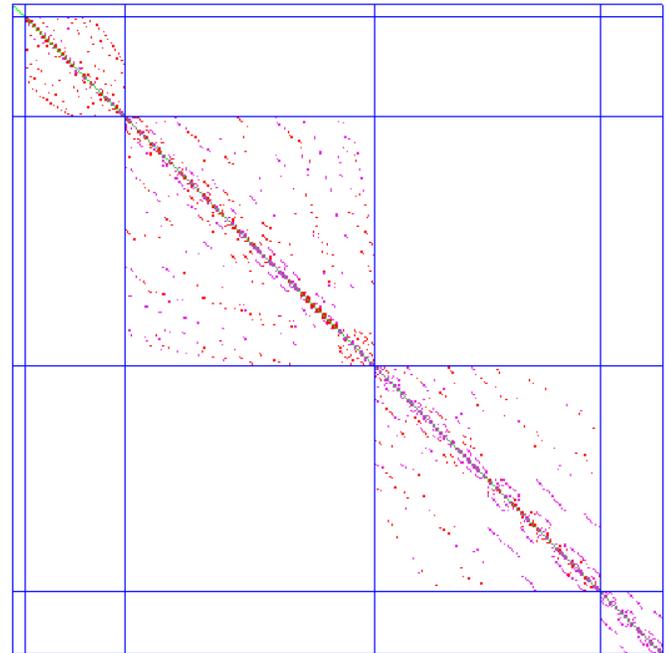
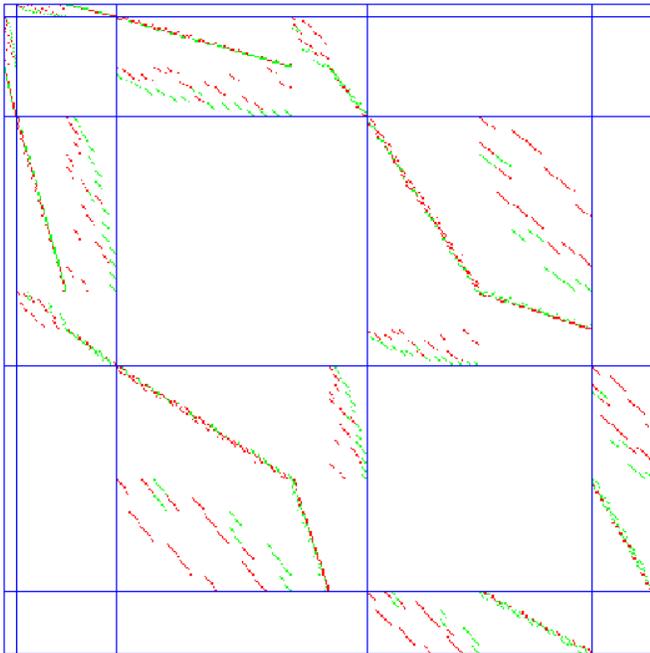
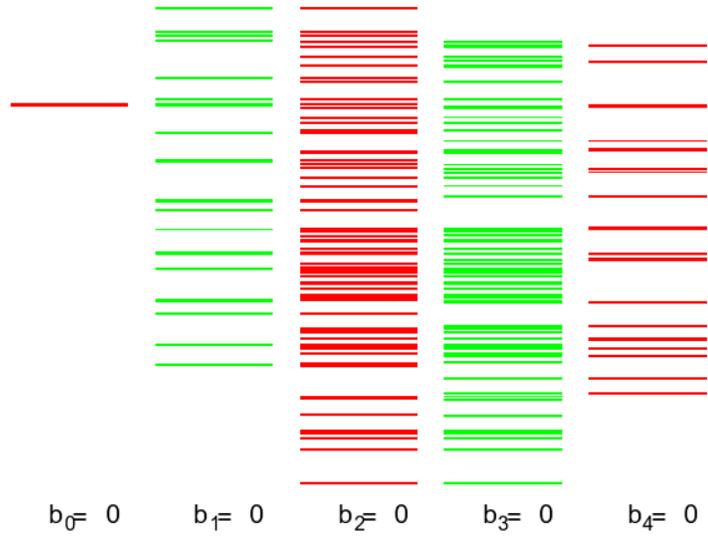


The figure shows the Dirac and Laplacian of the cylinder graph G . We see the spectrum of the interaction Laplacian on each of the 5 interaction form sectors $p = 0, \dots, p = 4$. We see also the super symmetry: the union of the spectra on even dimensional forms $p = 0, 2, 4$ is the same than the union of the spectra on odd-dimensional forms $p = 1, 3$.



$$v_0 = 7 \quad v_1 = 56 \quad v_2 = 140 \quad v_3 = 126 \quad v_4 = 35$$

$$\omega = 0 \quad \text{dim} = 2.$$



The figure shows the Dirac and Laplacian of the Möbius graph G . We see the spectrum of the interaction Laplacian on each of the 5 interaction sectors $p = 0, \dots, p = 4$. There is one important difference however: the Laplacian is invertible.

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