

# DEHN-SOMMERVILLE FROM GAUSS-BONNET

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ABSTRACT. We give a “zero curvature” proof of Dehn-Sommerville for finite simple graphs. It uses a parametrized Gauss-Bonnet formula telling that the curvature of the valuation  $G \rightarrow f_G(t) = 1 + f_0t + \dots + f_d t^{d+1}$  defined by the  $f$ -vector of  $G$  is the anti-derivative  $F$  of  $f$  evaluated on the unit sphere  $S(x)$ . Gauss Bonnet is then parametrized,  $f_G(t) = 1 + \sum_x F_{S(x)}(t)$ , and holds for all Whitney simplicial complexes  $G$ . The Gauss-Bonnet formula  $\chi(G) = \sum_x K(x)$  for Euler characteristic  $\chi(G)$  is the special case  $t = -1$ . Dehn-Sommerville is equivalent to the reflection symmetry  $f_G(t) + (-1)^d f_G(-1 - t) = 0$  which is equivalent to the same symmetry for  $F$ . Gauss-Bonnet therefore relates Dehn-Sommerville for  $G$  with Dehn-Sommerville for the unit spheres  $S(x)$ , where it is a zero curvature condition. A class  $\mathcal{X}_d$  of complexes for which Dehn-Sommerville holds is defined inductively by requiring  $\chi(G) = 1 + (-1)^d$  and  $S(x) \in \mathcal{X}_{d-1}$  for all  $x$ . It starts with  $\mathcal{X}_{(-1)} = \{\{\}\}$ . Examples are simplicial spheres, including homology spheres, any odd-dimensional discrete manifold, any even-dimensional discrete manifold with  $\chi(G) = 2$ . It also contains non-orientable ones for which Poincaré-duality fails or stranger spaces like spaces where the unit spheres allow for two disjoint copies of manifolds with  $\chi(G) = 1$ . Dehn-Sommerville is present in the Barycentric limit. It is a symmetry for the Perron-Frobenius eigenvector of the Barycentric refinement operator  $A$ . The even eigenfunctions of  $A^T$ , the Barycentric Dehn-Sommerville functionals, vanish on  $\mathcal{X}$  like  $22f_1 - 33f_2 + 40f_3 - 45f_4 = 0$  for 4-manifolds  $G$  with  $\chi(G) = 2$ .

## 1. GAUSS BONNET

**1.1.** The category of **finite abstract simplicial complexes**  $G$  requires only one axiom:  $G$  is a set of non-empty sets closed under the operation of taking finite non-empty subsets. The  **$f$ -vector** of  $G$  is  $f = (f_0, f_1, \dots, f_d)$  of  $G$ , where  $f_k$  is the set of sets in  $G$  with  $k + 1$  elements. The  **$f$ -function** of  $G$  is  $f_G(t) = 1 + \sum_{k=0}^d f_k(G)t^{k+1}$ . If  $G$  is the Whitney complex of a graph  $(V, E)$  the **unit sphere**  $S(x)$  for  $x \in V(G)$  is the unit sphere in that graph. For any  $G$ , let  $F_G(t) = \int_0^t f_G(s) ds$  denote the anti-derivative of  $f_G$ . The curvature valuation to  $f$  is the anti-derivative of  $f$  evaluated on the unit sphere:

**Theorem 1** (Gauss-Bonnet).  $f_G(t) - 1 = \sum_{x \in G} F_{S(x)}(t)$ .

*Proof.* If every  $k$ -simplex  $y$  in  $S(x)$  carries a charge  $t^{k+1}$ , then  $f_G(t)$  is the total charge. Because every  $k$ -simplex  $y$  in  $S(x)$  defines a  $(k + 1)$ -simplex  $z$  in  $G$ , the simplex  $z$  in  $S(x)$  carries a charge  $t^{k+2}$ . It contains  $(k + 2)$  zero-dimensional points, which were simplices in  $G$ . Distributing the charge equally to the points, gives each

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a charge  $t^{k+2}/(k+2)$ . The curvature  $F_{S(x)}(t)$  at  $x$  adds up all the charges of the simplices attached to  $x$ . □

**1.2.** There is code at the end allowing to experiment. For  $t = -1$ , we get a classical Gauss-Bonnet statement

$$\chi(G) = \sum_{x \in G} K(x),$$

where  $K(x) = F_{S(x)}(-1)$  is the Levitt curvature, the discrete analogue of the **Gauss-Bonnet-Chern curvature** in the continuum. An explicit formula for  $K(x)$  with  $f_{-1} = 1$  is

$$(1) \quad K(x) = \sum_{k=-1}^d (-1)^k \frac{f_k(S(x))}{k+2}.$$

It appeared first in [23]. For the continuum proof, see [6].

**1.3.** By differentiation of Gauss-Bonnet with respect to  $t$  we get:  $f'_G(t) = \sum_x f_{S(x)}(t)$ . For  $t = -1$  in particular, this gives an identity seen in [20].  $f'_G(-1) = \sum_{x \in G} 1 - \chi(S(x))$  which is a trace of the Green function operator  $g = L^{-1}$  with  $L_{xy} = 1$  if  $x \cap y \neq \emptyset$  and  $L_{xy} = 0$  else, where the diagonal entries are  $g(x, x) = 1 - \chi(S(x))$ .

## 2. DEHN-SOMMERVILLE SYMMETRY

**2.1.** Given a complex  $G$ , the  **$h$ -function**  $h_G(x) = (x-1)^d f_G(1/(x-1))$  is a polynomial  $h_G(x) = h_0 + h_1x + \dots + h_dx^d + h_{d+1}x^{d+1}$  defining a  **$h$ -vector**  $(h_0, h_1, \dots, h_{d+1})$ . The **Dehn-Sommerville relations** assert that the  $h$ -vector is **palindromic**, meaning that  $h_i = h_{d+1-i}$  for all  $i = 0, \dots, d+1$ . Let us call a complex  $G$  **Dehn-Sommerville** if the Dehn-Sommerville relations hold for  $G$ .

**2.2.** For example, for the **icosahedron complex**  $G$  generated by the triangles  $\{\{1, 2, 5\}, \{1, 2, 6\}, \{1, 3, 4\}, \{1, 3, 5\}, \{1, 4, 6\}, \{2, 5, 9\}, \{2, 6, 1\}, \{2, 9, 10\}, \{3, 4, 8\}, \{3, 5, 11\}, \{3, 8, 11\}, \{4, 6, 12\}, \{4, 8, 12\}, \{5, 9, 11\}, \{6, 10, 12\}, \{7, 8, 11\}, \{7, 8, 12\}, \{7, 9, 10\}, \{7, 9, 11\}, \{7, 10, 12\}\}$ , with  $d = 2$  and  $f_G(t) = 1 + 12t + 30t^2 + 20t^3$  we have  $h_G(t) = 1 + 9t + 9t^2 + 1$ . The  $f$ -vector  $(12, 30, 20)$  defined the  $h$ -vector  $(1, 9, 9, 1)$ . The graph is Dehn-Sommerville. The Whitney complex  $G_1$  of the icosahedron has the  $f$ -vector  $(62, 180, 120)$  and  $h$ -vector  $(1, 59, 59, 1)$ .

**2.3.** As an other example, the **Möbius strip**  $G$  generated by  $\{\{1, 2, 5\}, \{1, 5, 8\}, \{2, 3, 6\}, \{2, 5, 6\}, \{3, 4, 7\}, \{3, 6, 7\}, \{4, 5, 8\}, \{4, 7, 8\}\}$ , with  $f$ -vector  $(8, 16, 8)$  gets the  $h$ -vector  $(-1, 3, 5, 1)$  which is not palindromic. The Möbius strip is not Dehn-Sommerville. Manifolds with boundary in general are not Dehn-Sommerville. But see [10] for a formula in that case. We will below point out that for manifolds with Dehn-Sommerville boundary  $2f_G - f_{\delta G}$  has the Dehn-Sommerville property, just by gluing two copies along a boundary.

**2.4.** Properties like cohomology or orientability are not needed. The only thing which matters is the Euler characteristic  $\chi(G)$  of the complex as well as whether the unit spheres and unit spheres of unit spheres etc are Dehn-Sommerville.

**2.5.** The following result holds for any simplicial complex:

**Theorem 2.** *The simplex generating function  $f_G(t)$  of  $G$  satisfies the symmetry  $f(t) + (-1)^d f(-1-t)$  if and only if  $G$  is Dehn-Sommerville.*

*Proof.* The palindromic condition of  $h$ -vector can be rephrased that the roots of the  $h$ -function  $h(t) = 1 + h_0 t + \dots + h_d t^{d+1} = (t-1)^d f(1/(t-1))$  are invariant under the involution  $t \rightarrow 1/t$ . This is equivalent that the roots of  $f$  are invariant under the involution  $t \rightarrow -1-t$  and so to the symmetry  $f(-1-t) = \pm f(t)$  for the  $f$ -function.  $\square$

**2.6.** This symmetry means that the function  $g(t) = f(t-1/2)$  is either **even** or **odd**. For example for the three sphere  $G$  with  $f$ -vector  $(8, 24, 32, 16)$  where  $f_G(t) = 1 + 8t + 24t^2 + 32t^3 + 16t^4$ , we have  $f_G(t-1/2) = 16t^4$  which is an even function.

**2.7.** An other example: for the **Barnette sphere** [1], a 3-sphere with  $f$  vector  $(8, 27, 38, 19)$  and  $f$ -function  $f(t) = 1 + 8t + 27t^2 + 38t^3 + 19t^4$  we have  $f(t-1/2) = 3/16 - 3t^2/2 + 19t^4$ , which is an even function.

**Corollary 1.** *If  $G$  is a discrete  $d$ -manifold of Euler characteristic 1 with boundary  $\delta G$ , where  $\delta G$  is a discrete  $(d-1)$  manifold with Euler characteristic  $(1 + (-1)^{d-1})$  then  $f_M(t) = 2f_G(t) - f_{\delta G}$  is Dehn-Sommerville, as  $f_M(t)$  is the  $f$ -function of a discrete  $d$ -manifold  $M$  without boundary.*

*Proof.* Glue to copies of  $G$  along  $\delta G$ . This gives a discrete  $d$ -manifold  $M$  without boundary with  $\chi(M) = 2\chi(G) - \chi(\delta G) = 2 - (1 + (-1)^{d-1}) = (1 + (-1)^d)$ . Every unit sphere is a  $(d-1)$ -sphere which is Dehn-Sommerville.  $\square$

**2.8.** While  $f_M$  is Dehn-Sommerville, even if  $f_{\delta G}$  is Dehn-Sommerville, they are of different parity and  $f + M + f_{\delta G} = 2f_G$  is no more Dehn-Sommerville. The even or odd sets of functions  $f$  which satisfy a given Dehn-Sommerville parity form each a linear space, but the union is not a linear space. If  $G \in \mathcal{X}_d$ , is of the even kind, then all unit spheres  $S(x)$  are of the odd kind and vice versa by the assumption on Euler characteristic and the fact that  $f_G(0) = 1$  and  $f_G(-1) = 1 - \chi(G)$ .

**2.9.** For example, for the Wheel graph  $W_9$  with  $C_9$  as boundary, we have  $f_G(t) = 1 + 10x + 18x^2 + 9x^3$  which does not satisfy the Dehn-Sommerville symmetry  $f(x) = \pm f(-1-x)$ . But  $2f_G - f_{\delta G} = 1 + 11x + 27x^2 + 18x^3 = 18(x+1/2)^3 - 5(x+1/2)/2$  does. It is the  $f$ -vector of a sphere  $M$  obtained by gluing two discs along the rim.

**2.10.** We have assumed that Dehn-Sommerville complexes must have the Euler characteristic of a  $d$ -sphere but we can not get rid of the assumption. The symmetry for  $f$  forces it:

**Corollary 2.** *If  $G$  is a complex with maximal dimension  $d$  and  $G$  satisfies Dehn-Sommerville, then  $\chi(G) = 1 + (-1)^d$ .*

*Proof.* We have  $f_G(0) = 1$ . The Dehn-Sommerville symmetry tells

$$f(-1) = (-1)^d f(0) = (-1)^d .$$

But  $f(-1) = 1 - \chi(G)$ .  $\square$

**2.11.** For the zero-sphere  $S^0 = \{\{1\}, \{2\}\}$  we have  $f_G(t) = 1 + 2t$  which satisfies  $f_G(-t - 1) = -f_G(t)$ . Since  $f_{G+H}(t) = f_G(t)f_H(t)$ , we immediately see that the **suspension**  $S_0 + G$  of a Dehn-Sommerville complex is Dehn-Sommerville. More generally:

**Corollary 3.** *If  $G$  and  $H$  are Dehn-Sommerville, then the join  $G + H$  is Dehn-Sommerville.*

**2.12.** While the join of a  $k$ -sphere and a  $l$ -sphere is always a  $k + l + 1$ -sphere, we in general do not get discrete manifolds, if we take the join of two discrete manifolds. The join can produce lots of examples of simplicial Dehn-Sommerville complexes which are not manifolds.

**2.13.** The **Barycentric refinement**  $G_1$  of a complex  $G$  is the order complex of  $G$ . It is more intuitive to think of  $G_1$  as the Whitney complex of the graph  $\Gamma(G)$  defined by  $G$ . Barycentric refinements are always Whitney complexes of graphs. The sets in  $G_1$  are the vertex sets of the complete sub-graphs of  $\Gamma(G)$ . The following statement can be reformulated algebraically as a commutation between two operations, the Barycentric refinement operation and the Dehn-Sommerville involution. But it is also a geometric statement:

**Proposition 1.** *If  $G$  is Dehn-Sommerville then its Barycentric refinement  $G_1$  is Dehn-Sommerville.*

*Proof.* This can be proven by induction with respect to dimension. Gauss-Bonnet implies that  $G$  satisfies Dehn-Sommerville if and only if it has the right Euler characteristic  $1 + (-1)^d$  and all unit spheres satisfy Dehn-Sommerville. The unit spheres of  $G$  are either spheres or Barycentric refinements of unit spheres of  $G$ . Both cases satisfy Dehn-Sommerville by induction.  $\square$

**Corollary 4.** *If two discrete  $d$ -manifolds are Dehn-Sommerville, so is their **connected sum** obtained by removing two unit balls, one in each and gluing the manifolds along their boundary spheres.*

*Proof.* The connected sum of  $G_1, G_2$  is again a  $d$ -manifold  $G$ . Assume the connected sum has been glued along the sphere  $H$ . For the Euler characteristic we have  $\chi_G = (\chi(G_1) - (-1)^d) + (\chi(G_2) - (-1)^d) - \chi(H) = (1 + (-1)^d) - (-1)^d + (1 + (-1)^d) - (-1)^d - (1 + (-1)^{d-1}) = 1 + (-1)^d$ .  $\square$

### 3. A CLASS OF GRAPHS

**3.1.** We define now a class  $\mathcal{X}$  of finite simple graphs. Define  $\mathcal{X}_{(-1)} = \{0\}$ , where  $0 = (V, G) = (\emptyset, \emptyset)$  is the empty graph. Assume  $\mathcal{X}_{d-1}$  is given already, define

$$\mathcal{X}_d = \{G \mid \chi(G) = 1 + (-1)^d, S(x) \in \mathcal{X}_{d-1}, \forall x \in V(G)\}.$$

**3.2.** The graphs in  $\mathcal{X}_d$  have **maximal dimension**  $d$  meaning that for  $G \in \mathcal{X}_d$ , the **clique number**, the vertex cardinality of the maximal complete subgraph of  $G$  is equal to  $d+1$ . For  $d = 0$ , there is only one graph in  $\mathcal{X}_0$ , the two point graph  $S^0 = 1 \cup 1$  which is the 0-sphere. For  $d = 1$ , any finite collection of cyclic graphs  $C_n$  with  $n \geq 4$  is in  $\mathcal{X}_1$ ; the property that unit spheres are two point graphs  $S^0$  forces this. A graph

in  $\mathcal{X}_1$  does not have to be connected. Actually, there are disconnected graphs in any dimension  $\mathcal{X}_d$ . For odd  $d$ , the space  $\mathcal{X}_d$  contains any discrete manifold, meaning any graph for which any unit sphere is in  $\mathcal{X}_{d-1}$ . For even  $d$ , we can even take two copies of  $d$ -dimensional projective planes as this satisfies the Euler characteristic condition. This holds also for  $d = 0$ , we can see  $1 = K_1$  as a zero-dimensional projective plane, the quotient of the 0-sphere under an involution without fixed points.

**3.3.** The **join**  $G + H$  of two graphs  $G = (V, E), H = (W, F)$  is defined as  $G + H = \{V \cup W, E \cup F \cup \{(a, b) \mid a \in V, b \in W\}\}$ . It imposes a monoid structure on the full category  $(\mathcal{G}, +, 0)$  of finite simple graphs, where the empty graph is the 0 element. A graph  $G$  is called a **sphere** if it is in  $\mathcal{X}$  and additionally has the property that there exists a vertex  $x$ , such that  $G - x$  (the graph without  $x$  and all connections to  $x$ ) is contractible. (Inductively, a graph  $G \in \mathcal{G}$  is called **contractible** if there exists a vertex  $x$  such that  $G - x$  is contractible, the induction assumption being that  $1 = K_1$  is contractible. ) Let us call  $\mathcal{S}$  the set of graphs which are spheres.

**Proposition 2.** *The class  $(\mathcal{X}, +)$  is a submonoid of  $(\mathcal{G}, +)$  containing the monoid  $(\mathcal{S}, +)$  of spheres.*

*Proof.* By definition 0 is assumed to be in  $\mathcal{X}$ . We only have to show that if  $G, H$  are in  $\mathcal{X}$ , then  $G + H$  are in  $\mathcal{X}$ . But this follows from the definition as  $S_{G+H}(x) = S(x) + H$  if  $x \in V(G)$  and  $S_{G+H}(x) = G + S(x)$  if  $x \in V(H)$ . Also the assumption on Euler characteristic works as  $\chi(G) = 1 - f_G(-1) \in \mathbb{Z}_2 = \{-1, 1\}$  and  $\chi(G+H) = f_G(-1)f_H(-1)$  is still in  $\{-1, 1\}$ .  $\square$

**3.4.** The class  $\mathcal{X}$  might be small when comparing it with the full category  $\mathcal{G}$  of finite simple graphs but it is rather large when looking at geometry. Given any smooth and compact  $d$ -manifold  $M$ . A **triangulation**  $G$  of  $M$  is called **nice** if every unit sphere  $S(x)$  of a vertex is a graph in  $\mathcal{S}_{d-1}$ . (Not every triangulation is nice. Given a nice triangulation of a 2-sphere for example, like an octahedron, placing additional vertices to the inside of each of the 8 triangles and connecting them to the three neighboring vertices is not a nice triangulation as there are now unit spheres  $S(x)$  which are triangles  $K_3$  which are not in  $\mathcal{S}_1$  as the maximal dimension of the graph of such a triangulation is 3.)

**Proposition 3.** *Any nice triangulation of a  $d$ -manifold with Euler characteristic  $1 + (-1)^d$  is in  $\mathcal{X}$ .*

*Proof.* This follows from the definition. The condition on Euler characteristic is in the assumption. The unit spheres  $S(x)$  of each vertex  $x$  are by definition in  $\mathcal{S}_{d-1}$  by the assumption of “nice” so that every unit sphere  $S(x)$  is in  $\mathcal{X}_{d-1}$ .  $\square$

**3.5.** Note that we don’t need orientability so that there are examples in  $\mathcal{X}_d$  which do not satisfy Poincaré-duality. Projective spaces are examples. We can glue a projective plane to a 2-sphere at a single point and get a Dehn-Sommerville space. Combining the two propositions gives lots of examples of graphs in  $\mathcal{X}$  which are not discrete manifolds. Take the suspension  $G + S^0$  of a **twin projective plane**  $G = \mathbb{P}^2 \cup \mathbb{P}^2 \in \mathcal{X}_2$ . This is not a 3-manifold, because there are two vertices for which the unit sphere is  $G$ .

**3.6.** A level surface in a discrete manifold is a discrete manifold by discrete Sard [17]. Is it true that the level surface of a Dehn-Sommerville space is a Dehn-Sommerville space? No, we can have already in a circle level surfaces which are not-Dehn-Sommerville. In general, we can realize any smooth compact discrete d-manifold as a iterated level surface of a discrete Dehn-Sommerville space because of the Whitney embedding theorem.

#### 4. BARYCENTRIC REFINEMENT

**4.1.** Let  $A$  be the **Barycentric refinement operator** defined by  $f(G_1) = Af(G)$ . If  $G$  has maximal dimension  $d$ , then the matrix  $A$  is a  $(d+1) \times (d+1)$  upper triangular matrix. It is explicitly given as  $A_{ij} = \text{Stirling}(i, j)i!$ , involving the Stirling numbers of the second kind. Since all eigenvalues  $\lambda_k = k!$  are distinct, the eigenvalues of  $A$  are an eigenbasis of  $A$  on the vector space  $V_d = \mathbb{R}^{d+1}$  which is isomorphic to the space  $P_d$  of polynomials of degree less or equal to  $d$ . The isomorphism is given by  $[a_0, a_1, \dots, a_d] \rightarrow a_0 + a_1t + \dots + a_d t^d$ . As an affine space, it is isomorphic to  $1 + a_0t + a_1t^2 + \dots + a_d t^{d+1}$  which is the form of an f-function of a complex.

**4.2.** The linear unitary reflection  $T(f)(x) = f(-1 - x)$  on polynomials defines an involution on  $\mathbb{R}^{d+1}$ . As an unitary reflection  $T^2 = Id$ , it has the eigenvalues 1 and  $-1$  which by the spectral theorem of normal operators define an eigenbasis even so the algebraic multiplicities are larger than 1 for  $d > 1$ . In analogy to  $\tilde{T}(f)(x) = f(-x)$ , we can call eigenfunctions of 1 **even functions** and eigenfunctions of  $-1$  **odd functions**.

**4.3.** Any eigenvector  $V$  of  $A^T$  defines a functional  $\phi_V$  on complexes. Most functionals  $\phi_V(G_n)$  explode when looking at Barycentric refinements  $G_n$  of  $G$ . There is just one functional  $\phi_1$  which stays invariant and this is the Euler characteristic. Since the matrix  $A$  is upper triangular and  $A^T$  lower triangular, the eigenbasis diagonalizing  $A$  is triangular too. We say an eigenvector is **even** if it has an odd number of non-zero entries.

**Lemma 1.** *The eigenbasis of  $A$  is also an eigenbasis of  $T$ : even eigenvectors are eigenfunctions of  $T$  to the eigenvalue 1 and odd eigenvectors of  $A$  are eigenfunctions of  $T$  to the eigenvalue  $-1$ .*

*Proof.* As the linear operators  $T$  and the Barycentric operation  $A$  commute. They therefore have the same eigenbasis.  $\square$

**4.4.** We know from linear algebra that the eigenvectors  $V_k$  of  $A^T$  and the eigenvectors  $W_k$  of  $A$  have the property that  $V_k W_l = c_{kl} \delta_{k,l}$  meaning that if they are normalized, then they define dual coordinate systems. We think about eigenvectors of  $A^T$  as functionals. Functionals in the even eigenspace of  $T$  are zero on even functions etc. This gives us convenient Dehn-Sommerville invariants:

**Corollary 5.** *For even  $d$ , the even eigenvectors of  $A^T$  and for odd  $d$ , the odd eigenvectors of  $A^T$  define functionals which are zero on the class  $\mathcal{X}_d$ .*

**4.5.** This was Theorem (1) in [19], where already the idea of proving Dehn-Sommerville via curvature has appeared and multi-variate versions of Dehn-Sommerville were given, answering a open problem of Gruenbaum [7] from 1970. The current approach is much simpler. In multi-dimensions, the Dehn-Sommerville symmetry just has to hold for each of the variables appearing in the simplex generating function  $f(t_1, \dots, t_m)$ . The proof in higher dimensions is identical using Gauss-Bonnet.

**4.6.** An **edge refinement** of a graph cuts an edge  $e = (a, b)$  into two by adding a new vertex  $c$  in the middle and connecting the new vertex to the intersection of spheres at  $a$  and  $b$ . More formally, we remove the edge  $(a, b)$ , and adding new edges  $(a, c), (c, b)$  as well as  $\{(c, z) \mid z \in S(a) \cap S(b)\}$ . Edge refinements preserve discrete manifolds. More generally:

**Proposition 4.** *If  $G$  is in  $\mathcal{X}_d$  and  $e$  is an edge in  $G$ , then the edge refinement is in  $\mathcal{X}_d$ .*

*Proof.* The effect of the operation on the  $f$ -vector can be split into two parts. The first one is to increase  $f_0$  and  $f_1$  by 1 (which means adding  $t+t^2$  to  $f_G(t)$ ). Then we add  $t f_{S(a) \cap S(b)} + 2t^2 f_{S(a) \cap S(b)}$ , because every  $k$ -simplex in  $S(a) \cap S(b)$  defines a new  $(k+1)$ -simplex connecting to  $c$  and two new  $(k+2)$ -simplices connecting  $S(a) \cap S(b)$  to  $(a, c)$  and  $(b, c)$ . Now, the set of functions satisfying the Dehn-Sommerville symmetry form a linear space. The claim follows as the added part  $t + t^2 + t f_{S(a) \cap S(b)} + 2t^2 f_{S(a) \cap S(b)}$  satisfy the Dehn-Sommerville symmetry by induction because the space  $S(a) \cap S(b)$  is in  $\mathcal{X}_{d-2}$  if  $G \in \mathcal{X}_d$ .  $\square$

## 5. REMARKS

**5.1.** For Dehn-Sommerville, see for example chapter nine in [8] for convex polytopes. It is also covered in [3] where we read: *these relations had already been found by Max Dehn by 1905 for the dimensions 3,4,5; they were known in all dimensions by Duncan Sommerville by 1927 [27], but were then forgotten until they were rediscovered by Victor Klee in 1963..* The relations have been extended to larger classes of polytopes. An example of recent work is [2]. More literature is [11, 26, 25, 24, 5, 9, 10, 29].

**5.2.** The **Levitt curvature** for Euler characteristic Formula (1) appeared in [23]. We have rediscovered that formula  $\chi(G) = \sum_x K(x)$  in the introduction to [12], an article which had focused on geometric graphs (discrete manifolds). It surprises that **higher dimensional curvature in the discrete is so elegant**, especially if one compares to the continuum, where one has to refer to Pfaffians of curvature expressions to get to the general Gauss-Bonnet-Chern theorem (see [6]). In the continuum, the **Euler curvature** is not even defined for odd-dimensional manifolds. In the continuum, it is zero for odd-dimensional manifolds as we see here again as it is then a special case of the Dehn-Sommerville equations.

**5.3.** The Taylor expansion of the parametrized Gauss-Bonnet formula at  $t = 0$  gives a **generalized handshake formula**  $f_k(G) = \sum_{x \in G} V_{k-1}(S(x))/(k+1)$  which by linearity produces Gauss-Bonnet formulas for any discrete valuation  $X(G) = \sum_k X_k f_k(G)$  and especially for Euler characteristic  $\chi(G) = \sum_k (-1)^k f_k(G)$ . One can also just define  $f(t) = 1 + \sum_{k=0}^d X_k f_k t^{k+1}$  and its anti derivative. The Gauss-Bonnet

formula is the same. For example, for  $X(G) = v_1(G)$ , where  $f(t) = 1 + v_1 t^2$ , the curvature is  $K(x) = \deg(v)/2$ . Gauss-Bonnet is then **Euler handshake formula**, the **fundamental theorem of graph theory**. More generally, we have for  $v_k(G)$  the curvature is  $K(x) = f_{k-1}(S(x))/(k+1)$ .

**5.4.** In [12], we first noticed experimentally that the curvature is zero for odd-dimensional geometric complexes but we could not prove it yet at that time. These zero curvature relations were later proven with discrete integro-geometric methods in [14, 15] by seeing curvature as an average of Poincaré-Hopf indices when integrating over all functions or spaces of colorings (locally injective functions on the vertex set). The connection to Dehn-Sommerville emerged especially in the work about **Wu characteristic** [19]. So, Dehn-Sommerville conditions appeared for us three times independently: first as a zero curvature condition for odd-dimensional discrete manifolds, then as Barycentric invariants (eigenvectors of  $A^T$ ), then as a symmetry for the roots of simplex generating function  $t \rightarrow f_G(t)$ . In each case, we were always unaware of the Dehn-Sommerville connection at first. We hope that this note makes clear how all these concepts (curvature, Barycentric refinement and root symmetry) are related.

**5.5.** The classical **Dehn-Sommerville valuations** are

$$X_{k,d} = \sum_{j=k}^{d-1} (-1)^{j+d} \binom{j+1}{k+1} v_j(G) + v_k(G).$$

If the vectors  $X_{0,d}, \dots, X_{d-2,d}$  are written as row vectors in a matrix  $X_d$ , we have

$$X_2 = [ 2 \quad -2 ], X_3 = \begin{bmatrix} 0 & 2 & -3 \\ 0 & 2 & -3 \end{bmatrix}, X_4 = \begin{bmatrix} 2 & -2 & 3 & -4 \\ 0 & 0 & 3 & -6 \\ 0 & 0 & 2 & -4 \end{bmatrix}.$$

We have mentioned before (like [21]) that the curvature of  $X_{k,d}$  is  $K(x) = X_{k-1,d-1}(S(x))$ . But it is less obvious there. The reason is the combinatorial identity

$$X_{k+1,d+1}(l+1)/(l+1) = X(k,d)(l)/(k+2).$$

But it also implies that the Dehn-Sommerville curvatures are all zero for a geometric graph. Use Gauss-Bonnet and induction using the fact that the unit sphere of a geometric graph is geometric and that for  $d = 1$ , a geometric graph is a cyclic graph  $C_n$  with  $n \geq 4$ . For such a graph, the Dehn-Sommerville valuations are zero.

**5.6.** Gauss-Bonnet and Dehn-Sommerville can be generalized to multi-valuate valuations like **Wu characteristic**  $\omega(G) = \sum_{x \sim y} \omega(x)\omega(y)$  with  $\omega(x) = (-1)^{\dim(x)}$ . The Wu characteristic is then  $1 - f(-1, -1)$  where

$$f(t, s) = 1 + \sum_{k,l} f_{kl}(G) t^{k+1} s^{l+1}$$

is the **multivariate simplex generating function**. Here,  $f_{kl}(G)$  is the  **$f$ -matrix**, counting the number of intersecting  $k$ -dimensional and  $l$ -dimensional simplices.

**5.7.** The curvature of Wu characteristic is then  $F_G(t, s) = \int_0^t f(r, s) dr$ . Gauss-Bonnet reads

$$f_G(t, s) = \sum_{x \in G} F_{S(x)}(t, s)$$

and especially  $\omega(G) = \sum_{x \in G} K(x)$ , where  $K(x)$  is the Wu curvature.

**5.8.** While investigating Barycentric limits [18, 16], an other angle to Dehn-Sommerville appeared. We first did not see the connection between Barycentric invariants and Dehn-Sommerville. The Barycentric refinement operator  $A_d$  was first explored empirically by looking at the best linear operator implementing the map  $f(G) \rightarrow f(G_1)$  (brute force data fitting with hundreds of random graphs) and were surprised that the fitting would lead to an exact formula. After getting the formula for  $A$  and proving it, we learned that it is “well known”. It appears in [28, 24, 9].

**5.9.** The value  $g(x, x) = 1 - \chi((x))$  is the **Green function**, the diagonal entries of the inverse  $g = L^{-1}$  of the unimodular connection matrix  $L$  defined as  $L(x, y) = 1$  if  $x \cap y \neq \emptyset$  and  $L(x, y) = 0$  else. The Green function entries  $g(x, y)$  are potential energy values between two simplices  $x, y$ . We called  $f'_G(-1) = \sum_{x \in G} (1 - \chi(S(x))) = \text{tr}(L - L^{-1})$  the **Hydrogen functional** because of some affinity with the quantum mechanical Hamiltonian of the Hydrogen atom, where the potential part involves the kernel of the inverse of the kinetic part (the  $1/r$  potential of course being determined through the Laplacian  $\Delta$  in three dimensional space).

**5.10.** The **energy theorem** assures that the total potential energy  $\sum_{x,y} g(x, y)$  is the Euler characteristic  $\chi(G)$ , which is defined as the **super trace**  $\text{str}(L) = \sum_x \omega(x)L(x, x)$  and agrees with the super trace  $\sum_x \omega(x)g(x, x)$  of  $g = L^{-1}$ . The entries  $\omega(x)L(x, x) = (-1)^{\dim(x)}$  and  $\omega(x)g(x, x)$  are integers. They are the Poincaré-Hopf indices [13] of the function  $h(x) = \dim(x)$  or  $h(x) = -\dim(x)$  which are **colorings** of the graph  $\Gamma(G)$  defined by the simplicial complex  $G$ .

**5.11.** From the energy theorem and Gauss Bonnet we can express  $d/dt \log(f_G(t))$  at  $t = -1$  through the connection operator  $L$ . Let  $E$  be the matrix which has everywhere 1.  $\frac{d}{dt} \log(f(t))_{t=-1} = \text{tr}(L^{-1})/\text{Tr}(L^{-1}E)$ . Proof: From Gauss Bonnet, we have  $\frac{d}{dt} \log(f_G(t)) = \frac{f'_G}{f_G} = \sum_x \frac{f_{S(x)}(t)}{f_G(t)}$ . For  $t = -1$ , we have  $\sum_x \chi(S(x))/\chi(G) = \text{tr}(L^{-1})/\chi(G) = \text{tr}(L^{-1})/\text{Tr}(L^{-1}E)$ .

**5.12.** The involutive symmetry  $T(f) = \pm f(-1 - t)$  given by the Dehn-Sommerville condition implies **root pairing** for  $f$ . This article has started with such an observation. We noticed that for even-dimensional spheres, there is always root with  $\text{Re}(t) = -1/2$  and that the roots are reflection symmetric with respect to  $t = -1/2$ . A simplicial complex is defined to be a  **$d$ -sphere** if every unit sphere is a  $(d - 1)$ -sphere and removing one vertex renders the complex contractible. This inductive definition is primed by the empty complex 0 being the  $(-1)$ -sphere. There are various operations which preserve  $d$ -spheres. We observe that for spheres the roots of  $f$  pair up to  $-1$  in the odd-dimensional case and do so also in the even-dimensional case with the remaining roots.

**5.13.** For example, for the icosahedron graph  $G$  with  $f$ -vector  $(12, 30, 20)$  and simplex generating function  $f(x) = 1 + 12x + 30x^2 + 30x^3$ , the roots are  $-1/2, -1/2 \pm \sqrt{15}/10$  which is the set  $\{-0.5, -0.8873\dots, -0.1127\dots\}$ . We were initially interested in the algebraic structure of these polynomials and saw  $f(x) = (1 + 2x)(1 + 10x + 10x^2)$ . Now this factorization happened for any two-dimensional polyhedron. A random one may have the  $f$ -vector  $(112, 330, 220)$  leading to the  $f$ -function  $1 + 112x + 330x^2 + 220x^3 = (1 + 2x)(1 + 110x + 110x^2)$ . Then again, it appeared for any 4-sphere like one with the  $f$ -vector  $(13, 61, 134, 140, 56)$  leading to  $f(x) = 1 + 13x + 61x^2 + 134x^3 + 140x^4 + 56x^5 = (1 + 2x)^3(1 + 7x + 7x^2)$ . Of course, discovering this is very exciting when not knowing that it is already a known phenomenon (when making the connection with Dehn-Sommerville).

**5.14.** There are simplicial complexes outside  $\mathcal{X}$  which are Dehn-Sommerville. Similarly as zero Euler curvature implies zero Euler characteristic for even-dimensional manifolds, zero Euler characteristic does not necessarily mean zero curvature. The zero Dehn-Sommerville curvature condition is **sufficient** for the complex to be Dehn-Sommerville, but it is **not necessary**. There are complexes which are Dehn-Sommerville, but which are **not Dehn-Sommerville flat**. We give an example in the illustration section.

## 6. EXAMPLES

**6.1. Examples.** For  $d = 3$ , we have  $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 6 & 14 \\ 0 & 0 & 6 & 36 \\ 0 & 0 & 0 & 24 \end{bmatrix}$ . The eigenvec-

tors are  $[0, 0, 0, 1]^T, [0, 0, -1, 2]^T, [0, 22, -33, 40]^T, [-1, 1, -1, 1]^T$ . The eigenvector  $[-1, 1, -1, 1]$  to the eigenvalue 1 is the Euler characteristic. The second and last eigenvector leads to the Dehn-Sommerville invariants  $f_2 - 2f_3 = 0$  and  $f_0 - f_1 + f_2 - f_3 = 0$ .

For  $d = 4$ , we have  $A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 6 & 14 & 30 \\ 0 & 0 & 6 & 36 & 150 \\ 0 & 0 & 0 & 24 & 240 \\ 0 & 0 & 0 & 0 & 120 \end{bmatrix}$ . From the eigenvectors  $[0, 0, 0, 0, 1],$

$[0, 0, 0, -2, 5], [0, 0, 19, -38, 55], [0, -22, 33, -40, 45], [1, -1, 1, -1, 1]$ , the second and last produce Dehn-Sommerville invariants:  $2v_3 - 5v_4 = -0$  and  $22v_2 - 33v_3 - 40v_4 + 45v_5 = 0$ .

**6.2.** For  $d = 0$ , where  $h = -1 + t + v_0$ , the condition is  $v_0 = 2$  implying that the zero-dimensional graph must have 2 vertices.

For  $d = 1$ , where  $f = (v_0, v_1)$  gives the number of vertices, edges and triangles, then  $f = 1 + v_0t + v_1t^2$  and  $h = (1 - v_0 + v_1) + (v_0 - 2)t + t^2$ . The Dehn-Sommerville condition is  $v_0 = v_1$  meaning  $\chi(G) = 0$ . Note that we can attach hairs and even arbitrary trees to a circular graph and still have  $v_0 = v_1$  satisfied. This shows, that at least in one dimension, the Dehn-Sommerville relations can hold for a larger class of complexes than  $\mathcal{X}$  to be defined below.

For  $d = 2$ , the conditions give  $v_1 = 3v_0 - 6, v_2 = -4 + 2v_0$ . This is equivalent to  $v_0 - v_1 + v_2 = 2, 2v_1 = 3v_2$ . This means that Euler characteristic is 2 and that every edge meets two triangles.

For  $d = 3$ , the condition is equivalent to  $\chi(G) = v_0 - v_1 + v_2 - v_3 = 0$  and  $22v_1 - 33v_2 + 40v_3 = 0$ . We will see in a moment how to get the more intuitive Barycentric expressions through eigenvectors of the Barycentric refinement operators.

**6.3.** The root pairing property was already mentioned in [22]. We found this while investigating the statistics of the simplex cardinality distribution in simplicial complexes. The **root pairing statement** is obviously true for 0-dimensional spheres. If a zero dimensional complex has  $n$  points, then the generating function of the  $f$ -vector is  $1 + nt$ . This has a root  $-1/2$  if and only if the complex has exactly  $n = 2$  points, which means that  $G$  has to be a 0-sphere. Let us also mention the  $(-1)$ -dimensional complex which is the empty complex. In that case, the function is  $f = 1$  which has no roots. Root pairing still works, there are just no pairs.

**6.4.** For 1-dimensional complexes with  $n$  vertices and  $m$  edges, we have the generating function  $1 + nt + mt^2$ . The Euler characteristic is  $n - m = \chi(G)$ . The roots are  $-n \pm \sqrt{-4m + n^2}/(2m)$ . The sum of the roots is  $-n/m$ . This is  $-1$  if and only if  $n = m$ , meaning that we need  $\chi(G) = 0$ . Beside circular complexes, there are many complexes like **sun graphs** for which  $n = m$ . We can attach arbitrary trees to the circular graph for example and still have the property. There is a sphere complex which is not the Whitney complex of a graph, which is  $G = \{\{1, 2\}, \{2, 3\}, \{3, 1\}, \{1\}, \{2\}, \{3\}\}$  where  $n = 3, m = 3$  and where the roots become complex. We see confirmed here that roots are not real if and only if the complex is not the Whitney complex of a graph.

**6.5.** For 2-dimensional complexes the simplex generating function is  $1 + nt + mt^2 + lt^3$ , in order to have a root  $-1/2$ , we need  $n = (8 - l + 2m)/4$ . For a 2-manifold, we have  $2m = 3l$ . The two equations give  $m = 3n - 6, l = 2n - 4$  implying  $\chi(G) = n - m + l = 2$ . Actually, for two dimensional complexes, the two equations  $2m = 3l, \chi(G) = n - m + l = 2$  imply that  $f(-1/2) = 0$ . This in particular holds for two disjoint copies of the projective plane.

## 7. ILLUSTRATIONS

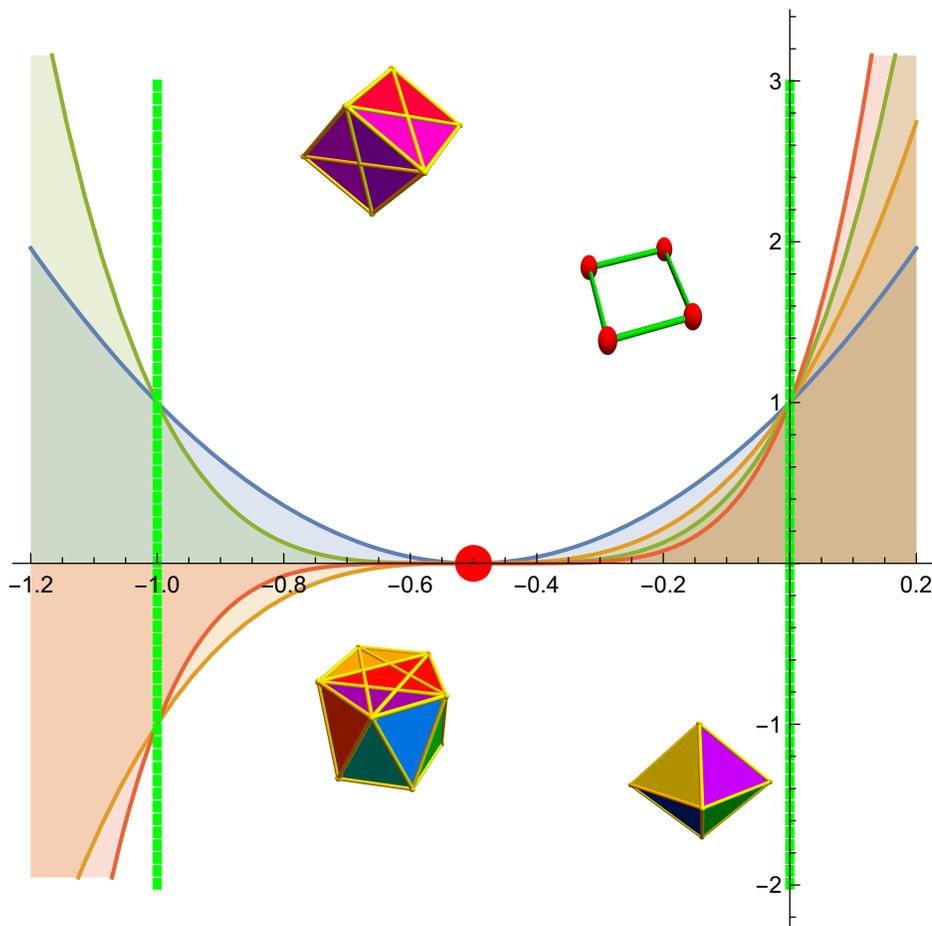


FIGURE 1. The functions  $f_G$  for the smallest spheres  $S^1 = C_4 = S^0 + S^0$ ,  $S^2 = C_4 + S^0 = S^0 + S^0 + S^0$  (the octahedron),  $S^3 = S^2 + S^0 = S^1 + S^1$  (the three sphere),  $S^4 = S^3 + S^0 = 5 * S^0$  (the four sphere), which are all cross polytopes. The index generating function  $f_G(t)$  of  $G = S^0$  is  $1 + 2t$ . so that  $f_{S^d}(t) = (1 + 2t)^{d+1}$ . We then observed experimentally that all spheres satisfy the symmetry  $f(x) + (-1)^d f(-1 - x) = 0$ , then linked it to Dehn-Sommerville.

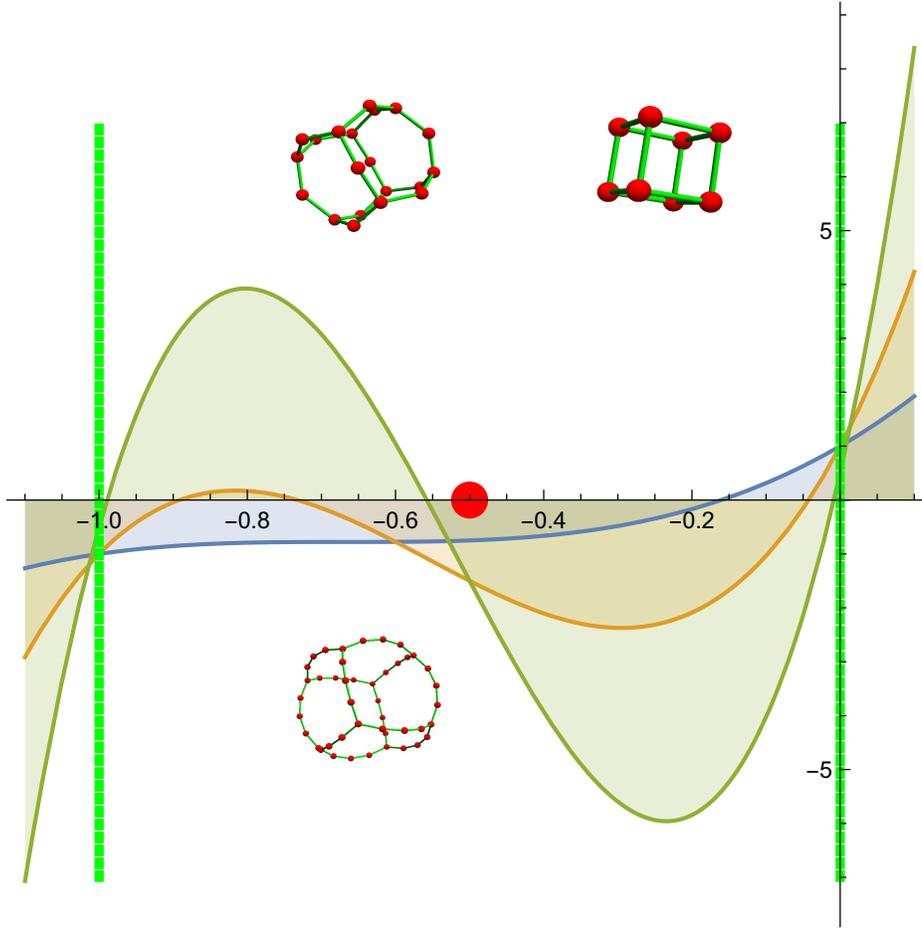


FIGURE 2. If the faces are included to a cube graph we get a **CW-complex** which models a discrete 2-sphere. Its generating function is  $f_G(x) = 1 + 8x + 12x^2 + 6x^3$ . It does not satisfy Dehn-Sommerville. It also has non-real roots. After Barycentric refinements however, the roots become real. We see  $f_{G_1}(x) = 1 + 26x + 60x^2 + 36x^3$  and  $f_{G_2}(x) = 1 + 122x + 336x^2 + 216x^3$  (we plotted  $f_{G_2}/2$ ). The coefficients  $[122, 336, 216]$  are already aligned quite well with the Perron-Frobenius eigenvector  $[1, 3, 2]^T$  to the Barycentric refinement operator  $A_2$  in dimension 2 which defines a function having only real roots. In general we see that the Perron-Frobenius functions  $a_1t + \dots + a_nt^{d+1}$  for the Perron-Frobenius eigenvector to the  $(d+1) \times (d+1)$  matrix  $A_d$  always has only real roots. It looks like a simple calculus/linear algebra problem, but we can not prove this yet. It would imply that for sufficiently large Barycentric refinement of any CW complex and especially simplicial complexes, the roots of  $f_{G_n}$  are real for large enough  $n$ .

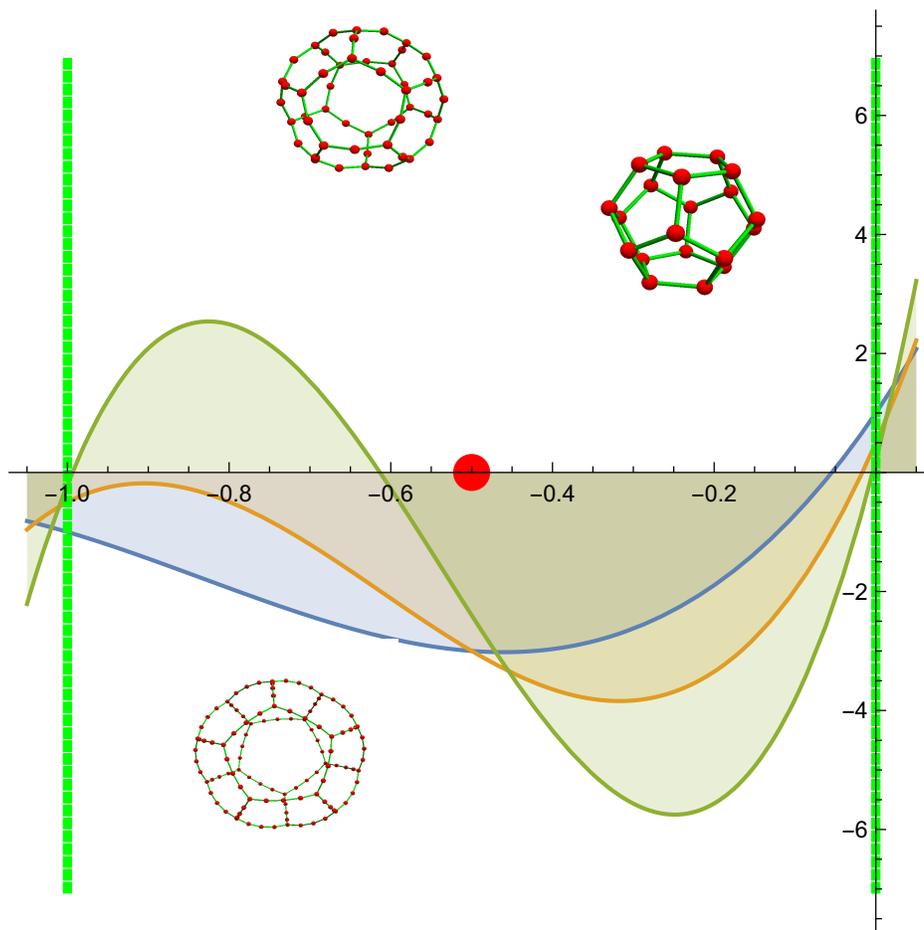


FIGURE 3. Also the **dodecahedron** (when seen as a CW-sphere and not a 1-dimensional graph, which it is when seen as a simplicial complex), has non-real roots for  $f_G(t) = 1 + 20t + 30t^2 + 12t^3$ . But here also  $f_{G_1}$  has non-real roots. Only  $f_{G_2}$  for the second Barycentric refinement  $G_2$  starts to have real roots. As the Perron-Frobenius eigenvector produces a function  $f$  which satisfies the Dehn-Sommerville symmetry, we get roots for  $f_{G_n}$  which are more and more symmetric. Also this was just observed experimentally at first. The linear algebra of the eigenvectors of the Barycentric refinement operators  $A_d$  explains this. Indeed, as we show here, Dehn-Sommerville for complexes of the type  $\mathcal{X}_d$  is a manifestation for a symmetry one has in the Barycentric limit. Since the involutory symmetry (duality) in the limit has eigenvalues 1 or  $-1$ , only half of the Barycentric invariants are Dehn-Sommerville invariants.

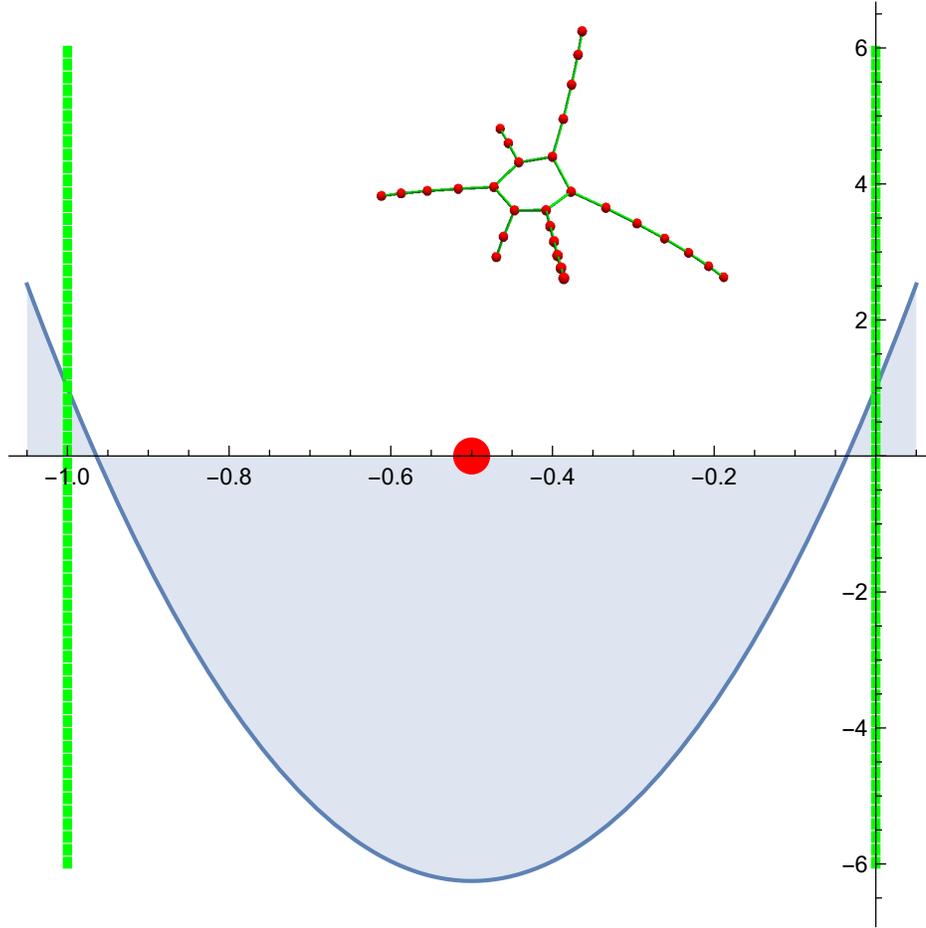


FIGURE 4. **Sun graphs** are 1-dimensional complexes which satisfy Dehn-Sommerville, even-so they are only varieties, not manifolds. As they have the same number of vertices than edges, we have  $f(t) = 1 + nt + nt^2$  which satisfies  $f(-1 - t) = f(t)$ . It is an example of a 1-variety. A  $d$ -variety is a complex  $G$  for which all unit spheres are  $d - 1$  varieties. Like manifolds, it starts with the induction that the empty complex is a  $-1$  variety but unlike for manifolds, we do not insist that unit spheres are  $(d - 1)$ -spheres. In this example,  $f_G(t) = 1 + 29t + 29t^2$ . The roots  $-1/2 \pm \sqrt{25/116}$  are symmetric with respect to  $\text{Re}(t) = -1/2$ . We have Dehn-Sommerville symmetry.

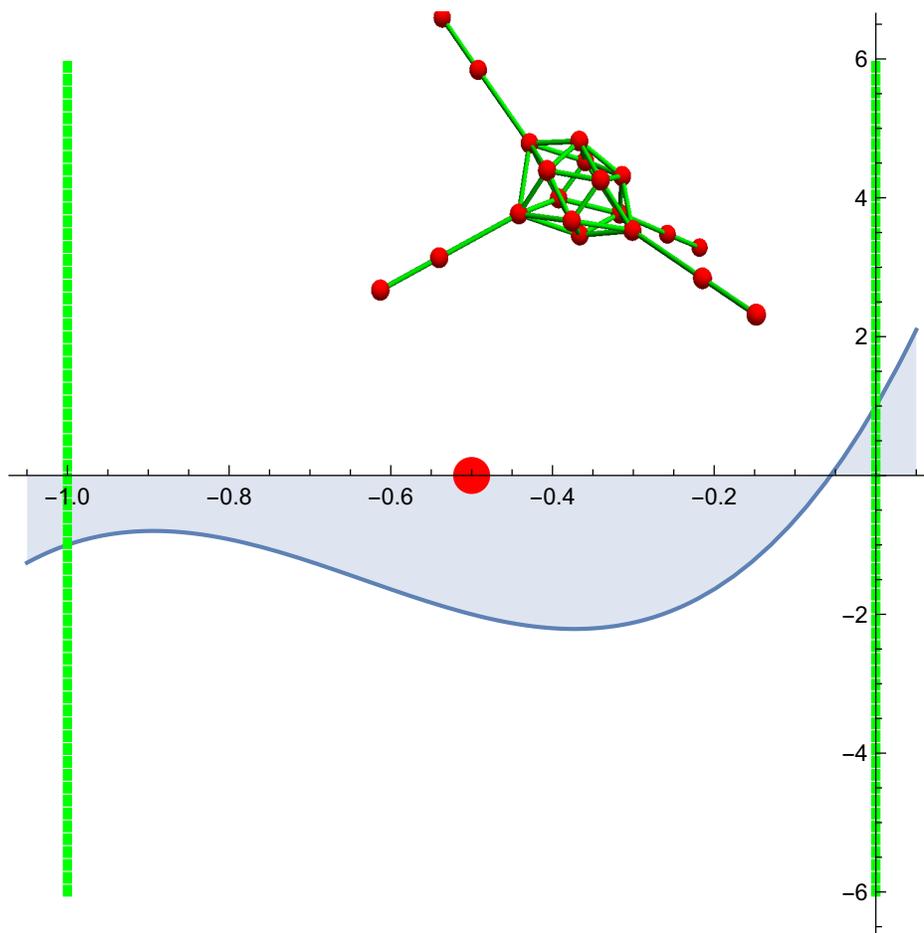


FIGURE 5. When adding hairs to a 2-sphere, the Dehn-Sommerville property gets destroyed. The complex  $G$  shown here is a simplicial complex but it is not pure. Its inductive dimension is  $47/30 = 1.56667$ . Its average simplex cardinality  $f'_G(1)/f_G(1)$  is  $156/79 = 1.97468\dots$  for the function  $f_G(t) = 1 + 20t + 38t^2 + 20t^3$ . The function  $f_G$  does not honor the Dehn-Sommerville symmetry  $f(t) = \pm f(-1-t)$ . We have  $f(-1-t) = -1 - 4t - 22t^2 - 20t^3$ . What happens is that the unit spheres do not satisfy Dehn-Sommerville. There are unit spheres which are a disjoint union of a 1-sphere and a point which does not satisfy Dehn-Sommerville. This means that the Dehn-Sommerville curvatures are not zero. The complex is not Dehn-Sommerville flat.

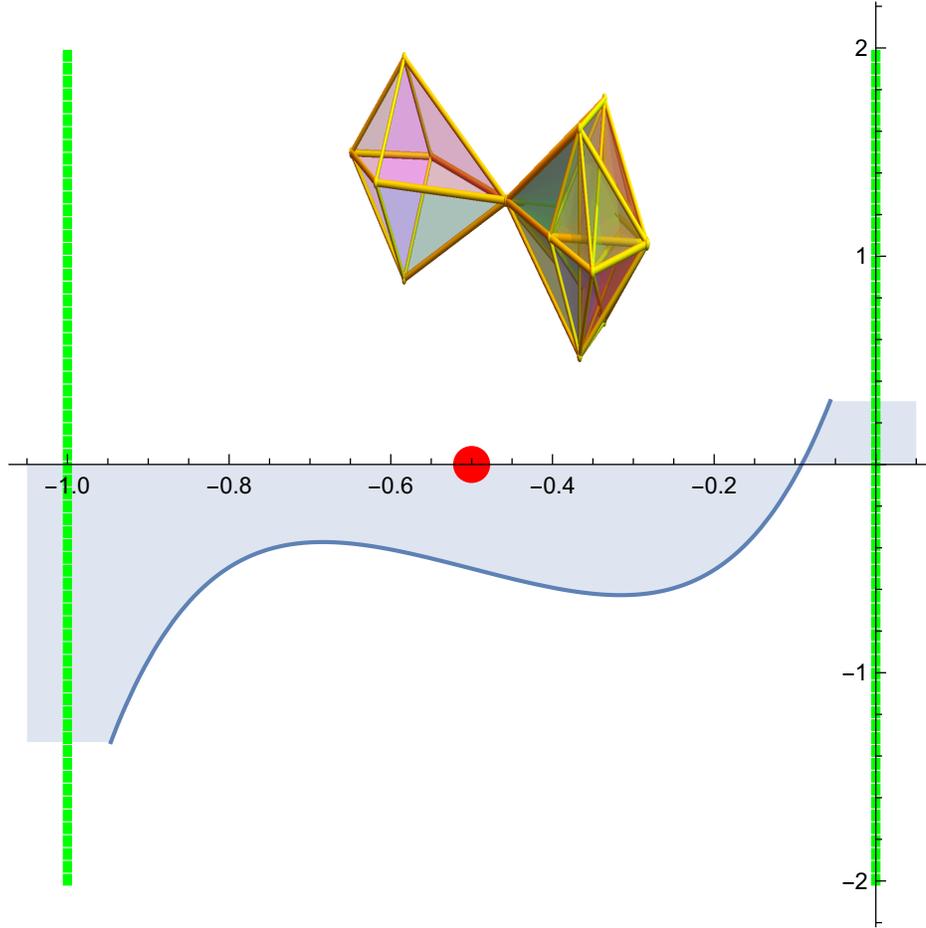


FIGURE 6. The Dehn-Sommerville property gets destroyed with disjoint sums as well as most connected sums (we need to glue along spheres). We see here the simple connected sum  $G$  of a 2-sphere  $O$  (an octahedron graph) and a 4-sphere  $C_4 + O$  joined at a vertex  $v$ . The unit sphere  $S(v)$  is a disjoint union of a 1-sphere  $C_4$  and a 3-sphere  $C_4 + C_4$ . This disjoint union does not satisfy Dehn-Sommerville. By Gauss-Bonnet (since all other points are Dehn-Sommerville flat), also  $G$  is not Dehn-Sommerville. It is almost,  $f + 1/2$  would satisfy the Dehn-Sommerville symmetry.

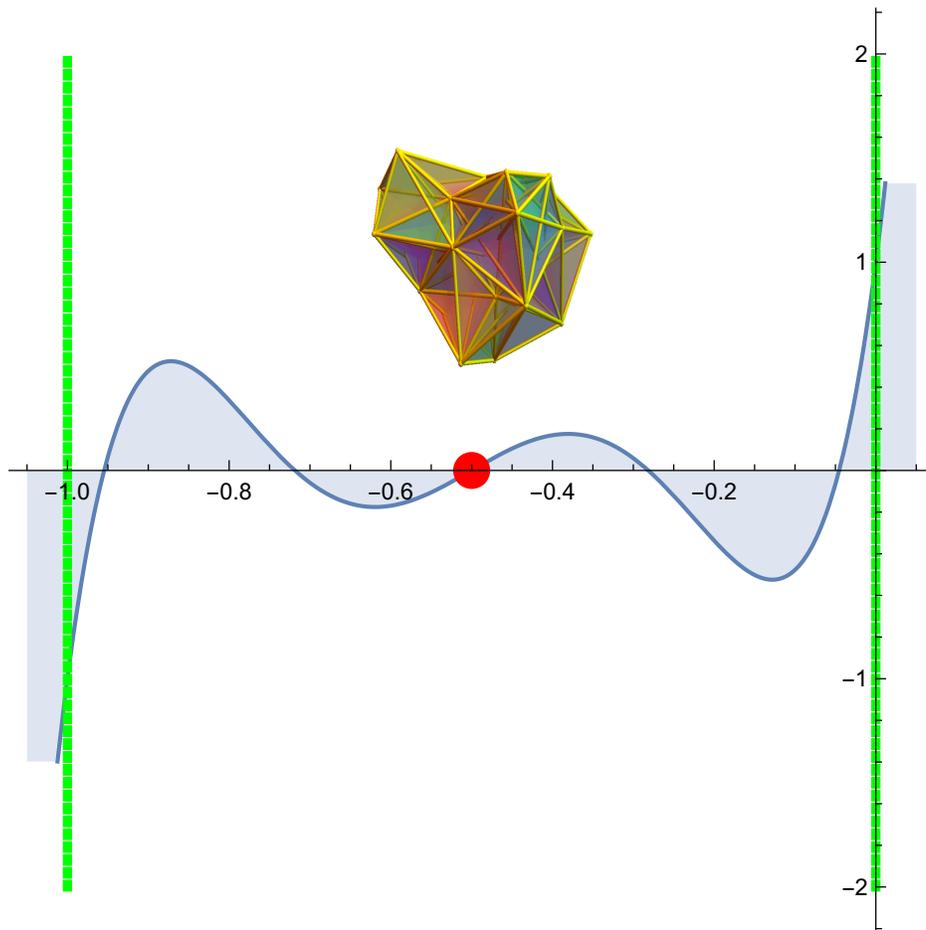


FIGURE 7. We see a random four sphere  $G$ . It is Dehn-Sommerville of course. As for any even dimensional sphere, there is a root  $t = -1/2$  for the simplex generating function  $f_G(t)$ .

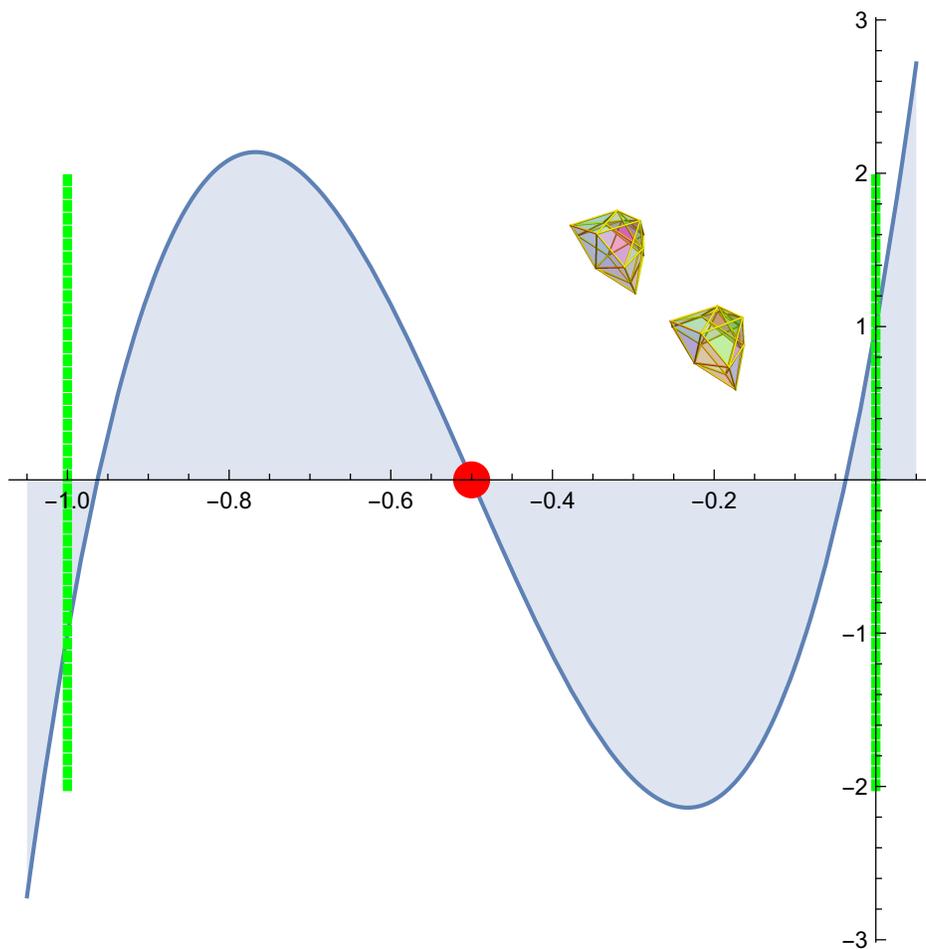


FIGURE 8. We see the disjoint union of two projective planes. Any even dimensional manifold of Euler characteristic 2 satisfies the Dehn-Sommerville condition. So also  $G = \mathbb{P}^2 \cup \mathbb{P}^2$ . We have  $f_G(t) = (1+2x) * (1 + 28x + 28x^2)$ . The Betti-vector is  $(b_0, b_1, b_2) = (2, 0, 0)$ . Obviously, Poincaré duality is failing for  $G$  as  $G$  is non-orientable. Still, Dehn-Sommerville is intact.

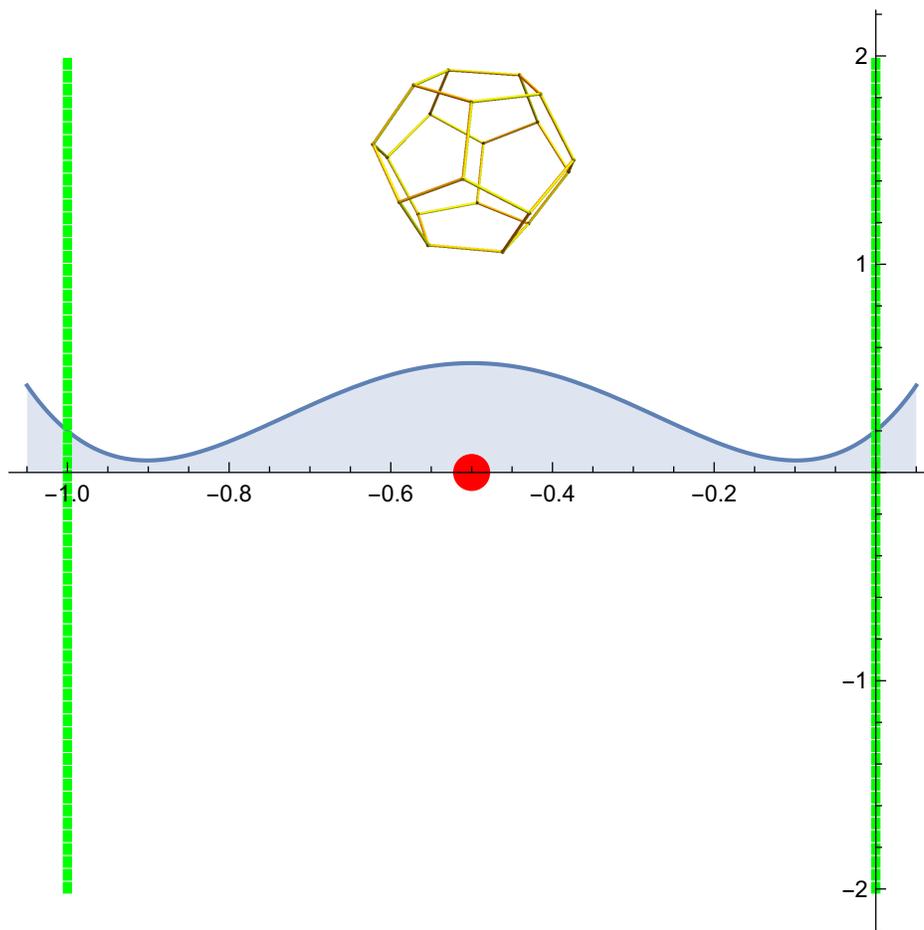


FIGURE 9. We see the graph of  $f_G(t) = 1 + 16t + 106t^2 + 180t^3 + 90t^4$ , where  $G$  is a **Poincaré sphere complex** with 16 zero-dimensional simplices, found in [4]. All 4 roots of  $f_G$  are complex. As a 3-manifold with zero Euler characteristic,  $G$  must be Dehn-Sommerville. Indeed,  $f_G(t - 1/2) = \frac{21}{8} - 29t^2 + 90t^4$  is an even function. A consequence of the Dehn-Sommerville property, the roots of the  $f$ -function  $f_G(t)$  of a 6 or even 8 dimensional sphere are always **algebraic**, the reason being that  $g(t) = f(t - 1/2)$  can then be written as  $g(t) = h(t^2)$ , where  $h$  has degree 3 or 4 which has algebraic solutions. We have not studied the question when the roots of  $f_G(t)$  are algebraic yet.

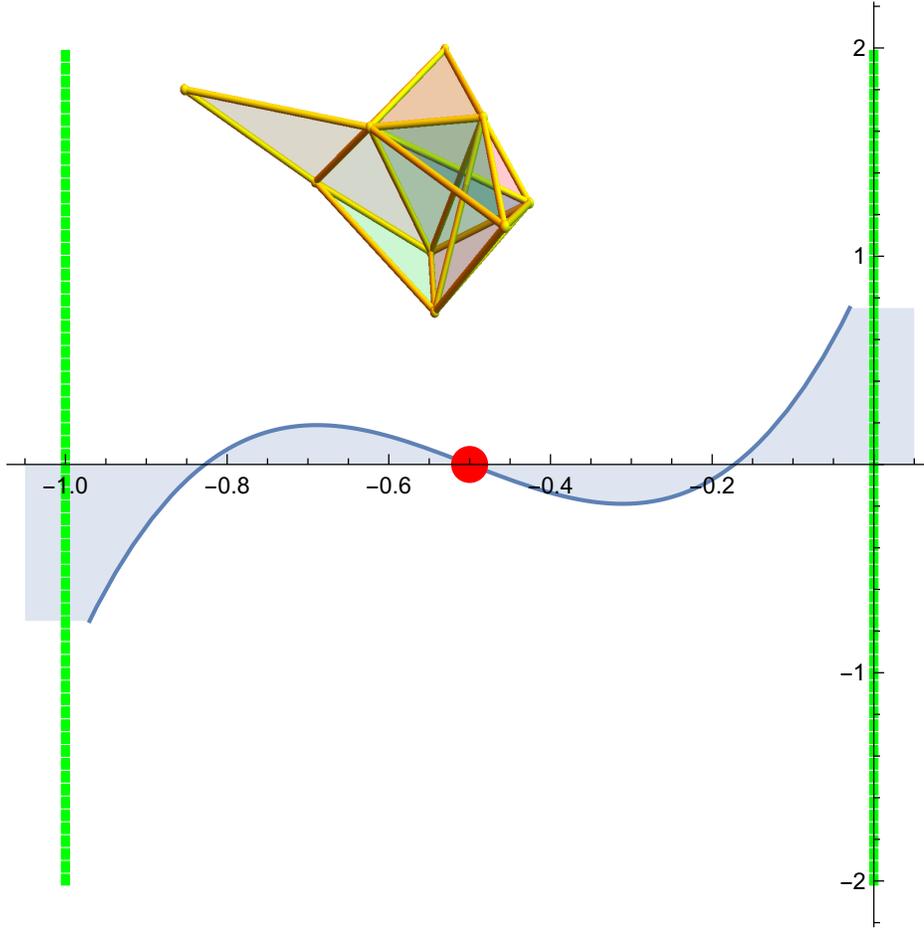


FIGURE 10. We see a graph obtained by poking around randomly in Erdos-Renyi spaces looking for Dehn-Sommerville graphs. This example has the simplex generating function  $f_G(t) = 1 + 9t + 21t^2 + 14t^3$  but it is not a 2-sphere. It is not a manifold but has the Betti numbers  $(b_0, b_1, b_2) = (1, 0, 1)$  of the 2-sphere. The graph has inductive dimension 2. It is interesting for us because it is an example which is not Dehn-Sommerville flat. It shows that there are Dehn-Sommerville complexes for which some unit spheres are not Dehn-Sommerville. The complex is **Dehn-Sommerville non-flat**. In other words, having zero Dehn-Sommerville curvatures (unit spheres are Dehn-Sommerville) is only sufficient and not necessary for  $G$  to be Dehn-Sommerville.

## 8. CODE

8.1. Here is Mathematica code (see ArXiv version to copy paste) which computes  $F_{S(x)}(t)$ , then adds up to  $f_G(t)$ .

```

UnitSphere[s_ , a_]:=Module[{b=NeighborhoodGraph[s , a] ,
  If[Length[VertexList[b]]<2,Graph[{}],VertexDelete[b,a]]];
UnitSpheres[s_]:=Module[{v=VertexList[s]} ,
  Table[UnitSphere[s , v[[k]]] , {k,Length[v]}];
ErdoesRenyi[M_ , p_]:=Module[{q,e,a},V=Range[M];
  e=EdgeRules[CompleteGraph[M]]; q={};
  Do[If[Random[]<p,q=Append[q,e[[j]]]] , {j,Length[e]}];
  UndirectedGraph[Graph[V,q]];
CliqueNumber[s_]:=Length[First[FindClique[s]]];
ListCliques[s_ , k_]:=Module[{n,t,m,u,r,V,W,U,l={},L},L=Length;
  VL=VertexList;EL=EdgeList;V=VL[s];W=EL[s];m=L[W];n=L[V];
  r=Subsets[V,{k,k}];U=Table[{W[[j,1]],W[[j,2]]} , {j,L[W]}];
  If[k==1,l=V,If[k==2,l=U,Do[t=Subgraph[s,r[[j]]];
  If[L[EL[t]]==k(k-1)/2,l=Append[l,VL[t]]] , {j,L[r]}]];l];
Whitney[s_]:=Module[{F,a,u,v,d,V,LC,L=Length},V=VertexList[s];
  d=If[L[V]==0,-1,CliqueNumber[s]];LC=ListCliques;
  If[d>=0,a[x_]:=Table[{x[[k]]} , {k,L[x]}];
  F[t_ , l_]:=If[l==1,a[LC[t,1]] , If[l==0,{} ,LC[t,1]]];
  u=Delete[Union[Table[F[s,1] , {1,0,d}]] , 1]; v={};
  Do[Do[v=Append[v,u[[m,1]]] , {1,L[u[[m]]]}] , {m,L[u]}] , v={};v];
Fvector[s_]:=Delete[BinCounts[Map[Length,Whitney[s]]] , 1];
Ffunction[s_ , x_]:=Module[{f=Fvector[s] , n},n=Length[f];
  If[Length[VertexList[s]]==0,1,1+Sum[f[[k]]*x^k , {k,n}]];
DehnSommerville[s_]:=Module[{f},Clear[x];f=Ffunction[s,x];
  Simplify[f] == Simplify[(f /. x->-1-x)];
Curvature[s_ , x_]:=Module[{g=Ffunction[s,y]} ,
  Integrate[g , {y,0,x}]];
EulerChi[s_]:=Module[{f=Fvector[s]} ,
  -Sum[f[[k]](-1)^k , {k,Length[f]}];
Curvatures[s_ , x_]:=Module[{S=UnitSpheres[s]} ,
  Table[Curvature[S[[k]] , x] , {k,Length[S]}];

s=ErdoesRenyi[16 , 0.4];
{Ffunction[s,x] , Curvatures[s,x]}
{Ffunction[s , x] , 1+Total[Curvatures[s , x]]}
{EulerChi[s] , -Total[Curvatures[s,x]] /. x->-1}

threesphere=UndirectedGraph[Graph[{1->3,1->4,1->5,1->6,1->7,
1->8,3->2,3->5,3->6,3->7,3->8,4->2,4->5,4->6,4->7,4->8,5->2,
5->7,5->8,6->2,6->7,6->8,7->2,8->2}]];
Print[DehnSommerville[threesphere]];
Print[DehnSommerville[StarGraph[10]]];
Print[DehnSommerville[WheelGraph[10]]];
Print[DehnSommerville[CompleteGraph[{3 , 3}]]];
Print[DehnSommerville[CycleGraph[5]]];
    
```

## 8.2. And here are the Barycentric invariants

```

A[n_]:=Table[StirlingS2[j,i]*i!,{i,n+1},{j,n+1}];
Invariants[n_]:=Eigenvectors[Transpose[A[n]]];
MatrixForm[Transpose[Reverse[Invariants[4]]]]

Fcrosspolytop[n_]:=Delete[CoefficientList[(1+2x)^(n+1),x],1]
Binvariants=Invariants[4];f=Fcrosspolytop[4];(*4-sphere*)
Print[f];
Print[MatrixForm[Binvariants]];
Binvariants[[2]].f
Binvariants[[4]].f

```

## 9. QUESTIONS

**9.1.** (A) One open question is: for which complexes are there non-real roots of  $f$ ? We measure that for all simplicial  $G$ , the roots of  $f_{G_n}$  for Barycentric refinements  $G_n$  of  $G = G_0$  are real and contained in the open interval  $(-1, 0)$  if  $n$  is large enough. This is a still unsolved concrete calculus problem as it requires to find roots of explicitly given polynomials defined by Perron-Frobenius eigenvectors of the Barycentric refinement operator.

**9.2.** We have never seen that non-real roots from  $f$  appear after a few Barycentric refinements. Non-real roots can appear for Whitney complexes of random graphs, for non-Whitney complexes like **homology spheres**  $G$  like the one with  $f = 1 + 16t + 106t^2 + 180t^3 + 90t^4$  or the **Barnette 3-sphere** with  $f = 1 + 8t + 27t^2 + 38t^3 + 19t^4$  or then the boundary sphere of a simplex like the **tetrahedral sphere** with  $f = 1 + 4t + 6t^2 + 4t^3$  or sphere-CW-complexes like the **cube**  $f = 1 + 8t + 12t^2 + 6t^3$ , the roots of  $f$  can become complex.

**9.3.** (B) An other question is to see how large the set of Dehn-Sommerville complexes are if we look at all graphs with  $n$  vertices. For  $n = 1, 2, 3$  there are none, for  $n = 4$ , there is only the cyclic graph, for  $n = 5$ , we have only cyclic  $C_5$  of  $C_4$  with a hair. When fishing randomly in the pool of Erdős-Rényi graphs, we come up empty in general. The probability of having a Dehn-Sommerville complex must be very small. One can wonder whether the probability is exponentially small in  $n$  or super exponentially small in  $n$ .

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