AN ELEMENTARY DYADIC RIEMANN HYPOTHESIS

OLIVER KNILL

ABSTRACT. The connection zeta function of a finite abstract simplicial complex G is defined as $\zeta_L(s) = \sum_{x \in G} \lambda_x^{-s}$, where λ_x are the eigenvalues of the connection Laplacian L defined by $L(x, y) = 1$ if x and y intersect and 0 else. (I) As a consequence of the spectral formula $\chi(G) = \sum_x (-1)^{\dim(x)} = p(G) - n(G)$, where $p(G)$ is the number of positive eigenvalues and $n(G)$ is the number of negative eigenvalues of L, both the Euler characteristic $\chi(G) = \zeta(0) - 2i\zeta'(0)/\pi$ as well as determinant $\det(L) = e^{\zeta'(0)/\pi}$ can be written in terms of ζ. (II) As a consequence of the generalized Cauchy-Binet formula for the coefficients of the characteristic polynomials of a product of matrices we show that for every one-dimensional simplicial complex G, the functional equation $\zeta_{L^2}(s) = \zeta_{L^2}(-s)$ holds, where $\zeta_{L^2}(s)$ is the Zeta function of the positive definite squared connection operator L^2 of G. Equivalently, the spectrum σ of the integer matrix L^2 for a 1-dimensional complex always satisfies the symmetry $\sigma = 1/\sigma$ and the characteristic polynomial of L^2 is palindromic. The functional equation extends to products of one-dimensional complexes. (III) Explicit expressions for the spectrum of circular connection Laplacian lead to an explicit entire zeta function in the Barycentric limit. The situation is simpler than in the Hodge Laplacian $H = D^2$ case [20], where no functional equation was available. In the connection Laplacian case, the limiting zeta function is a generalized hypergeometric function which for an integer s is given by an elliptic integral over the real elliptic curve $w^2 = (1+z)(1-z)(z^2 - 4z - 1)$, which has the analytic involutive symmetry $(z, w) \to (1/z, w/z^2)$.

1. INTRODUCTION

1.1. Zeta functions bind various parts of mathematics. They can be defined dynamically, spectrally, geometrically or arithmetically. In a dynamic setup, one considers prime periodic points of a dynamical system like closed geodesic loops or automorphisms. In an analytic

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setup, the zeta function is defined from eigenvalues of an operator, usually the Laplacian. In arithmetic cases, one considers Dedekind zeta functions of an algebraic number field and more generally at L-functions.

1.2. The simplest zeta functions are elementary in the sense that they are given by finite sums and defined by a self-adjoint integer valued matrix. In the context of simplicial complexes, the Bowen-Lanford zeta functions $\zeta_{BL}(s) = 1/\det(1 - sA)$ appeared, where A is an adjacency matrix. It is a rational function, and a special case of an Artin-Mazur-Ruelle zeta function, where the system is a subshift of finite type defined by A. From any selfadjoint matrix A one can also define a spectral zeta function. In the case of a positive definite matrix with eigenvalues λ_k, this is an entire function $\sum_k \exp(a_k s)$ with real $a_k = \log(\lambda_k)$. Both cases are relevant when we look at the connection Laplacian L of a finite abstract simplicial complex. We know there that $\zeta_{BL}(-1) = (-1)^{f(G)}$, where $f(G)$ is the number of odd dimensional simplices in G [16]. For zeta functions on graphs, see [32]. In the present paper however, we look at the spectral zeta function of a complex G. It determines the Euler characteristic and in certain cases the cohomology of the geometry. We then focus on mostly on the 1-dimensional case, where we have more symmetry, in particular a functional equation.

1.3. The archetype of all zeta functions is the **Riemann zeta function** $\sum_{n=1}^{\infty} n^{-s}$. It is associated to the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ and its Dirac operator $D = i\partial_x$ with spectrum $\mathbb{Z}$ as $De^{inx} = -ne^{inx}$. The spectral picture is normalized by taking $\zeta_{D^2}(s/2)$ and disregarding the zero eigenvalue. The relation to the arithmetic of rational primes is given by the Euler formula $\prod_p (1 - p^{-s})^{-1}$ so that $\zeta(s)$ is the Dedekind zeta function of the field of rational numbers $\mathbb{Q}$. Riemann looked at the Chebyshev function $\sum_{n \leq x} \Lambda(n)$ where Λ is the Mangoldt function and gave the Riemann-Mangoldt formula $\psi(x) = x - \sum_w \frac{x^w}{w} - \log(2\pi) - \frac{1}{2}\log(1 - x^{-2})$, where the sum is taken over all non-trivial roots w and the remaining expressions are due to the single pole respectively the trivial zeros $-2, -4, -6, \ldots$. Pairing complex conjugated non-trivial roots $w_j = a_k + ib_j = |w_j|e^{i\alpha_j}, \overline{w}$ gives $f_j(x) = e^{\log(x)a_j} 2\cos(\log(x)b_j - \alpha_j)/|a_j + ib_j|$, functions playing the tunes of the music of the primes. As the Riemann hypothesis is widely considered an important open problem in mathematics, the topic has been exposited in many places [4, 5, 7, 8, 28, 27, 34, 33, 25].

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1.4. When replacing the circle $\mathbb{T}$ with a finite circular graph C_n, Zeta functions become entire or rational functions. In the case of spectral zeta functions we have entire functions. Unlike for the Riemann zeta function, no analytic continuation is required. The Barycentric limit C_{2^n} is not the compact topological group $\mathbb{T}$ but the profinite compact topological group $\mathbb{D}_2$ of dyadic integers. We will see that for all 1-dimensional complexes with connection Laplacian L, the functional equation $\zeta(s) = \zeta(-s)$ holds for $\zeta(s) = \zeta_{L^2}(s/2)$. In the Barycentric limit, we then have still an entire zeta function.

1.5. While for circular graphs C_n, the roots of $\zeta(s)$ are symmetric to the imaginary axes, in the Barycentric limit, all eigenvalues appear on the imaginary axes. We have an explicit limiting function but currently still lack a reference which confirms that all roots are on the imaginary axes. Unlike in the circle case, the distribution of the roots in the dyadic case is very regular. This could be related to the fact that the Pontryagin dual of the dyadic group of integers $\mathbb{D}_2$ (which is a subgroup of the dyadic numbers $\mathbb{R}_2$ is the Prüfer group $\mathbb{P}_2 = \mathbb{R}_2/\mathbb{D}_2$ which is a divisible Abelian group (meaning $n\mathbb{P}_2 = \mathbb{P}_2$ for every integer $n \geq 1$), while the Pontryagin dual $\mathbb{Z}$ of the circle group $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ is not divisible.

1.6. The dyadic analogy is to pair up the spaces as follows: $\mathbb{R} \leftrightarrow \mathbb{R}_2$, $\mathbb{Z} \leftrightarrow \mathbb{D}_2$ and $\mathbb{T} = \mathbb{R}/\mathbb{Z} \leftrightarrow \mathbb{P}_2 = \mathbb{R}_2/\mathbb{D}_2$. The Riemann zeta function belongs to the non-discrete compact topological group $\mathbb{T}$ with non-divisible dual group $\mathbb{Z}$ which features interesting primes while the Dyadic zeta function belongs to the discrete compact topological group $\mathbb{D}_2$ with divisible dual group $\mathbb{P}_2$ harbors no interesting primes.

1.7. While we deal here with a spectral zeta function, also the Bowen-Lanford Zeta function $\zeta(z) = 1/\det(1 - zA)$ of a finite matrix plays a role for simplicial complexes. Using $N_n = \text{tr}(A^n)$, counting periodic paths, one can write $-\log(\det(1 - zA)) = -\text{tr}(\log(1 - zA)) = \sum_{n=1}^{\infty} z^n N_n/n$. If A is the adjacency matrix of a graph, then N_n is the number of periodic points of size n of the subshift of finite type, the natural Markov process defined by the graph. A summation over all possible prime orbits x of length $|x|$ gives $\zeta(z) = \prod_{x \text{ prime}}(1 - z^{|x|})^{-1}$. With $p = e^{|x|}$ and $s = -\log(z)$, this is Ihara type zeta function $\prod_{x \text{ prime}}(1 - p(x)^{-s})^{-1}$ looks a bit like the Riemann zeta function $\prod_p(1 - p^{-s})^{-1}$.

1.8. A continuum limit for finite zeta functions on simplicial complexes can be obtained by taking Barycentric limits $\lim_{n \to \infty} G_n$ of the complex $G = G_0$. The Barycentric refinement G_{n+1} of a complex G_n is

the Whitney complex of the graph, where G_n is the set of vertices and two vertices are connected, if one is contained in the other. It leads to a limiting almost periodic Laplacian on the compact topological group $\mathbb{D}_2$ of dyadic integers. For the connection Laplacian, the limiting Laplacian has a "mass gap". In the dyadic case, the Laplacian remains nice and invertible even in the continuum limit. The Barycentric limits in the higher dimensional case $\dim(G) \geq 2$ are not well understood, but if we take products [19] of 1-dimensional Barycentric limits, we get higher dimensional models of spaces for which the physics is trivial in the sense that all Green function values $L^{-1}(x,y)$ are still bounded. Also, the functional equation extends to products of 1-dimensional simplicial complexes. And also in the Barycentric limit, we have $\zeta_{G \times H}(s) = \zeta_G(s)\zeta_H(s)$ because $(\sum_k \lambda_k^{-s})(\sum_l \mu_k^{-s}) = \sum_{k,l}(\lambda_k \mu_l)^{-s}$ in the finite dimensional case.

1.9. Similarly as with the limit Hodge Laplacian, where the Barycentric zero-locus is supported on the line $\text{Re}(s) = 1$, we see here that the roots of the limit connection Laplacian are supported on the imaginary axes $\text{Re}(s) = 0$. The functional equation definitely simplifies the analysis. Whatever Laplacian we have in mind (like the Hodge Laplacian, the connection Laplacians or adjacency matrices), the limiting density of states is universal and only dimension dependent [14, 15]. While this universality holds in any dimension, only the 1-dimensional case, it appears to be "integrable" in the sense that the expressions of the density of states are explicit. In higher dimensions we see density of states measures which have gaps and possibly singular parts. Also the roots of the zeta function do not appear to obey a linear functional equation any more. We will explore in a further paper that there are truly higher dimensional cases (not only products of one dimensional cases), where the operator L can be deformed to have a functional equation as in one dimension.

1.10. When looking at finite dimensional approximations of a limiting case, we have to define convergence. The measure supported by the roots of the zeta function converges in the following sense: take a compact set K and put a Dirac measure 1 on each root of the Zeta function to get a measure μ_n for each Barycentric refinement G_n of G. We will show that μ_n restricted to K converges weakly to a limiting measure μ on K. As we know the function f which has μ as the zero-locus measure, we expect the measure μ to sit on the imaginary axes $\text{Re}(z) = 0$. As we have shown in the Hodge paper [20], every accumulation point lies there on the axes $\text{Re}(s) = 1$. This corresponds to $\text{Re}(s) = 1/2$ for

the Dirac case. Some relation of the finite dimensional and Riemann zeta case has been pointed out in [9].

1.11. Spectral zeta function are defined for any geometric object with a Laplacian; in particular they are defined for manifolds or simplicial complexes. An inverse spectral problem is to read off the cohomology of the geometry from ζ. We have seen that the spectrum of L does not determine the cohomology, even for 1-dimensional complexes [18]. We will see however in future work that the cohomology can easily be read of from the spectrum of 1-dimensional complexes which are Barycentric refinements. The connection is very direct: the eigenvalues $\lambda = 1$ correspond to eigenfunctions supported on a zero-dimensional connected component of the complex: as a Barycentric refinement of a 1-dimensional complex is bipartite, we can take a 2-coloring and take a coloring of the vertices with with 1 and -1. This is an eigenfunction as every edge intersects both a value 1 and -1 canceling. The eigenvalues -1 correspond to eigenfunctions supported on the 1-dimensional parts of the complex. A basis for the eigenspace of $\lambda = -1$ can be given by picking generators of the fundamental group of the complex and color the edges of that loop alternatively with 1 or -1. In some sense, the connection case entangles the zero and one form eigenvalues nicely.

2. Related literature

2.1. The spectral zeta function considered here is geometrically defined. It is completely unrelated to a zeta function defined in [31] where for any finite field like $F = \mathbb{F}_p$, denoting the absolute value of a "number" $n \in F[t]$ is $|F|^{\deg(n)}$, the zeta function is $\zeta(s) = \sum_{n \in F[t]_1} |n|^{-s}$ where $F[t]_1$ is the set of monomials in $F[t]$. There is then an explicit formula $\zeta(s) = 1/(1 - p^{s-1})$.

2.2. Also completely unrelated is a zeta function for a simplicial complex G with zero-dimensional components $\{1, \ldots, n\}$, defined in [3]. Given a finite field $F = F_{q^k}$ one can look at the projective $(n-1)$-space P over F and let for $x \in G$ the set $V(x, F_{q^k})$ of points in P which are supported in x. As a union of linear subspaces, this is a projective variety. The zeta function is then $Z_G(t) = \exp(\sum_{k \geq 1} |V(x, F_{q^k})| t^k / k)$. One of their results is that $Z_G(t) = \prod_{j=0}^{d} (1 - q^j t)^{-f_j}$, where d is the dimension of G and $f_j = \chi(G_j)$, where G_j is the j'th co-skeleton, the Whitney complex of the graph with vertex set $\{x \in G | \dim(x) \geq j\}$ in which two elements are connected if one is contained in the other.

2.3. In [9] (apparently unaware of [20] which was never submitted to a journal), the Hodge spectral function $\zeta_n(s) = 4^{-s} \sum_{k=1}^{n} \sin^{2s}(\pi k/n)$ of circular graphs, (which is $\zeta_n(2s)$ in (2) of [20]) and $\zeta_Z(s) = \frac{4^{-s}\Gamma\left(\frac{1}{2}-s\right)}{\sqrt{\pi}\Gamma(1-s)}$ called there spectral function of the Z are considered. It is the function $c(2s)$ on page 6 of [20]. They then look at the entire completion $\xi_Z(s) = 2^s \cos(\pi s/2)\zeta_z(s/2)$ and note (Theorem 0.2 in [9]) that it satisfies the functional equation $\xi_Z(s) = \xi_Z(1-s)$. Their Theorem 0.3 is close to Theorem 10 and Proposition 9 in [20]. Also some integer values are computed in both papers.

2.4. In [9] is a nice statement which shows that ζ_n has some relation with the Riemann zeta function: introducing

$$h_n(s) = (4\pi)^{s/2}\Gamma(s/2)n^{-s}(\zeta_n(s/2) - n\zeta_Z(s/2)$$

the authors there show first that that

$$\lim_{n\to\infty} |h_n(1-s)/h_n(s)| = 1$$

for every s in the critical strip $0 < Re(s) < 1$ for which $\zeta(s) \neq 0$. Then they show that the statement holds for all s in the critical strip, if and only the Riemann hypothesis is true. This is hardly a prospective path to prove the Riemann hypothesis because the condition essentially just probes in a fancy way whether $\zeta(s)$ is zero or not. It is still very interesting as it relates the Riemann hypothesis with an error in concrete Riemann sums of a concrete function parametrized by a complex parameter s.

3. The Morse index

3.1. Any selfadjoint matrix A defines a spectral zeta function. If λ_j are the non-zero eigenvalues of an invertible matrix A then its spectral Zeta function is defined as

$$\zeta(s) = \sum_k \lambda_k^{-s} .$$

It satisfies $\zeta'(0) = -\sum_j \log(\lambda_j) = -\log(\prod_j \lambda_j) = -\log(\det(A))$. It is this relation which allows to regularize determinants in geometric settings where the usual determinant does not make sense. This Minakshisundaram - Pleijel approach is used in differential geometry for Laplacians of a manifold M. In all the spectral settings related to ζ functions, one disregards the zero eigenvalues, and deal so with pseudo determinants. Zeta functions was one motivation to study them better in [13].

3.2. Like determinants or traces, also the pseudo determinant is one of the coefficients of the characteristic polynomial $p(x) = \det(A - xI)$. Actually, in some sense, the characteristic polynomial is already a zeta function as $p(A, z) = (-z)^n/\zeta(1/z)$ for $\zeta(z) = 1/\det(1 - zA)$. The roots of the characteristic polynomial are the reciprocals of the poles of ζ. The analogy puts the zeros of a zeta function in the vicinity of the importance of eigenvalues of a matrix.

3.3. If $M = \mathbb{T}$ is the circle then the spectral zeta function is the classical Riemann zeta function and $\zeta'(0) = -\log(2\pi)/2$. The Dirac operator id/dx has the regularized determinant $\det(D) = \sqrt{2\pi}$ and its square $H = D^2$ satisfies $\det(H) = \det(D^2) = \det(D)^2 = 2\pi$. The determinant of the Laplacian $H = -d^2/dx^2$ on the circle is the circumference of the circle. The zeta regularization shows that one can make sense of the infinite diverging product $\prod_n n^2$. The classical Riemann example illustrates also how to wash away the ambiguities coming from negative eigenvalues by taking $\zeta_{A^2}(s/2)$ rather than $\zeta_A(s)$. The example also illustrates the need to transition from determinants to pseudo-determinants. Riemann however did not look at the zeta function as a spectral zeta function but as a tool to investigate primes.

3.4. A finite abstract simplicial complex G is a finite set of non-empty sets which is closed under the process of taking finite non-empty subsets. The connection Laplacian L of a finite abstract simplicial complex G is the $n \times n$ matrix L which satisfies $L(x, y) = 1$ if two simplices $x, y \in G$ intersect and where $L(x, y) = 0$ otherwise. For the zeta function ζ_L of the connection Laplacian L, we have

Proposition 1. $\zeta'_L(0) = -i\pi n(G)$, where $n(G)$ is the number of negative eigenvalues of L.

Proof. The unimodularity theorem $|\det(L)| = 1$ implies $\text{Re}(\log(L)) = 0$ [16]. Now $\text{trIm}(\log(L)) = i\pi(n(G))$, where $n(G)$ is the number of negative eigenvalues of L. The fact that $n(G)$ is the number of odd-dimensional simplices was proven in [18]. $\square$

Corollary 1 (Euler from Zeta). $\chi(G) = \zeta(0) - 2i\zeta'(0)/\pi$

Proof. We have $\zeta(0) = p(G) + n(G)$ and $i\zeta'(0)/\pi = n(G)$ so that $\chi(G) = p(G) - n(G) = \zeta(0) - 2i\zeta'(0)/\pi$. $\square$

3.5. If we think of L as a Hessian matrix, then the number of negative eigenvalues of L is a Morse index of L. We have now, collecting the already established equality of odd-dimensional simplices and negative eigenvalues:

Proposition 2. *The Morse index $n(G)$ of L is equal to the number of odd-dimensional simplices in G. The Poincaré-Hopf type index $(-1)^{n(G)}$ satisfies $(-1)^{n(G)} = \exp(-i\zeta'_L(0)) = \det(L(G))$.*

Proof. The fact that $(-1)^{n(G)} = \det(L(G))$ is the unimodularity theorem. □

3.6. There is an obvious ambiguity when defining a zeta function of a general self-adjoint matrix: if λ is a negative number, then $\lambda^{-s} = e^{-s\log(\lambda)}$ requires to chose a branch of the logarithm. If λ is real and positive, there is no ambiguity. But even if we chose a definite branch for negative λ, the corresponding zeta function is not so nice. This is illustrated in Figure (1) where the pictures show the case of a matrix with positive and negative spectrum and to the right the zeta function of the square. To remove any ambiguities, we can look at

$$\zeta(s) = \zeta_{A^2}(s/2)$$

as the normalization of the spectral zeta function of a self-adjoint matrix A. In this paper, we compute in all pictures with $\zeta_{A^2}(s)$, skipping the factor 2.

3.7. The normalization contains less information because we lose information which eigenvalues are negative and positive. The analytic properties are nicer however for the normalization. As we have just seen, we can recover the Euler characteristic of a simplicial complex G from ζ_L and the trace of L but we have not figured out yet whether it is possible to recover $\chi(G)$ from the regularized $\zeta_{L^2}(s/2)$ rather than from $\zeta(L)$ as in Corollary (1). The normalization is actually something quite familiar: the traditional Riemann zeta function is a normalization in this respect already as it ignores the negative eigenvalues of the selfadjoint Dirac operator $D = i\partial_x$ on the circle. The Zeta function $\zeta_{D^2}(s/2)$ is then the sum $\sum_{n\geq 1} n^{-s}$ of Riemann.

3.8. For $G = K_3 = \{\{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$, the eigenvalues of L are $\sigma(L) = \{5.511,\ 1.618,\ 1.618, -0.7525,\ -0.618,\ -0.618,\ 0.241\}$ so that

$$\zeta_L(s) = (-0.7525)^{-s} + 2(-0.618)^{-s} + 0.241^{-s} + 2(1.618^{-s}) + 5.511^{-s}$$

satisfying $\zeta(0) = \text{tr}(I) = 7$ and $\zeta'(0) = -3\pi i$. The normalized $\zeta_{L^2}(s)$ has $\zeta(1) = \text{tr}(L^2) = 25$. But now, $\zeta'(0) = 0$ and $\det(L^2) = 1$. Unlike in the 1-dimensional case, the spectral symmetry fails, $\zeta_{L^2}(-1) = 37$. Because of unimodularity, the zeta values are integers for all integer s. In this case $\{\zeta(-3), \ldots, \zeta(3)\}$, it is the list $\{28063, 937, 37, 7, 25, 313, 5131\}$.
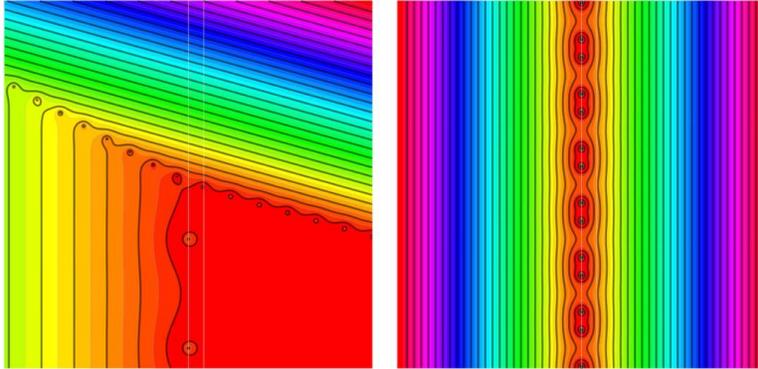
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FIGURE 1. To the left we see the level curves of $|\zeta_L(s)| = |1 + (1 - \sqrt{2})^{-s} + (1 + \sqrt{2})^{-s}|$ for $G = K_2$. On the right is the normalized case $\zeta_{L^2}(s) = 1 + a^{-s} + a^s$ with $a = (1 + \sqrt{2})^2$. For simplicity, we don't divide s by 2.
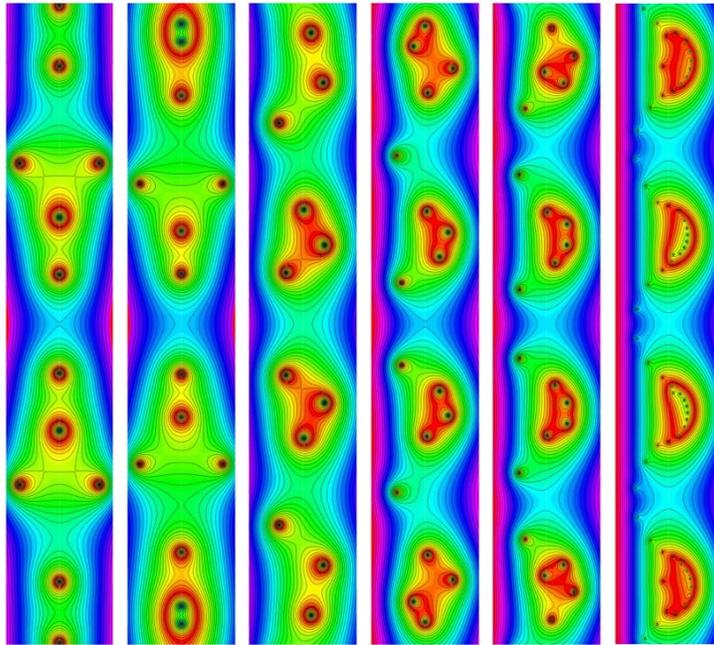

FIGURE 2. The level curves of $|\zeta_{L^2}(s)|$ for C_4, figure eight $F_8 = C_4 \dot\cup C_4$ and K_3, K_4, K_5 and K_{12} on $\{s = x + iy \mid |x| < 1, |y| < 6\}$. The first two cases show the spectral symmetry $\zeta(s) = \zeta(-s)$.

4. THE PRODUCT CASE

4.1. If $G \times H$ is the product of two simplicial complexes G, H in the strong ring, then we know that the spectra of G and H multiply. This immediately implies that the zeta functions multiply. This holds in full generality for all simplicial complexes G, H:

Corollary 2. *If $\zeta_G(s)$ and $\zeta_H(s)$ are the zeta functions of two simplicial complexes G and H, then $\zeta_{G \times H}(s) = \zeta_G(s)\zeta_H(s)$.*

In the subring of the strong ring generated by 1-dimensional simplices, one has

Corollary 3. *An arbitrary product of 1-dimensional simplicial complexes satisfies the functional equation.*

If we think of an element of the strong ring as a "number", then the spectral symmetry applies to the subring generated by 1-dimensional "primes". We turn to the symmetry.

5. THE FUNCTIONAL EQUATION

5.1. A finite abstract simplicial complex G is 1-dimensional, written $\dim(G) = 1$, if its clique number is 2. This means that all $x \in G$ have cardinality $|x| \leq 2$ but that there exists at least one simplex x with $|x| = 2$. Any such complex is the 1-skeleton complex of a finite simple graph. (Most graph theory books view graphs as 1-dimensional simplicial complexes. We like to use graphs also in higher dimensional cases as the Whitney complex is a natural complex.) The next theorem therefore can be seen as a result in old-fashioned graph theory which looks at graphs as 1-dimensional simplicial complexes. It will be proven in the next section:

Theorem 1 (Functional equation). *For a 1-dimensional complex, we have $\zeta_{L^2}(-s) = \zeta_{L^2}(s)$.*

5.2. It follows from the definition of the zeta function $\zeta(s) = \sum_k \lambda_k^{-s}$ that this statement is equivalent to:

Corollary 4 (Spectral symmetry). *The spectrum of the square L^2 satisfies $\sigma(L^2) = \sigma(L^{-2})$.*

5.3. Using the sign notation of (1), these two statements are again equivalent to

Corollary 5 (Palindromic characteristic polynomial). *If p_k are the coefficients of the characteristic polynomial of L^2 belonging to a 1-dimensional complex, then $p_{n-k} = p_k$.*

5.4. The statements can be rephrased also that the union of eigenvalues of the multiplicative group generated by G in the ring of simplicial complexes produce a finitely generated Abelian group. We can look at the rank of this group hoping to get a combinatorial invariant.

5.5. As a comparison, we know that the Dirac operator $D = d + d^*$ defined by incidence matrices d has a spectrum with an additive symmetry $\sigma(D) = -\sigma(D)$. But this is hardly a good analogy because the additive symmetry for D is true for all simplicial complexes, while the functional equation symmetry we are currently look at only holds in the 1-dimensional case. Because both the connection matrix L as well as the Dirac operator D have both positive and negative spectrum, we like to compare L and D and see L^2 as an analog of the Hodge operator $H = D^2$. There are more and more indications which support this point of view like the exciting formula $H = L - L^{-1}$ (which holds for a suitable basis) we will cover elsewhere. We called the operator $L - L^{-1}$ the Hydrogen operator [17].

5.6. As before, we proceed inductively by building up the complex as a CW-complex. First start with a zero- dimensional complex which is built by cells attached to -1-dimensional spheres, the empty complexes. As the dimension of G to 1, we now only need to understand what happens if we add a 1-dimensional edge. As mentioned above, the spectral symmetry is equivalent to a palindromic characteristic polynomial in which we use the notation

$$(1) \quad p(x) = \det(A - xI) = p_0(-x)^n + \cdots + p_k(-x)^{n-k} + \cdots + p_n \ ,$$

so that $p_0 = 1, p_1 = \operatorname{tr}(A)$ and $p_n = \det(A)$ and where the largest non-zero element p_k is the pseudo determinant of A. The validity of the functional equation is equivalent to establishing the palindromic symmetry $p_k = p_{n-k}$ for $A = L^2$.

5.7. Palindromic polynomials are also called self-reciprocal. They are of some importance in other parts of mathematics. Here are some connections: they play a role in coding theory [11], symplectic geometry or knot theory. A palindromic polynomial is the characteristic polynomial of a symplectic matrix. A result of Seifert assures that a polynomial is the Alexander polynomial of some knot if and only if it is monic and reciprocal and $p(1) = \pm 1$ [30]. Even degree monic integer polynomials are exactly the symplectic polynomials [24]. Since $(1 + x)$ always is a factor if the degree is odd, we can always write $p(x) = x^m q(x + 1/x)$ or $p(x) = (1 + x)x^m q(x + 1/x)$ for a polynomial q of degree m. This illustrates a general fact that one can compute the roots of palindromic
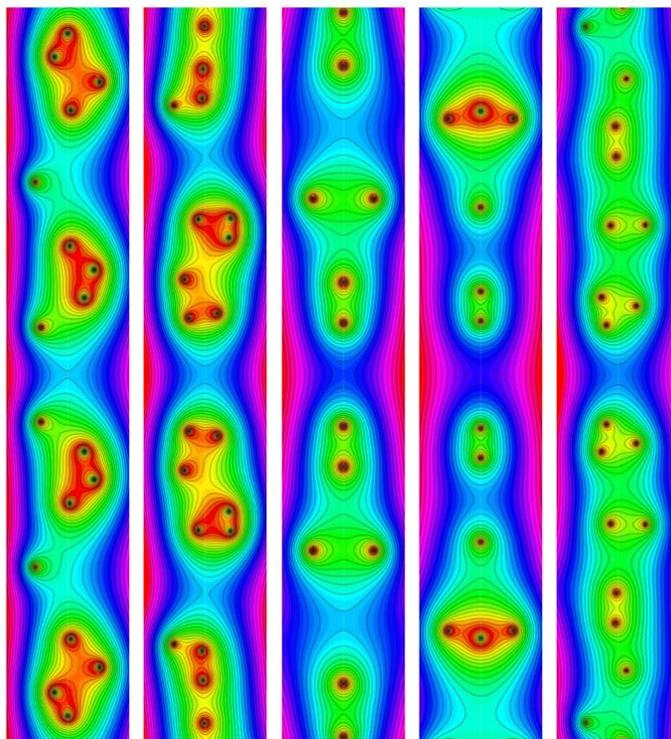
FIGURE 3. The zeta function of the 5 platonic solids Tetrahedron, Octahedron, Cube, Dodecahedron and Icosahedron (equipped with the Whitney complex). The cube and Dodecahedron are both 1-dimensional (as the graphs have no triangles) and so satisfy the spectral symmetry.

polynomials up to degree 9 with explicit formulas [23]. Also every minimal polynomial of a Salem number is palindromic [30].

5.8. The complex $G = \{\{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{1,4\}, \{2,3\}, \{3,4\}\}$ belongs to the circular graph C_4. The characteristic polynomial of L^2 is $p(x) = x^8 - 32x^7 + 316x^6 - 1248x^5 + 1926x^4 - 1248x^3 + 316x^2 - 32x + 1$. The quartic polynomial $q(x) = x^4 - 32x^3 + 312x^2 - 1152x + 1296$ has the property that $p(x) = x^4 q(x + 1/x)$. This illustrates that one can compute the roots of a palindromic polynomials up to degree 9. [23]

5.9. An amusing thing happens for cyclic polynomials p of $G_n = C_{2^n}$: the polynomial of $G_{n+1} = C_{2^{(n+1)}}$ has the polynomial p_n of C_{2^n} as a factor. Also q_{n+1}/q_n is a square of an irreducible polynomial of degree 2^n. This means that the eigenvectors and eigenvalues of G_n have an
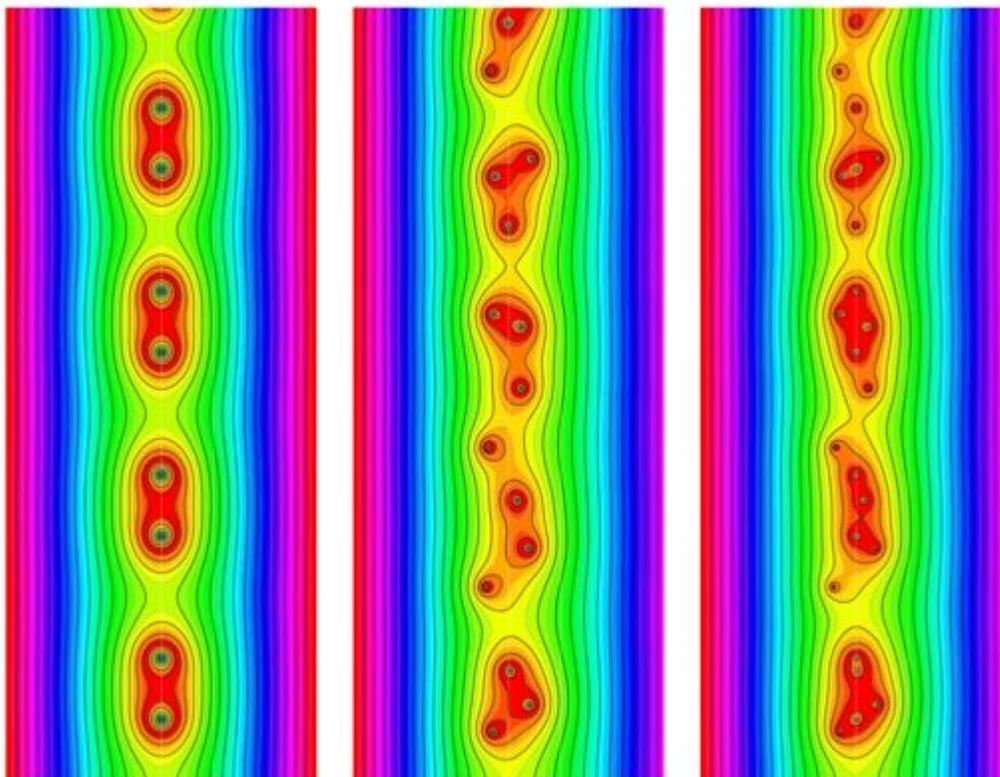
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FIGURE 4. The figure shows the contour plots of the zeta functions of the complete complexes $G = K_2$ and $H = K_3$ as well as of the product $G \times H$. The product is a 3-dimensional CW complex. The roots of the $\zeta_{G \times H}$ is exactly the union of the roots of ζ_G and ζ_H.

incarnation as eigenvalues and eigenvectors in the Barycentric refinement G_{n+1}. But this happens only for circular graphs, not for general 1-dimensional complexes.

5.10. It would be nice to have other classes where one can get limiting statements. One class to consider are the complete graphs K_n which are $n - 1$-dimensional complexes with $|G| = 2^n - 1$ simplices. We observe that $L(K_n)$ has only $2n - 1$ different eigenvalues. This could explain the periodic clumping along curves of the roots of the corresponding zeta functions. We also observe experimentally that the coefficients of the characteristic polynomial of $L^2(K_n)$ is asymptotically nice and
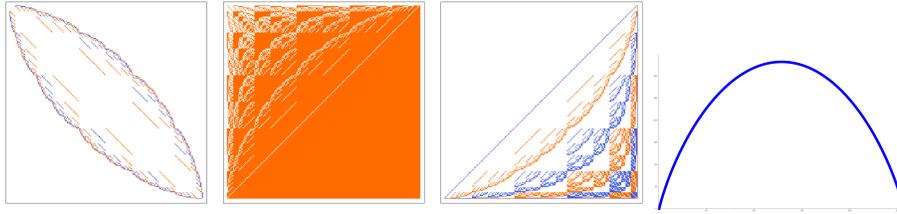
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FIGURE 5. The Dirac matrix D, the connection Laplacian L and its inverse L^{-1} for $G = K_8$ are $(2^8-1) \times (2^8-1)$ matrices. The last graph shows the coefficient list $k \to \log |p_k(L^2(K_8))|$ of the characteristic polynomial of L^2.

smooth. The case of the 1-dimensional skeleton of K_n is illustrated at the end of the article.

6. CIRCULAR GRAPHS

6.1. Like in the Hodge case, also for connection operators everything is explicit if the complex is a circular graph. The eigenvalues λ_j of the connection Laplacian L are obtained from the eigenvalues μ_j of the Hodge Laplacian H by a multiplicative symmetrization $\lambda_j - 1/\lambda_j = \mu_j$:

Lemma 1. *The connection Laplacian L of a circular graph C_n is a $2n \times 2n$ matrix L which has the eigenvalues $\lambda_k = f^\pm(\mu_k)$, where $\mu_k(k/n)$ with $\mu(x) = f^\pm(4\cos^2(\pi x))$ and $f^\pm(x) = (x \pm \sqrt{x^2 + 4})/2$.*

Proof. The matrix $M = L - L^{-1}$ has the eigenvalues $\mu_k = \lambda_k - \lambda_k^{-1}$. If the simplices of the complex are ordered as $G = \{\{1\}, \{1,2\}, \{2\}, \ldots, \{n-1,n\}, \{n\}, \{n,1\}\}$, then $M = 2 + Q + Q^*$, where $Qu(n) = u(n+2)$ is shift by 2. Fourier theory gives the eigenvalues $2 + 2\cos(2\pi k/n) = 4\cos^2(\pi k/n)$. $\square$

6.2. As in the Hodge case, the eigenvalues of M are $\{4\sin^2(\pi k/n)\}_{k=1}^n$ but appear with algebraic multiplicity 2 at least each. In the Hodge case, the limiting density of states is the equilibrium measure on the real line segment $I = [0, 4]$ in the complex plane. We can see the density of states of the connection Laplacian as the pull back of the map $T(z) = z - 1/z$. It is the equilibrium measure on the union of two intervals $J = [1, 2 + \sqrt{5}]$ and $[-1, -2 + \sqrt{5}]$. The map T maps the intervals J onto the single interval I.

6.3. The list of eigenvalues of L is the same when taking $4\sin^2(\pi k/n)$ rather than $4\cos^2(\pi k/n)$. This is more convenient as it relates directly to the Hodge case: where the eigenvalues were $\lambda_k = g(k/n)$ with $g(x) =$

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$4\sin^2(\pi x)$. We see that the map $T(x) = x + 1/x$ maps the density of states measure on $\sigma(L)$ to the density of states measure of $\sigma(H)$.

7. Pythagoras

7.1. Given two arbitrary $m \times n$ matrices F, G, the coefficient p_k of the characteristic polynomial of the $n \times n$ matrix $F^T G$ satisfies the identity

$$p_k = \sum_P \det(F_P G_P) ,$$

where F_P is a sub-matrix of F in which only rows from $I \subset \{1, \cdots, n\}$ and columns from $J \subset \{1, \cdots, n\}$ are taken. This generalized Cauchy-Binet result was first found and proven in [13] and is the first formula for the characteristic polynomial of a product of arbitrary two matrices. The classical Cauchy-Binet theorem is the special case for p_k when $k = \min(m, n)$, where $\det(F^T G) = \sum_{|P|=k} \det(F_P)\det(G_P)$. In the even more special case $k = n = m$, it is the familiar product formula for determinants. A consequence of the generalized Cauchy-Binet result [13] is:

Lemma 2 (Generalized Pythagoras). *Given a matrix L, then*

$$p_k(L^T L) = \sum_{|P|=k} \det((L^T)_P)\det(L_P) .$$

7.2. Note that $(L^T)_P$ is not the same than $(L_P)^T$. The formula for the coefficients p_k implies for a self-adjoint matrix L, that the coefficients p_k of the characteristic polynomial of L^2 satisfies some Pythagorean identities. It implies also the formula $p_k(A) = \text{tr}(\Lambda^k A)$ [29]. In the even more special case $k = 1$, where $p_1(A)$ is the trace, one has then the Hilbert-Schmidt identity $\text{tr}(L^T L) = \sum L_{ij}^2$ is a version of Pythagoras justifying the name "generalized Pythagoras".

7.3. In [18] we have looked at the deformation

$$\tilde{K}(t) = \begin{bmatrix} L_{11} & L_{12} & . & . & . & L_{1n} & tL_{1,x} \\ L_{21} & L_{22} & . & . & . & . & tL_{2,x} \\ . & . & . & . & . & . & tL_{3,x} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ L_{n1} & . & . & . & . & L_{nn} & tL_{n,x} \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} .$$

We look here at the more symmetric deformation

$$
K(t) = \begin{bmatrix}
L_{11} & L_{12} & . & . & . & L_{1n} & tL_{1,x} \\
L_{21} & L_{22} & . & . & . & . & tL_{2,x} \\
. & . & . & . & . & . & tL_{3,x} \\
. & . & . & . & . & . & . \\
. & . & . & . & . & . & . \\
. & . & . & . & . & . & . \\
L_{n1} & . & . & . & . & L_{nn} & tL_{n,x} \\
tL_{x1} & tL_{x2} & \ldots & \ldots & \ldots & tL_{xn} & 1
\end{bmatrix}.
$$

It interpolates $L = K(0)$, the connection Laplacian of the complex G with $K(1)$, the connection Laplacian of the complex $G \cup_A \{x\}$ in which a new cell x is attached to G along A. As for $\tilde{K}(t)$ we have $\det(K(t)) = \det(K)(1 - 2t^2)$ which is consistent with the fact that adding an odd-dimensional simplex changes the sign of the determinant of the connection matrix L, a fact which is true for all simplicial complexes.

Lemma 3 (Coefficients of characteristic polynomial). *The coefficients $q_k(t)$ of the characteristic polynomial of $K_{ij}(\sqrt{t})$ are all linear t. The coefficients $p_k(t)$ of the characteristic polynomial of $K_{ij}(t)^2$ are of maximal degree 4 in t.*

Proof. This follows from the Pythagoras formula and the fact that every minor $\det(K_P(t))$ is linear or quadratic in t. The reason for the later is that every 1-dimensional path the directed sub-graph defined by P can enter or leave x only once. It also can be seen directly when doing the Laplace expansion of the determinant with respect to the last column. Therefore, each term $\det(K_P(t))^2$ is a mostly quartic function in t and only coefficients $c_1 + c_2 t^2 + c_4 t^4$ appear. $\square$

Now there are lots of such functions starting with the value 0 at $t = 0$, it appears like a miracle that all these functions have again a root at $t = 1$. But this is exactly what happens in the 1-dimensional case. This will be explained in the next section.

8. PROOF OF THE FUNCTIONAL EQUATION

8.1. In order to show the spectral symmetry it suffices to prove the palindromic relation

$$
p_k = \sum_{|P|=k} \det(L_P)^2 = \sum_{|P|=n-k} \det(L_P)^2 = p_{n-k} \, .
$$

Every pattern $P = I \times J$ of size k corresponds to a set of directed 1-dimensional closed directed sub-graphs of G. The formula for p_k is

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then a "path integral" over all directed paths of length k. Using the complementary pattern $\overline{P} = \overline{I} \times \overline{J}$ satisfying $I \cup \overline{I} = J \cup \overline{J} = \{1, \ldots, n\}$, and $I \cap \overline{I} = J \cap \overline{J} = \emptyset$, we can write the claim also as

$$\delta_k(t) = \sum_{|P|=k} \det(L_P)^2 - \det(L_{\overline{P}})^2 \ .$$

What happens if we deform the operator L to an augmented one using the operator $K(t)$? While the palindrome difference deformation function

$$\delta_k(t) = p_k(K(t)) - p_{n-k}(K(t))$$

is not constant zero for $t \in [0, 1]$, we can show that $\delta_k(t)$ is zero for $t = 1$ again, proving thus the palindromic property.

Proposition 3 (Artillery Proposition). *If G is a 1-dimensional simplicial complex and a new edge $e = (a, b)$ is added to G, leading to the deformation operator $K(t)$, then each palindrome difference function satisfies*

$$\delta_k(t) = C_k t^2 (1 - t^2) \ ,$$

where C_k is an integer.

8.2. The connection Laplacian is $L = I + A$, where I is the identity matrix and A is the adjacency matrix of the connection graph G' of G, the graph which has the simplices of G as vertices and where two vertices $x, y \in G$ are connected if they intersect. As used in [16], the determinant of L, the Fredholm determinant of A can be interpreted as a signed sum over all possible 1-dimensional oriented closed subgraphs (unions of closed oriented circular paths), where fixed points are counted as closed paths of length 1 and closed loops $a \to b \to a$ along an edge $e = (a, b)$ are counted as closed paths of length 2. The sign of each path is according to whether it has even or odd length. This is in accordance to the Leibniz definition of determinants; every oriented closed path corresponds to permutation in the determinant and the decomposition into different connectivity components exhibit the cycle structure of the permutation. Figures (14)-(17) in [16] illustrate this geometric interpretation for concrete graphs, where all oriented closed subgraphs are drawn.

8.3. The coefficient p_k of the characteristic polynomial of L^2 also has a path integral interpretation. It depends on the pattern $P = I \times J$ however, whether paths in it can be closed or not. If $I = J$, then every connected component in the path is closed. If $I \cap J = \emptyset$, then no closed path components appear. The formula $p_k(L^2) = \sum_{|P|=k} \det(L_P)^2$ shows that we have to look at all paths of length k now. The length of a connected

path component $\{v_1, v_2, \ldots, v_m\}$ is is m if $v_1 \neq v_m$ and $k-1$ if $v_m = v_1$, a case where the path is closed. A vertex $\{v\}$ alone is considered a path of length 1 and a path $\{v_1, v_2, v_1\}$ is a path of length 2. The reason for these assumptions is it that the connection matrix L has 1 in the diagonal so that, if it is interpreted as an adjacency matrix, we are allowed to loop at a simplex, but then that path of length 1 is isolated. Paths contributing to a minor in $P = I \times J$ with $|P| = k$ are given by a collection of k tuples $\{v_1, \ldots, v_k\}$ with k different elements v_j where transitions $v_i \in I \to v_{i+1} \in J$ or $v_k \in I \to v_1 \in J$ are allowed. We can think of I as the exit set and J the entry set. This allows especially transitions $v_i \to v_i$ if $i \in I \cap J$.

8.4. There are three type of patterns P:

Class A) $P = I \times J$ satisfies $e \notin I, e \notin J$. It represents paths of length k which never enter, nor leave e.

$\overline{A}$) Patterns $\overline{P}$, where P is in A).

Class B) $P = I \times J$ has the property that exactly one of the subset I, J contains e. These paths add to the t^2 contributions in the polynomial $p_k(t)$.

$\overline{B}$) Patterns $\overline{P}$ where P is in B).

Class C) $P = I \times J$ has the property that $e \in I, e \in J$. This produces the t^4 contributions in $p_k(t)$.

$\overline{C}$) Patterns $\overline{P}$, where P is in C).

8.5. Now we can use induction with respect to the number of edges. There are two things to show: the constant part is zero and the t^2 and t^4 contributions have opposite sign. The induction assumption is the case when G is zero dimensional with no edge. But then, K is the identity matrix and the coefficients of the characteristic polynomial are Binomial coefficients which are palindromic. The induction foundation is established.

(i) All the constant parts go away: the paths not hitting the edge e in class A) and class $\overline{A}$ cancel. We have to show that the contribution of patters $|P| = k$ which $e \notin P$ is equal to the contribution of $|\overline{P}| = k$

never hiting e. We can formulate this as $A - \overline{A} = 0$.

Proof of i): If $P = I \times J$ has $e \notin I, e \notin J$, we can take e away and look at the complex G before adding e, where we know the statement already to hold.

8.6. (ii) To see the symmetry between t^2 and t^4 we have to show that the contribution of paths of length which either leave or only enter e minus the contribution of paths of length $n - k$ which either only leave or only enter e agrees with the difference of the number of paths of length k which leave and enter e, minus the contribution of paths of length $n - k$ which leave and enter e. We can formulate this as $B - \overline{B} = C - \overline{C}$.

Proof of ii): There exists an other edge f in G different from e. Otherwise, we are in the induction foundation case. We split now the sum

$$\sum_{|P|=k} \det(L_P)^2 - \det(L_{\overline{P}})^2$$

into three parts. Look first at all patterns $P = I \times J$, where $f \in I, f \in J$. For those patterns the t^2 and t^4 contributions are the same by induction: it is the sum when the edge f has been taken away. Now look at all the patterns $P = I \times J$, where $f \notin I, f \notin J$. But this is the same sum as before just with a negative sign, and it is therefore again symmetric in t^2 and t^4. Finally, look at

$$U = \sum_{|I \times J|=k, f \in I, f \notin J} \det(L_P)^2 - \det(L_{\overline{P}})^2$$

and

$$V = \sum_{|I \times J|=k, f \notin I, f \in J} \det(L_P)^2 - \det(L_{\overline{P}})^2$$

These two add up to zero $U + V = 0$ because of symmetry.

It follows that $p_k(L^2) = 0 + ct^2 - ct^4$ for some constant c, which is the claim of the artillery proposition.

8.7. Remarks:
a) Where did the assumption enter that G is 1-dimensional? It was in the induction part, where the one dimensionality of the complex mattered. When we add the first two-dimensional cell and omit one of the edges, then we don't deal with a simplicial complex any more.
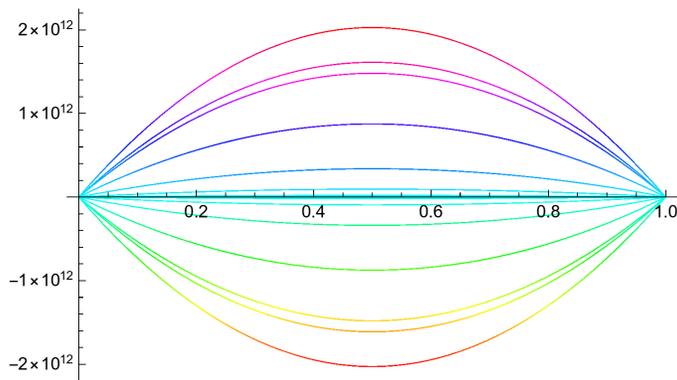**b)** We can look in general at the deformation of the Green function

FIGURE 6. We see the palindromic difference $\delta_k(t) = p_k(K(\sqrt{t})) - p_{n-k}(K(\sqrt{t}))$ of the coefficients of the characteristic polynomial of the deformation $K(t)$, when adding an other edge. The artillery proposition assures that all trajectories hit the ground simultaneously at $t = 1$. When taking time $\sqrt{t}$ we see parabolas.

entries $g_t(x, y) = K(t)^{-1}(x, y)$. Using the Cramer's formula, one can see that $g_t(x, y)(1-2t^2)$ is a quadratic polynomial in t for all x, y. Since the Euler characteristic changes by -1 as we add one single edge also the total energy $\sum_{x,y} g_t(x, y)$ changes by -1. In the one dimensional case, we have complete control about the Green functions and Green function changes. This will also allow us to see that $L - L^{-1}$ is the Hodge Laplacian.

8.8. While all $\delta_k(\sqrt{t})$ are quadratic functions and the induction assumption assures $\delta_k(0) = 0$ for all k, we have for a general complex (dropping the assumption that it is necessarily 1-dimensional), that the second root of $\delta_k(t)$ depends on k. For 1-dimensional complexes however, $\delta_k(1) = 0$ for all k assuring that also the enlarged complex keeps the palindromic property. For general complexes, the choreography falls out of sync. We need to deform the operator L to synchronize them again.

8.9. Example: take the complex $G = \{\{1\}, \{2\}, \{3\}, \{1, 2\}\}$ which is not connected and has a single vertex not connected to a K_2 component. We have $L = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$. Now, we add the new edge connecting the single vertex 3 to the rest. This gives the deformation

$$K(t) = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & t \\ 0 & 0 & 1 & 0 & t \\ 1 & 1 & 0 & 1 & t \\ 0 & t & t & t & 1 \end{bmatrix}.$$ Now, $K(1)$ is the connection Laplacian of

$H = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}\}$. The list of coefficients of the characteristic polynomial of $K(0)$ is the palindrome $(1, 9, 22, 22, 9, 1)$ while the list of coefficients of the characteristic polynomial of $K(1)$ is the palindrome $(1, 15, 49, 49, 15, 1)$.

8.10. To illustrate the computation of coefficients p_k, let us take the simpler example of the transition $G = \{\{1\}, \{2\}\}$ to $G_e = \{\{1\}, \{2\}, \{1, 2\}\}$, where $K(t) = \begin{bmatrix} 1 & 0 & t \\ 0 & 1 & t \\ t & t & 1 \end{bmatrix}$. We have $p_{K(t)}(x) = 1(-x)^3 + (3 + 4t^2)x^2 + (3 + 4t^2)(-x) + (1 - 2t^2)^2$ so that the coefficients deform as $p_0(t) = 1, p_1(t) = p_2(t) = 3 + 4t^2, p_3(t) = (1 - 2t^2)^2$. Now, as $n = 3$ we have $\delta_0(t) = p_0(t) - p_n(t) = 4t^2(1 - t^2), \delta_1(t) = p_1(t) - p_2(t) = 4t^2(1 - t^2)$. This illustrates the artillery proposition even so G was 0-dimensional. The patterns with $|P| = 1$ are the 1×1 submatrices

$$P \in \{[1], [0], [t], [0], [1], [t], [t], [t], [t]\}.$$

The sum of the squares of the determinants is $3 + 4t^2$. The patters with $|P| = 2$ are the (2×2)-submatrices

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & t \\ 0 & t \end{bmatrix}, \begin{bmatrix} 0 & t \\ 1 & t \end{bmatrix}, \right.$$
$$\begin{bmatrix} 1 & 0 \\ t & t \end{bmatrix}, \begin{bmatrix} 1 & t \\ t & 1 \end{bmatrix}, \begin{bmatrix} 0 & t \\ t & 1 \end{bmatrix},$$
$$\left. \begin{bmatrix} 0 & 1 \\ t & t \end{bmatrix}, \begin{bmatrix} 0 & t \\ t & 1 \end{bmatrix}, \begin{bmatrix} 1 & t \\ t & 1 \end{bmatrix} \right\}$$

of $K(t)$. It leads to the minors $1, t, -t, t, 1-t^2, -t^2, -t, -t^2, 1-t^2$ whose sum of the squares is $3 + 4t^4$. Finally, there is one matrix $K_P = K$ if $|P| = 3$ having the minor $\det(K) = 1 - 2t^2$ leading to $p_3(t)$.

9. CONVERGENCE

9.1. Given a 1-dimensional simplicial complex $G = G_0$, we can look at its n'th Barycentric refinement G_n. Let $\zeta_n(s)$ denote the normalized connection zeta function of G_n. That is $\zeta_n(s) = \sum_k \lambda_k^{-s}$, where λ_k are the eigenvalues of $L^2(G_n)$. Let $\mu(G_n)$ denote the zero locus measure of the zeta function ζ_n. It is a measure summing up the Dirac masses of the roots of ζ_n. We say that μ_n converges weakly to the imaginary

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axes, if for every compact set K away from the imaginary axes, we have $\mu_n(K) = 0$ for large enough n. In our case, the convergence is nicer as we have a definite entire limiting function $z(s)$.

Theorem 2 (The limiting roots). *For every 1-dimensional complex G, the roots of ζ_{G_n} converge weakly to the roots of an explicit limiting function $z(s)$. The functions $\zeta_{G_n}/|G_n|$ converge to the entire function $z(s)$ uniformly on compact subsets.*

9.2. For the analysis, it is enough to look at the circular case: for every 1-dimensional simplicial complex G there exists a finite integer $m(G)$ such that m cutting or gluing procedures produce a circular graph. 2) As shown in [14, 15], if a complex G is modified to a complex H by changing maximally m columns and m rows only, then the eigenvalues λ_j of G and eigenvalues μ_j of H satisfy $\sum_j |\lambda_j^2 - \mu_j^2| \leq Cm(G)$, where $m(G)$ depends on G only and C does not depend on the choice of the Laplacian (like $C = 4$ for the connection Laplacian of a 1-dimensional complex). For a constant $C(s)$ such that $|\sum_j |(\lambda_j^2)^s| - \sum_j |\mu_j^2)^s|| \leq C(s)m$. This follows from the Lidskii-Last inequality $||\mu - \lambda||_1 \leq \sum_{i,j=1}^n |A - B|_{ij}$ for the eigenvalues μ_j and λ_j of two arbitrary selfadjoint matrices A, B of the same size. This means $\zeta_{G_n}(s)/|G_n| - \zeta_{H_n}(s)/|H_n|$ goes to zero uniformly on compact sets for s.

9.3. For the circular case, one can give explicit expressions. Define the function

$$Z(x, s) = \left(f_+(4\sin^2(\pi x))^2\right)^s + \left(f_-(4\sin^2(\pi x))^2\right)^s ,$$

where $f_\pm(y)$ are the solutions x of the equation $x - 1/x = y$. The function $1 - 1/x$ maps the two intervals $[-1, 2 - \sqrt{5}]$ and $[1, 2 + \sqrt{5}]$ to the interval $[0, 4]$ and $f_\pm$ are the inverses. We have seen that the eigenvalues λ_k^s of $L^2(C_n)$ satisfy $\{\lambda_k^s + 1/\lambda_k^s = Z(k/n, s)\}_{k=1}^n$. The normalized Zeta function $\zeta(s) = \sum_{k=1}^n \lambda_k^s$ of C_n satisfies now

$$\zeta_{C_n}(s) = \sum_{k=1}^n Z(\frac{k}{n}, s) .$$

The roots of $\zeta_n(s)$ converge in the limit $n \to \infty$ to the roots of $z(s)$. Indeed, $\zeta_n(s)/n \to z(s)$.

9.4. We see that the zeta function is n times a Riemann sum for the integral
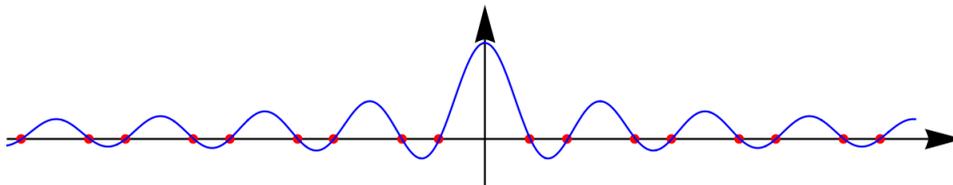
$$z(s) = \int_0^1 Z(x)^s \, dx .$$

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FIGURE 7. The limiting zeta function is here restricted to the imaginary axes so that we can plot the graph of $t \to z(it)$ on $t \in [-10, 10]$ and mark the roots in that interval.

We have seen in the Hodge case already that for smooth functions, the error $\sum_{k=1}^{n} f(k/n) - n \int_0^1 f(x) \, dx$ goes to zero. This implies that if $\int_0^1 f(x) \, dx$ is non-zero, then for large enough n the sum $\sum_{k=1}^{n} f(k/n)$ can not be zero.

9.5. Using a change of variables $v = \sin(\pi x)^2$ we can write

$$z(it) = \int_0^1 \text{Re} \frac{2 \left(\sqrt{4v^2 + 1} + 2v \right)^{2it}}{\pi \sqrt{1 - v} \sqrt{v}} \, dv$$

or in the form d) of theorem (3).

9.6. The limiting function $z(s)$ can be described better by turning the picture by 90 degrees so that all roots become real. The function $t \to z(it)$ is now a function which is real on the real axes. It is given in part c) of theorem (3). The roots of the limiting zeta function are the roots of this function.

9.7. Hypothesis: All the roots of $t \in \mathbb{C} \to w(t)$ are on the real line.

As this is an explicit generalized hypergeometric series, we don't expect this to be difficult. We have just not found any reference yet, nor an argument which assures that. Let us thus express the limiting function in various ways:

Theorem 3 (limiting zeta function). *We can rewrite the function in various ways:*
a) *Hypergeometric Series:* $\zeta(2s) = \pi \, _4F_3 \left(\frac{1}{4}, \frac{3}{4}, -s, s; \frac{1}{2}, \frac{1}{2}, 1; -4 \right).$

$$\zeta(2s, z) = \pi \sum_{n=0}^{\infty} \frac{(1/4)_n (3/4)_n (-s)_n, (s)_n}{(1/2)_n, (1/2)_n, (1)_n} \frac{z^n}{n!}$$

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analytically continued to $z = -4$.
b) *Fourier Transform: we have a cos-transform of a smooth measure on an interval*

$$\zeta(it) = \int_0^{\log(2+\sqrt{5})} \cos(ty) \sqrt{\frac{\cosh(y)\coth(y)}{2 - \sinh(y)}}\, dy \ .$$

c) *Original expression:*

$$\zeta(it) = \int_0^1 2\cos(t\log((\sqrt{4\sin^4(\pi x) + 1} + 2\sin^2(\pi x))^2))\, dx \ .$$

d) *Simplest integral*

$$\zeta(it) = \int_0^1 \frac{2\cos\left(2t\log\left(\sqrt{4v^2 + 1} + 2v\right)\right)}{\pi\sqrt{1-v}\sqrt{v}}\, dv$$

e) *Abelian integral: if $a = 2 + \sqrt{5}$, then*

$$\zeta(s) = \int_1^a \frac{(z^2 + 1)\, z^{-1-s}\, (1 + z^{2s})}{2\sqrt{(1+z)(1-z)(z-a)(z-1/a)}} = \int_1^a F(z)\, dz \ .$$

9.8.

Proof. The function $f(x) = \log(\sqrt{4\sin^4(\pi x) + 1} + 2\sin^2(\pi x))$ is a 1-periodic, non-negative function with roots at 0 and 1. It is monotone from 0 to $1/2$ For any smooth 1-periodic non-negative function f we can look at the complex function $w(z) = 2\int_0^{1/2}\cos(zf(x))\, dx$. For general periodic f, the transformed function has non-real roots but in our case we can make a substitution: $u = f(x)$ gives $x = -\arcsin(\sqrt{2\sinh(u)}/2)/\pi = g(u)$. The integral is now the cos-transform $(1/(2\pi))\int_0^{\log(2+\sqrt{5})}\cos(zu)h(u)\, du$, where h is the function

$$h(y) = \cosh(y)/\sqrt{(2 - \sinh(y))\sinh(y)}$$

on $I = [0, \log(2 + \sqrt{5})] \subset [0, \pi/2]$. We can rewrite as in part **b)** of the theorem. In other words, the limiting zeta function is 4π times the cos-transform of the probability measure μ with density $h(y)$ on I. It is not true that for a general probability measure μ on the real line with compact support, the cos-transform of μ is an entire function with only roots on the real axes.

9.9. Let $a = 2 + \sqrt{5}$ and $p(x) = (1 + z)(1 - z)(z - a)(z - 1/a)$. Then

$$\zeta(it) = \int_1^a \frac{(z^2 + 1)\, z^{-1-it}\, (1 + z^{2it})}{2\sqrt{p(z)}}\, dz = \int_1^a F_z(s)\, dz \ .$$

Because p is a polynomial of degree 4 it is a generalized Abelian integral. Technically, for every integer $s = it$, it can be expressed as an Abelian integral integrating over a closed loop on the real elliptic Riemann curve $w^2 = p(z)$ with $p(z) = (1 + z)(1 - z)(z^2 - 4z - 1)$. (Nowadays, elliptic curves are mostly written as cubic curves, in particular in Weierstrass form, but historically, the real quartic forms have prevailed (by Abel, Jacobi, Gauss and even Euler earlier [1]). There was a revival of quartic normal forms in particular in cryptology, like curves in Jacobi quartic form, where the addition formulas are particularly elegant). The polynomial p can be written as $p(z) = z^2((z - 1/z)^2 - 4(z - 1/z))$. The elliptic curve not only has the anti-analytic symmetry $(z, w) \to (\bar{z}, \bar{w})$ but also features the analytic involution $(z, w) \to (1/z, w/z^2)$. This involution will play a role in the future.

9.10. It follows that for every integer s, the value is an elliptic integral. For every rational $s \in \mathbb{Q}$, we express the value as a hyperelliptic integral for a higher degree polynomial. For general s, it is just the value of a generalized hyperelliptic function. The Abelian connection especially implies that $\zeta(it)$ is a period for every rational $t \in \mathbb{Q}$. This fact again contributes to the impression that the Dyadic story is "integrable" as Abelian integrals are everywhere in integrable dynamical systems. For the classical Riemann zeta function, we only know that the integers larger than 1 are periods. (The class of periods is rather mysterious as one does not even know an explicit example of a number (an explicit expression like $e^\pi + \sqrt{2}$) which is not a period evenso a cardinality argument shows that almost all are [22].)

9.11. Here is an other piece of information which came up when trying to exclude roots away from the axes of the hyperelliptic function. The logarithmic derivative of the function $F_z(s)$ simplifies to

$$\frac{F_z'(s)}{F_z(s)} = \log(z)(1 - \frac{2}{(1 + z^{2s})}) .$$

It would be nice to use some contour integral to exclude roots away from the imaginary axes. But the logarithmic derivative does not simplify any more nicely after the integral of F_z over z is performed.

9.12. As the polynomial $p(x)$ is palindromic, we can make a transformation $u = x + 1/x$. This gives

$$\sqrt{\frac{-(1 + z^2)^2}{1 + 4z - 2z^2 - 4z^3 + z^4}} \frac{1}{2z} dz = \frac{1}{\sqrt{4u - u^2}} du$$

and we get the integrals

$$\int_0^4 ((u+\sqrt{4+u^2})/2)^s/\sqrt{4u-u^2} \; du + \int_0^4 ((u-\sqrt{4+u^2})/2)^s/\sqrt{4u-u^2} \; du \; .$$

This gives

$$\zeta(s) = \pi \, _4F_3\left(\frac{1}{4}, \frac{3}{4}, -\frac{s}{2}, \frac{s}{2}; \frac{1}{2}, \frac{1}{2}, 1; -4\right).$$

We use the notation of [2, 21]. By definition, this is the (analytically continued value) of the series

$$\zeta(2s, z) = \sum_{n=0}^{\infty} \frac{(1/4)_n (3/4)_n (-s)_n (s)_n}{(1/2)_n (1/2)_n (1)_n} \frac{z^n}{n!}$$

for $z = -4$. It uses the Pochhammer notation $(a)_0 = 1, (a)_1 = a, (a)_n = a(a+1)\cdots(a+n-1)$, which is also called a raising factorial. Now, this is a sum of polynomials of the form $c_n(s)_n(-s)_n$ but the series is understood only by analytic continuation; for all hypergeometric functions of type $_4F_3$, the radius of convergence in z is 1. A hypergeometric function is by definition a sum $\sum_n c_n$, if c_{n+1}/c_n is a rational function in n [21]. $\qquad\square$

10. REMARKS

10.1. The Barycentric limit of a 1-dimensional complex is naturally the dyadic group $\mathbb{D}_2$ of integers. Similarly as the circle $\mathbb{T}$ is the Pontryagin dual of the integers $\mathbb{Z}$, the group $\mathbb{D}_2$ is the natural dual to the Prüfer group $\mathbb{P}_2$, the group of 2-adic rationals modulo 1. The analogy is to compare $\mathbb{D}_2$ with $\mathbb{T}$. Both are compact and a continuum, having a dual which is not-compact and countable. But $\mathbb{D}_2$ is quantized in the sense that it features a smallest translation. It is a world in which the Planck constant is "hard wired" in. The arithmetic in the dyadic case is simpler as there are no interesting primes in the dual group $\mathbb{P}_2$. It is no surprise therefore that the Zeta story is simpler. This realization also should squash any speculation that there is a direct bridge between the "dyadic zeta function" and the "rational zeta function". As [20] and especially [9] show in the Hodge Laplacian case, the finite dimensional cases can approximate the Riemann zeta function and the speed of approximation plays a role for the location of the roots.

10.2. One can get to the dyadic picture via ergodic theory. When performing a Barycentric refinements of random Jacobi matrix L defined over any measure preserving dynamical system (X, T, m), the renormalization map [12] allows to write $L = D^2 + c$ with an other Laplacian D defined over a renormalized system (Y, S, n). The renormalization
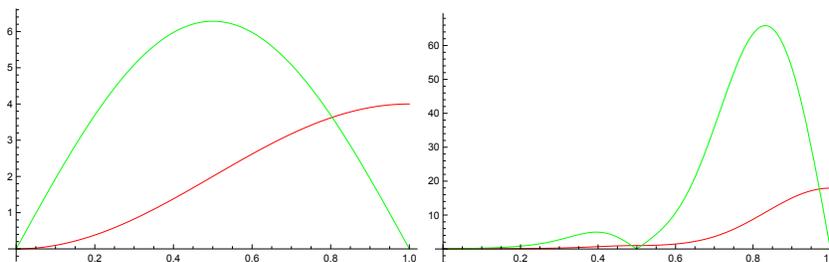
FIGURE 8. The limiting function $f(x) = \lambda_{[nx]}$ and its derivative in the Hodge case $H = D^2$ and then for the connection Laplacian case L^2. Both limiting functions are exactly known.

of the base dynamical system $T \to S$ is almost trivial as it is the 2:1 integral extension [10], where one takes two copies of the probability space and have S map one to the other and then use T to go back to the first. This map is a contraction in a complete metric space of measure preserving dynamical systems and has by the Banach fixed point theorem a unique fixed point T. This von-Neumann-Kakutani system T is also known under the name "adding machine". The ergodic theory, T is completely understood because the discrete spectrum in ergodic theory allows to rewrite the system as a group translation. The Koopman operator U on the Hilbert space $L^2(\mathbb{D}_2)$ has discrete spectrum which is the Prüfer group. By general facts in ergodic theory [6], the system is now measure theoretically equivalent to a group translation on $\mathbb{D}_2$. The limiting Hodge operators which are Jacobi matrices in finite dimensions have the spectrum on the Julia set of quadratic maps. See also [26, 26].

10.3. Both the Hodge Laplacian H as well as the connection Laplacian L have a Barycentric limit, an almost periodic operator on the group of dyadic integers. In both cases, the limiting operator is the same as long as we start with a 1-dimensional complex. The limiting connection operator is invertible. It has a "mass gap". The dyadic group of integers is compact but discrete. It is the dyadic analogue of the circle but unlike the circle $\mathbb{T}$ which has a continuum of translations, there is a smallest translation on $\mathbb{D}_2$. The limiting Riemann zeta function behaves differently than the Riemann zeta function of the circle and appears more approachable. In the Hodge case, we have only a vague intuition about the limiting distribution of roots.
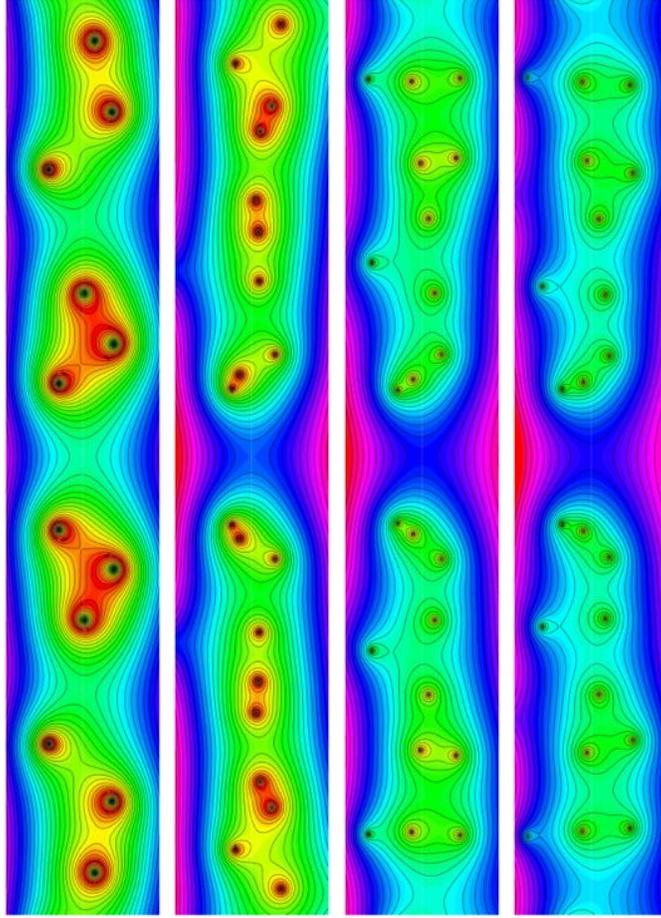
FIGURE 9. The spectrum of Barycentric refinements G_i of the triangle $G_0 = K_3$. We don't know the density of states of $L(G_i)$ in the Barycentric limit yet. We also don't know the limiting distribution of the zeros of $\zeta(G_i)$ but expect it to converge to a limiting measure in the complex plane.
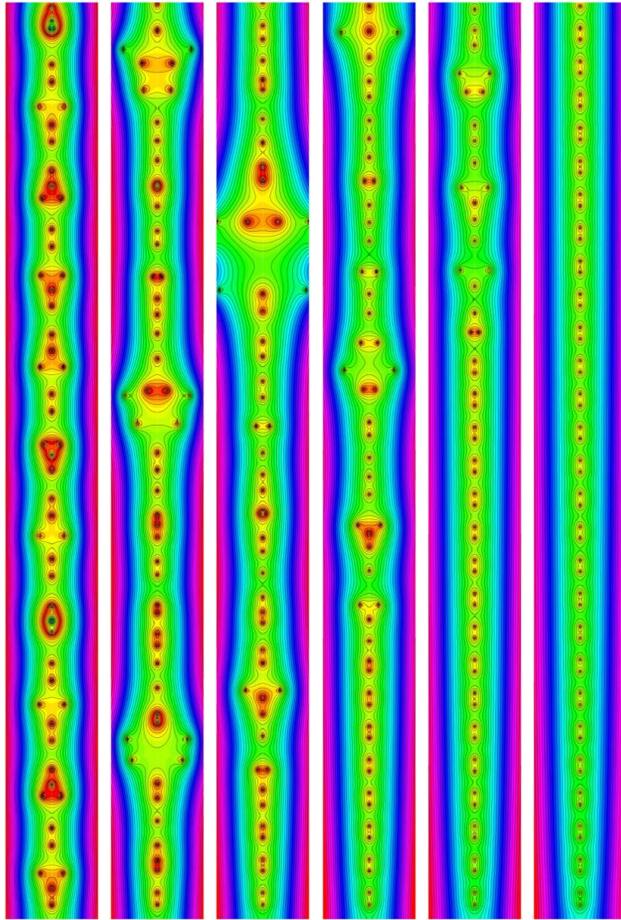
FIGURE 10. We see the level curves of $|\zeta_G(s)|$ for the first few Barycentric refinements of a discrete circle $G = C_4, C_8, C_{16}, C_{32}, C_{64}, C_{128}$. We see in each of the 6 cases the rectangle $[-3, 3] \times [0, 60]$. The roots converge more and more to the imaginary axes, the line of symmetry of the functional equation.

11. Questions

- We still need a reference or argument which assures that the entire function

$$\zeta(it) =_4 F_3 \left(\frac{1}{4}, \frac{3}{4}, -it, it; \frac{1}{2}, \frac{1}{2}, 1; -4 \right)$$

has all roots on the real axes. We tried to rewrite this function in various ways, but could place it in a class of functions which have real roots. Finite Pochhammer sums like $\sum_k a_k(s)_n(-s)_n$ or cos-transforms of measures on the real line can in general have non-real roots too. We tried contour integration and Sturmian criteria so far. Why was it possible in a similar Hodge case that roots are absent for $\mathrm{Re}(s) < 0$? The reason was that we had Riemann sums of an explicit function $g(s) = \int_0^1 4 \sin^s(\pi x) \, dx$ for which

$$g(s) = \frac{4\Gamma(\frac{1+s}{2})}{\sqrt{\pi}\Gamma(\frac{s}{2}+1)}$$

reveals that there are no roots because the Gamma function has no roots.

- We would like to have explicit expressions for the roots. We would also like to know in the case of each $G = C_n$, at which place the first pair of roots away on the imaginary axes appears. We have not investigated whether there are any circular graphs C_n for which all roots are on the imaginary axes. A "lock-in" situation, meaning the existence of an n_0 such that for all $n > n_0$, all roots are on the imaginary axes would be a big surprise but it is not excluded at this point. An other open question is whether there are higher dimensional complexes for which all roots of the zeta function are on the imaginary axes. We can give products of simplicial complexes for which that happens but these are not simplicial complexes any more.

- In which cases does the spectrum of L^2 determine the spectrum of L? We know that in general, it does not. How large can the multiplicity of eigenvalues become? We see that is rare that negative and positive eigenvalues cancel each other when adding the spectrum. We looked experimentally for cases where $\lambda_k + \lambda_l = 0$ and found only very few. So far, an example of a simplicial complex G with f-vector $(14, 36, 27, 9)$ was found, where two eigenvalues $\pm(1 + \sqrt{5})/2$ appear. The case of non-simple spectrum $\lambda_k = \lambda_l$ happened so far only for $\lambda = 1$ or

$\lambda = -1$, which for 1-dimensional complexes linked to the 0-dimensional and 1-dimensional cohomology.

- In general, given any simplicial complex G, the complex G generates a monoid in the strong ring for which the union of the spectrum is a multiplicative subgroup of R. As a finitely presented Abelian group, it has a Prüfer rank. What is this rank? Is this rank invariant under Barycentric refinements?

- There is more about meaning of the eigenvalues 1 or -1. Their multiplicity are important combinatorial invariants: if G is a Barycentric refinement of a 1-dimensional complex, then the eigenvalues 1 belong to the zero'th cohomology $H^0(G)$, the eigenvalues -1 belong to the first cohomology $H^1(G)$. The eigenfunctions to the eigenvalue 1 are supported on the zero-dimensional part, the eigenfunctions to the eigenvalue -1 are supported on the 1-dimensional part. We have seen earlier examples showing that one can not hear the cohomology. Yes, but these examples were not Barycentric refinements. Indeed, the eigenfunctions define a vertex or edge coloring and require the graph to be bipartite. Every Barycentric refinement of a 1-dimensional complex is trivially bipartite, the vertex dimension providing a 2-coloring.

- For two or higher dimensional simplicial complexes, we have no idea yet how to get the cohomology from the spectrum or zeta function or whether it is accessible from the roots in the Barycentric limit. Exploring this numerically is harder as the number of simplices explodes much faster than in the 1-dimensional case.

- A matrix L is called totally unimodular, if every invertible minor in L is unimodular, meaning $\det(L_P) \in \{-1, 0, 1\}$. An interesting question is in which cases the connection matrix L of a 1-dimensional complex is unimodular. It is already not for circular graphs but appears so for linear graphs.

- It would be nice to have an analogue statement relating the Riemann sum approximation $\zeta_n(s)/n - \zeta(s)$ asymptotics with the existence of roots of $\zeta(s)$ at a point s.

12. MATHEMATICA CODE

12.1. The code can be grabbed by looking at the LaTeX source on the ArXiv. Each of the blocks is self-contained. We first compute $\zeta(C_n)$ of a circular graph and compare with the explicit formulas for

the eigenvalues. Then we compare the zeta function value with the
limiting value.

```
(* A)    Explicit Connection spectrum in circular case          *)
Generate[A_]:=Delete[Union[Sort[Flatten[Map[Subsets,A],1]]],1];
M=26; A=Table[{k,Mod[k,M]+1},{k,M}]; G=Generate[A]; n=Length[G];
L=Table[If[DisjointQ[G[[k]],G[[l]]],0,1],{k,n},{l,n}];
EV=Sort[Eigenvalues[1.0*L]]; W=EV^2;Z[s_]:=Sum[W[[k]]^(-s/2),{k,n}]/n;
mu[k_]:=4 Sin[Pi k/M]^2; Q=Sort[Table[1.0 mu[k],{k,M}]];
f1[y_]:=(y + Sqrt[4 + y^2])/2; f2[y_]:=(y - Sqrt[4 + y^2])/2;
V=Sort[Flatten[{Map[f1,Q],Map[f2,Q]}]]; Print[V==EV];
X[t_]:=HypergeometricPFQ[{1/4,3/4,-t/2,t/2},{1/2,1/2,1},-4];
T=10*Random[]; Print[X[I T]-Z[I T]]    (* How close to limit? *)
```

Now, we first generate a random 1-dimensional complex, then check
the functional equation.

```
(* B)    Functional equation for one dimensional complexes       *)
Generate[A_]:=Delete[Union[Sort[Flatten[Map[Subsets,A],1]]],1];
sort[x_]:=Sort[{x[[1]],x[[2]]}]; Gr=RandomGraph[{9,20}];
A=Union[Map[sort,EdgeList[Gr]]]; G=Generate[A]; n=Length[G];
L=Table[If[DisjointQ[G[[k]],G[[l]]],0,1],{k,n},{l,n}];
V=Sort[Eigenvalues[1.0*L]];
Print[Total[Chop[Sort[V^2]]-Sort[1/V^2]]]
```

Here we collect various expressions for the limiting Zeta function which
were mentioned in the text.

```
(* C)    Rewriting the limiting Zeta Function                    *)
NI=NIntegrate; Clear[U,V,W,p,P,h,H,X,T];
U[t_]:=NI[Cos[t*Log[(2*Sin[Pi*x]^2+Sqrt[1+4*Sin[Pi*x]^4])]], {x,0,1}];
V[t_]:=NI[2*Cos[t*Log[2*u^2+Sqrt[1+4*u^4]]]/(Pi*Sqrt[1-u^2]),{u,0,1}];
W[t_]:=NI[2*Cos[t*Log[2*v+Sqrt[1+4*v^2]]]/(2Pi*Sqrt[v(1-v)]),{v,0,1}];
p[z_]:=(1+z)(1-z)(z^2-4z-1); a=1;b=2+Sqrt[5];
P[t_]:=Re[NI[(1+z^2)(z^(I*t)+z^(-I*t))/(2Pi*z*Sqrt[p[z]]), {z,a,b}]];
h[y_]:=Sqrt[Cosh[y] Coth[y]/(2-Sinh[y])];
H[t_]:=NI[Cos[t*y]*h[y],{y,0,Log[2+Sqrt[5]]}]/Pi;  (* Cos Transform *)
X[t_]:=Re[HypergeometricPFQ[{1/4,3/4,-I t/2,I t/2},{1/2,1/2,1},-4]];
T=10*Random[];   Print[{U[T],V[T],W[T],P[T],H[T],X[T]}];
```

For a general G we compute $\chi(G)$ and $\det(L)$ of L using the zeta func-
tion $\zeta(G)$. Also illustrated is the energy theorem $\chi(G) = \sum_x \sum_y g(x, y)$
where $g = L^{-1}$.

```
(* D)    Euler Chi and Det from Zeta                             *)
Generate[A_]:=Delete[Union[Sort[Flatten[Map[Subsets,A],1]]],1]
R[n_,m_]:=Module[{A={},X=Range[n],k},Do[k:=1+Random[Integer,n-1];
  A=Append[A,Union[RandomChoice[X,k]]],{m}]; Generate[A]];
G=R[10,12];n=Length[G]; Dim=Map[Length,G]-1;
Fvector=Delete[BinCounts[Dim],1]; Vol=Total[Fvector];
L=Table[If[DisjointQ[G[[k]],G[[l]]],0,1],{k,n},{l,n}];
Pos=Length[Position[Sign[Eigenvalues[1.0*L]],1]];
Bos=Length[Position[Flatten[Map[OddQ,Map[Length,G]]],True]];
Fer=n-Bos; Euler=Bos-Fer; Fred=Det[1.0*L]; Fermi=(-1)^Fer;
Energy=Round[Total[Flatten[Inverse[1.0 L]]]];
Checks={Energy,Fred,Pos,Energy==Euler,Fred==Fermi,Pos==Bos,Vol==n}
```

```
EV=Sort[Eigenvalues[1.0*L]]; zeta[s_]:=Sum[EV[[k]]^(-s),{k,n}];
a=Chop[ I zeta'[0]/Pi ]; Print[{Chop[a],Fer}];
Print[{Euler,Chop[zeta[0]-2 I zeta'[0]/Pi]}];
```

To illustrate the proof of the proposition we compute the minors of the deformation $K(t)$, then the coefficients of the characteristic polynomial p. The polynomial p is also computed directly. In both cases, the function $\delta_k(t)$ appearing in the artillery proposition is given.

```
(* E) Illustrating the artillery Proposition            *)
G={{1},{2},{3},{1,2},{2,3}}; n=Length[G];
L=Table[If[DisjointQ[G[[k]],G[[l]]],0,1],{k,n},{l,n}]; K=L;
Do[If[(k==n||l==n)&&k+l!=2n,K[[k, l]]=t K[[k, l]]],{k,n},{l,n}];
m=Table[Flatten[Minors[K,k]],{k,0,n}];
q=Table[m[[k+1]].m[[k+1]],{k,0,n}]; Phi=Simplify[q-Reverse[q]];
p=-CoefficientList[CharacteristicPolynomial[K.K,x] /. x->-x,x]
phi=Simplify[p-Reverse[p]]; Print[Phi-phi]; Print[phi];
```

Finally, let us look at the Riemann sum approximation and especially the Friedli-Karlsson function $h_n(s)$ [9]. We check numerically $h_n(1 - s)/h_n(s)$ goes to one. Equivalent to the Riemann hypothesis is that the convergence holds for all s in the critical strip.

```
(* F)    Riemann Sum in Hodge case                      *)

h[n_,s_]:=Module[{z,Z},
  z=2^(-s) Sum[1/Sin[Pi k/n]^s,{k,n-1}]; (* discrete zeta function *)
  Z=Gamma[1/2-s/2]/(2^s*Sqrt[Pi]*Gamma[1-s/2]); (* limiting zeta    *)
  (4 Pi)^(s/2) Gamma[s/2] n^(-s)*(z-n Z)];     (* asymptotic        *)
n=1000; s=Random[]+400 I Random[]; Print[Abs[h[n,1-s]/h[n,s]]];
```

In our case, with the connection Laplacian, the approximation is much better. While before, one had to compare the value at s with the value at $1 - s$, we have here total symmetry $s \to -s$ also in the finite case. It appears as if $|\zeta_n/n - \zeta|$ is incredibly good. The following experiment probes whether the error is of order $1/n^3$.

```
(* G)    Riemann Sum in connection case                  *)

H[n_,s_]:=Module[{m=2n,Z,z,q}, q[x_]:=Abs[x/2]^s;
Z=HypergeometricPFQ[{1/4,3/4,-s/2,s/2},{1/2,1/2,1},-4];   (* Dyadic*)
z=Sum[y=4Sin[Pi k/n]^2;u=Sqrt[4+y^2];q[y+u]+q[y-u],{k,n}];
Abs[m^2(z-m Z)]]; s=Random[]+10*I Random[]; Print[H[10000,s]];
```

The asymptotic error as a function of of n and s still needs to be investigated. Also interesting would be to know what happens in the random case, in the large n limit of some Erdös-Rényi spaces $E(n,p)$, where the Whitney complex of a graph with n vertices is taken in which each edge is turned on with probability p. The pictures for small n suggest that the zero-locus measure could converge to a continuous measure for $n \to \infty$.
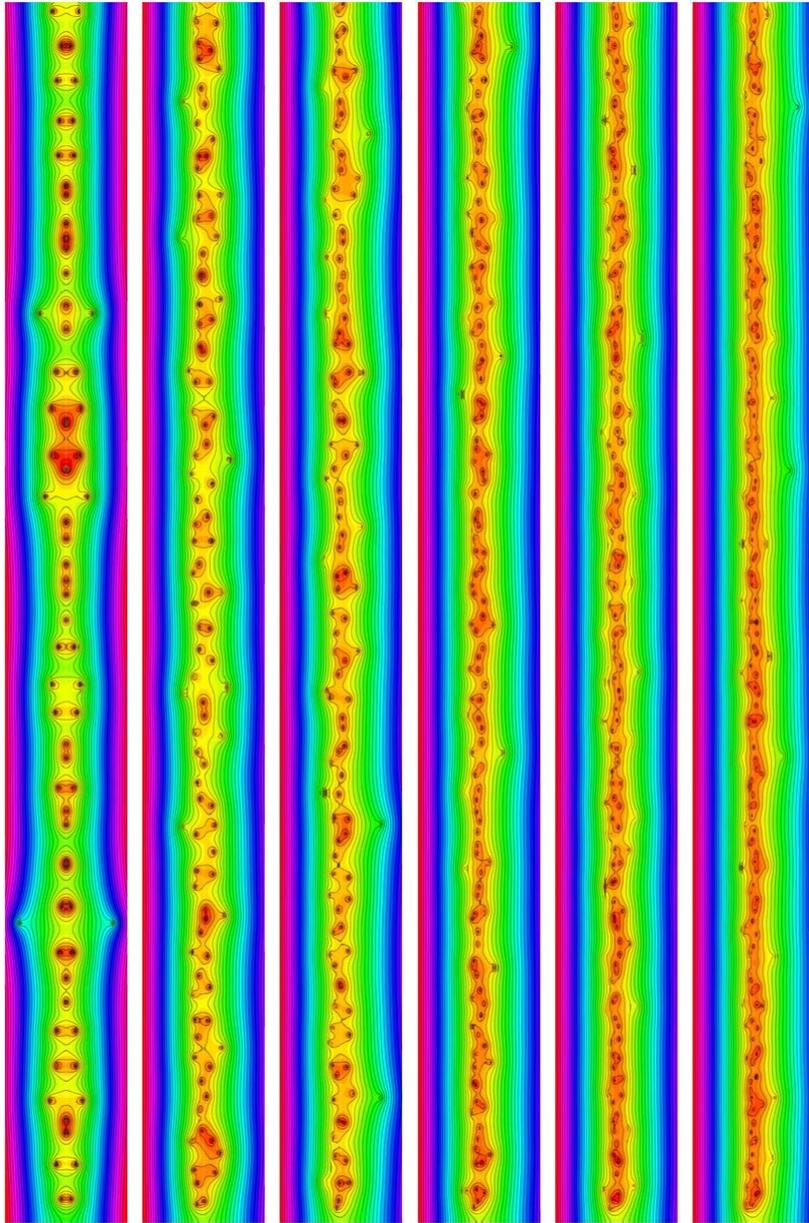
FIGURE 11. The level curves of $|\zeta_{L^2}|$ for the Whitney complexes of some Erdös-Rényi type graphs in $E(20, p)$, where the edge probability is $p = 0.1, 0.2, \ldots, 0.6$. In this case, the complexes had $70, 108, 169, 288$, and 600 simplices.
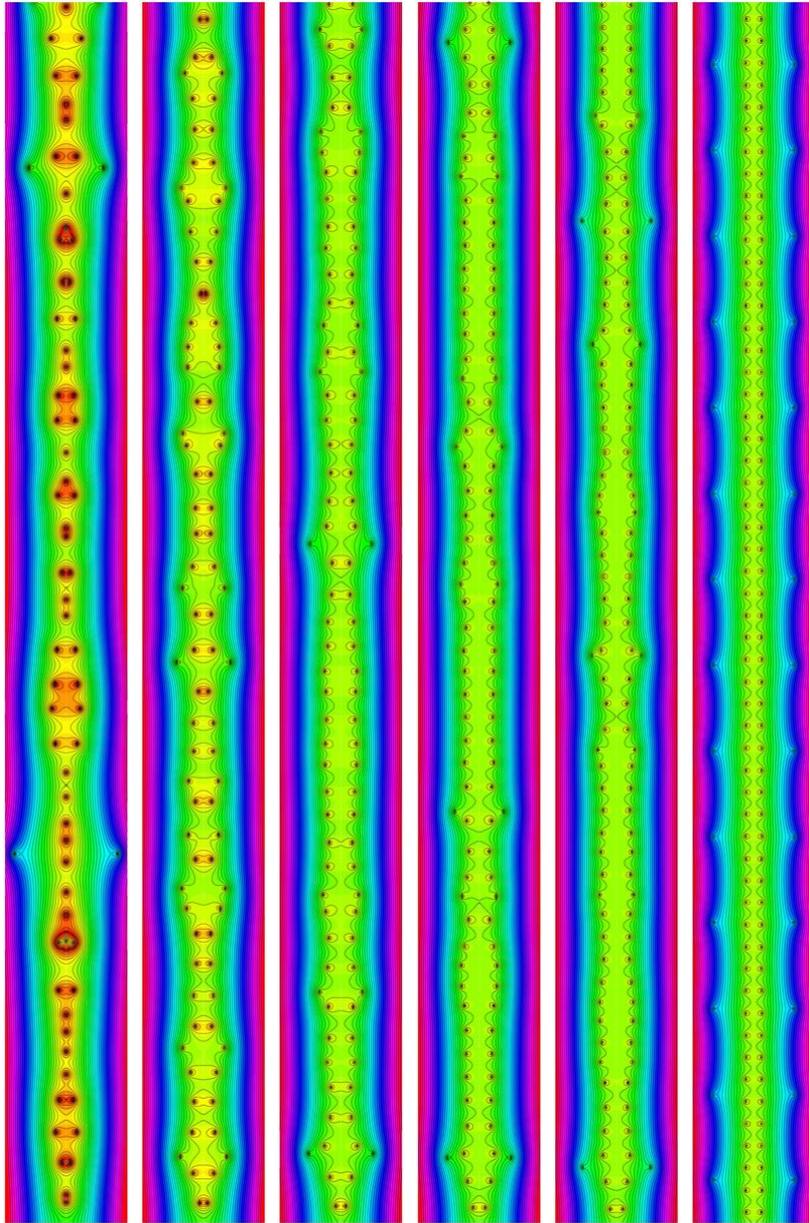
FIGURE 12. The level curves of $|\zeta_{L^2}|$ for the 1D skeleton complexes of graphs in $E(20, p)$ with $p = 0.1, 0.2, \ldots, 0.5$ and $p = 1$. In this case, the complex size is $|V| + |E| = 20 + 190p$. We see the spectral symmetry.

References

[1] N.L. Alling. *Real Elliptic Curves*, volume 54 of *Mathematics Studies*. North Holland, 1981.

[2] W.N. Bailey. *Generalized Hypergeometric Series*, volume 32 of *Cambridge Tracts in Mathematics and Mathematical Physics*. Stechert-Hafner Service Agency, 1964.

[3] A. Björner and K.S. Sarkaria. The zeta function of a simplicial complex. *Israel Journal of Mathematics*, 103:29–40, 1998.

[4] E. Bombieri. Prime territory: Exploring the infinite landscape at the base of the number system. *The Sciences*, pages 30–36, 1992.

[5] J.B. Conrey. The Riemann Hypothesis. *Notices of the AMS*, pages 341–353, 2003.

[6] I.P. Cornfeld, S.V.Fomin, and Ya.G.Sinai. *Ergodic Theory*, volume 115 of *Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen*. Springer Verlag, 1982.

[7] J. Derbyshire. *Bernhard Riemann and the Greatest Unsolved Problem in Mathematics*. Joseph Henry Press, 2003.

[8] M. du Sautoy. *The Music of the Primes*. Harper Collins, 2003.

[9] F. Friedli and A. Karlsson. Spectral zeta functions of graphs and the Riemann zeta function in the critical strip. *Tohoku Math. J. (2)*, 69(4):585–610, 2017.

[10] N.A. Friedman. *Introduction to Ergodic Theory*. Van Nostrand-Reinhold, Princeton, New York, 1970.

[11] D. Joyner and T. Shaska. Self-inversive polynomials, curves, and codes. In *Algebraic curves and their fibrations in mathematical physics and arithmetic geometry*, volume 703 of *Communications in Contemporary Mathematics*. AMS, 2016.

[12] O. Knill. Renormalization of of random Jacobi operators. *Communications in Mathematical Physics*, 164:195–215, 1995.

[13] O. Knill. A Cauchy-Binet theorem for Pseudo determinants. *Linear Algebra and its Applications*, 459:522–547, 2014.

[14] O. Knill. The graph spectrum of barycentric refinements.
http://arxiv.org/abs/1508.02027, 2015.

[15] O. Knill. Universality for Barycentric subdivision.
http://arxiv.org/abs/1509.06092, 2015.

[16] O. Knill. On Fredholm determinants in topology.
https://arxiv.org/abs/1612.08229, 2016.

[17] O. Knill. On a Dehn-Sommerville functional for simplicial complexes.
https://arxiv.org/abs/1705.10439, 2017.

[18] O. Knill. One can hear the Euler characteristic of a simplicial complex.
https://arxiv.org/abs/1711.09527, 2017.

[19] O. Knill. The strong ring of simplicial complexes.
https://arxiv.org/abs/1708.01778, 2017.

[20] O. Knill. The zeta function for circular graphs.
http://arxiv.org/abs/1312.4239, December 2013.

[21] R. Koekoek. Hypergeometric functions. Lecture notes TU Delft.

[22] M. Kontsevich and D. Zagier. Periods. *Publications, IHES*, 2001.

[23] P. Lindstrom. *Galois Theory of Palindromic Polynomials*. University of Oslo, 2015. Master's Thesis at the University of Oslo.

[24] D. Maraglit and S. Spallone. A homological recipe for pseudo-Anosovs. *Mathematical Research Letters*, 14:853–863, 2007.

[25] B. Mazur and W. Stein. *Prime Numbers and the Riemann Hypothesis*. Cambridge University Press, 2016.

[26] M.F.Barnsley, J.S.Geronimo, and A.N. Harrington. Almost periodic jacobi matrices associated with julia sets for polynomials. *Commun. Math. Phys.*, 99:303–317, 1985.

[27] D. Rockmore. *Stalking the Riemann Hypothesis*. Pantheon Books, 2005.

[28] K. Sabbagh. *Dr. Riemann's Zeros*. Atlantic books, 2003.

[29] B. Simon. *Trace Ideals and their Applications*. AMS, 2. edition, 2010.

[30] C. Smyth. Survey article: Seventy years of Salem numbers. https://arxiv.org/abs/1408.0195v3, 2015.

[31] T. Tao. *Structure and Randomness*. AMS, 2008.

[32] A. Terras. *Zeta functions of Graphs*, volume 128 of *Cambridge studies in advanced mathematics*. Cambridge University Press, 2011.

[33] R. van der Veen and J. de Craats. *The Riemann hypothesis*. MAA Press, 2015.

[34] M. Watkins. *The Mystery of the Prime Numbers*. Liberalis, Washington, USA, 2010.

DEPARTMENT OF MATHEMATICS, HARVARD UNIVERSITY, CAMBRIDGE, MA, 02138