KÜNNETH FORMULA IN GRAPH THEORY
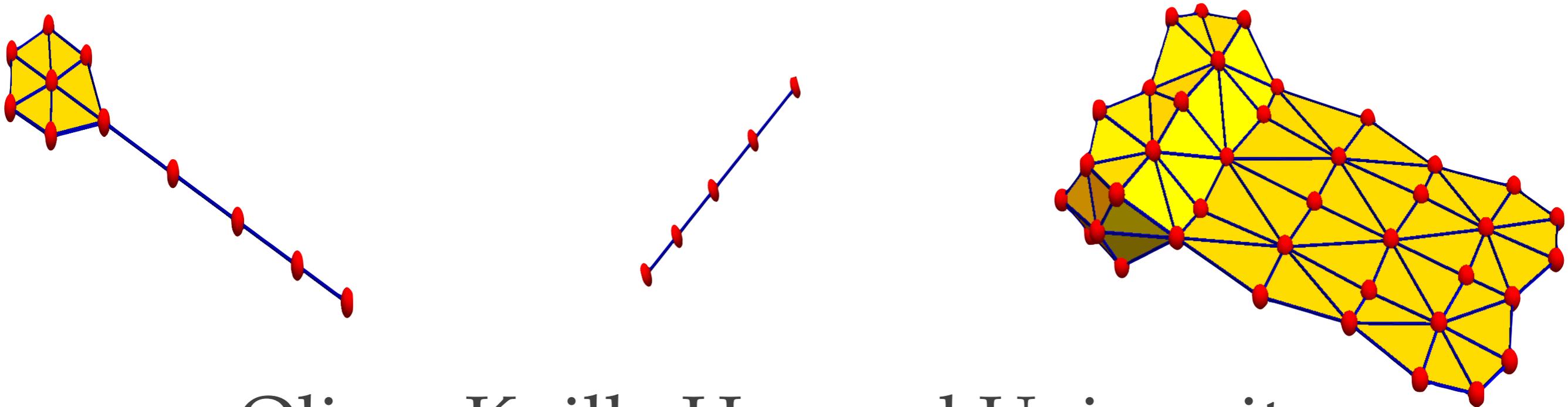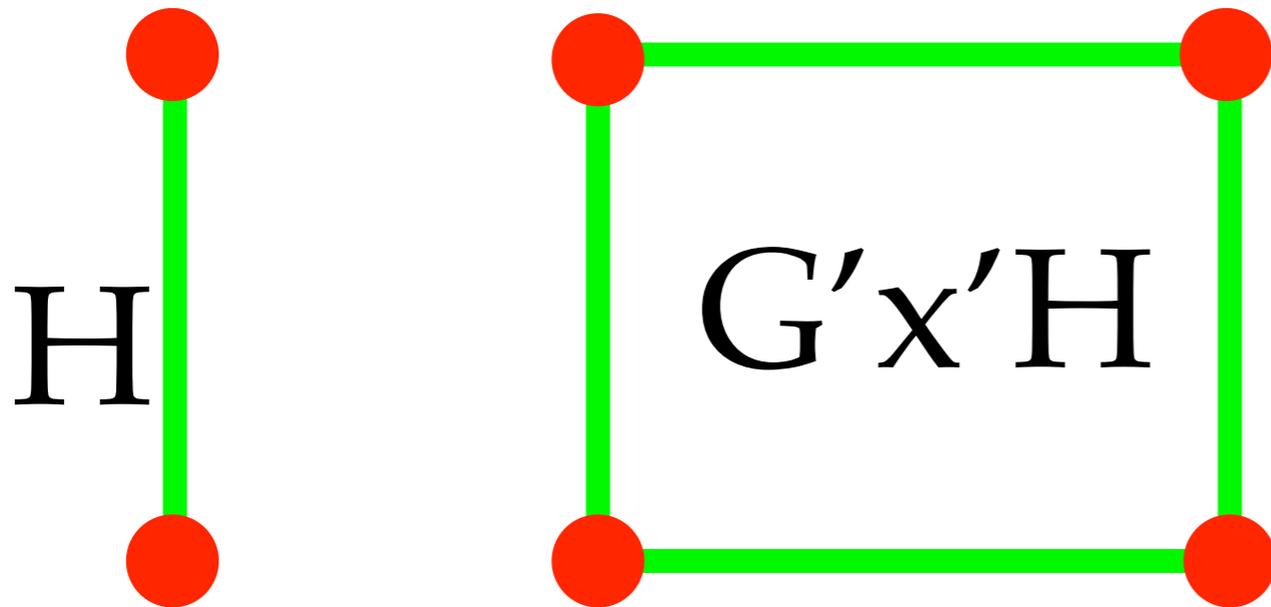

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May 27, 2014

http://arxiv.org/abs/1505.07518

CLASSICAL PRODUCT

H

G'x'H

dim(H)=1

G

dim(G 'x'H)=1   dim(G)=1

no Künneth
formula

$p(G,x)=1$

$p(H,x)=1$ Poincaré
polynomial

$p(G\ 'x'\ H,x)=1+x$

not Euler Charact.
multiplicative

$X(G)=X(H)=1$

$X(G\ 'x'\ H) = 0$

not dimension
additive

THE CHAIN RING

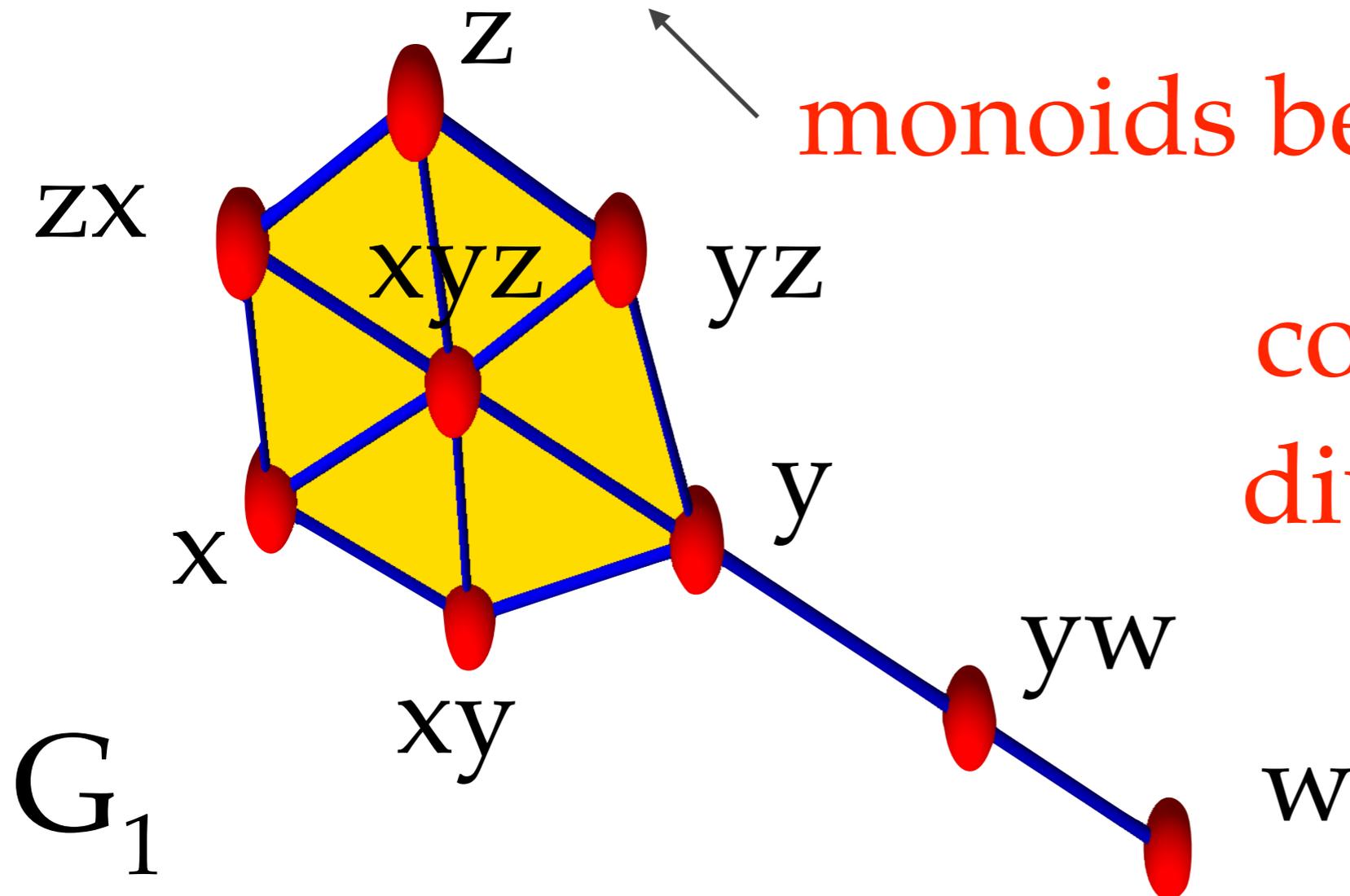G finite simple graph
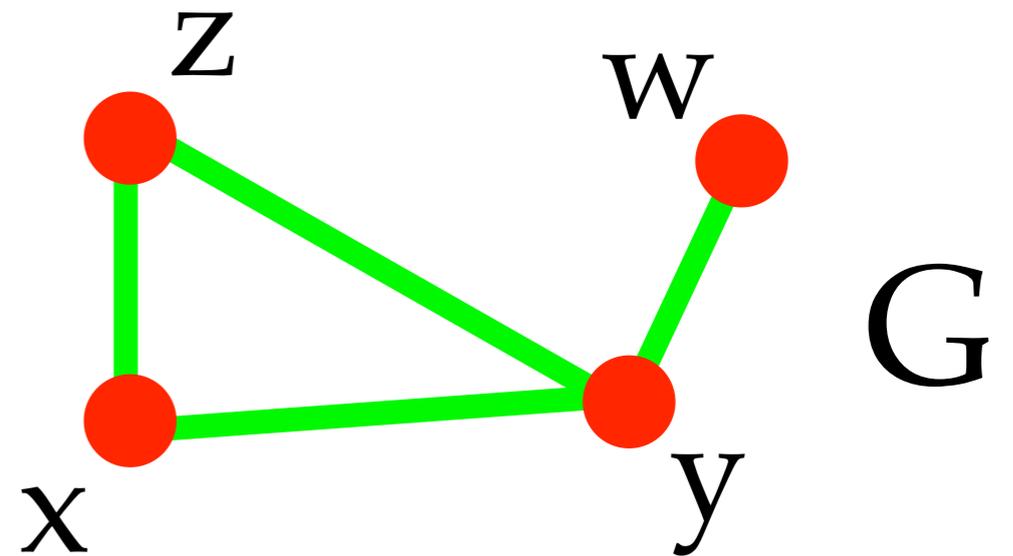
g = x+y+z+w+xy+yz+zx+yw+xyz

monoids in g

in ring Z[x,y,z,w]

DIVISOR GRAPH

$g = x+y+z+w+xy+$
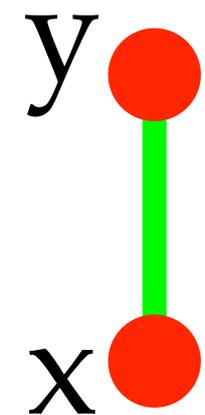$yz+zx+yw+xyz$
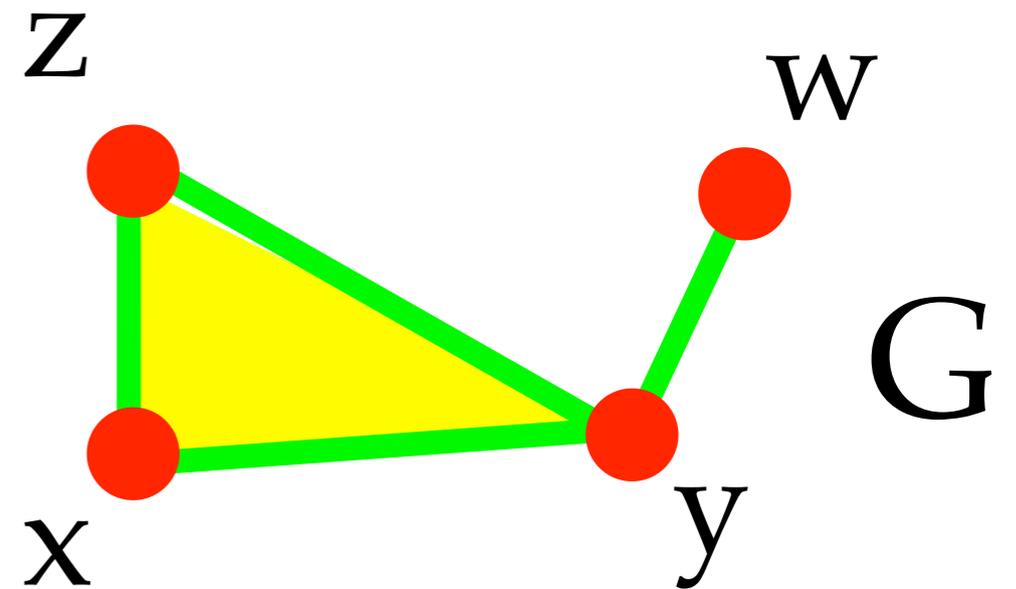
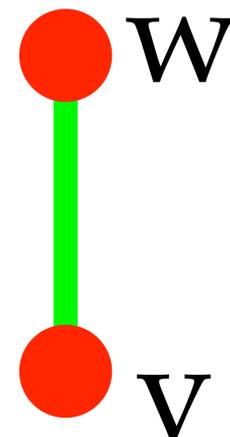monoids become vertices

connected if one
divides the other

THE PRODUCT

y ● — ● G

x ●

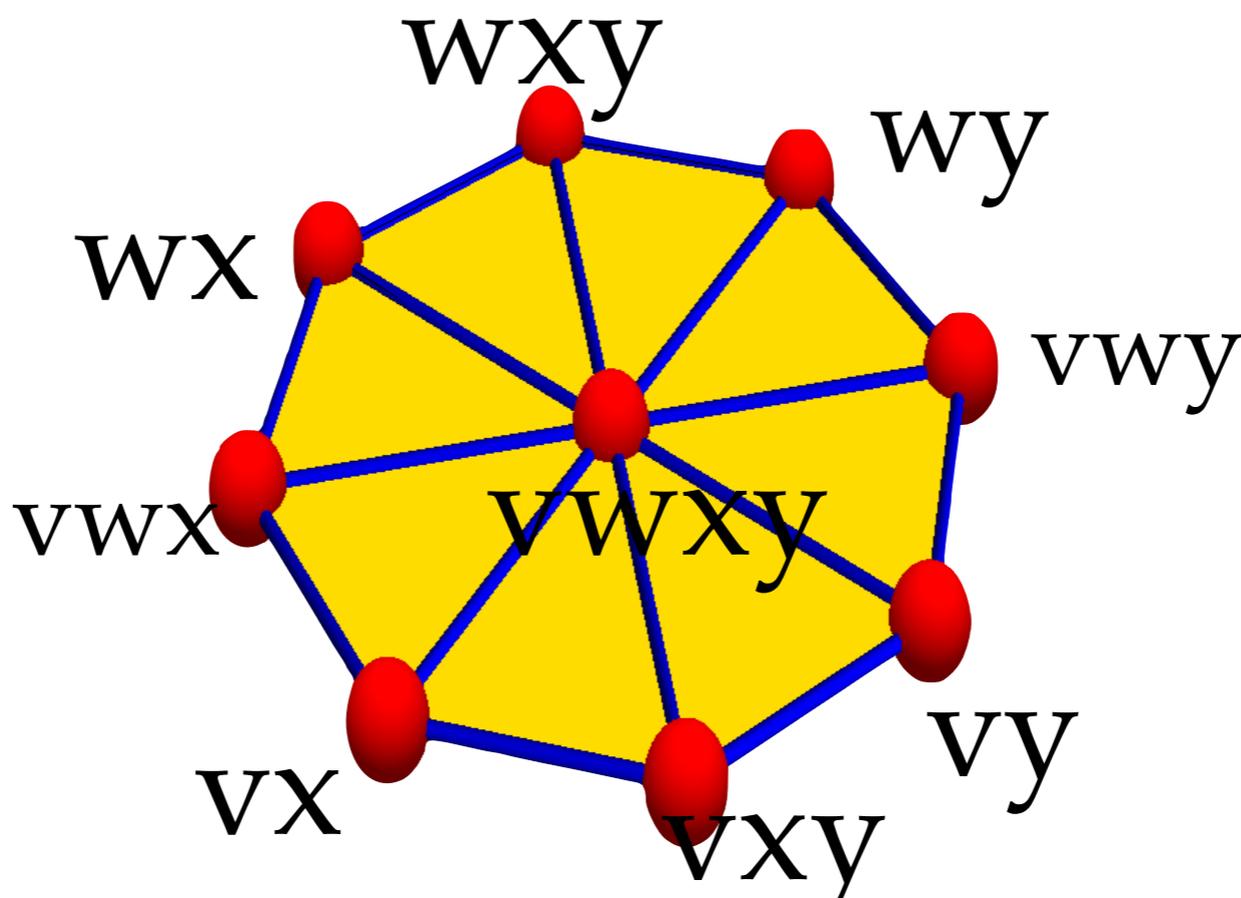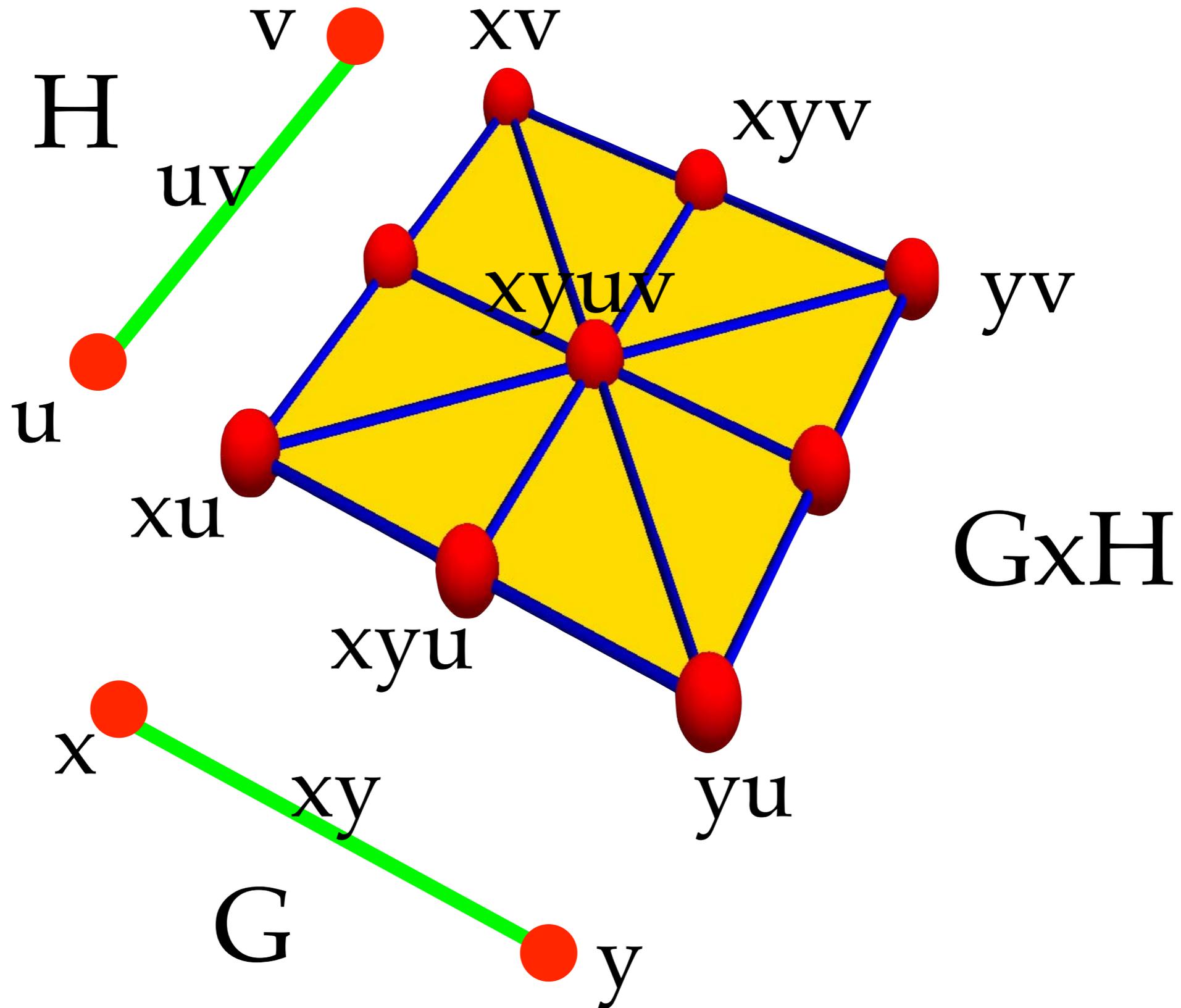$$g = x+y+xy$$
$$h = v+w+vw$$

H ● w

● v

$$g\,h = vx + wx + vwx + vy +$$
$$wy + vwy + vx\,y + wxy + vwxy$$

GxH

SQUARE

LARGER SQUARE

EXAMPLE

G x H = G x H

DIMENSION

G x H = G x H

dim=1.4666

dim=1.333

dim=2.702
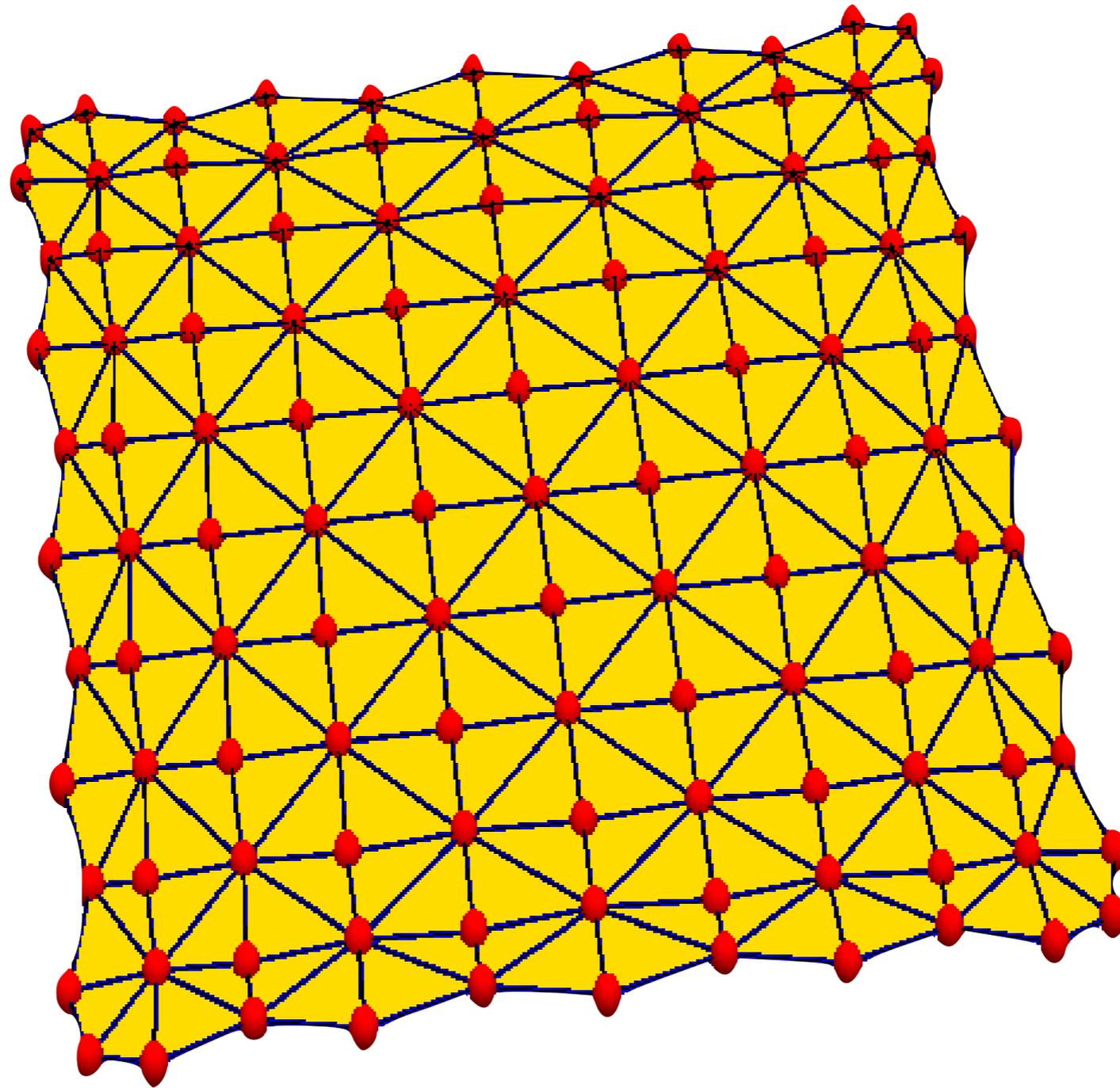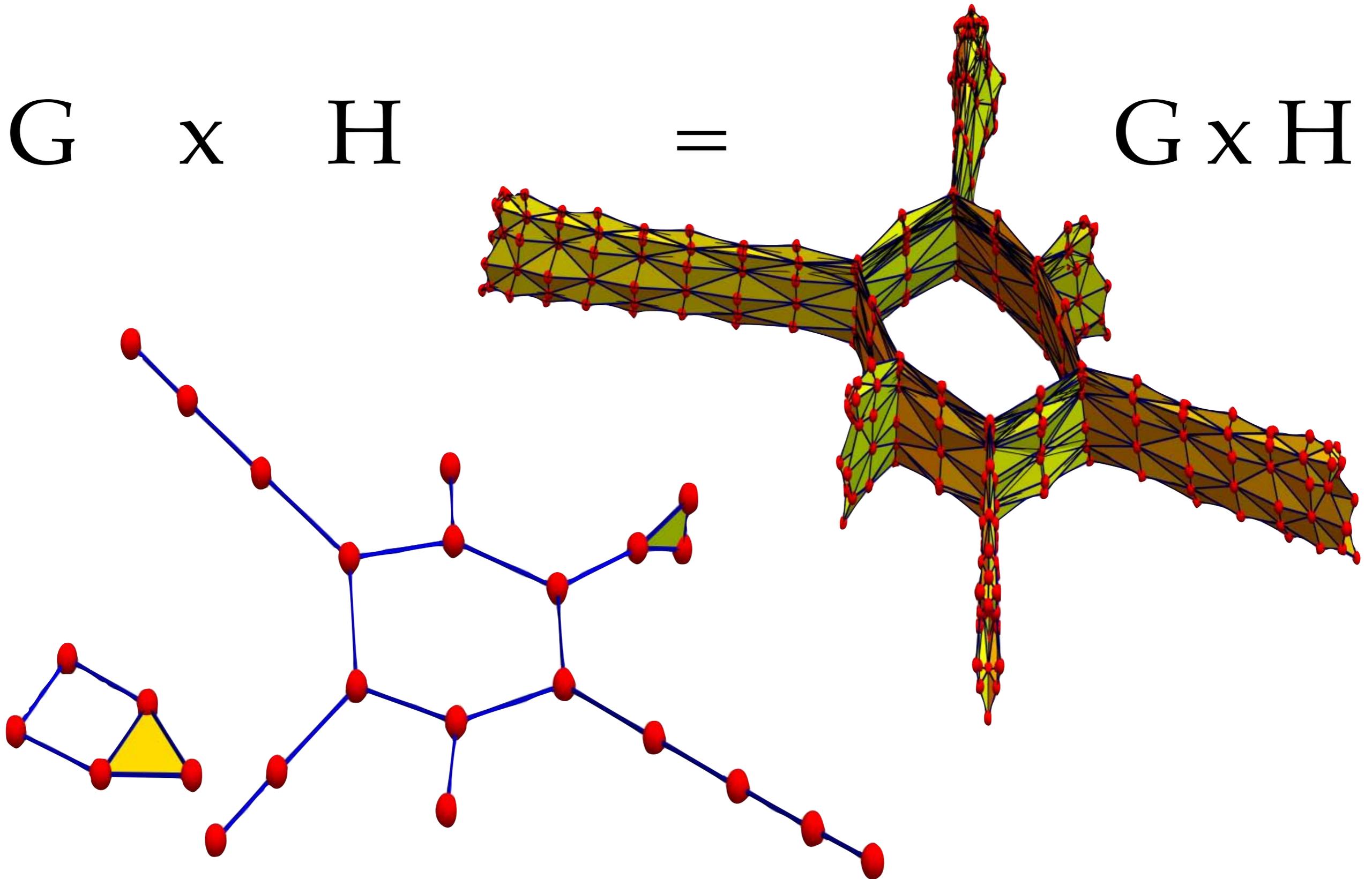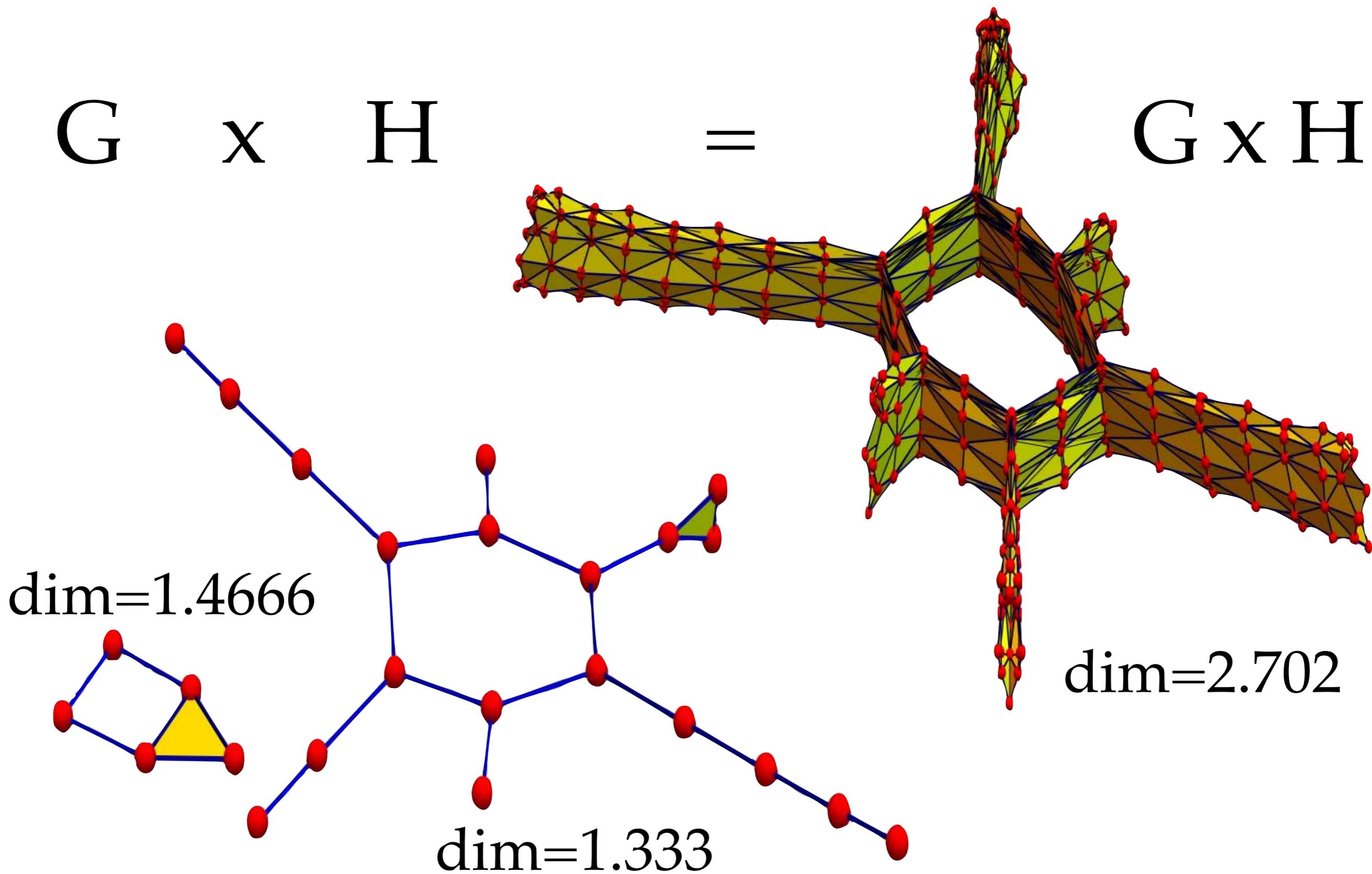
COHOMOLOGY

$$H^2(G \times H) = H^1(G) \oplus H^1(H)$$
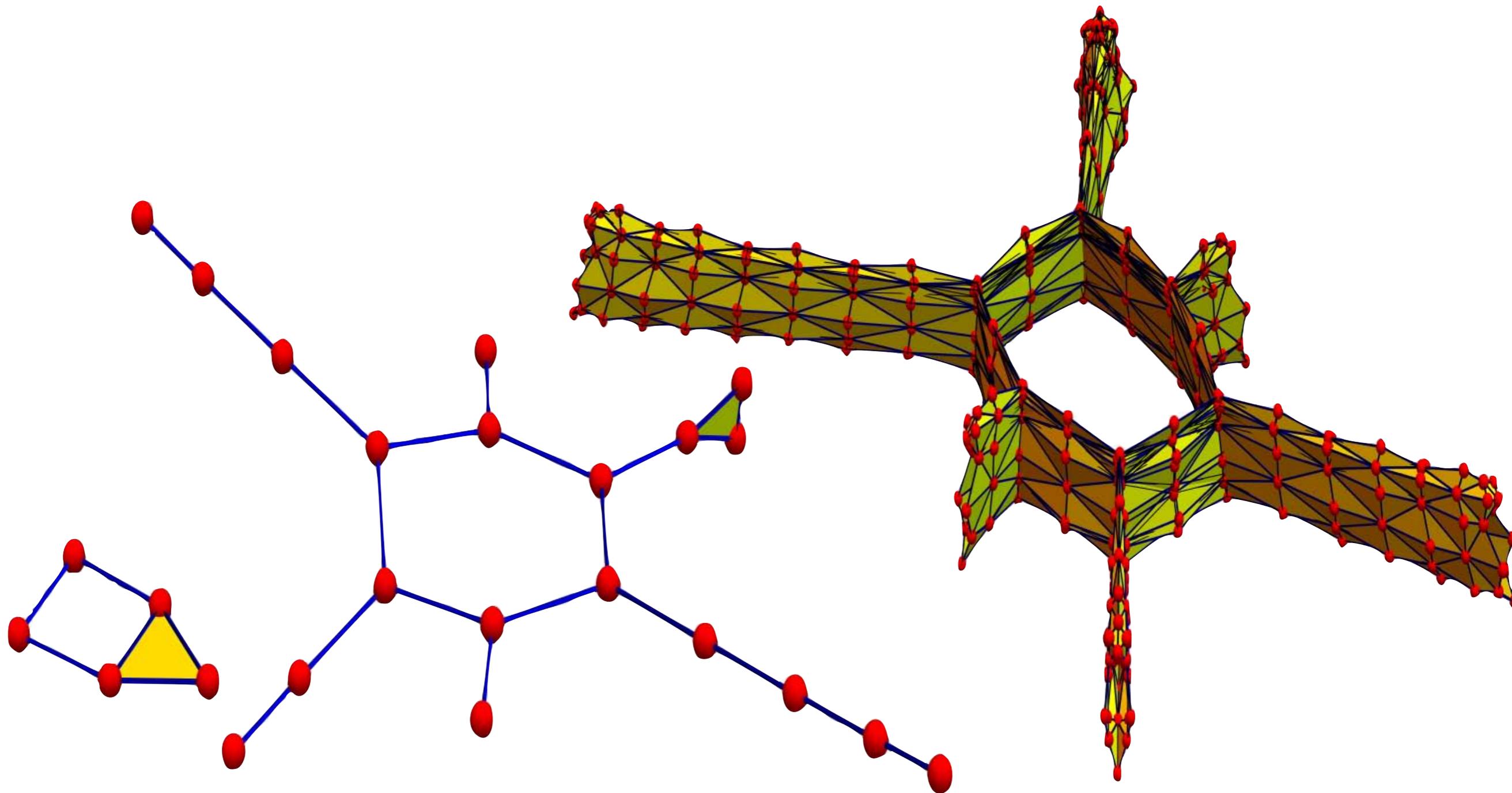
EULER CHARACTERISTIC

EULER AND POINCARÉ POLYNOMIALS

$$q(G,x) = \sum_k v_k(G)x^k \qquad q(G,x) = \sum_k \bar{v}_k(G)x^k$$

Euler

$$p(G,x) = \sum \dim(H^k(G))x^k$$

Poincaré

$$\chi(G) = q_{dR}(G,-1)$$
$$= q(G,-1) = p(G,-1)$$

Euler characteristic

Euler-
Poincaré

G,H arbitrary
finite simple
graphs

new:

$$q_{dR}(GxH,x) = q(G,x)\, q(H,x) \qquad p(GxH,x) = p(G,x)\, p(H,x)$$
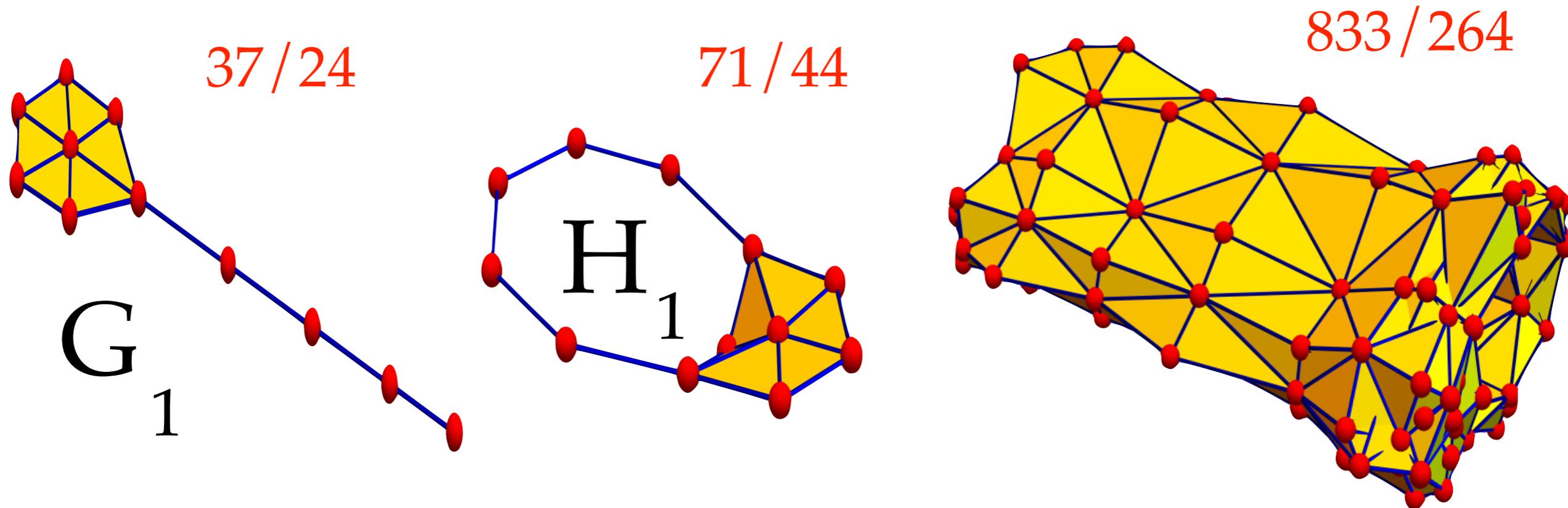
DIMENSION

DIMENSION LEMMA
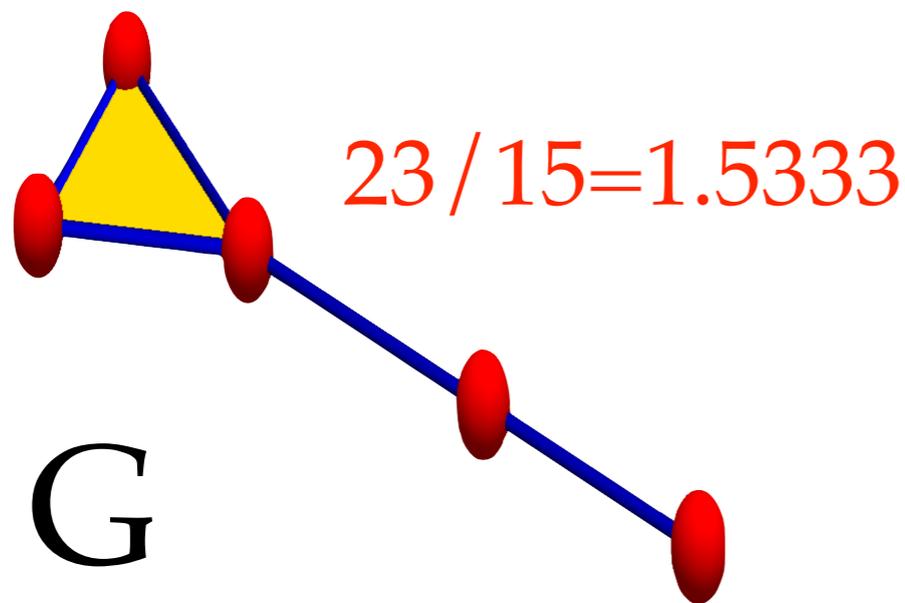
$23/15=1.5333$

$71/44=1.6136$

G
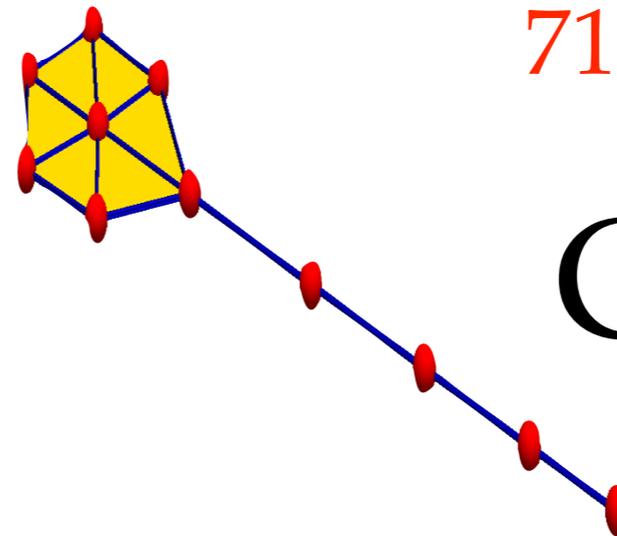
$GxK_1=G_1$

$$\dim(G_1) \geq \dim(G)$$

higher dim simplices spawn more new vertices
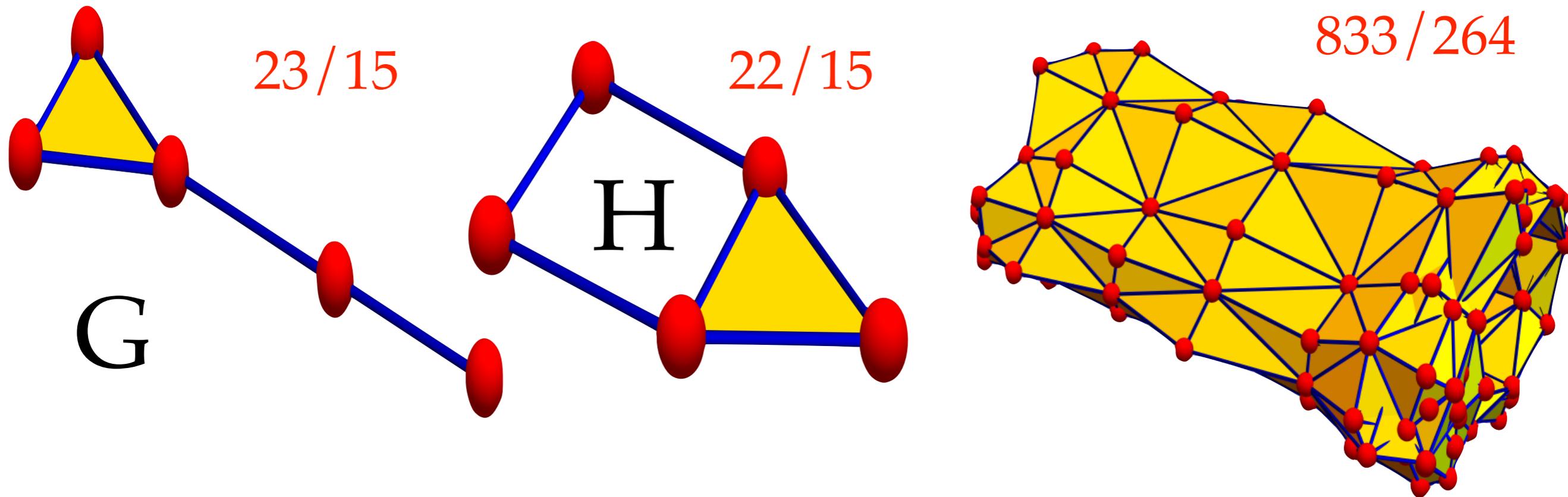
DIMENSION FORMULA

$37/24$

$71/44$

$833/264$

G_1

H_1

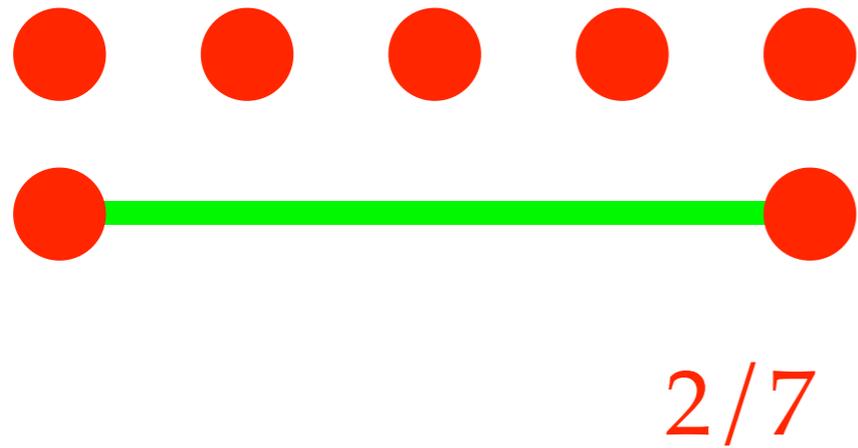$$\dim(G_1) + \dim(H_1) = \dim(G \times H)$$
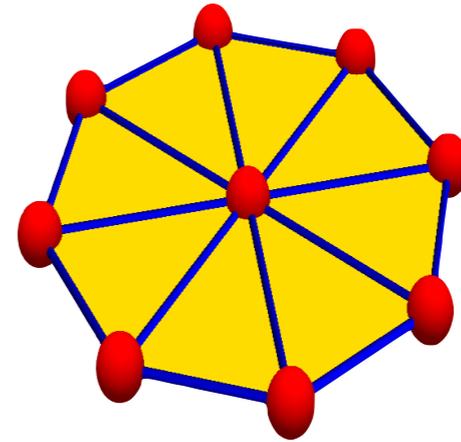
holds even locally!

SUPER ADDITIVITY

$$\dim(G) + \dim(H) \leq \dim(G \times H)$$
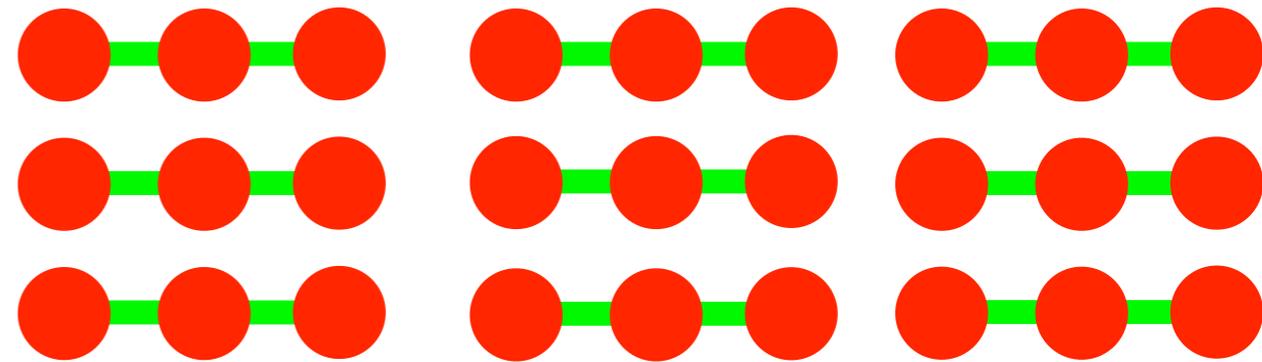
Same formula as for Hausdorff dimension!

EXAMPLE

2/7
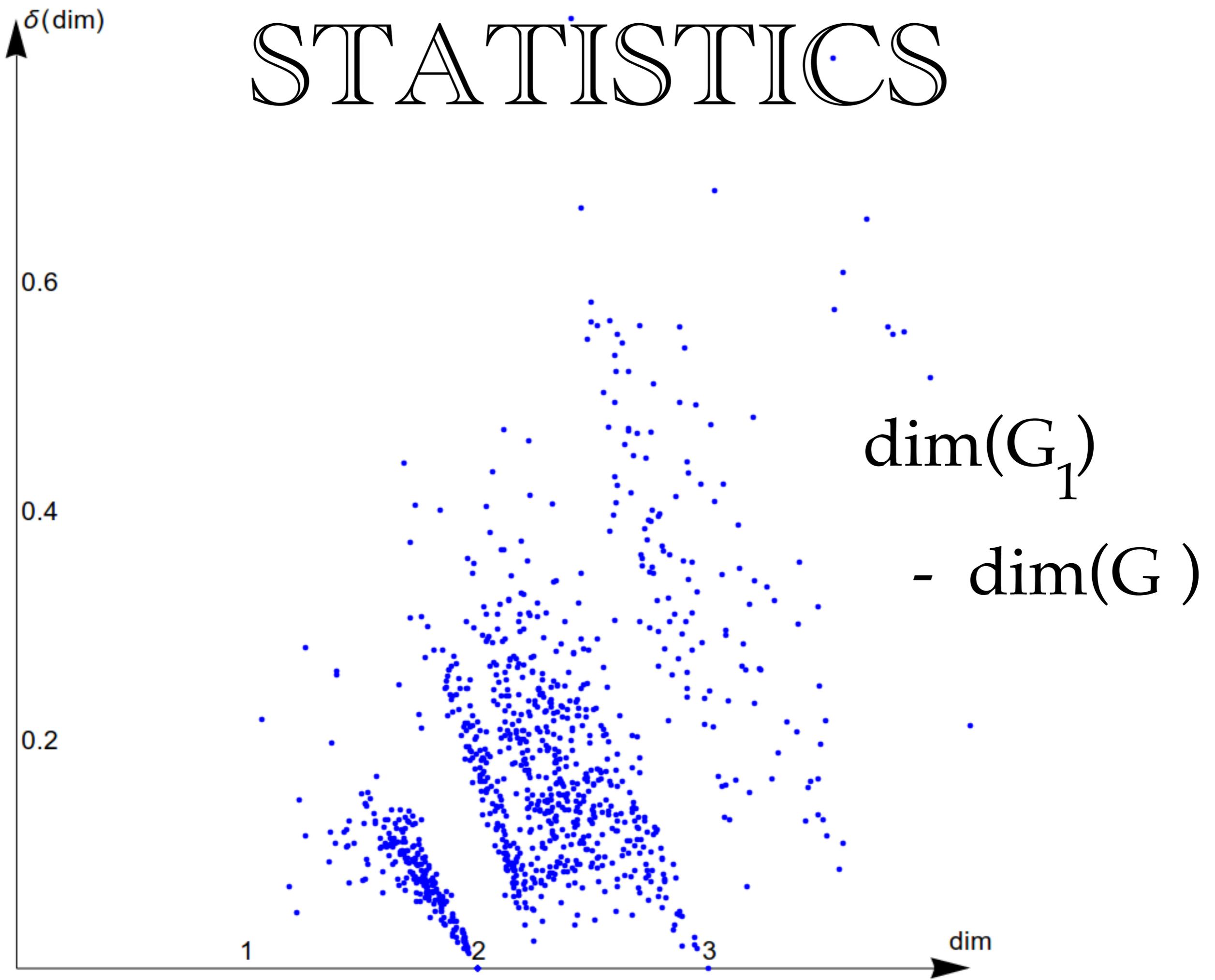
16/13

G =H

G x H

Exists graphs with dimension close to 0
whose product have dimension close to 1

STATISTICS
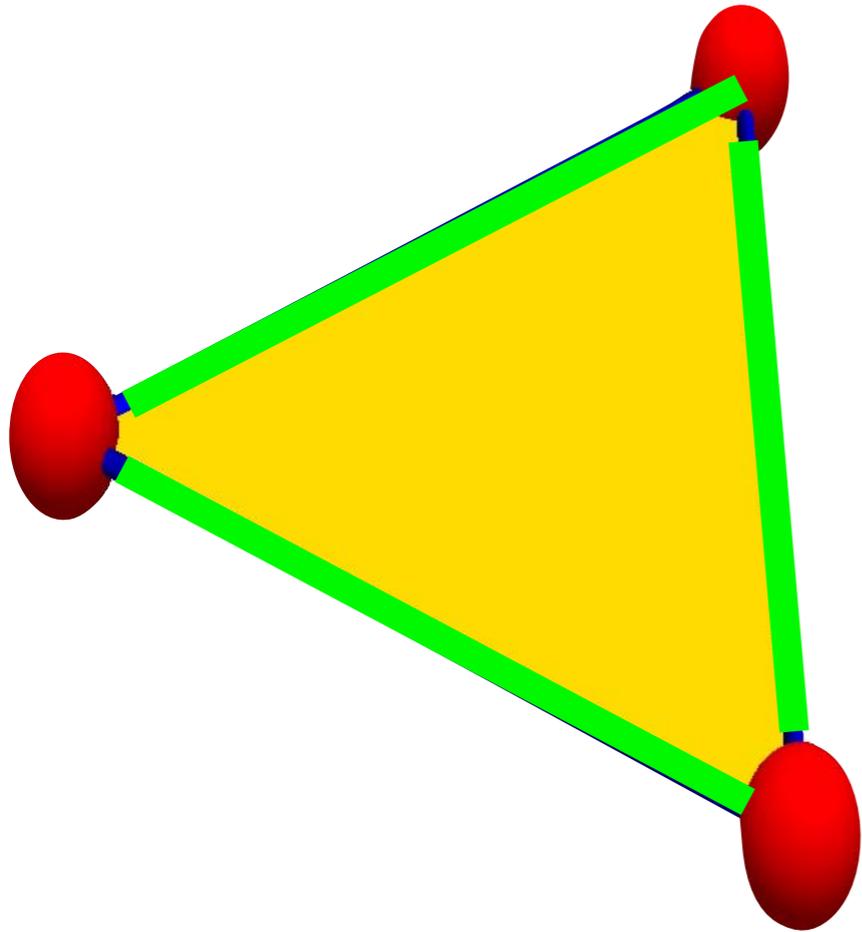
EVOLUTION

$G = G_0$
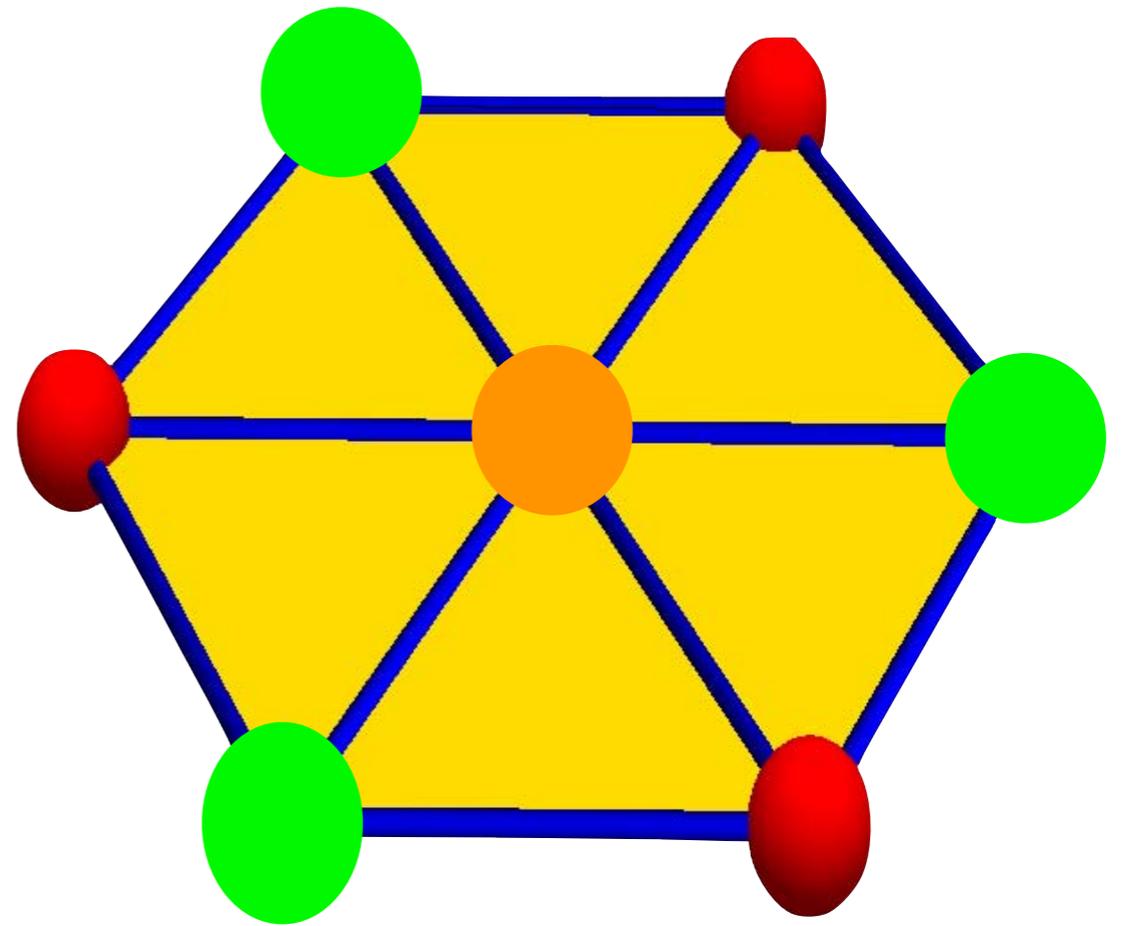
G_1

G_2

G_3
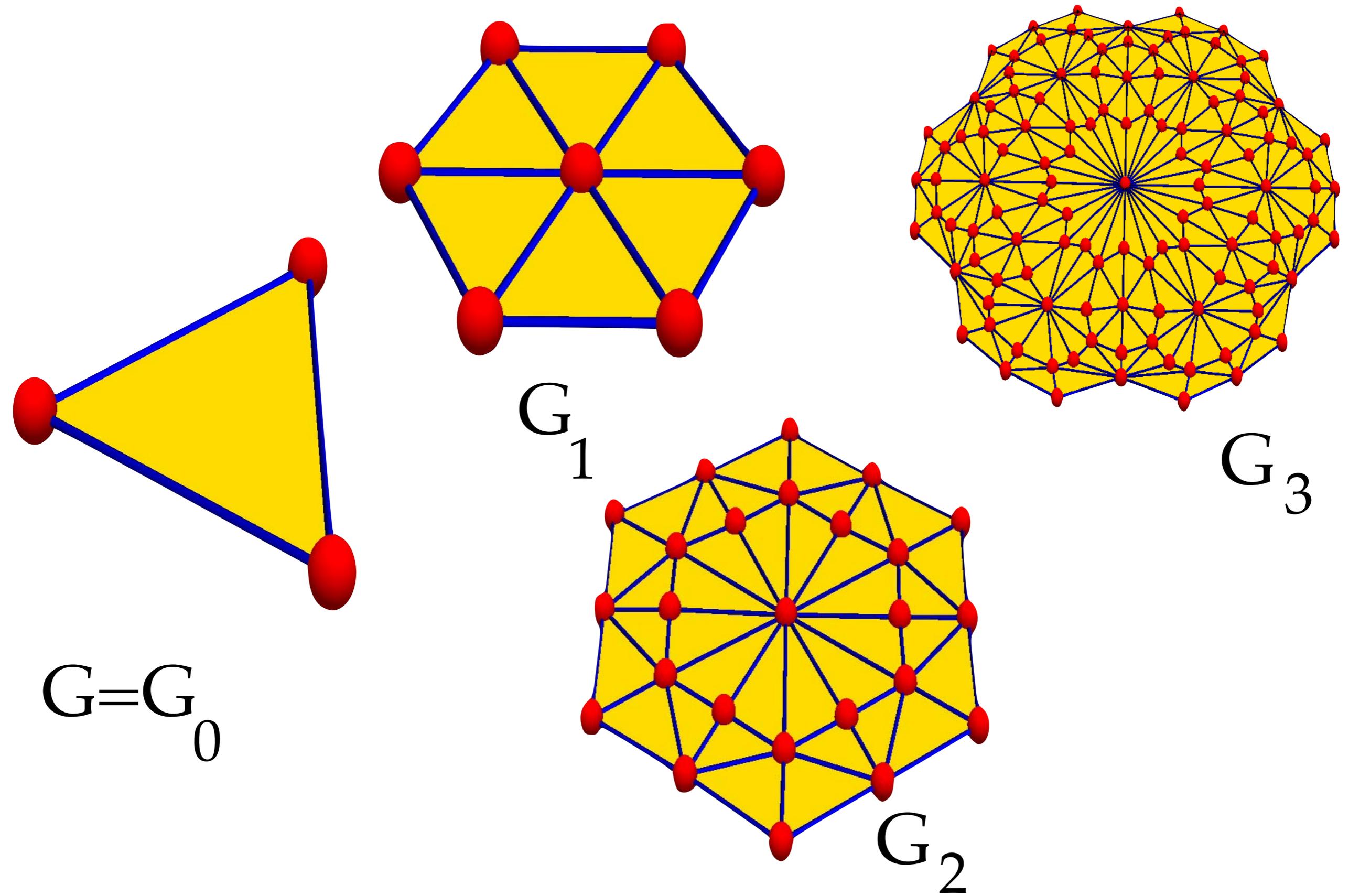
CHROMATOLOGY

COLORING BY DIM

$c(G)=3$

$c(G_1)=3$

$$c(G_1) \leq c(G)$$

GEOMETRIC GRAPHS

TORUS

geometric

not geometric
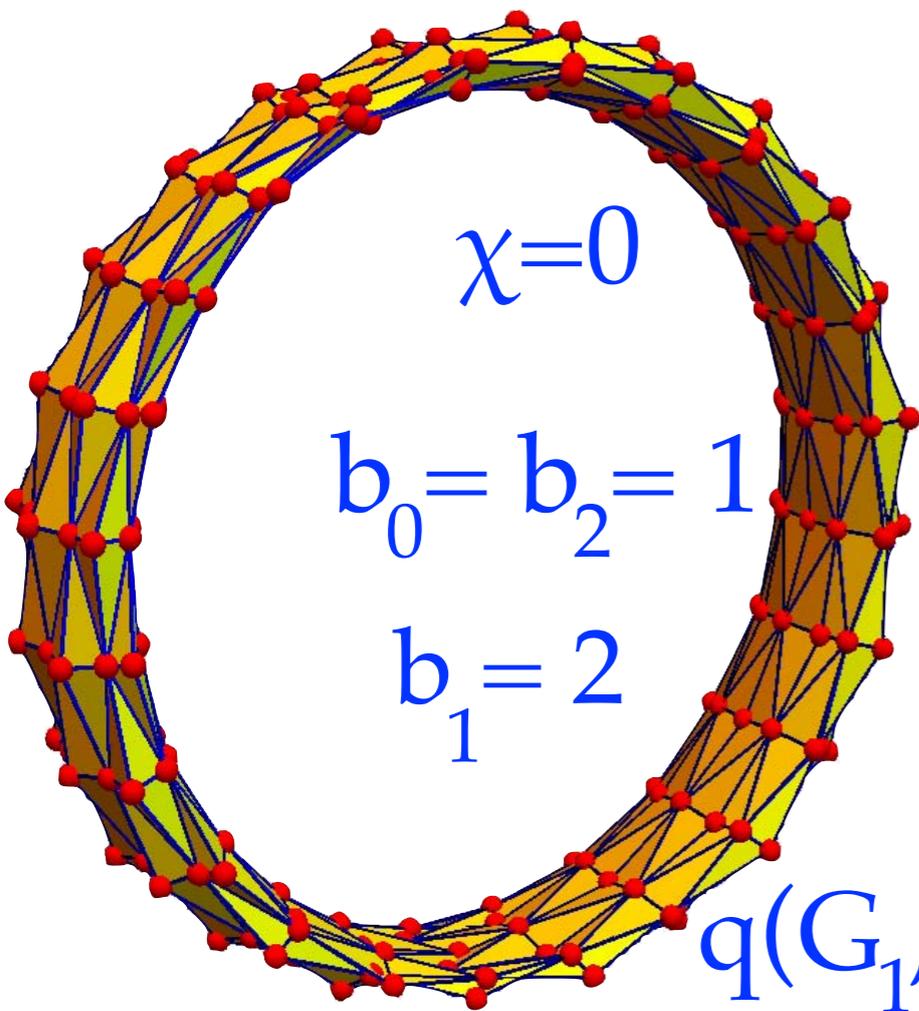
$v_0 = 240$

$v_1 = 720$

$v_2 = 480$

$\chi = 0$

$\bar{v}_0 = 240$

$\bar{v}_1 = 480$

$\bar{v}_2 = 240$

$b_0 = b_2 = 1$

$b_1 = 2$

$v_0 = 60$

$v_1 = 120$

$\chi = 0$

$b_0 = 1, b_2 = 0$

$b_1 = -61$

$q(G_1, x) = 24 + 24x$

$q(H_1, x) = 10 + 10x$
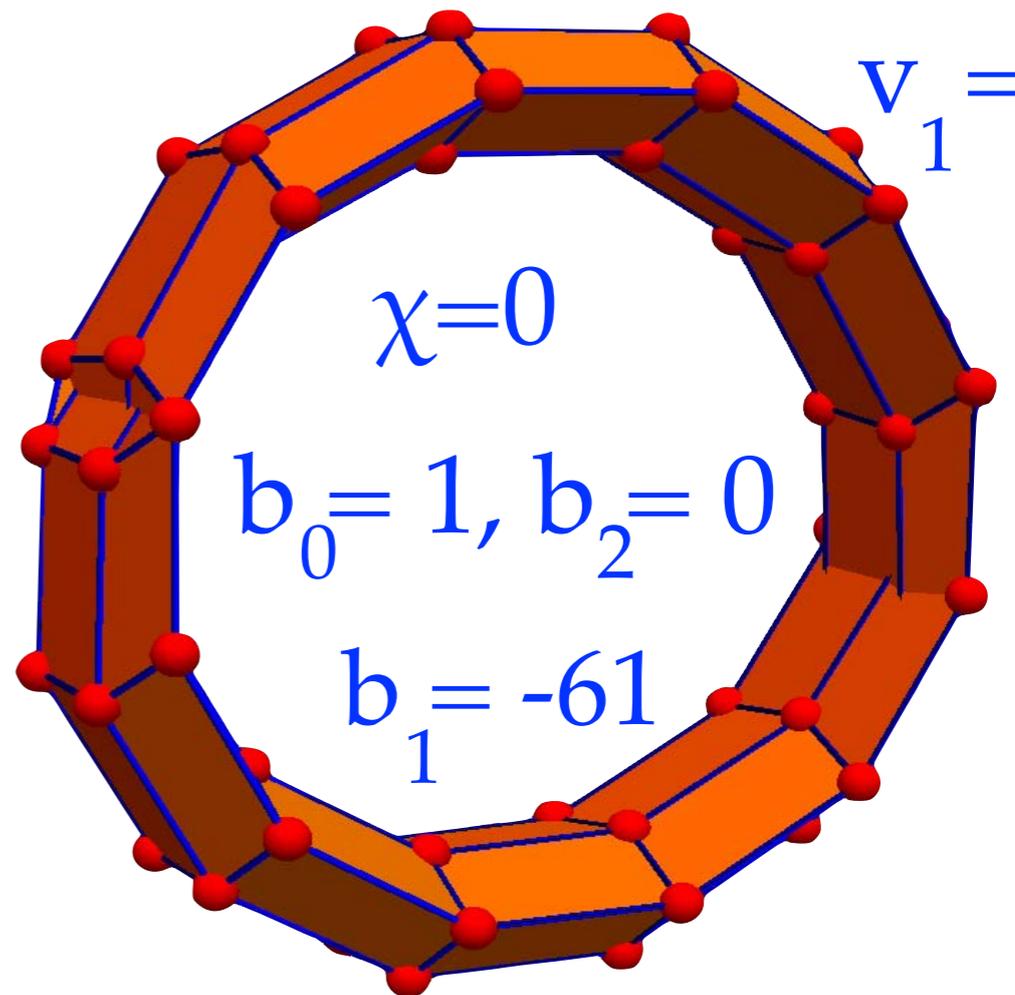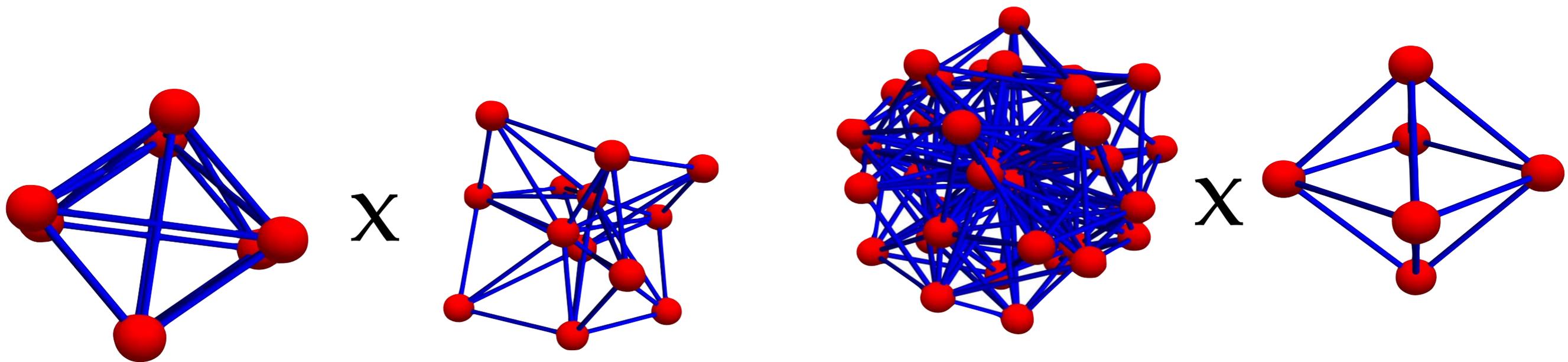
$G \times H = C_{12} \times C_5$

$C_{12} \,'x'\, C_5 = G \,'x'\, H$

HOMOTOPY GROUPS

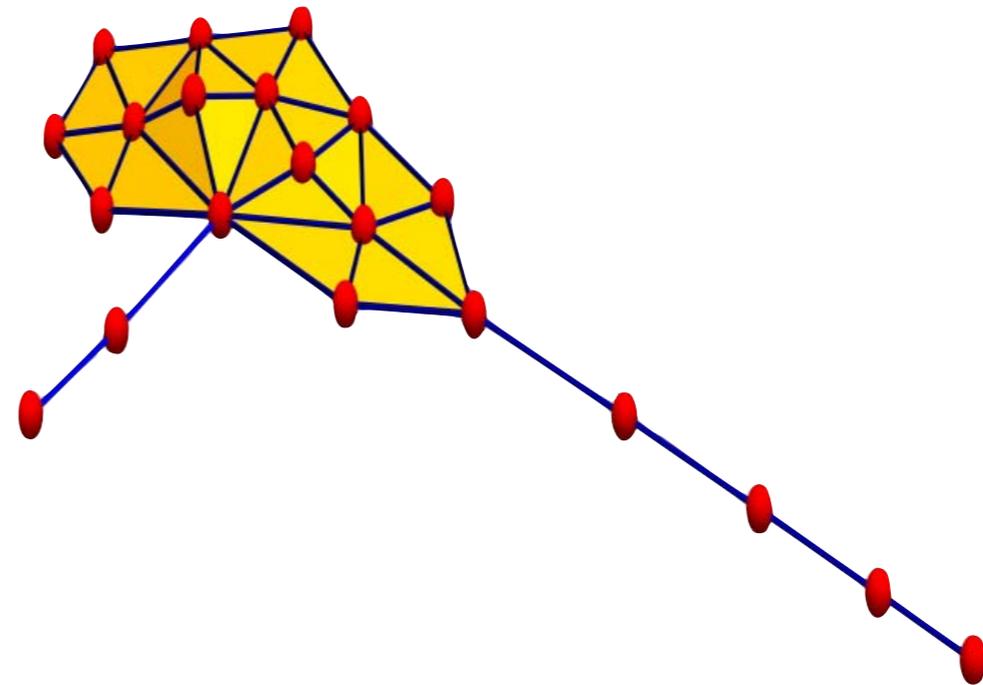Have different cohomology but same homotopy groups
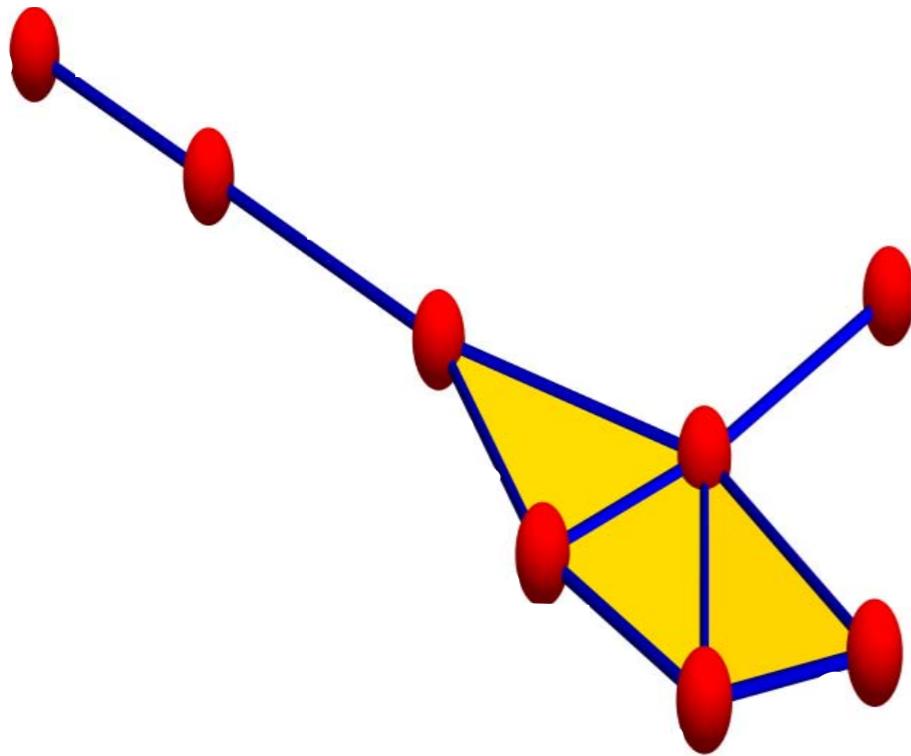

$$S^3 \times P^2 \qquad P^3 \times S^2$$

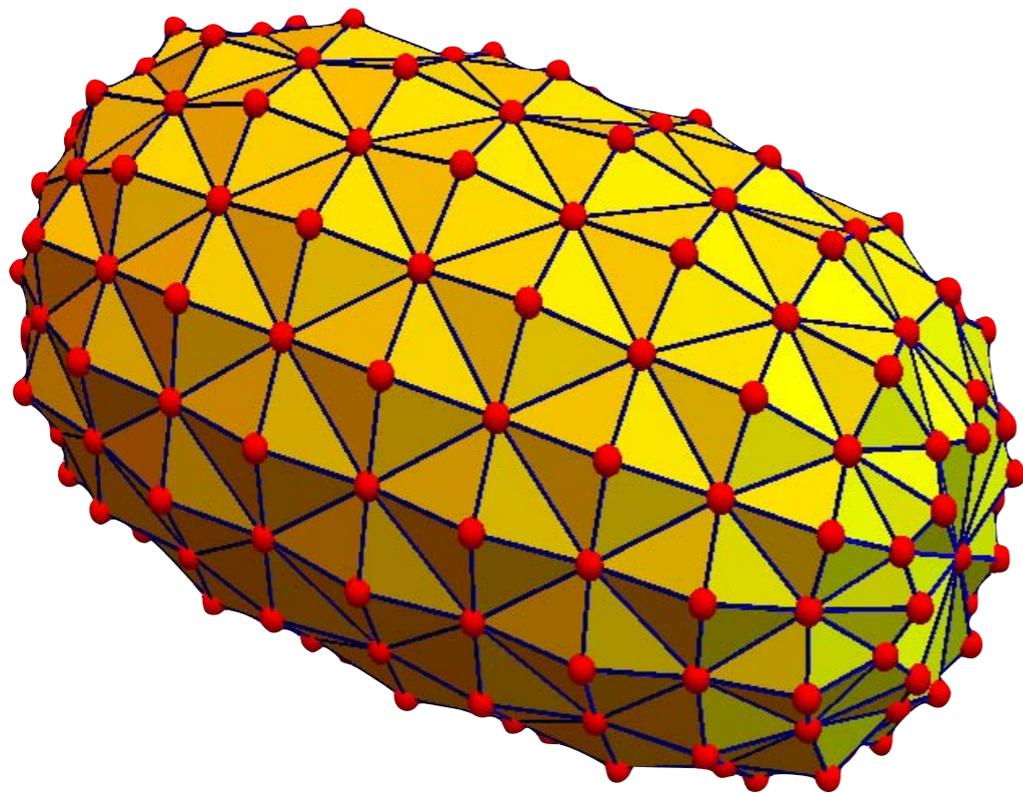# COLORING

Question: $c(G) \geq c(G_1)$

JOIN

use interval
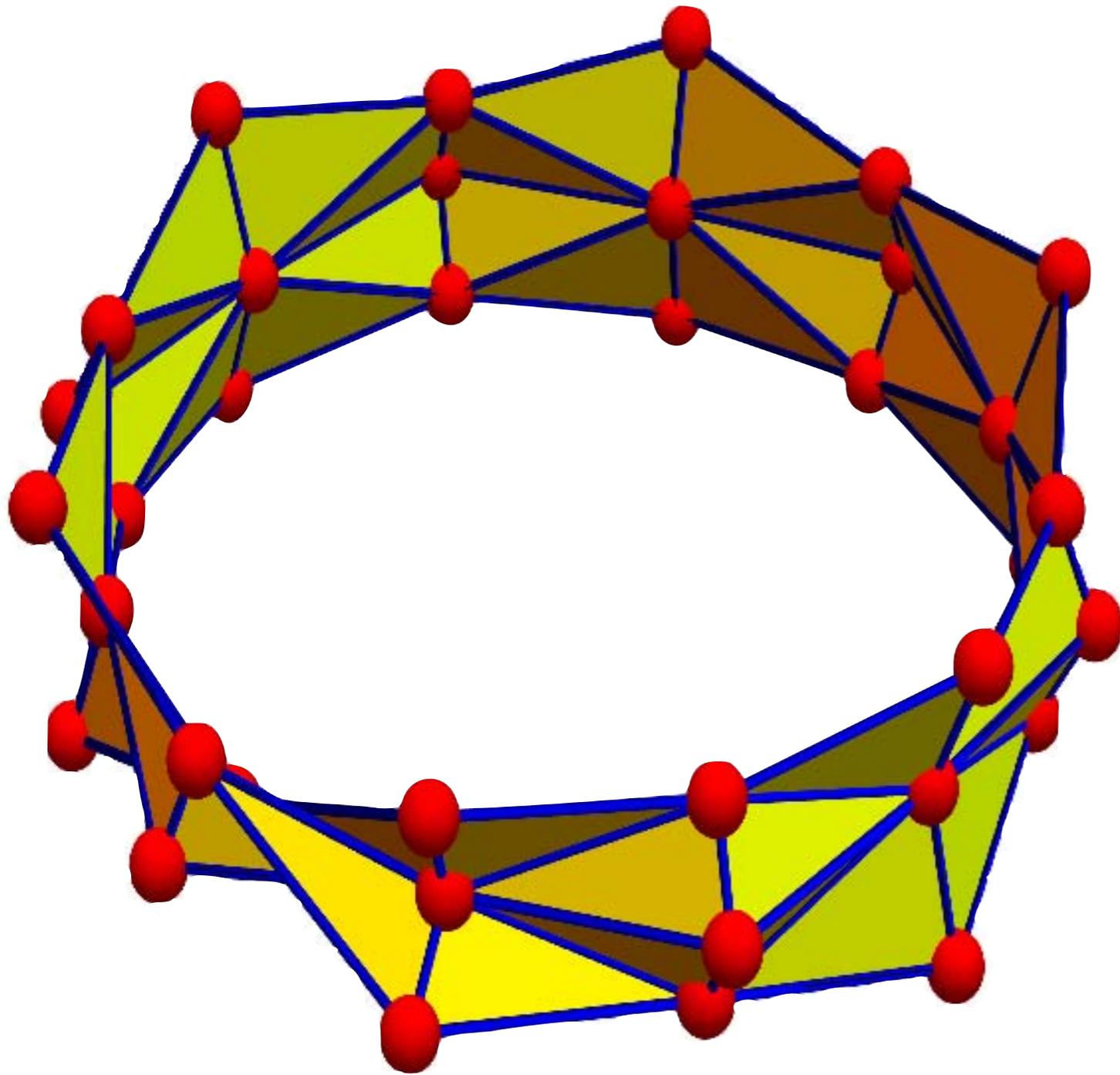
$$* = *_n$$

$$I = L_n$$

$$G * H = G \times H \times I / \sim$$
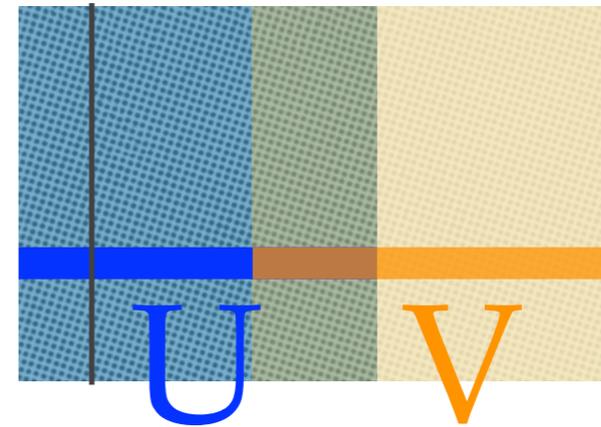
$$(x, y_1, 0) \sim (x, y_2, 0)$$

$$(x_1, y, 1) \sim (x_2, y, 1)$$

$$C_9 *_4 S_0$$

suspension

BUNDLES

fibre

locally, have product

We can glue locally trivial bundles

U   V

THE END