1a Hatsumon: Exp and Log **Oliver Knill, 10/23/2003**

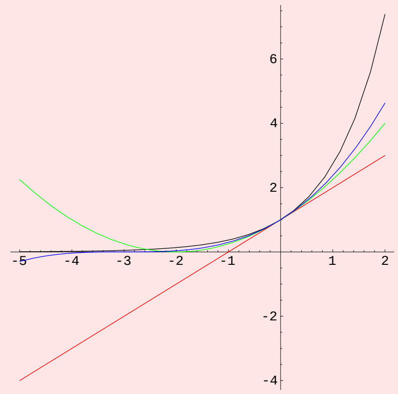
In this Hatsumon you discover a short, elegant and independent way to introduce the functions log and exp and to establish their properties. This Hatsumon (= "hook question") could also make sense in a precalculus course.

Definition.

Define the exponential function $\exp(x)$ as the limit of the polynomials

$$\boxed{\exp_n(x) = (1 + x/n)^n}$$

when n goes to infinity.



Preliminary problems:

1) Each $\exp_n(x)$ for $n > 0$ is a function which is monotone in x.
2) $\exp_0(x) = 1$, $\exp'_n(x) = (1 + x/n)^{n-1}$.
3) So $\exp'(x) = \exp(x)$ and $\exp'(x)$ is monotone also.
4) $\exp_n(x + y) - \exp_n(x)\exp_n(y)$ goes to 0 for n to infinity.
5) $\exp_n(x)$ is a polynomial in x which is positive on $[-n, n]$.
6) For fixed x and large n, we have $\exp_n(x) < \exp_m(x)$ for $m > n$.

Use this information to prove the following **Theorem:**

a) $\exp(x)$ is a function defined for all x on the real line.
b) $\exp(x)$ is bounded and monotone in x.
c) The formula $\boxed{\exp'(x) = \exp(x)}$ holds.
d) The function exp is smooth: all derivatives exist.
d) The identity $\boxed{\exp(x + y) = \exp(x)\exp(y)}$ holds.

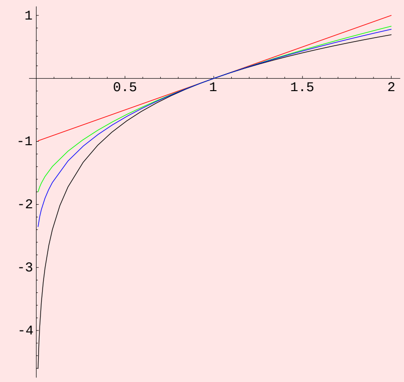
Challenge: estimate the difference $\exp_n(x) - \exp(x)$.

Definition.

Define the **logarithmic function** $\log(y)$ for positive y as the limit of

$$\boxed{\log_n(y) = n(y^{1/n} - 1)}$$

when n goes to infinity.



Preliminary problems:

1) Each function $\log_n(x)$ for $n > 0$ is monotone in x.
2) $\log_n(x)$ is the inverse of $\exp_n(x)$.
3) $\log'_n(x) = 1/x^{(1-1/n)}$ converges to the function $1/x$ for $n \to \infty$.
4) $\log_n(xy)$ is close to $\log_n(x) + \log_n(y)$.
5) $\log'_n(x)$ has a power series using the Binomial formula $(1 + x)^\alpha$.
6) For fixed x and large n, we have $\log_n(x) > \log_m(x)$ for $m > n$.

Use this information to prove the following **Theorem:**

a) $\log(x)$ is a function defined for all positive x.
b) $\log(x)$ is monotone in x on the positive real axes.
c) The formula $\log'(x) = 1/x$ holds.
d) The function $\log(x)$ is smooth on $x > 0$.
e) $\log(x)$ is the inverse of $\exp(x)$
f) the formula $\log(xy) = \log(x) + \log(y)$ holds for $x > 0, y > 0$.

Challenges:
a) give explicite estimates of the difference $\log_n(x) - \log(x)$.
b) use the above building blocks to optimze the proofs.