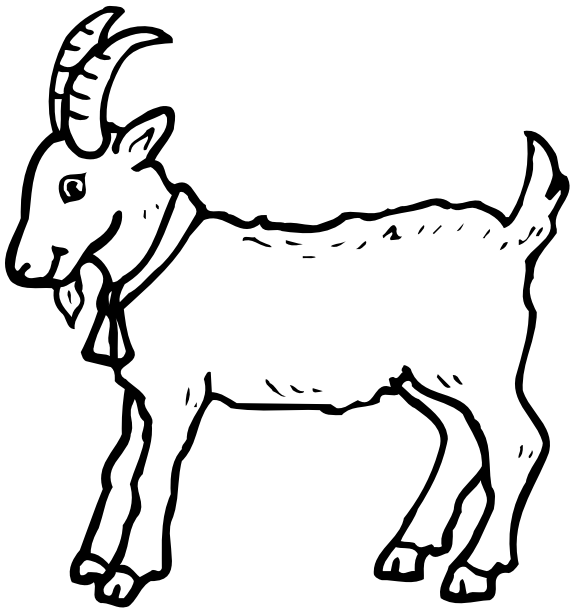
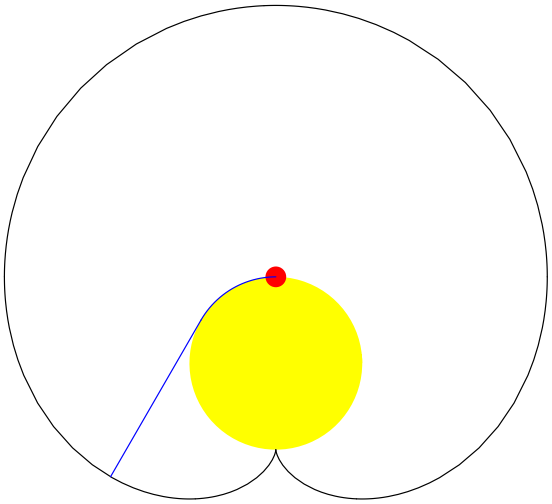
1B Hatsuma: Billy the goat **O. Knill, 7/24/2003**

Dear 1b students,
note that this problem is likely to be too difficult for you to solve now. During your calculus track, you will be taught the necessary tools, which will allow you to attack this problem and make it routine. For now, just think about the problem and find places, where Calculus topics like differentiation or integration are needed to understand the problem. Here it is:

> Billy the goat is attached with a rope of length π to a point at a circular silo of radius 1. What is is the area which Billy can reach?

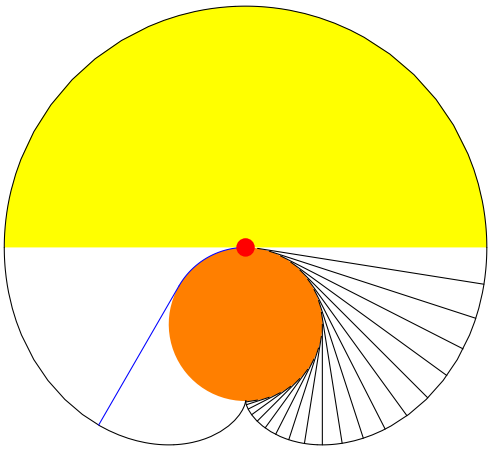
You might also want to "solve" the problem experimentally, by constructing the region approximately on a sheet of and counting the number of grid points which Billy can reach. Discuss the problem with your friends. Ask "senior" students, how they would solve the problem. Challenge them, whether they can do it on a scrap paper during lunch. Play around with the problem. We will discuss the problem in the first class.





Background: The lower part of the curve is called the **circle involute**. It is the **involute** of the circle. It was first studied by Huygens, in the context of building a clock without a pendulum. Having precise clocks on ships was crucial for navigation.

Solution. The upper half circle has area $\pi^3/2$. The rest can be split into two identical regions, each of which can be split into a Riemann sum of triangles of height $\pi - t$ and base $dt(\pi - t)$ and area $(\pi - t)^2/2$. The area is $\pi^3/2 + 2\int_0^\pi (\pi - t)^2/2\, dt = 5\pi^3/6 = 25.8....$



A Pitfall of an advanced student.

- The region above that tangent is half a disc of radius π and has therefore area $\pi^3/2$.
- The area of the rest including silo is $2\int_0^\pi r(t)^2/2\, dt$, where $r(t)$ is the distance from 0. The curve is $(x(t), y(t)) = (\sin(t), \cos(t)) + (\pi - t)(\cos(t), -\sin(t))$. It satisfies $\int_0^{\pi/2} r(t)^2/2\, dt = (4\pi + \pi^3/3)$. Both sides together have an area $3\pi + 5\pi^3/6$.

Problem: the parameter t is not the angular parameter. Can one fix this approach?

Difficulty. Is this a 1b problem? Without discussion, this problem is difficult for a 1b student. Math21a students who routinely deal with curves, tangents, parameterization and integration in polar coordinates could get mislead. The problem can therefore also make senior students thinking. It could be solved routinely using **Green's theorem** and could appeal 21a students also. One of the difficulties for beginners is to realize that there is no routine "area under a graph" formula which does the job.

Origin. This is a famous problem called **"Goat problem"** or **"Bull thethering problem"**. (M.E. Hoffman, "An application of Curvature", American Mathematical Monthly, 105, 55-58, 1988).

Sometimes the goat is inside inside the silo on a shorter rope. Students could spot the problem online. However, solutions offered online are usually quite incomprehensive.

Variations.

- Instead of asking for the area, one could ask, how the accessible region looks like if the rope becomes smaller or larger.
- With a very long rope, the goat decides to circle around the silo in one direction, always keeping the rope tight. How does the curve look like?
- How would the accessible region for Billy look like, if the silo is a square of width and length 1 and billy has a rope of length 2 attached to one of the corners?