§7. Applications to the Theory of Univalent Functions.

1. The method of extremal length is well suited for the determination of univalent (schlicht) functions with certain extremal properties. The well-known distortion theorems which ensue are usually established for the class of all univalent functions in the unit circle which satisfy a normalizing condition at the origin. With the methods we are using it is preferable to consider a more general situation, where the functions are given in an annulus and their values are studied in relation to a second annulus, as explained further below. The familiar results for the circle will then follow as a limiting case.

As a preparation we introduce two invariants attached to the configuration formed by a doubly-connected region and an interior point. Let the region Ω be represented by the annulus $r_1 < |z| < r_2$, and let us denote the inner contour by Γ_1, the outer by Γ_2. The interior point z_0 may be taken to lie on the positive real axis. Two classes of curves will interest us: 1) the class $\{\gamma^*\}$ of connected, closed curves which separate Γ_1 from Γ_2 and the point z_0; 2) the class $\{\gamma\}$ of connected arcs, beginning and ending on Γ_1, which separate z_0 from Γ_2. We wish to determine the extremal lengths $\lambda^*(z_0) = \lambda_\Omega\{\gamma^*\}$ and $\tilde{\lambda}(z_0) = \lambda_\Omega\{\gamma\}$. Note that the latter is identical with the invariant $\tilde{\lambda}_\Omega(z_0, \Gamma_1)$ defined in §6.

The evaluation of these extremal lengths is a problem very similar to the ones solved in §6. However, instead of using an auxiliary transformation we prefer this time to make use of the symmetry alone, a method which could also have been applied in the preceding cases.

We draw the real diameter and denote by E_1 the segment $(-r_2, -r_1)$, by E_2' and E_2'' the segments (r_1, z_0) and (z_0, r_2). The upper halves of Γ_1, Γ_2 and Ω are denoted by Γ_1^+, Γ_2^+, Ω^+. It is claimed that

$$(56) \qquad \begin{aligned} \lambda^*(z_0) &= 2\lambda_{\Omega^+}(E_1, E_2') \\ \tilde{\lambda}(z_0) &= 2\lambda_{\Omega^+}(\Gamma_1^+, E_2'') . \end{aligned}$$

For the proof, choose first an arbitrary ρ in Ω and define in Ω^+ a new metric $\rho + \bar{\rho} = \rho(z) + \rho(\bar{z})$. Any curve γ' which joins E_1 and E_2' within Ω^+ forms, together with its symmetric image, a curve γ^*. Similarly, a curve γ'' which joins Γ_1^+ and E_2'' gives rise to a γ. Hence $L_{\rho+\bar{\rho}}\{\gamma'\} \geq L_\rho\{\gamma^*\}$ and $L_{\rho+\bar{\rho}}\{\gamma''\} \geq L_\rho\{\gamma\}$, while $A_{\rho+\bar{\rho}}(\Omega^+) \leq 2A_\rho(\Omega^+) + 2A_{\bar{\rho}}(\Omega^+) = 2A_\rho(\Omega)$. From this we derive

$$2\,\frac{L_{\rho+\bar{\rho}}\{\gamma'\}^2}{A_{\rho+\bar{\rho}}(\Omega^+)} \geq \frac{L_\rho\{\gamma^*\}^2}{A_\rho(\Omega)}$$

$$2\,\frac{L_{\rho+\bar{\rho}}\{\gamma''\}^2}{A_{\rho+\bar{\rho}}(\Omega^+)} \geq \frac{L_\rho\{\gamma\}^2}{A_\rho(\Omega)} ,$$

and consequently

$$\begin{aligned} \lambda^*(z_0) &\leq 2\lambda_{\Omega^+}(E_1, E_2') \\ \tilde{\lambda}(z_0) &\leq 2\lambda_{\Omega^+}(\Gamma_1^+, E_2'') . \end{aligned}$$

Take now an arbitrary ρ in Ω^+ and extend it by symmetry to the lower half. Given any curve γ^* or γ, we reflect the part that lies in the lower half-plane over the real axis. The resulting curve has the same length, in the metric ρ, as the original, and it contains two arcs joining E_1 and E_2' or Γ_1^+ and E_2'' (Fig. 10). Hence $L_\rho\{\gamma^*\} \geqq 2L_\rho\{\gamma'\}$ and $L_\rho\{\gamma\} \geqq 2L_\rho\{\gamma''\}$, while $A_\rho(\Omega) = 2A_\rho(\Omega^+)$. We obtain

$$\frac{L_\rho\{\gamma^*\}^2}{A_\rho(\Omega)} \geqq 2\,\frac{L_\rho\{\gamma'\}^2}{A_\rho(\Omega^+)}$$

$$\frac{L_\rho\{\gamma\}^2}{A_\rho(\Omega)} \geqq 2\,\frac{L_\rho\{\gamma''\}^2}{A_\rho(\Omega^+)}$$

and

$$\lambda^*(z_0) \geqq 2\lambda_{\Omega^+}(E_1, E_2')$$

$$\tilde{\lambda}(z_0) \geqq 2\lambda_{\Omega^+}(\Gamma_1^+, E_2'') .$$

The relations (56) are thus proved.

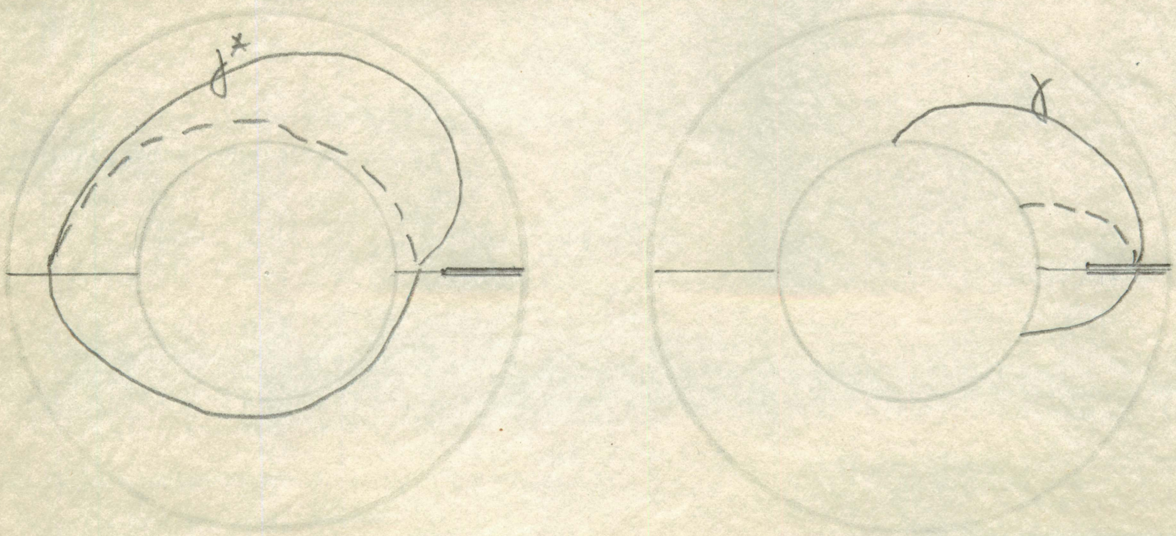

Fig. 10

As a simplification the extremal distances may be referred to the whole annulus. It is in fact obvious that $2\lambda_{\Omega^+}(E_1, E_2') = \lambda^*_{\Omega}(\Gamma_1, \Gamma_2 + E_2'')$ and $2\lambda_{\Omega^+}(\Gamma_1^+, E_2'') = \lambda_{\Omega}(\Gamma_1, E_2'')$, so that (56) can be written in the form

$$\lambda^*(z_0) = \lambda^*_{\Omega}(\Gamma_1, \Gamma_2 + E_2'') \tag{57}$$
$$\tilde{\lambda}(z_0) = \lambda_{\Omega}(\Gamma_1, E_2'') \ .$$

There is of course no need to compute these invariants further, but the following remark ~~as to~~ concerning the second invariant is pertinent. The extremal distance is evaluated by means of a harmonic function with constant values on Γ_1 and E_2'' whose normal derivative vanishes on Γ_2. Because of the symmetry the normal derivative is also zero along the whole real diameter exclusive of E_2. It follows that $\tilde{\lambda}(z_0)$ will not change if an arbitrary cut along the negative real axis is added to the contour Γ_2.

2. We are now ready to apply these results to a univalent function $w = f(z)$ in Ω. The images of Γ_1, Γ_2 and Ω are denoted by Γ_1', Γ_2', Ω', and it is no limitation of the generality to assume that Γ_1' lies inside of Γ_2'. In the w-plane we consider a second annulus Λ: $\rho_1 < |w| < \rho_2$ whose inner and outer contours are called C_1, C_2 respectively. We are interested in two general conditions ~~as~~ pertaining to the relative position of Ω' with respect to Λ: 1) Γ_1' and Γ_2' separate C_1 from C_2; 2) C_1 separates Γ_1' from Γ_2' and Γ_2' separates C_1 from C_2. Under each of these assumptions, ~~how much~~ what can be said about the position of $w_0 = f(z_0)$, where z_0 is a given point in Ω?

Under the first hypothesis Ω' is contained in Λ , and we must have $r_2/r_1 \leqq \rho_2/\rho_1$ as an expression of the fact that the extremal distance from Γ_1' to Γ_2' cannot exceed the extremal distance from C_1 to C_2 . This is trivial, but if we make use of the correspondence between z_0 and w_0 the principle of majoration yields more. In fact, every closed curve which separates Γ_1' from Γ_2' and w_0 automatically separates C_1 from C_2 and w_0 . Hence, by majoration and conformal invariance,

$$\lambda^*(w_0) \leqq \lambda^*(z_0) , \tag{58}$$

where the invariants are understood to be taken with respect to the regions Λ and Ω .

This inequality can be written in a more convenient form if we make use of the extremal function $F(z)$ which maps Ω onto a region obtained from Λ by a radial incision from the outer contour. The function is unique if we require that the cut lie on the positive real axis with its tip corresponding to $z = r_2$. Since $F(z)$ is positive on the positive real axis, it also maps Ω with a cut from $|z_0|$ onto Λ with a cut from $F(|z_0|)$. Replacing z_0 and w_0 by their absolute values, and applying conformal invariance to the right side of the first relation (57), we obtain from (58)

$$\lambda^*(|w_0|) \leqq \lambda^*(F(|z_0|)) ,$$

where both invariants are now taken with respect to Λ . Since $\lambda^*(\rho)$ is clearly a decreasing function of ρ we finally obtain

$$|f(z_0)| \geqq F(|z_0|) , \tag{59}$$

and by its very form this inequality is the best possible. In particular, ~~as~~ on letting z_0 ~~tends~~ tend to the outer boundary we are able to conclude that the shortest distance from Γ_2' to the origin is at least equal to $F(r_2)$.

An opposite inequality,

$$|f(z_0)| \leqq \rho_1\rho_2/F\left(\frac{r_1 r_2}{|z_0|}\right) , \tag{60}$$

is obtained by inversion in both planes. For the sake of completeness it should be shown that all values between the lower and upper bound can be taken. This is easily seen by consideration of the functions $F_\theta(z) = F(z)^\theta F_0(z)^{1-\theta}$, $0 \leqq \theta \leqq 1$, where $F_0(z) = \rho_1\rho_2/F(\frac{r_1 r_2}{z})$. The values lie in Λ , and a study of the change of $\arg F_\theta(z)$ on Γ_1 and Γ_2 reveals that $F_\theta(z)$ is univalent and preserves the relative position of the contours. As θ varies from 0 to 1 the values of $F_\theta(z_0)$ for a positive z_0 cover the whole segment from the maximum (60) to to the minimum (59) .

We turn now to the second hypothesis, which states that Γ_1', C_1, Γ_2', C_2 follow each other in this order, one outside the other. It cannot be inferred that $r_2/r_1 \leqq \rho_2/\rho_1$, but we shall at first make this additional assumption. We are interested only in the maximum of $|f(z_0)|$, and since this maximum is clearly $> \rho_1$ we may suppose that w_0 lies in Λ .

The image of any ~~curve~~ arc in Ω ~~which separates z_0 from Γ_2 contains an arc separating w_0 from C_2~~ of the class whose extremal length was denoted by $\bar\lambda(z_0)$ contains an arc of the corresponding class with respect to ~~[illegible]~~ w_0 and C_2. Hence, by the principle of majoration,

$$\tilde{\lambda}(w_0) \leqq \tilde{\lambda}(z_0) .$$

We note now that the function $-F(-z)$ maps Ω with a cut along the positive real axis from $|z_0|$ to the outer boundary onto Λ with two cuts, one on the negative axis from $-F(r_2)$ to $-\rho_2$ and one on the positive axis from $-F(-|z_0|)$ to ρ_2. According to a previous remark $\tilde{\lambda}(-F(-|z_0|)) = \tilde{\lambda}(|z_0|)$, and we obtain

$$\tilde{\lambda}(|f(z_0)|) \leqq \tilde{\lambda}(|F(-|z_0|)|)$$

or, in view of the monotonic character,

$$|f(z_0)| \leqq |F(-|z_0|)| . \tag{61}$$

Again, this inequality cannot be improved.

We recall that (61) was proved and has a meaning only if $r_2/r_1 \leqq \rho_2/\rho_1$, for only then does the function $F(z)$ exist. Our two main results are gathered in the following theorem:

<u>Theorem</u>. <u>Suppose that</u> $w = f(z)$ <u>is univalent in</u> $r_1 < |z| < r_2$ <u>and preserves the orientation of the contours. Then, if the image of this annulus lies in</u> $\rho_1 < |w| < \rho_2$ <u>and separates the bounding circles,</u>

$$|f(z)| \geqq F(|z|) ,$$

where the extremal function $F(z)$ maps the two annuli onto each other except for a radial cut to the outer boundary of the second annulus whose tip corresponds to $z = r_2$.

If, on the other hand, the mapping is such that the image of $|z| = r_2$ lies between $|w| = \rho_1$ and $|w| = \rho_2$, and the image of $|z| = r_1$ lies inside the first circle, and if we suppose that $r_2/r_1 \leqq \rho_2/\rho_1$, then

$$|f(z)| \leqq |F(-|z|)|,$$

with the same function $F(z)$ as above.

In both cases we may allow ρ_2 to become infinite. $F(z)$ will then map $r_1 < |z| < r_2$ onto a region bounded by $|w| = \rho_1$ and an infinite radial cut, and the inequalities subsist.

3. The case of a univalent function in $|z| < r_2$, normalized by the conditions $f(0) = 0$, $|f'(0)| = 1$, can now be handled very easily. For greater generality we may again assume that $|f(z)| < \rho_2$, letting $\rho_2 = \infty$ correspond to the unrestricted case. The existence of a normalized function implies $\rho_2 \geqq r_2$.

We draw a small circle $|z| = r_1$, and denote by ρ_1' the minimum, by ρ_1'' the maximum of $|f(z)|$ on this circle. As r_1 tends to zero $\rho_1' \sim r_1$ and $\rho_1'' \sim r_1$. It is clear that $f(z)$ satisfies our first hypothesis with $\rho_1 = \rho_1'$ and the second hypothesis with $\rho_1 = \rho_1''$. Moreover, for sufficiently small r_1 the condition $r_2/r_1 \leqq \rho_2/\rho_1$ is fulfilled. Hence, by the results of the preceding section,

$$F_1(|z|) \leqq |f(z)| \leqq |F_2(-|z|)|,$$

where F_1 and F_2 are the extremal functions for $\rho_1 = \rho_1'$ and $\rho_1 = \rho_1''$. As r_1 tends to zero it is not hard to see that F_1 and F_2 both tend to the same limit function $F(z)$. This function is normalized, real on the real axis, and maps $|z| < r_2$ onto $|w| < \rho_2$ with a cut along the positive axis. From the preceding inequality we obtain in the limit

$$F(|z|) \leqq |f(z)| \leqq |F(-|z|)|, \tag{62}$$

a relation which is part of the conventional distortion theorem. In the simplest case $r_2 = 1$, $\rho_2 = \infty$ it takes the explicit form

$$\frac{|z|}{(1+|z|)^2} \leqq |f(z)| \leqq \frac{|z|}{(1-|z|)^2}. \tag{63}$$

Usually these inequalities are accompanied by corresponding relations for $|f'(z)|$ and $|f'(z)|/|f(z)|$. A more refined treatment would have yielded these results as well, but we feel that our method, which we consider more important than the results, has been made sufficiently clear as it is.

4. We wish to complete our treatment by a discussion of the situation which arises under our second hypothesis when $r_2/r_1 > \rho_2/\rho_1$. This case is rather elementary, and we shall not need the notion of extremal length.

It is not hard to make a reasonable guess ~~as to~~ concerning the extremal function which will make $|f(z_0)|$ a maximum. Assuming $z_0 > 0$ we conjecture that the extremal function $\Phi(z)$ maps the annulus Ω, under preservation of symmetry with respect to the real axis, onto a region bounded by $|w| = \rho_2$ and an arc of the circle $|w| = \rho_1$, the center of this arc being at $w = \rho_1$. The existence of such a function is obvious, for as the arc increases from a point to the whole circle, the extremal distance of the contours will decrease from ∞ to $\frac{1}{2\pi}\log \rho_2/\rho_1$, and must hence pass through the value $\frac{1}{2\pi}\log r_2/r_1$.

The function $\Phi(z)$ is zero at a point ξ on the negative axis. The quotient $f(z)/\Phi(z)$ is regular in Ω except, at most, for a simple pole at ξ, and its boundary values are $\leqq 1$ in absolute value. We make now use of an interesting observation, due to R. M. ~~M.~~ Robinson [ ], namely that such a function remains $\leqq 1$ on the ray opposite to the pole. To see this, we form the function $f(z)\overline{f(\bar z)}/\Phi(z)^2$, suggestion $f+\bar f$ which is real on the real axis, and consider the region, surrounding ξ, in which it is of absolute value $\geqq 1$. We maintain that this region cannot be doubly-connected, and hence cannot meet the positive axis. In fact, on each contour the variation of the argument of our function is twice the change of $\arg f(z)/\Phi(z)$ and hence a multiple of 4π. On the other hand, this variation is necessarily negative. Hence, with two contours the total change would be $\leqq -8\pi$, in violation of the fact that the function has only a double pole. We conclude that $|f(z)| \leqq \Phi(z)$ on the positive real axis, and, more generally,

$$|f(z)| \leqq \Phi(|z|) \tag{64}$$

for any position of z.

The reader is encouraged to find, in a similar way, upper and lower bounds for a univalent function which maps $|z| = r_1$ inside of $|w| = \rho_1$ and $|z| = r_2$ outside of $|w| = \rho_2$. There are two cases, depending on whether one allows the function to become infinite or not.

5. As a last related problem we treat a question raised and answered by Teichmüller [ ]. Suppose that the inner complement of a doubly-connected region Ω contains the circle $|w| = \rho_1$ and a certain point w_1, while the outer complement contains $|w| = \rho_2$ and a point w_2. For given absolute values $|w_1|, |w_2|$, what is the greatest possible extremal distance between the contours of Ω?

Suppose that Ω can be mapped onto the annulus $r_1 < |z| < r_2$, and consider also a map of the region Ω_1 contained between the inner contour of Ω and the outer circle $|w| = \rho_2$ onto an annulus $r_1' < |\zeta| < r_2'$. In the latter mapping, let ζ_2 be the point corresponding to w_2 (Fig. 11).

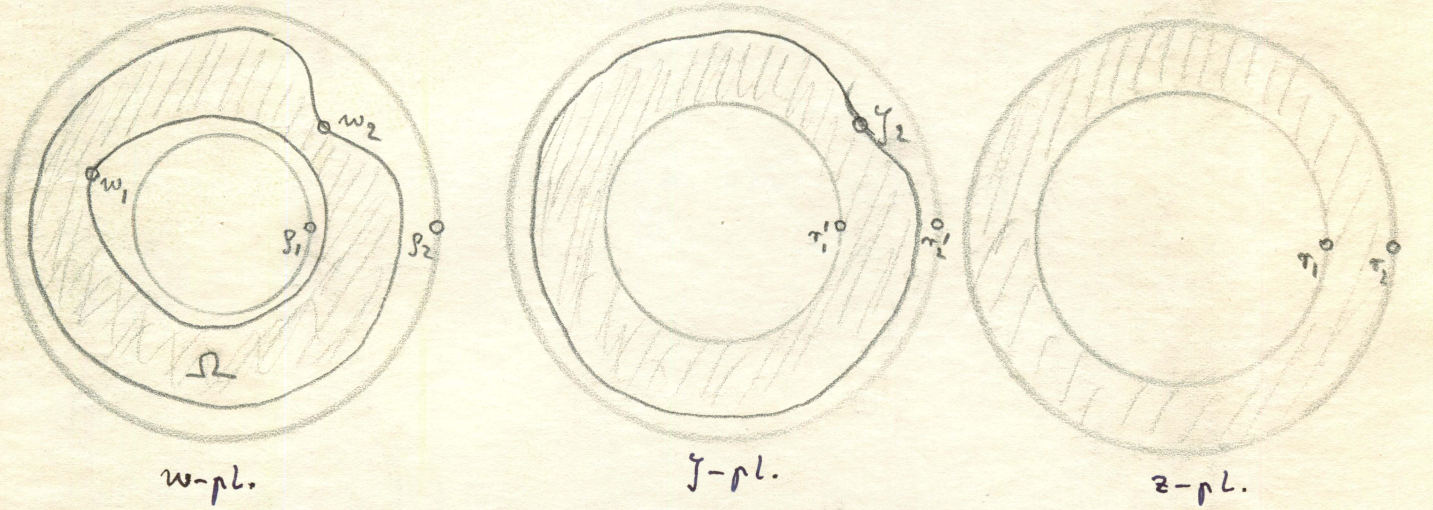

Fig. 11

There are two extremal mappings to be considered: 1) the function $\zeta = F(z)$ which maps $r_1 < |z| < r_2$ onto $r_1' < |\zeta| < r_2'$ with a radial slit from the outer circle; 2) the function $w = F_1(\zeta)$ which maps $r_1' < |\zeta| < r_2'$ onto $\rho_1 < |w| < \rho_2$ with a slit from the inner circle. Both functions are chosen so as to be positive on the positive real axis, negative on the negative real axis.

By the first part of the Theorem of no. 2 , applied to the mapping from the z-plane to the ζ-plane, we conclude that

$$|\zeta_2| \geqq F(r_2) .$$

Similarly, the same result, as modified by inversions in both planes, yields

$$|w_1| \leqq F_1(r_1')$$

when applied to the mapping of the ζ-annulus onto Ω_1 . To this latter mapping the second part of the same theorem is also applicable and gives

$$|w_2| \geqq -F_1(-|\zeta_2|) .$$

Together, the three inequalities obviously imply

$$|w_2| \geqq -F_1(-F(r_2))$$
$$-|w_1| \geqq -F_1(-F(-r_1)) .$$

The composite function $-F_1(-F(z))$ is readily seen to map $r_1 < |z| < r_2$ onto the given annulus in the w-plane with two cuts, one on the positive axis from the outer circle and one on the negative axis from the inner circle. Our inequalities express that these cuts extend at least to the points $|w_2|$ and $-|w_1|$ respectively. Consequently, the ratio r_2/r_1

is a maximum when Ω is the region obtained by drawing the cuts exactly to these points.[1)]

The simplest and most interesting case arises when $\rho_1 = 0$ and $\rho_2 = \infty$. It is then known only that Ω separates 0 and w_1 from ∞ and w_2. The maximum extremal distance is the one between the

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[1)] Footnote added 1957:

The last reasoning should perhaps be spelled out a little more clearly. Let us write $-F_1(-F(r_2)) = m_2$, $F_1(-F(-r_1)) = m_1$. These are positive numbers, and we have found that Ω is conformally equivalent to the region obtained by cutting $\rho_1 < |w| < \rho_2$ along the real axis from ρ_2 to m_2 and from $-\rho_1$ to $-m_1$. Consider more generally the same annulus $\rho_1 < |w| < \rho_2$ with slits to M_2 and $-M_1$, and let $R_2:R_1$ be the ratio in a conformally equivalent annulus. This ratio is a certain function of M_1, M_2, say $R_2:R_1 = \Phi(M_1,M_2)$. It is clearly an increasing function of M_2 and a decreasing function of M_1. Therefore the inequalities that we have proved imply $r_2:r_1 = \Phi(m_1,m_2) \leqq \Phi(|w_1|,|w_2|)$. This is the precise statement of the theorem.

segments $(-|w_1|, 0)$ and $(|w_2|, \infty)$. This is a universal increasing function of the ratio $\rho = |w_2|/|w_1|$ which we shall denote by $\Lambda(\rho)/2$. Without the factor $1/2$ it represents the extremal distance of the segments with respect to the upper half-plane. Since complementary intervals have reciprocal extremal distances, we obtain the functional relation $\Lambda(\rho)\Lambda(1/\rho) = 1$ and, in particular, $\Lambda(1) = 1$. This last result implies a rather interesting consequence: <u>Any doubly-connected region whose contours have an extremal distance</u> $> 1/2$ <u>contains the periphery of a circle around any point of the inner complement</u>.

Still following Teichmüller we proceed to deduce the best possible form of an inequality originally due to Ahlfors [ ]. A simply-connected region Ω in the z-plane is mapped onto the strip $0 < \eta < 1$ of the ζ-plane. It is assumed that $\xi = -\infty$ corresponds to a boundary point or prime end to the left of $x = x_1$, while $\xi = +\infty$ corresponds to a point to the right of $x = x_2$. On each vertical line of abscissa x from the interval (x_1, x_2) there exists at least one cross-cut θ_x of Ω which separates these two prime ends. Assuming, merely for the sake of simplicity, that there is just one θ_x we denote its length by $\theta(x)$. In §3.4 it was proved that the extremal distance of θ_{x_1} θ_{x_2} is at least equal to

$$\int_{x_1}^{x_2} \frac{dx}{\theta(x)} .$$

Denote now the maximum of ξ on θ_{x_1} by ξ_1 and the minimum of ξ on θ_{x_2} by ξ_2 (fig. 12). We wish to derive a lower bound for $\xi_2 - \xi_1$. To this end we reflect the whole configuration in the

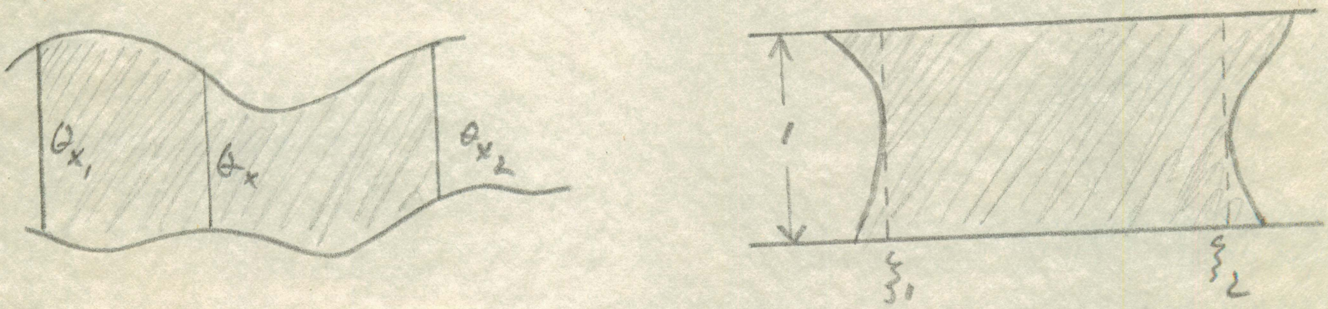

Fig. 12

ζ-plane over the real axis and transform the double strip by means of the function $w = e^{\pi\zeta}$. The symmetric images of θ_{x_1} and θ_{x_2} will then bound a doubly-connected region to which our preceding result can be applied. Because of the symmetry the extremal distance of the contours is exactly half the extremal distance of θ_{x_1} and θ_{x_2}. We may hence conclude that

$$\int_{x_1}^{x_2} \frac{dx}{\theta(x)} \leqq \Lambda\left(e^{\pi(\xi_2-\xi_1)}\right) \tag{65}$$

or, introducing the inverse function of $\Lambda(\rho)$,

$$\xi_2 - \xi_1 \geqq \frac{1}{\pi} \psi \Lambda^{-1}\left(\int_{x_1}^{x_2} \frac{dx}{\theta(x)}\right). \tag{66}$$

ψ log

This is the precise form of Ahlfors' inequality.

For $\xi_1 = \xi_2$ (65) yields the interesting relation

$$\int_{x_1}^{x_2} \frac{dx}{\theta(x)} \leqq 1 ,$$

where x_1 and x_2 are now to be interpreted as the minimum and maximum of x on the inverse image of a segment with constant ξ . Naturally, all these inequalities represent best possible results.

In most applications a weaker but more manageable form of (66) is preferable. For this and other purposes we shall, in the next section, derive upper and lower bounds for $\Lambda(\rho)$ in terms of elementary functions.

6. The extremal distance of two arcs on a circle or two segments with respect to the circle or half-plane is a function only of the cross-ratio of the four endpoints. If the arcs are (a,b) and (c,d) , in positive cyclic order, we write

$$\rho = - \frac{c-b}{a-b} : \frac{c-d}{a-d} .$$

This is a positive number, and the arcs can be transformed into the segments $(-1,0)$ and (ρ,∞) of the real axis. The extremal distance of the arcs is hence $\Lambda(\rho)$.

We transform the arcs to the unit circle, choosing the linear transformation so as to give the new arcs $(-\theta,\theta)$ and $(\pi-\theta, \pi+\theta)$. Computing the cross-ratio we have $\rho = \cot^2\theta$. The extremal distance with respect to the circle is twice the extremal distance with respect to the whole plane. The latter is greater than the extremal distance of the orthogonal circles constructed over the given arcs (fig. 13) . When these are transformed into concentric circles, the ratio of the radii is found to be $\cot^2 \theta/2$. Hence we obtain

$$(67) \qquad \Lambda(\rho) \geqq \frac{2}{\pi} \log \cot \frac{\theta}{2} .$$

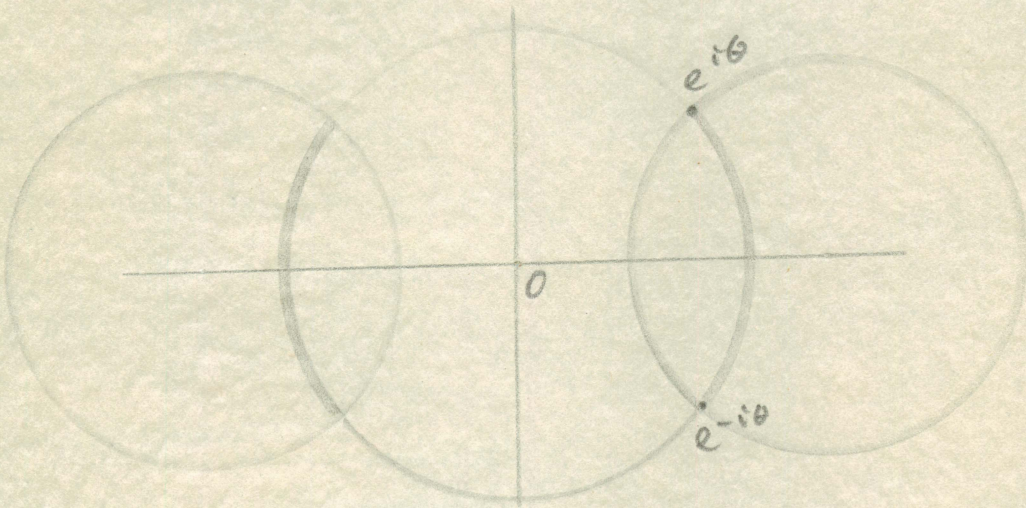

Fig. 13

In order to obtain an upper bound we note first that $\frac{1}{4}\Lambda(\rho)$ is ~~also~~ the extremal distance between the right arc and the imaginary axis. Now map the whole plane outside of the right hand arc onto the outside of a circle enclosing the point 1 which is orthogonal to the unit circle and symmetric with respect to the real axis. Letting 0 and ∞ remain fixed, we retain symmetry with respect to the unit circle and the real axis. Simple computations show that the orthogonal circle will intersect the unit circle at the points of argument $\pm \theta/2$, and that the image of the imaginary axis will lie entirely in the ~~left~~ right half-plane. For this reason $\frac{1}{4}\Lambda(\rho)$ is less than the extremal distance between the orthogonal circle and the imaginary axis. Using the value given above we find

(68) $$\Lambda(\rho) \leqq \frac{2}{\pi} \log \cot \frac{\theta}{4} .$$

These inequalities are reasonably accurate for large ρ . For small values we replace them by

(69) $$\frac{2}{\pi} \log \operatorname{tg} \frac{\pi+\theta}{4} < 1/\Lambda(\rho) < \frac{2}{\pi} \log \operatorname{tg} \frac{\theta}{2} ,$$

drawn from the functional equation $\Lambda(\rho)\,\Lambda(1/\rho) = 1$.

Inverting the inequality (67) we obtain, very easily,

(70) $$\rho < \sinh^2 \frac{\pi}{2} \Lambda(\rho),$$

and from (68) , after some computation,

(71) $$\rho > \frac{1}{4} \left(\sinh \frac{\pi}{2} \Lambda(\rho) - \frac{1}{\sinh \frac{\pi}{2} \Lambda(\rho)} \right)^2$$

the last inequality being valid only for $\sinh \frac{\pi}{2} \Lambda(\rho) \geqq 1$. From (70) and (71) we can derive simpler inequalities

(72) $$\Lambda(\rho) - C_1 < \frac{1}{\pi} \log \rho < \Lambda(\rho) + C_2$$

valid as soon as $\Lambda(\rho)$ is greater than a third constant C_3 . Numerical values can be assigned to the three constants, but are of comparatively little interest.

Returning ~~Reaching back~~ to (66) we can now conclude that

$$\xi_2 - \xi_1 > \int_{x_1}^{x_2} \frac{dx}{\theta(x)} - C_1 \tag{73}$$

as soon as

$$\int_{x_1}^{x_2} \frac{dx}{\theta(x)} > C_3 .$$

This inequality has well-known applications to the theory of entire functions.