Lie groups

Gleason 1957

Ch I. Analytic functions

1.1. All variables real.

$p \in E^n \qquad p = \langle p_1, \dots, p_n \rangle$

$F : E^n \to E^m$

$F(p) = \langle F_1(p), \dots, F_m(p) \rangle$

1.2. Power series $\sum_{i_1 \dots i_n = 0}^{\infty} a_{i_1 \dots i_n} x_1^{i_1} \dots x_n^{i_n}$

or $\sum (a)(x)$

Conv. means abs. conv.

1.3. Def. F real def. on part of E^n.

Anal. if $F(x) = \sum (a)(x-p)$ repr. F in a neighb. of p.

1.4. Lemma. $\sum \rho_1^{i_1} \dots \rho_n^{i_n} x_1^{i_1} \dots x_n^{i_n}$

conv. for $|x_k| < \frac{1}{\rho_k}$ and equals

$\Pi (1 - \rho_k x_k)^{-1}$

Lemma: Term-wise diff. of geom. series

1.5. Lemma. $\sum (a)(x)$ defines anal. fn. at 0 $\Longleftrightarrow \exists\, C$ and ρ such that

$$|a_{i_1 \ldots i_n}| \leq C \rho^{i_1 + \ldots + i_n}$$

1.6. Lemma: Term wise diff.

Notation: $D_k F = \dfrac{\partial F}{\partial x_k}$

1.7. Uniqueness of power series.

1.8. Mean-value theorem

$$F(x_1 \ldots x_n) = \sum_{i_1 + \ldots + i_n \leq m} \frac{x_1^{i_1} \ldots x_n^{i_n}}{i_1! \ \ldots \ i_n!} (D_1^{i_1} \ldots D_n^{i_n} F)(0, \ldots 0)$$

$$+ \sum_{i_1 + \ldots + i_n = m+1} \quad '' \qquad '' \quad (\theta x_1, \ldots \theta x_m)$$

for some $0 < \theta < 1$

Lemma.)

1.9. F anal. at 0 if and only if for some C, ρ

$$(*) \qquad |F^*(\rho)| \leq C\rho^m\, m!$$

in a neighb. (m is total degree of derivative)

Proof. If F anal. it is dominated by $\sum C\rho^{i_1+\dots+i_n} x_1^{i_1}\dots x_n^{i_n}$. Follows

$$|F^*(\rho)| \leq {}^*\text{Der. of } \prod (1-\rho x_i)^{-1}$$

$$\leq C\rho^m \cdot m!$$

If (*) and $|x_i| < \frac{1}{2n\rho}$ then remainder in 1.8

$$\leq C\rho^{m+1} (\max |x_i|)^{m+1} \sum_{i_1+\dots+i_n=m+1} \frac{(m+1)!}{i_1!\dots i_n!}$$

$$\leq \frac{C}{(2n)^{m+1}} n^{m+1} < \frac{C}{2^{m+1}} \rightarrow 0$$

1.10. Cor. If F is anal. at 0 and Taylor series conv. on N, then F anal on Int N. (rather on the comp. of 0)

1.11. Lemma. F anal. at $\langle a_1, \dots a_n \rangle$, G anal. at $\langle b_1, \dots b_m \rangle$ and $G(b_1, \dots b_m) = a_1$ then

$$H(y_1 \dots y_m, x_2 \dots x_n)$$

$$= F(G(y_1 \dots y_m), x_2 \dots x_n)$$

is anal. at $\langle b_1, \dots b_m, a_2 \dots a_n \rangle$ and power series is obtained by substitution.

1.12. If F anal. then

$G = F(x_1, \dots x_{n-1}, x_{n-1})$ is anal.

1.13 Let $\Phi_1, \ldots, \Phi_n$ be anal. of p var. at $\langle b_1, \ldots b_p \rangle$. Let $a_i = \Phi_i(b_1, \ldots b_p)$. Let $F_1, \ldots F_n$ be anal in $n+p$ var. near $\langle a_1 \ldots a_n\ b_1 \ldots b_p \rangle$.

[Then system

$$\frac{dy_i}{dt} = F_i(y_1 \ldots y_n, h_1, \ldots h_p)$$

$$y_i(0) = \Phi_i(h_1, \ldots, h_p)$$

has unique solution]

Then there exist functions $y_1, \ldots y_n$ in $p+1$ variables anal at $\langle 0, b_1, \ldots b_p \rangle$ such that

$$(D_1 y_i)(t, h_1, \ldots, h_p) = F_i(y_1(t, h_1, \ldots h_p) \ldots y_n(t, h_1 \ldots h_p))$$

and $y_i(0, h_1, \ldots h_p)$

$$= \Phi_i(h_1 \ldots h_p)$$

in a suitable neighborhood of $\langle 0, b_1, \ldots b_p \rangle$.

In this neighborhood the functions y_i are unique in the class of functions with cont. derivative in the first argument.

Consider system

$$\begin{cases} \dfrac{dy_i}{dt} = F_i(y_1, \dots y_n) = \sum (\&)(y) \\ y_i(0) = 0 \end{cases}$$

Try solutions

$$y_i(t) = \sum_{k=0}^{\infty} a_{i,k} t^k .$$

If solution exists we can compute coeff. one at a time. Hence

Exist. $\Rightarrow$ unicity in class of anal. functions.

Take comparison eq.

$$\frac{dZ_i}{dt} = \sum B_{i,j_1 \cdots j_n} Z_1^{j_1} \cdots Z_n^{j_n}$$

when $B_{i,(j)} > |b_{(j)}|$, $Z_i(0) = 0$.

Inductively, $|a_{i,k}| < A_{i,k}$

Now choose, if $|b_{i(j)}| \leq C\rho^m$

we take $B_{i(j)} = C\rho^{j_1 + \cdots + j_n}$

Now eqn is

$$(DZ_i)(t) = C \frac{1}{(1-\rho Z_1)\cdots(1-\rho Z_n)}$$

$Z_1 = \cdots Z_n = Z$ is solution

if $\dfrac{dZ}{dt} = \dfrac{C}{(1-\rho Z)^n}$

$$Z(t) = \frac{1}{\rho}\left(1 - (1+(n+1)Ct)^{\frac{1}{n+1}}\right)$$

Proves existence

Unicity proof for ~~diff.~~ cont. fns.

We can find M and a convex neighborhood of O such that

$$a \in N \;,\; b \in N$$

$$\sum_1^n |F_i(a_1 \ldots a_n) - F_i(b_1 \ldots b_n)| \leq M \sum_1^n |a_i - b_i|$$

Suppose $y_i(t)$, $z_i(t)$ are two solutions.

$$y_i(t) = \int_0^t F_i(y)\, ds$$

$$z_i(t) = \int_0^t F_i(z)\, ds$$

$$|y_i - z_i| \leq M \int_0^t \sum |y_i - z_i|\, |ds|$$

Can choose u so that

$$|t| < u \Rightarrow y \in N,\; z \in N$$

Then

$$K = \max_{|t| \leq u} \mu(t)$$

$$\mu(t) = \sum |y_i(t) - z_i(t)|$$

Prove $\quad \mu(t) \leq \frac{K M^\alpha}{\alpha!} |t|^\alpha \quad$ for all α.

Suppose for β

$$|\mu| \leq M \int K \frac{M^\beta |s|^\beta}{\beta!} ds$$

$$= K \frac{M^{\beta+1} |t|^{\beta+1}}{(\beta+1)!} \quad \cdots \longrightarrow 0.$$

Hence $\max \mu = 0$

Introduce parameters

$$DY_i = \sum b_{i(j)} Y_1^{j_1} \cdots Y_n^{j_n}$$

and compared with

$$DY_i = \sum B_{i(j)} \overline{\phantom{Y_1^{j_1} \cdots Y_n^{j_n}}}$$

$B_{i(j)}$ have to be power-series which dominate the $b_{i(j)}$

1.14 Lemma. Suppose that the real functions $b_1, \ldots b_n$ are defined on $S - \{q\}$ (S topol. space, $q \in \overline{S - \{q\}}$). a_{ij} are def. on S and cont. at q, $\det a_{ij}(q) \neq 0$. If $c_1, \ldots c_n$ are constants such that

$$\sum a_{ij}(p) b_j(p) = c_i$$

for $p \neq q$, then $\lim\limits_{p \to q} b_j(p)$ exists

$$= \sum A_{ki} c_i , \quad (A_{ki}) = (a_{ij})^{-1}.$$

Proof. det $a_{ij}(p)$ bdd away from 0 in a neighb. N.

Then

$$b_j(p) = \sum A_{hi}(p)\, c_i$$

for $p \in N$. Here A_{hi} are cont. at q.

1.15. Theorem.

Suppose $F_1, \ldots F_n$ are anal. of n variables on ~~$\mathcal{D}$ and $p \in$ int D.~~ E and $p \in$ Int E. Suppose that the Jacobian at p is $\neq 0$. Then the mapping F is a homeomorphism of a neighb. $N(p)$ onto a neighb. of $q = F(p)$, and furthermore the inverse mapping F^{-1} is defined by functions analytic at q.

$$F^{-1} : (y_1, \ldots y_n) \to (g_1(y), \ldots g_n(y))$$

analytic

Proof. We first show that F is 1-1 on a small neighb.

Select N convex, closed compact so that Jacobian is not zero even if each partial deriv. is taken at a diff. point of N.

a and b distinct in N.

$$F_i(a) - F_i(b) = \sum (a_j - b_j) D_j F_i (p_i)$$

for p_i between a and b.

Since det. is not zero, all $F_i(a) - F_i(b)$ are not zero and mapping is 1-1.

Next, prove that $F(N)$ covers a neighb. of q.

$B = \mathrm{Bd}\, N$

$q \notin F(B)$

q has pos. dist. 2ρ from $F(B)$.

Then $F(N)$ covers any point within ρ of q.

Suppose dist $(a, q) < \rho$.

Form $s(x_1, \dots x_n) = \sum \left(F_j(x) - a_j\right)^2$

on N.

Has a min. for z, say.

1) z is boundary pt, not true.

2) or, $(D_i s)(z) = 0$ all i

$$\sum D_i F_j(z) \left(F_j(z) - a_j\right) = 0$$

$$\Longrightarrow \quad F_j(z) = a_j$$

Prove that inverse is anal.

G_i are cont. and dif. on ~~that $F(N)$~~, a convex open set N_2

Let $h_1, \dots h_n$ be parameters

and $c = F(a) \in N_2$

$$H_i(t) = G_i(c_1 + th_1, \dots, c_n + th_n)$$

$$c_i + t_0 h_i = F_i\left(H_1(t_0), \dots H_n(t_0)\right)$$

$$c_i + t_1 h_i = F_i\left(H_1(t_1) \; - \; - \; \right)$$

$$(t_1 - t_0) h_i = \sum_j D_j F_i(\xi_i), \quad \xi \in N$$

$$\cdot \left(H_j(t_1) - H_j(t_0)\right)$$

By lemma we get existence

of $DH_j =$

$$DH_i = \sum_p A_{ip} h_p \quad \text{at } t_0$$

$$A_{ip} = \text{inv. of} \quad D_j F_i(H_i)$$

These are system of diff. equations for the H_i which are anal. in the parameter h_p.

Initial condit

$$H_i(0) = a_i$$

$g_i(c+th)$ is anal in

$t, h_1, \ldots, h_p$

Hence $g_i(x)$ anal. near a.

2. Covering spaces.

2.1. Def. of locally conn. sp.

2.2. A space is loc. conn. if and only if each comp. of an open set is open.

2.3. Def. Arc-wise connected

Any point has a largest arc-wise connected component.

2.4. Def. Locally arc-wise conn.

Connected and loc. arc-wise conn. $\Longrightarrow$ connected.

[If S is complete metric space and locally conn. it is locally arc-wise connected.

Is important but difficult theorem. Also for compact.

2.5. Simply connected if given a cont. map of unit circle into S can be extended continuously over the unit disk.

2.6. A set $X \subset S$ is relatively simply connected if every circle in X is homotopic to a constant in S.

2.7. A space is semi-locally simply connected if every point has a relatively simply connected neighborhoods.

(Stick to metric spaces from now on).

The radius of relative simple connectivity at $x \in S$

$$\sigma(x) = \text{lub}\,\{r \mid r\text{-sphere rel. simply conn.}\}$$

The open sphere of radius $\sigma(x)$ is rel. simply conn.

$$|\sigma(x) - \sigma(y)| \leqq \rho(x,y)$$

2.8. S a conn. loc. arcwise conn. and some fixed pt. $s_0 \in S$.

Consider class ~~$\mathcal{F}$~~ F of all cont. fns on $[0,1]$ to S with $f(0) = s_0$.

F becomes a top. space.

if

$$\rho(f,g) = \text{lub}\ \rho(f(t), g(t)).$$

An arc in F is $s \to f_s$

or as a mapping of $[0,1] \times [0,1] \to S$

$$\langle s,t \rangle \to f_s(t).$$

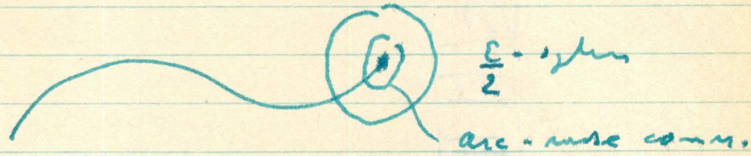
Homotopies in S are arcs in F.

Let $\rho : F \to S$

be given by $\rho(f) = f(1)$

ρ is <u>continuous</u>: clear.

ρ is <u>open</u>: Given f and $\varepsilon > 0$ we must show that the image of the ε-neighb. of f contains a neighb. of $\rho(f) = f(1)$



To prove, consider

N = arc-wise conn. comp. of the $\frac{\varepsilon}{2}$-sphere about $f(1)$

Given $q \in N$, join $f(1)$ to q by an arc α

$f(t) \in N$ for $u \leq t \leq 1$.

Set $g(t) = f(t)$ for $t \leq u$

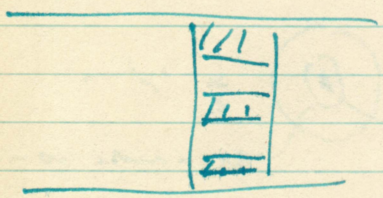
$$g(t) = f(u + 2(t-u)) \quad \text{for } u \leq t \leq \frac{1+u}{2}$$

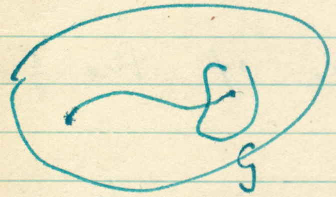
$$g(t) = \alpha\left(\frac{t - \frac{1+u}{2}}{\frac{1-u}{2}}\right) \quad \text{for } \frac{1+u}{2} \leq t \leq 1$$

$\rho(f, g) \leq \varepsilon$

2.9. If S is semi-locally simply connected, then for any open $\mathcal{G} \subset S$ the arc-components of $p^{-1}(\mathcal{G})$ are open in F.

<u>Proof:</u>





Given an arc f with $f(1) \in \mathcal{G}$ i.e. $f \in p^{-1}(\mathcal{G})$ we must find $\delta > 0$ such that if h is an arc with $\rho(f, h) < \delta$ then h is in the same arc component of $p^{-1}(G)$ which means that h can be deformed into f with the end point remaining in G.

Let $\eta = \min_t \sigma(f(t))$ (See 2.7)

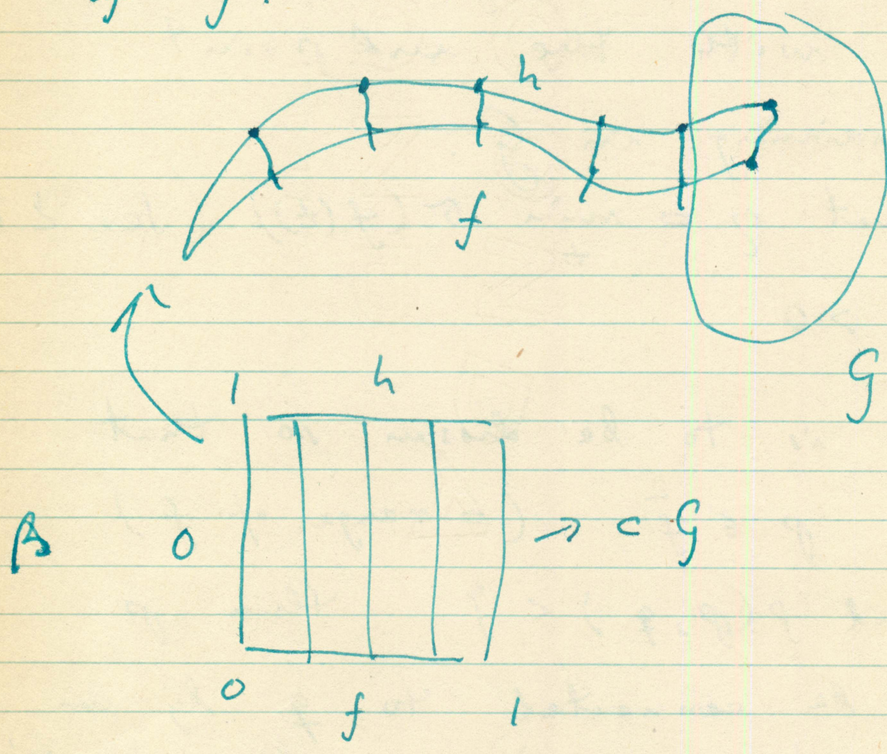
$\eta > 0$

ξ is to be chosen so that if $p \in \bar{f}$ (= range of f) and $\rho(p, q) < \xi$ then p can be connected to q by an arc which remains in the $\frac{1}{2}\eta$-sphere about p.

For each point this can be done by local ~~connectivity~~ arc wise connectivity, and by compactness can be done uniformly.

The ξ-sphere about $f(1)$ is arc-wise connected in $G \cap \frac{1}{2}\eta$-sphere about $f(1)$

Now we prove assertion for any arc h within G of f.



Choose n so that $\rho(f(t), f(t')) < \frac{\varepsilon}{2}$

if $|t - t'| < \frac{1}{n}$

Connect $f(\frac{i}{n})$ to $h(\frac{i}{n})$ by arc

in $\frac{1}{2}\varepsilon$-region about $f(\frac{i}{n})$

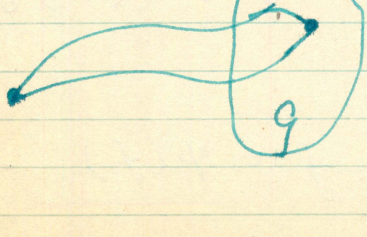
Each of the rectangular perimeters are loops in the z-sphere about $f(\frac{i}{n})$.

This region is rel. simply connected and the mapping β can be extended over each rectangle.

Then $f_s(t) = \beta(s,t)$ defines the required arc-wise connection.

2.10. If G is relatively simply connected ~~and arc-wise connected~~ and $s \in G$, then for any arc-component A of $p^{-1}(G)$

~~$p^{-1}G$~~ $p^{-1}(s) \cap A$ is arc wise conn.

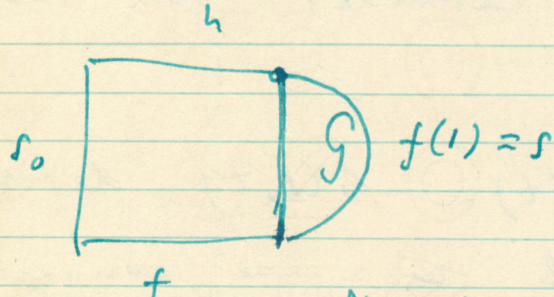


(means if you can deform with end point remaining in G, then you can do it with fixed end point)

Proof. f and h two arcs

$$f(1) = h(1) \quad , \quad f, h \in A$$

There is a cont. map of $[0,1] \times [0,1]$



Mapping can be extended over G.

2.11. Let $\mathcal{U}$ be the set of all arc components of sets $p^{-1}(s)$, $s \in S$.

Clearly there are unique mappings $\psi : F \to \mathcal{U}$ and $\varphi : \mathcal{U} \to S$ such that

$$p = \varphi \circ \psi \quad \text{and} \quad f \in \psi(f)$$

Introduce topology into U by definition

$X \subset U$ is open $\Longleftrightarrow$

$\psi^{-1}(X)$ is open in F

Now φ and ψ are continuous

Trivial for ψ

Let G be open in S.

$\varphi^{-1}(G)$ is open $\Longleftrightarrow$

$\psi^{-1} \circ \varphi^{-1}(G)$ is open

$= p^{-1}(G)$ because p is continuous.

φ ~~and ψ are~~ is open

Let X be open in U

$\varphi(X) = \varphi \circ \psi \circ \psi^{-1}(X)$

$= p \circ \psi^{-1}(X)$ open

because p is open

Separation axiom T_2 (Hausd.)

If $p, q \in U$ and $\varphi(p) \neq \varphi(q)$ choose disjoint neighb. ~~of~~ P, Q of $\varphi(p)$, $\varphi(q)$ and then $\varphi^{-1}(P)$ and $\varphi^{-1}(Q)$ satisfy conditions.

If $\varphi(p) = \varphi(q)$, I will prove more than separation:

If G is rel. simply conn. neighb. of $\varphi(p)$ then ~~p, q~~ $\varphi^{-1}(G)$ falls into disjoint open sets, each mapped 1-1 homeomorphically onto G.

(Replace p by r)

$r \neq q \qquad \varphi(r) = \varphi(q)$

Form $p^{-1}(G)$ which breaks into arc-components each of which is open in F and each of which intersects $p^{-1}(\varphi(r))$ in an arc-component of $p^{-1}(\varphi(r))$

Each arc-comp. of $p^{-1}(G)$ is a union of arc components of points $p^{-1}(s)$, i.e. a union of points of U and thus set is open by defin. of open.

Furthermore, for each $s \in S$ only one arc-comp. of $p^{-1}(s)$ touches a given arc-comp. of $p^{-1}(G)$. This gives us φ 1-1 on each such set.

Finally, φ is a homeomorphism when restricted to such a set; for it is continuous and open, because the restriction of an open mapping to an open set is open.

2.12. Let α be any arc in S. Let $u_1 \in U$ and $\varphi(u_1) = \alpha(0)$. There is a unique arc α^* in U with $\alpha^*(0) = u_1$ and $\varphi \circ \alpha^* = \alpha$.

Proof. Unicity. Suppose α^* and α^{**} have properties, set $E = \{t \mid \alpha^*(t) = \alpha^{**}(t)\}$

E is closed by continuity

E is open because φ is a local homeomorphism.

Theorem follows.

Existence.

Special case $u_1 = u_0$, the class of the trivial arc f_0 ($f_0(t) = s_0$)

Let the arc be β

Define arc β' in F by

$$[\beta'(s)](t) = \beta(st)$$

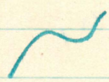
$$\rho(\beta'(s)) = [\beta'(s)](1) = \beta(s)$$

so $\rho \circ \beta' = \beta$ $\qquad \beta'(0) = f_0$

Then an arc satisfying the requirement is $\psi \circ \beta' = \beta^*$

for then $\varphi \circ \psi \circ \beta' = \rho \circ \beta' = \beta$

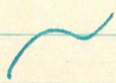
Observe $\beta^*(1) = \psi(\beta)$

s_0 

General case:

Let $g \in u_1$.

g is an arc

s_0 

from s_0 to $\rho(g) = \varphi(u_1)$

Set
$$\beta(t) = \begin{cases} g(2t), & 0 \le t \le \frac{1}{2} \\ \alpha(2t-1), & \frac{1}{2} \le t \le 1 \end{cases}$$

Can apply previous construction to get β^*. Now we can set

$$\alpha^*(t) = \beta^*\left(\tfrac{1}{2} + \tfrac{1}{2}t\right)$$
$$\gamma^*(t) = \beta^*\left(\tfrac{1}{2}t\right)$$

$$\varphi \circ \alpha^* = \alpha, \quad \varphi \circ \gamma^* = g$$

also $\gamma^*(0) = u_0$

Hence γ^* is the unique arc covering g as in the special case.

Therefore $\gamma^*(1) = \psi(g) = u_1$

also $\gamma^*(1) = \alpha^*(0)$

hence $\alpha^*(0) = u_1$

and α^* is the required arc.

2.13. Let α & β be arcs in S with the same endpoints and homotopic with fixed end points.

If α^* and β^* cover α and β with $\alpha^*(0) = \beta^*(0) = u$,

then $\alpha^*(1) = \beta^*(1)$.

Proof Special case

$$\alpha^*(0) = \beta^*(0) = u_0$$

In this case we have seen

$$\alpha^*(1) = \varphi(\alpha)$$

$\beta^*(1) = \varphi(\beta)$. The hypotheses mean that α and β are conn. by an arc in $p^{-1}(\alpha(1))$. Therefore $\varphi(\alpha) = \varphi(\beta)$.

General case: Adjoin an arc $g \in U_1$ in front of α and β getting arcs α_1, β_1. These new arcs satisfy cond. of special case and therefore

$$\alpha_1^*(1) = \alpha^*(1) = \beta_1^*(1) = \beta^*(1).$$

2.14. Let $\mathcal{Q}$ be any comm. loc. arc-wise connected simply conn. space, and let θ be a cont. mapping of $\mathcal{Q}$ onto S. Suppose $q_1 \in \mathcal{Q}$ and $u_1 \in \mathcal{U}$ are such that

$\theta(q_1) = \varphi(u_1)$.

Then there exist a unique contin. map θ^* of $\mathcal{Q} \to \mathcal{U}$ such that

$$1) \quad \theta^*(q_1) = u_1$$

$$\theta = \varphi \circ \theta^*$$

<u>Proof.</u> Any point $q \in \mathcal{Q}$ can be connected to q_1 by an arc α. Let $\beta = \theta \circ \alpha$, β^* such that $\varphi \circ \beta^* = \beta$ and $\beta^*(0) = u_1$.

Now let $\theta^*(q) = \beta^*(1)$.

Because $\mathcal{Q}$ is simply connected result does not depend on the choice of α.

It is clear that $\varphi \circ \theta^* = \theta$

and $\theta^*(q_1) = U_1$

We must prove θ^* continuous.

Take $q_2 \in Q$. Let N be a small neighbhood of $u_2 = \theta^*(q_2)$. We may assume that N is arc-wise conn. and mapped homeomorphically by φ.

Then $\theta^{-1}(\varphi(N))$ is a neighbh of q_2 in Q. Any point q in a neighb. ~~of~~ M of q_2 can be conn. by an arc to q_2 which lies in $\theta^{-1}(\varphi(N))$. The θ-image of this arc lies in $\varphi(N)$ and when raised to U lies in N, i.e. $\theta^*(q) \in N$.

(Have omitted to show that $(\theta \circ \alpha)^* = \theta^* \circ \alpha$)

2.15. U is simply connected

<u>Proof.</u> Special case: Let α^* be an arc in U with $\alpha^*(0) = \alpha^*(1) = u_0$.

Write $\alpha = \varphi \circ \alpha^*$

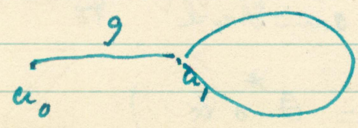
$$u_0 = \alpha^*(1) = \psi(\alpha)$$

Hence α is in the arc component of $p^{-1}(s_0)$ as is f_0.

This means α can be deformed into f_0 in S.

The homotopy can be raised to U

<u>General case</u>



2.16. Let S be a conn., loc. arc-wise connected, loc. relatively simply conn. metric space. There exists a space $\mathcal{U}$ and a mapping φ of $\mathcal{U}$ onto S with properties:

1) $\mathcal{U}$ is conn., loc. arc-wise conn., simply connected

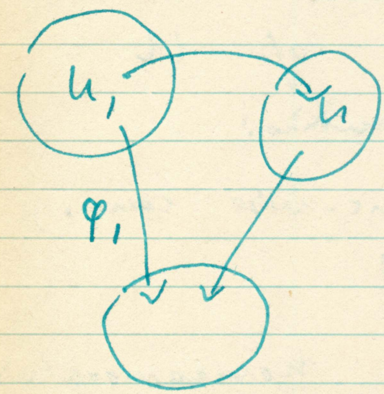
2) φ is a local homeomorphism

3) If Q is a conn. loc. arc-wise conn. simply conn. space and θ maps Q into S, then θ can be factored in $\theta = \varphi \circ \theta^*$ where θ^* is a map of Q into $\mathcal{U}$. The factorization is unique if $\theta^*(q_1)$ is specified.

This pair $(\mathcal{U}, \varphi)$ is unique within isomorphism, that is if $(\mathcal{U}_1, \varphi_1)$ have properties 1) + 3) then there is a homeomorphism ξ of $\mathcal{U}_1$ onto $\mathcal{U}$ such that

$\varphi_1 = \varphi \circ \xi$.

Proof.



Since U_1 is simply connected we can write

$$\varphi_1 = \varphi \circ \xi \quad \text{where}$$

s ξ is a continuous mapping $U_1 \to U$.

We can also factor $\varphi = \varphi_1 \circ \eta$

$\eta : U \to U_1$.

We can arrange that

$$\xi(\eta(u_0)) = u_0 .$$

$$\varphi = \varphi \circ \xi \circ \eta$$

Apply unicity of 3) to the case $\Phi = U$, $\theta = \varphi$. Then $\xi \circ \eta$ = identity. The same argument works other way and shows that ξ and η are inverse homeomorphisms.

2.17. Consider all cont. maps α of U into U such that

$\varphi \circ \alpha = \varphi$.

These maps have composition property.

For each $u_1 \in \varphi^{-1}(u_0)$ there is exactly one α with

$\alpha(u_0) = u_1$

If $(\alpha \circ \beta) u_0 = u_0$ then $\alpha \circ \beta =$ identity.
Shows that inverses exist.

We get a <u>group</u> the Poincaré group or fundamental group (with reference point u_0)

2.18. Consider pairs V, ψ which satisfy 1) except for simple connectivity and 3) of 2.16. Such a pair is called a <u>covering space</u> of S

A specified covering space ~~of~~ of S, s_0 is a triple V, ψ, v_0 such that $\psi(v_0) = s_0$.

Compare V, ψ, v_0 and U, φ, u_0. Can factor

$$\varphi = \psi \circ \bar{\varphi} \qquad \text{with}$$

$$\bar{\varphi}(u_0) = v_0 .$$

Now $U, \bar{\varphi}$ is univ. cov. space of V and $U, \bar{\varphi}, u_0$ is spec. univ. cov space.

To prove 3) give $\Theta / Q \to V$

have $\varphi \circ \theta = \varphi \circ \xi$

and find $\theta = \bar{\varphi} \circ \xi$.

The functions $\alpha \mid U \to U$
such that $\bar{\varphi} \circ \alpha = \bar{\varphi}$ are a group
isom to the Poincaré group
of V and a subgroup of
Poincaré group of S

To reverse: Given a subgroup
H of the P. group of S
we take equiv. classes of U
under H ~~and~~ as points
of V.

2.23 If S is connected, loc. ~~conn.~~ arc-wise conn. and loc. rel. simply conn., then if S is simply conn. in sense of Chevalley, then it is simply conn.

2.25. If α is an arc in S, V, f a covering space of S, there is an $\varepsilon > 0$ such that any arc β in S with the same end points and distance $< \varepsilon$ from α and homotopic to α, within ε-nghd when raised with the same initial point has the same final point.

Proof. Choose $0 = t_0 < t_1 \dots < t_n = 1$ so that $\alpha([t_{i-1}, t_i]) \subset N_i$, an open neighborhood which is evenly covered.

Now take $\varepsilon < \inf_i \; \inf_{t_{i-1} \leq t \leq t_i} \rho(\alpha(t), \tilde{N}_i)$

($\tilde{N}_i$ means complement)

Let β satisfy the conditions for this ε.

We have $\beta^*(0) = \alpha^*(0)$.

Consider the component of $f^{-1}(N_i)$ which contains $\beta^*([t_{i-1}, t_i])$.

By induction on i this is the same component which contains $\alpha^*([t_{i-1}, t_i])$

Trivial for $i=1$

If true for $i=k$, then deform

of α into β carrying $\alpha(t_k)$

along an arc in N_i to the

point $\beta(t_k)$. This arc is also

in N_{k+1}. Therefore, ~~when~~ if

P_{k+1} is the comp. of $f^{-1}(N_{k+1})$

containing $\alpha[t_k, t_{k+1}]$, then

P_{k+1} contains an arc which projects

into arc from $\alpha(t_k)$ to $\beta(t_k)$.

This arc connects $\alpha^*(t_k)$ to

$\beta^*(t_k)$ in P_k, and hence

$\beta^*[t_k, t_{k+1}] \subset P_{k+1}$.

2.26. Any homotopy is made up of short homotopies

If α_s is an arc in F, then, by usual argument, the set of s for which α_s^* leads to the same end point $\alpha_0^*(1)$ is open and closed

<u>Theorem</u> If α and β are homotopic with ~~the~~ fixed end points in S, then if $\alpha^*(0) = \beta^*(0)$, then $\alpha^*(1) = \beta^*(1)$.

2.27. Let S be covering loc. arc conn., ~~s.~~ s.c. then S is s.c. in sense of Chevalley.

Proof. Let V, f be a covering space of S. To prove f is homeom. it is enough to show it is 1-1.

Let $f(v_0) = f(v_1)$. Since V locally homeo to S, V is loc. arcwise conn.

Can join v_0 to v_1 by an arc in V, say α^*.

$\alpha = f \circ \alpha^*$ is an arc with coincident end points.

α can be shrunk to a point

shows $v_0 = v_1$.

2.28 Chevalley's monodromy theorem.

Let S be s.c. (C). Assume that for each $p \in S$ there is a set E_p and for each $\langle p, q \rangle$ in a set $D \subset S \times S$ there is a function $\phi_p(q)$ mapping E_q into E_p. Assume further

1) D is an open conn. neighb. of the diagonal.

2) Φ_{pq} is one to one.

3) If $\langle p,q\rangle$, $\langle q,r\rangle$, $\langle p,r\rangle \subset D$ then $\Phi_{pq} \circ \Phi_{qr} = \Phi_{pr}$.

Let $p_0 \in S$ be fixed and $e^*_{p_0} \in E_{p_0}$.

There exists a unique function ψ on S such that

$$\begin{cases} \psi(p) \in E_p & \text{for all } p \\ \psi(p) = \Phi_{pq}(\psi(q)) & \text{for } \langle p,q\rangle \in D. \\ \psi(p_0) = e^*_{p_0} \end{cases}$$

Proof. [We may assume that the sets E_p are disjoint.] omit

Consider $V = \{\langle p, e_p\rangle \mid p \in S,\ e_p \in E_p\}$

Impose a topology by ~~setting~~ taking an elementary neighb. of

$\langle p, e_p \rangle$ as

$N \times N \subset D$

$N^* = \{ \langle q, e_q \rangle \mid \phi_{p,q}(e_q) = e_p , \ q \in N \}$

It is a neighborhood basis.

Now $f(\langle p, e_p \rangle) \to p$ gives cont. map of V, and every N with $N \times N \subset D$ is evenly covered.

Component of $\langle p_0, e^*_{p_0} \rangle$ is a covering space and hence f is 1–1.

Let $H = \{ \langle p, q \rangle \mid \psi_p = \phi_{p,q}(\psi_q), \ \langle p, q \rangle \in D \}$

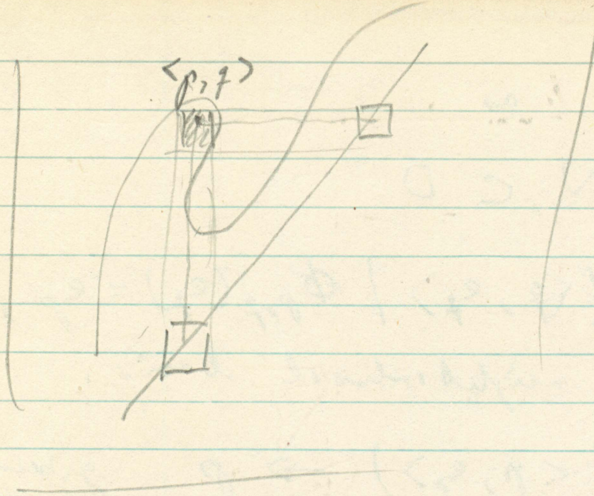
Prove H open and closed in D.
Obviously H contains diagonal.

Let $\langle p, q \rangle \in \bar{H} \cap D$

will prove $\langle p, q \rangle \in \operatorname{Int} H$

Means H open and closed rel. D

hence $H = D$



Choose U a conn. neighb. of p

such that $U \times U \subset D$ and $U \times V \subset D$

$V(q)$ so that $V \times V \subset D$

Then prove that $U \times V \subset H$

We know that there is a pt

$\langle u, v \rangle$ such that $\langle u, v \rangle \in H$, $u \in U$, $v \in V$

Let $\langle x, y \rangle \in U \times V$.

Since x, u are close we have

$$\psi_x = \phi_{xu}(\psi(u))$$

$$\psi_v = \phi_{vy}(\psi(y))$$

$$\psi_u = \phi_{uv}(\psi(v))$$

Since $\langle x, u \rangle$, $\langle u, v \rangle$, $\langle x, v \rangle \in D$

$$\phi_{xu} \circ \phi_{uv} = \phi_{xv}$$

$$\phi_{xv} \circ \phi_{vy} = \phi_{xy}$$

$$\phi_{xy} = \phi_{xu} \circ \phi_{uv} \circ \phi_{vy}$$

Gilt $\phi_{xy}(\psi(y)) = \psi(x)$

3. <u>Topological groups.</u>

I. $\rho(\sigma\tau) = (\sigma\rho)\tau$

II $\exists\, \varepsilon$ such that

a) $\varepsilon\alpha = \alpha\varepsilon = \alpha$ all α

b) $(\alpha)(\exists\,\delta)\ \alpha\delta = \varepsilon$

A top. space $\langle S, \mathcal{T}\rangle$ when $\mathcal{T}$ is a collection of subsets of S

~~If X is a finite subset of~~

~~the~~

(1) If $\mathcal{F}$ is a finite subset of $\mathcal{T}$ then

$$\bigcap_{X \in \mathcal{F}} X \in \mathcal{T}$$

(2) If $\mathcal{G}$ is any subset of $\mathcal{T}$, then

$$\bigcup_{X \in \mathcal{G}} X \in \mathcal{T}$$

Hausdorff axiom always assumed.

3.1.1 A top. group is a triplet $\langle G, T, M \rangle$ such that $\langle G, T \rangle$ is Hausdorff, $\langle G, M \rangle$ is a group and the mappings

$$\langle \sigma, \tau \rangle \longrightarrow \sigma\tau^{-1}$$

of $G \times G \to G$ is continuous.

G has four meanings.

3.1.2. Two top. groups G and H are isomorphic if there exists a homeomorphism ϕ of G onto H which is an algebraic isomorphism.

3.1.3. The translation operators L_σ and R_σ defined by

$$R_\sigma(\tau) = \tau\sigma$$
$$L_\sigma(\tau) = \sigma\tau$$

are cont. with cont. inv.

so that

$$\tau \to \sigma\tau\sigma^{-1}$$

is an automorphism of the top. gr.

3.2. <u>Subgroups</u>

3.2.1. Any alg. subgr. of G with the subspace topology is a top. group. This group is called a subgr. of G.

(Is <u>not</u> required to be closed).

3.2.2. If H is subgr. of G, then $\bar{H}$ is a subgr. of G.

<u>Proof.</u> Suff. to prove $\bar{H}$ closed under $\phi : (\sigma, \tau) \to \sigma\tau^{-1}$

$\sigma, \tau \in \bar{H}$

Choose directed system $\sigma_\alpha, \tau_\alpha \in H$ with $\sigma_\alpha \to \sigma$

$\tau_\alpha \to \tau$

By continuity $\sigma_\alpha \tau_\alpha^{-1} \to \sigma\tau^{-1}$

so that $\sigma\tau^{-1} \in \overline{H}$

If H is normal, abelian or central, so is $\overline{H}$.

3.2.3. Any discrete subgroup is closed.

Proof. Say that H is a discrete subgr. of G. There exists a neighb. N of ε in G with $N \cap H = \{\varepsilon\}$.

Choose neigh M of ε in G such that $M^{-1}M \subset N$

Say σ is a limit pt of H

We know σM is a neigh of σ

Can choose $\tau \in H \cap \sigma M$

If σ is not in H we can choose inf. many τ say $\tau_1 \neq \tau_2$

$\tau_1 = \sigma m_1$ $\qquad \tau_1^{-1}\tau_2 = m_1^{-1} m_2 \in M^{-1}M \subset N$

$\tau_2 = \sigma m_2$ $\qquad$ also in H. But that would give $\overline{\tau}_1 = \overline{\tau}_2$.

3.3. If we are given a family $\{G_\alpha\}$ of top. groups, then $\prod G_\alpha$ with the usual algebraic and topol. structures is a topological group, the <u>direct product</u>.

3.4. <u>The topology of a top. group.</u>

3.4.1 The ~~neighbor~~ set $\mathcal{N}$ of neighb. of ε satisfy

TGN I. If $N \in \mathcal{N}$ and $P \supset N$ then $P \in \mathcal{N}$.

TGN II. If $N, P \in \mathcal{N}$ then $N \cap P \in \mathcal{N}$.

III. $\bigcap_{\mathfrak{N}} N = \{\varepsilon\}$

IV. If $N \in \mathfrak{N}$, $\exists\, P \in \mathfrak{N}$ such that

$$PP^{-1} \subset N$$

V. If $N \in \mathfrak{N}$, $\sigma \in G$ then

$$\sigma N \sigma^{-1} \in \mathfrak{N}.$$

Theor. If G is an alg. group and $\mathfrak{N}$ a family of subsets satisfying TGN I–V, then there is a unique top. on G which makes G a topol. group with $\mathfrak{N}$ = family of ε-neighborhoods.

Proof. Let $\mathfrak{N}(\sigma) = \{N\sigma \mid N \in \mathfrak{N}\}$ be neighborhoods of σ. This shows uniqueness.

Verify four conditions:

1) If $U = U(\sigma)$ and $V \supset U$ then $V = V(\sigma)$.

2) Intersection $U(\sigma) \cap V(\sigma)$ is a neighb. of σ.

3) For all σ, $U \in \mathcal{N}(\sigma)$, $\sigma \in U$.

4) If $U = U(\sigma)$ then there is a $V = V(\sigma)$ such that U is a neighb. of every $\tau \in V$

Remark. Let $U \in \mathcal{N}$. Choose V so that $VV^{-1} \subset U$.
Then $V^{-1} \subset U$, $V \subset U^{-1}$.
Hence $U^{-1} \in \mathcal{N}$

Suppose that U is a given neighb of σ and $U = N\sigma$, $N \in \mathcal{N}$
Chose $PP^{-1} \subset N$

$$P^{-1} \in \mathcal{N} \implies P^{-1}\sigma \in \mathcal{N}(\sigma)$$

Take $V = P^{-1}\sigma$, $\tau \in V$

$P\tau \in \mathfrak{N}(\tau)$, $P\tau \subset PP^{-1}\sigma$

$\subset N\sigma = U$

Hence U is a neighb. of τ, proves 4).

Suppose $\rho\sigma^{-1} = \tau$

Let $W\tau$ be a neigh. of τ

Take $UU^{-1} \subset W$

$\tau^{-1}U\tau \subset \mathfrak{N}$

$\tau^{-1}U\tau\sigma$ is neighb. of σ

$U\rho$ " " " ρ.

$(U\rho)(\tau^{-1}U\tau\sigma)^{-1}$

$= U\rho\sigma^{-1}\tau^{-1}U^{-1}\tau = UU^{-1}\tau \subset W\tau$

Shows continuity.

We see that TGN $\overline{\text{III}}$

is T_1 for the top. space.

(regularity)

In fact, T_3 is satisfied.

If σ and V a nbgh of σ

then there is a neigh. U of σ

with $\overline{U} \subset V$

We consider only $\sigma = \varepsilon$

$V \in \mathfrak{N}$ $\quad U^{-1}U \subset V$

Say $\sigma \in \overline{U}$ then $U\sigma \cap U \neq \varphi$

can find $u_1 \sigma = u_2$, $u_1, u_2 \in U$

$$\sigma = u_1^{-1} u_2 \in U^{-1}U \subset V$$

Proves $\overline{U} \subset V$.

3.4.2. ~~We can~~

Axiom $T_0 \rightarrow T_3$, in a

top group without sep. axioms.

3.4.3. The uniform structure.

We define a unif. structure for G as follows:

A basis:

For each nbgh. U of e set

$$\alpha_U = \{\langle\sigma, \tau\rangle\} \ \sigma\tau^{-1} \in U\}$$

The set of all uniformities α_U is a basis for unif. structure

Called the <u>right</u> uniform structure.

Left would be $\{(\sigma, \tau), \ \sigma^{-1}\tau \in U\}$

are <u>not</u> the same.

Means: $\sigma \to \sigma^{-1}$ is not unif. continuous.

3.4.4. Every top group T_0 is also $T_{3½}$ or completely regular.

Given σ and a neighb. V of σ $\exists$ contin fn f such that $f(\sigma) = 1$

$f(\tau) = 0$, $\tau \notin V$.

3.4.5. If a top. group satisfies first ax. of countability, then it is metrizable.

In this case one gets a uniform structure with a countable basis. This implies metrizable, and further the metric can be chosen so that $\rho(\sigma, \tau) = \rho(\sigma\xi, \tau\xi)$.

3.5. Homomorphisms, representations & homogeneous spaces.

3.5.1. If φ is a cont. mapping of a top group G into a top Group H which is an algebraic homomorphism it is called a <u>representation</u>.

The kernel is then a closed normal subgroup.

If it is 1-1 it is called <u>faithful</u>.

3.5.2. If φ is also open then it is called a <u>homomorphism</u>.

3.5.3. Let H be a closed subgroup of G. Let G/H the set of left cosets of H. Let φ be the natural map of G onto G/H ($\sigma \to \sigma H$). There is a unique topology on G/H for which φ is continuous and open.

Proof. Let $\underline{X} \subset G/H$ be „open"
if $\varphi^{-1}(\underline{X})$ is open.

Null-set and whole space trivially open.
Other condition follows by

$$\varphi^{-1}(\bigcup X) = \bigcup \varphi^{-1}(X)$$
$$\varphi^{-1}(\bigcap X) = \bigcap \varphi^{-1}(X).$$

Separation axiom:

Given two distinct points
of G/H, say σH and τH.
~~We must prove~~

Since $\sigma \notin \tau H$, which is closed,
we can choose W as a nbhd.
of ε so that $W\sigma \cap \tau H = 0$.
Choose open U so that $U^{-1}U \subset W$

Then $U\sigma H$ and $U\tau H$
are open in G, containing σH
and τH.

$$\varphi^{-1}(\varphi(U\sigma H)) = U\sigma H$$
$$\varphi^{-1}(\varphi(U\tau H)) = U\tau H$$

Therefore $\varphi(U\sigma H) = S$
$\varphi(U\tau H) = T$

are open. They are disjoint

$U\sigma H \cap U\tau H = 0$

and then that S, T are disjoint.

$u_1 \sigma h_1 = u_2 \tau h_2$

$u_2^{-1} u_1 \sigma = \tau h_2 h_1^{-1}$ not poss.

If U is open, $\varphi(U)$ is open because

$\varphi^{-1}(\varphi(U)) = UH$ is open.

Unicity inherent in the construction.

3.5.4. <u>Lemma.</u> Let φ maps X onto Y, g map Y into Z and $f = g \circ \varphi$.

1) If f cont. and φ open then g is cont.

2) If f open and φ cont. then g is open.

First part.

Let U be open in Z

$f^{-1}(U)$ is open in X

$\varphi(f^{-1}(U)$ is open in Y

$= g^{-1}(U)$ by 1-1 ness

3.5.5. If f is a cont. mapping of G onto a top. space X which is constant on each coset of H, then f can be factored

$f = g \circ \varphi$ where g is cont. from $G/H \to X$

g exists trivially, and is cont. by 1) of lemma.

3.5.6. The mapping $\langle \tau, \sigma H \rangle \to \tau\sigma H$ of $G \times G \to G/H$ is continuous and open.

$\langle \sigma, \tau \rangle \to \sigma\tau \to \sigma\tau H$

We can consider $G \times G \to G \to G/H$

can be factored

$$G \times G \to G \times G/H \to G/H$$

3.5.7. If N is a normal subgroup then G/N is a top. group.

Proof: $G \times G \xrightarrow{\text{open}} (G/N) \times (G/N) \to G/N$

$$\langle \sigma, \tau \rangle \to \langle \sigma N, \tau N \rangle \to \sigma\tau^{-1} N$$

Also $\langle \sigma, \tau \rangle \to \sigma\tau^{-1} \to \sigma\tau^{-1} N$

cont

conclude continuous.

3.5.8. We see that every repr. of G with kernel N can be factored $f = g \circ \varphi$ where g is faithful, φ is nat. mapping.

If f is a homomorphism to begin with, then g is also open and therefore an isomorphism onto.

3.5.9. If f is an alg. homomorphism of G into H which is cont. at one point is cont. everywhere.

If f is open at one point it is open. ~~everywhere~~

By translation invariance of open sets in G and H.

3.6. Connectivity.

3.6.1. Any open subgroup of a top. group is closed.

Proof. Each coset is open, so the complement of the subgroup is open, and the subgroup is closed.

A subgroup is open if it has an interior point.

Cor. In a connected group the only open subgroup is the whole group.

3.6.2. The connected component of the identity is a closed characteristic subgroup, hence normal.

(Char. means invariant under automorphisms)

Proof. Char. is trivial

Say K = conn comp of ε

$\sigma \in K$

$K\sigma^{-1}$ is conn. contain ε

then $K \supset K\sigma^{-1}$

$K \supset KK^{-1}$

and K is a subgroup.

3.6.3.

If K is component of e, then G/K is totally disconnected. If G is locally conn., G/K is discrete.

3.6.4. In a connected group every neighborhood of e is a set of generators.

Proof. V is a neighb.

$\bigcup_{n=1}^{\infty} V^n$ is a group, for if V is symmetrical then inverses are obviously present.

Since it has an interior point it is open, and hence it is all of G.

Then from V, every element can be written $v_1 v_2 \cdots v_n$ with $v_i \in V$

3.6.5. In a connected group every totally disconnected normal subgroup is central, and therefore abelian.

<u>Proof.</u> Say $\sigma \in N$ (t.d.n.)

$\tau \to \tau^{-1}\sigma\tau$ maps $G \to N$

Hence image of G is connected, but therefore it is an one-element set $\{\sigma\}$ and hence

$\tau^{-1}\sigma\tau = \sigma$ for all $\tau \in G$.

3.7. Local properties.

3.7.1. A local algebraic homomorphism of a group G into a group H is a function φ defined on an open neighborhood V on G with values in H such that if

$x, y, xy \in V$, then $\varphi(x)\varphi(y) = \varphi(xy)$

local representation <u>cont.</u>

homomorphism cont. <u>open</u>

3.7.2. A local alg. homom. of G into H can be extended over all of G provided the domain of def. is connected and G is simply connected

Proof. We apply the monodromy theorem in Chevalley's form.

For each $p \in G$, $E_p = H$

$$D = \{ \langle p, q \rangle \mid pq^{-1} \in V \}$$

Because G and V are connected, then D is an open connected neighborhood of the diagonal.

$\Phi_{pq} \mid E_q \to E_p$ defined as

$$h \to f(pq^{-1})\, h$$

If $\langle p, q \rangle, \langle q, r \rangle, \langle p, r \rangle \in D$

$f(pq^{-1})\, f(qr^{-1}) = f(pr^{-1})$ and

thus $\Phi_{pq} \circ \Phi_{qr} = \Phi_{pr}$

Conclusion: There exists a unique ψ defined on $\mathcal{G}$ such that $\psi(\rho) \in H$ for all $\rho \in \mathcal{G}$ and $\psi(\varepsilon) = \varepsilon_H$

and

$$\psi(\rho) = \phi_{\rho q}\, \psi(q) \quad \text{if} \quad \rho q^{-1} \in V$$

$$= f(\rho q^{-1})\, \psi(q)$$

For $q = \varepsilon$ we get

$$\psi(\rho) = f(\rho)$$

so that ψ is an extension.

$$\psi(\rho) = \psi(\rho q^{-1})\, \psi(q) \quad \text{if} \quad \rho q^{-1} \in V$$

$$\psi(\sigma\tau) = \psi(\sigma)\, \psi(\tau) \quad \text{if} \quad \sigma \in V$$

$$\psi(\sigma_1 \cdots \sigma_k \tau) = \psi(\sigma_1) \cdots \psi(\sigma_k)\, \psi(\tau)$$

$$= \psi(\sigma_1 \cdots \sigma_k)\, \psi(\tau)$$

which proves ψ is a homom.

By 3.6.4.

3.7.3. If f is a local repr. or homomorphism, then the extension has the same property.

3.7.4. 1-1 may be lost. But, if f is locally 1-1, then kernel of φ is a discrete normal subgroup of G, because kernel has only e in V.

Thus kernel is also central.

Conversely, if N is a discrete normal subgroup of G, φ is the natural mapping of G onto G/N, then φ is locally an isomorphism.

3.7.5. If $(\mathcal{G}, f)$ is a simply comm. cov. space (C) of a top space H, then it is possible to define a unique group structure in $\mathcal{G}$ such that f is a homomorphism.

We say $(\mathcal{G}, f)$ is a covering group of H

<u>Proof.</u>

$$\begin{array}{ccc} \mathcal{G} \times \mathcal{G} & & \mathcal{G} \\ \downarrow (f, f) & & \downarrow f \\ H \times H & \xrightarrow[\phi_H]{} & H \end{array}$$

There is a cont. fn $\Phi_{\mathcal{G}}$ of $\mathcal{G} \times \mathcal{G}$ $\longrightarrow \mathcal{G}$ such that

$$\phi_{\mathcal{G}}(\varepsilon_{\mathcal{G}}, \varepsilon_{\mathcal{G}}) = \varepsilon_{\mathcal{G}}$$

and $$f \circ \phi_{\mathcal{G}}(\sigma, \tau) = f(\sigma) f(\tau)^{-1}$$

Because $\mathcal{G} \times \mathcal{G}$ is simply connected.

We have to prove that ϕ_g is a group composition.

Let $\sigma * \tau = \phi_g(\sigma, \phi_g(\varepsilon, \tau))$

$f(\sigma * \tau) = f(\sigma) f(\tau)$ immediate

For associate law, define

$h(\rho, \sigma, \tau) = (\rho * \sigma) * \tau$

$h'(\rho, \sigma, \tau) = \rho * (\sigma * \tau)$

$f \circ h = f \circ h'$ clear

and $h(\varepsilon_g, \varepsilon_g, \varepsilon_g) = h'(\varepsilon_g, \varepsilon_g, \varepsilon_g)$

$$\begin{tikzcd}
\mathcal{G} * \mathcal{G} * \mathcal{G} \arrow[d] & & \mathcal{G} \arrow[d] \\
H \times H \times H \arrow[rr] & & H
\end{tikzcd}$$

$h = h'$ by the unicity of raising of mappings.

3.8. Covering groups.

3.8.1. Let H be a top. group. A covering group of H is a pair $\langle G, f \rangle$ where G is a top. group and f a homomorphism of G onto H such that $\langle G, f \rangle$ is a covering space of H.

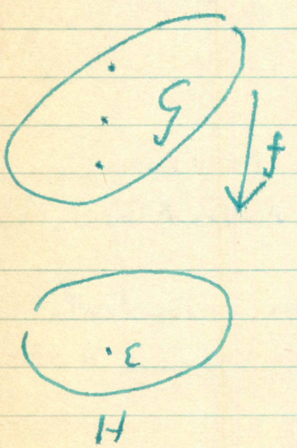
3.8.2. $\langle G_1, f_1 \rangle$ and $\langle G_2, f_2 \rangle$ are isom. covering groups if ~~there~~ exist φ isom. ~~mapping~~ G_1 onto G_2, one to one, so that $f_1 = f_2 \circ \varphi$

3.8.3. If H has a s.c. covering space it has a s.c. covering group and all s.c. covering groups are isomorphic.

3.8.4. The Poincaré group of a topological group (for which a simply conn. covering space can be defined) is Abelian.

<u>Proof.</u> Let H be the top. group. Let (G, f) be the simply conn. covering group.

Kernel of f is a discrete normal subgroup N, and therefore Abelian.



Let $L_n(\sigma) = n\sigma$ for $n \in N$

$$(f \circ L_n)\sigma = f(n\sigma) = f(\sigma)$$

$$f \circ L_n = f$$

and L_n belongs to Poincaré group. Π

The set of all L_n exhausts the Poincaré group, for given $n_1 \in N$ there is exactly one element of Π which maps

ε_g onto n_1. Therefore
this element is L_{n_1}.

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4. Manifolds   5. Groups.

4.1. Definition: An analytic manifold is a connected Hausdorff space ~~and~~ M together with a family P of cont. real valued functions (called the proper functions) such that

1) The domain of each proper function is an open subset of M

2) If U is an open subset of dom f ($f \in P$) then f/U is proper.

3) If $R \subset P$ is a set of consistent functions (i.e. $f, g \in R$ have equal

restriction to the intersection of their domains,) then P contains a function which is an extension of all members of $\mathcal{R}$.

4) If $f_1, \ldots f_k$ are proper functions with common domain and F is any analytic function on the range of $(f_1, \ldots, f_k)$ then $F \circ (f_1, \ldots, f_k)$ is proper.

5) For each point $p \in M$ there is a finite set of proper functions $x_1, \ldots, x_n$ having a common domain $\mathcal{D}$ containing p, such that

a) $(x_1, .. x_n)$ is a homeomorphism of $\mathcal{D}$ onto open set in $\mathcal{R}^n$.

b) If $g \in P$ and $\operatorname{dom} g \subset \mathcal{D}$ there exists an anal. function G such that $g = G \circ (x_1, .. x_n)$

4.1.2. Examples

R^n with proper functions = anal. function.

Comm.

Open subset of a manifold with restriction of proper fns.

Unit sphere S^{n-1}

$x_1^2 + \cdots + x_n^2 = 1$.

Proper functions = restriction of proper function in R^n.

4.1.3. Def. A system of proper functions $(x_1, \cdots x_n)$ which satisfies 5) is called a system of local coordinates at p.

If it is also a system of local coord. at each pt of its domain we call it local coord. on D.

4.1.4. (M, P) and (N, Q) are isomorphic if there exists a homeomorphism φ of M onto N such that $f \in P \iff f = g \circ \varphi$ for $g \in Q$

Amendment of manifold definition: One point $\{p\}$ with P all real functions on $\{p\}$.

4.1.5. Local definition.

Let M be a conn. Hausdorff space and Φ a collection of cont. functions ~~on open sets~~ with values in R^n such that each $\varphi \in \Phi$ is a homeomorphism of an open subset of M onto an open subset of R^n.

If domain $\varphi \cap$ domain ψ is not empty then $\psi \circ \varphi^{-1}$

has the form $(F_1, \ldots F_n)$
where $F_1, \ldots F_n$ are analytic.
The domains of the $\varphi \in \Phi$ cover M.

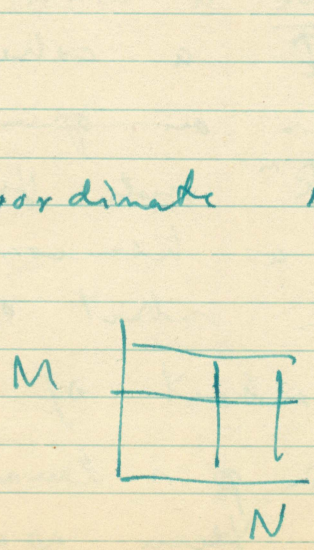
$\Longrightarrow$ Then there is a uniquely determined set P of real functions on open sets such that $\langle M, P \rangle$ is an anal. manifold and each $\varphi \in \underline{\Phi}$ is a local coord. system.

4.1.6. The direct product of two manifolds.

(M, P) (N, Q)

$M \times N$

Produce coordinate system by



We let $\overline{\Phi}$ be the set of functions obtained as follows. At any $\langle p, q \rangle \in M \times N$

$x_1, \dots x_m$ loc. coord. at p in M,

and $y_1, \dots y_n$ " —— at q in N

$\varphi = (x_1, \dots x_m, y_1, \dots y_n)$ is a local mapping.

4.2.1. Let $x_1, \dots x_n$ be a local coord. syst. at p, domain $\mathcal{D}$. Let $f_1, \dots f_m$ be proper functions defined at.

A n.a.s.c. that $f_1, \dots f_m$ (suitably restricted) should be a local coord is

1) $m = n$

2) if $f_i = F_i(x_1, \dots x_n)$

$$|D_j F_i| \neq 0 \text{ at } p.$$

1st suppose $f_1 \cdots f_m$ is a coord. system

Then $x_i = G_i \circ (f_1, \cdots f_m)$

with anal. G

$f_i = F_i \circ (G) \circ (f)$

$F \circ G$ is identity mapping on an open set in R^m

Similarly $G \circ F$ is identity map in R^n

usual proof by derivation

Corollary: All coord. system have the same number of functions.

L^{nd}. If cond. is satisfied there is an inverse so that $(F_1, \ldots F_n)$ effects a local homeomorphism.

4.2.2. Define dimension.

4.3. Let φ map an open subset of M into N. φ is called proper if $f\circ\varphi$ is proper on M for each f proper on N.

5. Analytic groups.

5.1.1 Definition. Suppose G is an alg. group and a manifold. Then G is an anal. group if

$$\langle \sigma, \tau \rangle \to \sigma\tau^{-1}$$

is a proper mapping of $G \times G \to G$.

~~5.1.2.~~ For each $\sigma \in G$, R_σ

$\tau \to \tau\sigma$ is an isomor. of manifold

of G with G.

5.1.2. Isomorphism of anal. groups.

5.1.3.

5.1.4. Comp. of local mult. fn.

$\Phi : \langle \sigma, \tau \rangle \to \sigma\tau$ is proper

Choose coord. $x_1, \dots, x_n$ at ε

$c = (0, \dots, 0)$

Let D be domain of (x).

Can choose E a neighb. of c such that $\Phi(E \times E) \subset D$.

We get coord. system of $E \times E$ as

$$\begin{cases} y_i = x_i \circ \pi_1 & (i = 1 \dots n) \\ y_{i+n} = x_i \circ \pi_2 & (i = 1, \dots n) \end{cases}$$

Since Φ is proper we can say

$x_i \circ \Phi$ proper on $G \times G$,

def. in $E \times E$

$x_i \circ \Phi = F_i \circ (y)$, F_i anal. in $2n$ variables.

$$x_i(\Phi(\sigma,\tau)) = F_i(y_1(\sigma,\tau), \ldots, y_{2n}(\sigma,\tau))$$

$$x_i(\sigma\tau) = F_i(x_1(\sigma), \ldots, x_n(\sigma), x_1(\tau), \ldots, x_n(\tau))$$

Power series expansion

$$x_i(\sigma\tau) = x_i(\sigma) + x_i(\tau) + \text{higher terms.}$$

4.4. Tangents and vector fields

4.4.1. Let (M, P) be an anal. manifold. Let $q \in M$ and let P_q be all P defined at q.

A tangent vector is a mapping T of P_q into R such that

1) If f is an extension of g, then $Tf = Tg$.

2) T is linear ~~when~~;
$T(\lambda f + g) = \lambda Tf + Tg$ when f and g have same domain.

3) $T(fg) = (Tf)g(q) + f(q)T(g)$.

(For local ring $A(q)$ of fun. anal. at q let $\underline{I}(q) =$ ideal of those vanishing at q

$$A(q) = R \oplus \underline{I}(q)$$

3) Says that T vanishes on $\underline{I}(q)^2$ and constants.)

The set of all T's is the dual space of $\underline{I}(q) / \underline{I}(q)^2$.

4.4.2. Let T_q be the set of all tangent vectors at q. T_q is a vector space over the reals. This vector space has the dim. n.

Proof. Let $x_1, \dots, x_n$ be local coord. at q.

We define n tangent vectors $X_1, \dots, X_n$ as follows:

$$f = F \circ (x)$$

$$X_i f = (D_i F) \circ (x(q)).$$

They are linearly indep. for

$$X_i x_j = \delta_{ij}$$

Given any tangent vector T we first show $Tx_i = 0$ for all i $\Longrightarrow$ $T = 0$.

1) For a constant function $Tf = 0$ because if $f = 1$ we have $ff = f$ and $Tf = 2Tf$ and hence $Tf = 0$.

2) $Tx_i = 0$, $f = F\circ(x) =$
$a + b_1x_1 + \cdots + b_nx_n + x_1G_1 + x_2G_2 + \cdots + x_nG_n$

Where $G_i(x)$ has no const. term.

Applying T we get $Tf = 0$.

Generally $T = \sum (Tx_i)X_i$

4.4.3. If $x_1 \cdots x_n$ are local coord. then we associate a basis of T_q for each q in the domain of (x_i) by $X_{i,q}\, x_j = \delta_{ij}$.

$$(X_{1,q}, \ldots, X_{n,q})$$

[$\bigcup_{q \in M} T_q = T$ has a topology in natural way.

Can be made analytic
This is the tangent bundle.
Makes a manifold of 2n dimensions.]

4.4.4. A *vector field* is a function on a part of M such that $V(q) \in T_q$

A *proper* vector field is one with open domain and such that Vf is proper if f proper

If V is a vector field and f is a proper function then
$V(q)f = g(q)$ will be denoted Vf

4.4.5 Can form $(XY)f = X(Yf)$ which is *not* a tangent vector

But $X_p Y - Y_p X$ is a tang. at p

Say that $X = \sum f_i X_i$

$Y = \sum g_i X_i$

Take h proper at p

$h = H \circ (x)$

$Y_q h = \sum g_i(q) (X_{i,q} h)$

$= \sum g_i(q) D_i(H)(x(q)) = \sum (G_i \circ D_i H) \circ x$

$Yh = \sum g (D_i H) x$

$X_q(Yh) = \sum_{ij} (f_j D_j (G_i \circ D_i H) x)$

$= \sum F_j [D_j (G_i D_i H)] \circ (x)$

$= \sum [F_j (D_j G_i)(D_i H) + F_j G_i (D_j D_i H)] \circ (x)$

$XYh - YXh = [F_j (D_j G_i) D_i H - G_j (D_j F_i) D_i H] x$

$= \sum (f_j X_j g_i) X_i h - (g_j X_j f_i) X_i h$

$$(XY - YX) fg$$

$$XY(fg) = X[Yf \cdot g + f \cdot Yg]$$

$$= XYf \cdot g + Yf \cdot Xg + Xf \cdot Yg + f\, XYg$$

$$(XY - YX) fg = (XY - YX) f \cdot g$$

$$+ f \cdot (XY - YX) g$$

4.4.6 Each T_q has a dual space $\mathcal{D}_q$

The dual bundle

Covector fields:

Propn θ covector field

V a vector field

if $g(q) = \theta(q)\, V(q)$

when V is proper.

4.4.7

If f is a proper function, then $T_q f$ depends linearly on $T_q \in \mathcal{T}_q$. Therefore there is a cotangent at q, i.e. an element of $\mathcal{D}_q$ for which

$$\theta_q T_q = T_q f$$

θ_q is called the <u>differential</u>.

df of f

The differential is <u>proper</u>.

Take V proper.

$$\theta_q V_q = V_q f \quad \text{is proper}$$

The diff. of the coordinates are a basis of $\mathcal{D}_q$ at each point, for

$$(dx_i)_q X_{j,q} = X_{j,q}\, x_i = \delta_{ij}$$

$$df = \sum (X_i f)\, dx_i$$

4.4.8

M a manifold dim n,
$f_1, \ldots f_n \in \mathfrak{P}_q$.

A n.s.c. that $f_1, \ldots f_n$ suitably restr. are a coord. system at q is that $(df_1)_q \ldots (df_n)_q$ span $\mathfrak{D}_q$

We know already that the cond. is, $f_i = F_i \circ (x)$

$$\det D_j F_i \circ (x(q)) \neq 0$$

$$(D_j F_i)(x(q)) = X_{j,q} f = (df)_q X_{j,q}$$

4.4.9. Effect of proper maps

$$\varphi : M \to N \qquad \text{proper}$$

This induces a mapping of the tangent spaces to M into tangent spaces to N.

$$T_q \to T_{\varphi(q)}$$

is defined as follows

$$[\varphi(X_q)]f = X_q(f \circ \varphi)$$

φ is a linear map of $T_q \to T_{\varphi(q)}$

φ^*, the transpose of φ, maps $\mathcal{D}_{\varphi(q)} \to \mathcal{D}_q$, namely

$$\varphi^*(\theta) \cdot X = \theta \cdot \varphi(X) \qquad \theta \in \mathcal{D}_{\varphi(q)}$$

$$\varphi^*(df_{\varphi(q)}) = [d(f \circ \varphi)]_q$$

4.4.10. A proper mapping $\varphi : M \to N$ is called <u>regular</u> at q if φ is 1-1 on T_q

4.4.11 If φ maps f.d. vector space A into B and φ^* is the transposed mapping $B^* \to A^*$, then

φ is 1-1 $\iff$ φ^* is onto

φ^* is 1-1 $\iff$ φ is onto

4.4.12 Let φ be a proper map $M \to N$, regular at p.

φ is 1-1 on T_p

φ^* is onto $\mathcal{D}_p$

Let $y_1, \dots y_n$ be coord. at $\varphi(p)$.

$(dy_1)_{\varphi(p)} \dots (dy_n)_{\varphi(p)}$ span $\mathcal{D}_{\varphi(p)}$

Therefore $\qquad \varphi^*(dy_i)_{\varphi(p)} = d(y_i \circ \varphi)_p$

span $\mathcal{D}_p$

We can choose a subset of the y's such that $y_i \circ \varphi$, $i = 1, \dots m$, are coordinates at p

Therefore φ is a homeomorphism into near p.

4.4.13. φ proper $M \to N$ induces onto mapping of $T_p \to T_{\varphi(p)}$

$\therefore$ φ^* is 1-1 from $\mathcal{D}_{\varphi(p)} \to \mathcal{D}_p$

If $y_1, \dots y_n$ are coord. at $\varphi(p)$ then $dy_1, \dots, dy_n$ are linearly indep in $\mathcal{D}_{\varphi(p)}$, and hence the $\varphi^*(dy_i)$ are lin. ind. in $\mathcal{D}_p$.

$d(y_i \circ \varphi)$ are lin. ind.

$(y_1 \circ \varphi) \cdots (y_n \circ \varphi), z_1, \dots z_{m-n}$

is a coord. system at p. The mapping is locally the same as proj. of E^m onto E^n. In this vicinity the mapping is <u>open</u>

4.4.14. φ proper $M \to N$ and φ iso T_p onto $T_{\varphi(p)}$, then φ is a homeomorphism onto near p.

4.4.15 If y is proper, domain y connected, $dy = 0$ everywhere, then y is constant.

In a coord. region can write

$y = Y \circ (x_1, \dots x_n)$

$dy = (D_i Y)(x_1 \dots x_n) \, dx_i$

Mean $(D_i Y)(x_1(q) \dots x_n(q)) = 0$

Therefore $D_i(Y)$ vanishes on an open set and Y is a const., y is loc. const. and y is const. by connectedness.

4.4.16. φ maps M into N and induces the zero map of each tang. space then φ is constant.

For any proper y on M

$$d(y \circ \varphi) = \varphi^*(dy) = 0$$

5.1.6 A Lie group is a locally conn. topological group such that the conn. component of the identity is isomorphic (as a top group) to an analytic group.

5.2. The Lie algebra of an anal. group

5.2.1 Invariant vector fields

φ_σ = left transl. by σ

$\varphi_\sigma \circ \varphi_\tau = \varphi_{\sigma\tau}$

φ_σ induces an iso of T_τ onto $T_{\sigma\tau}$. A left invariant vector field on G is a vector field V such that $\varphi_\sigma V = V$ for all $\sigma \in G$.

i.e. for all σ, τ $\varphi_\sigma V_\tau = V_{\sigma\tau}$.

Hence $V_\sigma = \varphi_\sigma V_\varepsilon$. Conversely, if V_ε is specified then $\varphi_\sigma V_\varepsilon$ defines a left invariant vector field.

5.2.2. A left invariant vector field is proper.

Take an arbitrary proper function f and $\tau \in$ dom f. Choose loc coord at τ