

Das Dichtigkeitstheorem von Polya

Problem: Betrachten Dirichletreihen

$$f(z) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n z}$$

was kann man über die Singul. auf d. konv. Abszisse aussagen?

Lemma

$$\{\lambda_n\} \subseteq \mathbb{C}$$

$$\lim_{n \rightarrow \infty} \frac{n}{\lambda_n} = D \quad \text{Dichte}$$

$$\exists c > 0 \quad |\lambda_n - \lambda_m| \geq c |n - m|$$

$$F(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{\lambda_n^2}\right)$$

$$\forall \varepsilon > 0 \quad r \rightarrow \infty$$

$$F(re^{i\theta}) = O\left(e^{(\pi D r |\sin \theta| + \varepsilon) r}\right)$$

$$1/F(re^{i\theta}) = O\left(e^{-\pi D r |\sin \theta| + \varepsilon r}\right)$$

$$1/|F'(\lambda_n)| = O\left(e^{2|\lambda_n|}\right)$$

$$(\exists c) F(re^{i\theta}) \leq c \cdot e^{(\pi D r |\sin \theta| + \varepsilon) r}$$

$$|re^{i\theta} - \lambda_n| \geq \frac{1}{8} c$$

$$n \rightarrow \infty$$

Corollar

$$\{\lambda_n\} \subseteq \mathbb{R}$$

$$\lim_{n \rightarrow \infty} \frac{n}{\lambda_n} = D$$

$$\exists c > 0 \quad |\lambda_{n+1} - \lambda_n| \geq c \quad \Rightarrow \quad \times$$

Satz

$$f(z) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n z}$$

$$\lim_{n \rightarrow \infty} \frac{n}{\lambda_n} = D$$

$$\exists c > 0 \quad |\lambda_{n+1} - \lambda_n| \geq c$$

\Rightarrow Auf d. konv. Abszisse $\exists \forall$ Intervallen mit Länge $\geq 2\pi D$ eine Singularität

Dichtigkeitstheorem v. Polya

Bew Satz 1

(unter Benutzung von Corollar, Abtragman Lindelöf, Annahme dass Dirichletreihe eine konv. Abszisse besitzt $x = x_0$)

o.B.d.A. konv. Abszisse $x = 0$

sonst (konv. Abszisse $x = x_0$) nehmen
$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n(z-x_0)}$$

$$= \sum_{n=1}^{\infty} \frac{a_n e^{+\lambda_n x_0}}{a_n} e^{-\lambda_n z}$$

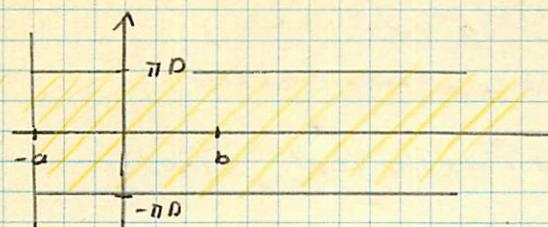
nehmen an, \exists Intervall I mit Länge $\geq 2\pi D$ ohne Singularität

o.B.d.A. $I \supseteq [-\pi D, \pi D]$

sonst nehmen wir Funktion
$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n z} - d$$
, was

nichts an der Beh. ändert.

$f(z)$ habe nun keine Singular auf $x=0$ $|y| \in B > \pi D$
 $\exists a > 0$, so dass $f(z)$ analytisch für $x \geq -a$ $|y| \in B$



Definieren $H(\omega) := \int_{-a}^{\infty} f(z) e^{\omega z} dz$ $\omega = u + iv$
 $u < 0$ sonst

schliefdefiniert $F(\omega)$

$$H(\omega) = \int_{-a}^{\infty} f(z) e^{\omega z} dz = \int_{-a-i\pi B}^{-a+i\pi B} f(z) e^{\omega z} dz + \int_{-a+i\pi B}^{\infty+i\pi B} f(z) e^{\omega z} dz$$

$$= \int_{-a}^{\infty} f(z) e^{\omega z} dz + \int_{-a-i\pi B}^{-a+i\pi B} f(z) e^{\omega z} dz + \int_{-a+i\pi B}^{\infty+i\pi B} f(z) e^{\omega z} dz$$

$$= \int_{-a}^{\infty} f(z) e^{\omega z} dz + \int_{-a-i\pi B}^{-a+i\pi B} f(z) e^{\omega z} dz - \sum_{k=1}^{\infty} a_k \frac{e^{(\omega-\lambda_k)(b+i\pi B)}}{\omega-\lambda_k}$$

Definieren $G(\omega) = H(\omega) \cdot F(\omega)$

ist eine ganze Funktion

$$G(\omega) = H(\omega) \cdot F(\omega) = \prod_{l=1}^{\infty} \left(1 - \frac{\omega^2}{\lambda_l^2}\right) \left(\sum_{k=1}^{\infty} a_k \frac{e^{(\omega-\lambda_k)(b+i\pi B)}}{\omega-\lambda_k} + \text{ganz} \right)$$

ganz, dann $= \prod_{l=1}^{\infty} \left(1 - \frac{\omega^2}{\lambda_l^2}\right) \cdot \text{ganz}$ so dass $1 - \frac{\omega^2}{\lambda_l^2} \in \text{ganz}$

$$= \prod_{l \neq k} \left(1 - \frac{\omega^2}{\lambda_l^2}\right) \sum_{k=1}^{\infty} (-a_k) \lambda_k + \omega \cdot e^{(\omega-\lambda_k)(b+i\pi B)}$$

$$G(\lambda_n) = + \sum_{k=1}^{\infty} a_k \prod_{\substack{l=1 \\ l \neq n}}^{\infty} \left(1 - \frac{\lambda_n^2}{\lambda_l^2}\right) \frac{(\lambda_n + \lambda_k)}{\lambda_n^2} e^{(\lambda_n - \lambda_k)(b+i\pi B)}$$

$$= + a_n \prod_{\substack{l=1 \\ l \neq n}}^{\infty} \left(1 - \frac{\lambda_n^2}{\lambda_l^2}\right) \frac{2}{\lambda_n}$$

$$= - a_n F'(\lambda_n) \quad \left(= \sum_{\substack{k=1 \\ k \neq n}}^{\infty} \prod_{\substack{l=1 \\ l \neq k}}^{\infty} \left(1 - \frac{\lambda_n^2}{\lambda_l^2}\right) \frac{-2\lambda_n}{\lambda_k^2} \right)$$

Schätzen nun H ab:

$$\omega = \rho e^{i\theta} \quad 0 < \theta < \frac{\pi}{2}$$

$$|H(\rho e^{i\theta})| \leq e^{-a\rho \cos\theta} \int_{-a}^{\infty} |f(z)| dz + e^{+b\rho \cos\theta - B\rho \sin\theta} \int_{-a-i\pi B}^{-a+i\pi B} |f(z)| dz$$

$$+ e^{b\rho \cos\theta - B\rho \sin\theta} \sum_{k=1}^{\infty} \frac{|a_k| e^{-b\lambda_k}}{B \sin\theta}$$

nach unten
b?

Können auch Reflektieren Weg unter reeller Achse wählen

$$\rightarrow H(\rho e^{\pm i\theta}) = 0 \quad \left(e^{-a\rho \cos\theta} + e^{b\rho \cos\theta - B\rho \sin\theta} \right) \quad \rho \rightarrow \infty$$

Es folgt aus dem Corollar d. Lemmas, also aus

$$\left. \begin{aligned} F(re^{i\theta}) &= O(e^{\pi D r \sin \theta + \varepsilon r}) \\ H(\rho e^{i\theta}) &= O(e^{-a \rho \cos \theta + b \rho \cos \theta - B \rho \sin \theta}) \end{aligned} \right\}$$

$$G(\rho e^{i\theta}) = O(e^{-a \rho \cos \theta + \pi D \rho \sin \theta + \varepsilon \rho + b \rho \cos \theta - (B - \pi D) \rho \sin \theta + \varepsilon \rho})$$

wählen nun $b = \frac{1}{2}(B - \pi D) \tan \theta$ ($B > \pi D$)
 $\Rightarrow B - \pi D = 2b \cot \theta$

$$G(\rho e^{i\theta}) = O(e^{-a \rho \cos \theta + \pi D \rho \sin \theta + \varepsilon \rho + e^{-b \rho \cos \theta + \varepsilon \rho}})$$

$$G(\rho e^{i\theta}) = O(e^{-\frac{a}{2} \rho \cos \theta + \pi D \rho \sin \theta + e^{-\frac{b}{2} \rho \cos \theta}}) \quad \exists \theta \text{ bel. kl.}$$

$\bullet D = 0 \quad \theta = \frac{1}{4} \pi \quad \ominus = \min(a, b) / 4$

$$G(\rho e^{i\theta}) = O(e^{-\delta \rho \cos \theta})$$

$\bullet D > 0 \quad \theta = \tan^{-1} \frac{a}{4\pi D} \quad \Rightarrow \pi D \sin \theta = \frac{1}{4} a \cos \theta$

$$G(\rho e^{i\theta}) = O(e^{-\frac{1}{2} a \rho \cos \theta + e^{-\frac{1}{2} b \rho \cos \theta}})$$

$$G(\rho e^{i\theta}) = O(e^{-\delta \rho \cos \theta})$$

(3)

$$d > 0 \Rightarrow e^{d\omega} G(\omega) = O(1) \quad |\omega| \rightarrow \infty \quad \arg \omega = \pm \theta$$

$$|\omega - \lambda_n| \geq \frac{1}{2} c \Rightarrow |H(\omega)| \leq A e^{A|\omega|} \quad \exists A$$

$$|G(\omega)| \leq A e^{A|\omega|} \quad \exists A \quad \text{cases Corollar}$$

$G(\omega)$ ganz \Rightarrow Maximum von $G(\omega)$ in $|\omega - \lambda_n| \leq \frac{1}{2} c$ wird angenommen in $|\omega - \lambda_n| = \frac{1}{2} c$

$$\Rightarrow |G(\omega)| \leq A e^{A|\omega|} \quad \forall \omega$$

$$\Rightarrow e^{d\omega} G(\omega) = O(e^{A|\omega|}) \quad \forall \omega$$

(3)

$$e^{d\omega} G(\omega) = O(1) \quad \arg \omega = \pm \theta$$

Phragmén-Lindelöf

$$e^{d\omega} G(\omega) = O(1) \quad \arg \omega \leq \theta$$

speziell $G(u) = O(e^{-du}) \quad u \rightarrow \infty$

(2) $G(\lambda_n) = -a_n F'(\lambda_n)$

$$\rightarrow a_n = O\left(\frac{e^{-d\lambda_n}}{|F'(\lambda_n)|}\right)$$

Corollar $a_n = O(e^{-d\lambda_n + \varepsilon \lambda_n})$

da ε beliebig:

$$a_n = O(e^{-d\lambda_n/2})$$

aber daraus würde folgen: Die Abzisse d. konv. Abszisse von $f(z)$ ist bei $-\frac{1}{2} d$ oder links davon

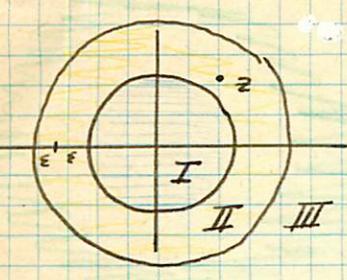
(Cohen: $\sigma_B = \overline{\lim}_{n \rightarrow \infty} \frac{\log |s_n|}{\lambda_n}$)

$$s_n = \sum_{\nu=0}^n a_\nu$$

Unklarheit: Lindelöf, Cohen S. 89 lim λ_n σ_1, b

Bew. d. Lemmas

$\varepsilon > 0$ fix z fix



- $z_n \in I \quad |z_n| \leq (1-\varepsilon)|z|$
- $z_n \in II \quad |z|(1+\varepsilon) < |z_n| < (1+\varepsilon)|z|$
- $z_n \in III \quad |z_n| \geq (1+\varepsilon)|z|$

$$|F(z)| = \left| \prod_{I} \prod_{II} \prod_{III} \right| \left| 1 - \frac{z^2}{z_n^2} \right|$$

$\frac{D}{z_n} \rightarrow D \rightarrow \# z_n \text{ in } II < 2\varepsilon|z|(D+d)$ $d > 0$
 $|z| \text{ gen. gross}$

$\rightarrow \prod_{II} \left| 1 - \frac{z^2}{z_n^2} \right| \leq \prod_{II} 3 \leq 3^{2\varepsilon|z|(D+d)}$

$< e^{4\varepsilon|z|(D+d)}$ (1)

nehmen von jetzt an an $\operatorname{Re}(z) > 0$ o.B.d.A

(2) $\left| \frac{1}{1 - \frac{z^2}{z_n^2}} \right| = \frac{1}{\left| 1 + \frac{z}{z_n} \right| \left| 1 - \frac{z}{z_n} \right|} \leq \frac{1}{\left| 1 - \frac{z}{z_n} \right|} = \left| \frac{z_n}{z - z_n} \right|$ $f. \text{ gen\u00fcg.}$
 $\text{grosse } n$

(3) $(\exists N) |z - z_n| \geq |z - z_N|$ (z_n mit minim. Abstand)

$|z_N - z_n| \leq |z - z_n| + |z - z_N| \leq 2|z - z_n|$

(3) $\Rightarrow \frac{1}{|z - z_n|} \leq \frac{2}{|z_N - z_n|}$

(2)(3) $\Rightarrow \left| \frac{1}{1 - \frac{z^2}{z_n^2}} \right| \leq \frac{2|z_n|}{|z_N - z_n|}$

Produkt \u00fcber alle n ohne $n=N$

$$\prod_{II} \frac{1}{\left| 1 - \frac{z^2}{z_n^2} \right|} \leq \prod_{II} \frac{2|z_n|}{|z_N - z_n|}$$

$M = \# z_n \text{ in } II$
 $|z_n - z_N| \geq c|n-N|$ (Voraussetz.)

$$\prod_{II} \frac{1}{\left| 1 - \frac{z^2}{z_n^2} \right|} \leq 2^M \left\{ (1+\varepsilon)|z| \right\}^M \prod_{II} \frac{1}{c|n-N|} \leq \left(\frac{3}{c}\right)^M |z|^M \prod_{II} \frac{1}{(n-N)}$$

$$\prod_{II} \frac{1}{(n-N)} \geq \left(\left[\frac{1}{2} (M-1) \right]! \right)^2$$

da n\u00e4chst. 2 Werte im Produkt gleich sind

$\Rightarrow \prod_{II} \frac{1}{\left| 1 - \frac{z^2}{z_n^2} \right|} \leq \left(\frac{3}{c}\right)^M |z|^M / \left(\left[\frac{1}{2} (M-1) \right]! \right)^2$

$\leq \left(\frac{3}{c}\right)^M |z|^M e^{3M} e^{-M \log M}$ Stirling

$= \left(\frac{3}{c}\right)^M e^{3M} e^{-M \log 2M}$ $n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$

$= \left(\frac{3}{c}\right)^M e^{3M} e^{-M \log \frac{2M}{e}}$ $n! \geq e^{M \log M} e^{2M}$

Wahl $M \leq 2\varepsilon |z| (D+\delta)$ und $x \log \frac{1}{x} < 1 \quad x > 0$ gilt

$$\frac{M}{|z|} \log \frac{|z|}{M} \leq \left(\frac{M}{|z|}\right)^{10} < \frac{2\varepsilon |z| (D+\delta)^{10}}{|z|^{10}} = (2\varepsilon (D+\delta))^{10}$$

$$\Rightarrow \prod_{II} \frac{1}{|1 - \frac{z^2}{2n^2}|} \leq \left(\frac{3}{\varepsilon}\right)^{2\varepsilon |z| (D+\delta)} e^{10\varepsilon |z| (D+\delta)} |z|^{10\varepsilon (D+\delta)^{10}}$$

\Rightarrow in rechtl. Halbebene $\Rightarrow \frac{1}{|1 - \frac{z^2}{2n^2}|} < \frac{2|z|}{|z-2n|}$

$|z|$ gross genug $\rightarrow |2n| < 2|z| \rightarrow \frac{1}{|1 - \frac{z^2}{2n^2}|} < \frac{4|z|}{|z-2n|}$

Es gilt auch $\frac{4|z|}{|z-2n|} \geq \frac{4|z|}{|z|+|2n|} \geq \frac{4|z|}{3|z|} > 1$

$$\Rightarrow \prod_{II} \frac{1}{|1 - \frac{z^2}{2n^2}|} \leq \frac{4|z|}{|z-2n|} \prod_{II} \frac{1}{|1 - \frac{z^2}{2n^2}|}$$

$$\Rightarrow \prod_{II} \frac{1}{|1 - \frac{z^2}{2n^2}|} \leq \left(\frac{3}{\varepsilon}\right)^{2\varepsilon |z| (D+\delta)} e^{10\varepsilon |z| (D+\delta)} e^{10\varepsilon |z| (D+\delta)^{10}}$$

④ $\prod_{II} \frac{1}{|1 - \frac{z^2}{2n^2}|} < e^{4\varepsilon |z| (D+\delta)}$

$$\Rightarrow \exists A \quad |z-2n| e^{-A|z| \varepsilon^{10} (D+\delta)^{10}} \leq \prod_{II} \frac{1}{|1 - \frac{z^2}{2n^2}|} \leq e^{A|z| \varepsilon^{10} (D+\delta)^{10}}$$

$D=0$ sei $\varepsilon = \frac{1}{2} \Rightarrow |z-2n| e^{-A|z| \varepsilon^{10}} = \prod_{II} \frac{1}{|1 - \frac{z^2}{2n^2}|} \leq e^{A|z| \varepsilon^{10}}$

$D>0$ $\delta = 1$ $B = A(D+1)^{10} \quad |z-2n| e^{-B|z| \varepsilon^{10}} \leq \prod_{II} \frac{1}{|1 - \frac{z^2}{2n^2}|} \leq e^{B|z| \varepsilon^{10}}$

④ $|z-2n| e^{-A|z| \varepsilon^{10}} \leq \prod_{II} \frac{1}{|1 - \frac{z^2}{2n^2}|} \leq e^{A|z| \varepsilon^{10}}$

haben jetzt Abschätzung in II
f. genug. grosse N_0 haben wir weil $\frac{n}{2n} \rightarrow D$

⑤ $\left| \frac{D^2 2n^2 - n^2}{2n^2} \right| < \frac{1}{8} D^2 \varepsilon^2 \quad n > N_0$

$2n \in I$ oder $III \quad |2n| \leq 2|z|$

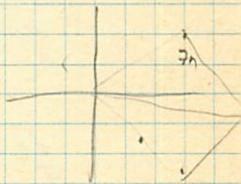
$|n^2 - z^2 D^2| \geq |D^2 2n^2 - z^2 D^2| - |n^2 - D^2 2n^2|$
 $|2n| - |z| \geq \varepsilon |z|$

$$\rightarrow |n^2 - z^2 D^2| \geq D^2 \varepsilon |z| |2n+z| - \frac{1}{8} D^2 \varepsilon^2 |2n|^2$$

$$\geq D^2 \varepsilon |z| \left(\frac{1}{2}|z| - \frac{1}{2} D^2 \varepsilon^2 |z|^2\right) = \frac{1}{2} D^2 \varepsilon |z|^2 (1 - \varepsilon)$$

ε klein

⑥ $\left| \frac{n^2 - D^2 z^2}{z^2} \right| \geq \frac{1}{4} D^2 \varepsilon$



⑤.⑥

$$1 - \varepsilon \leq \left| 1 + \frac{z^2(2n^2 D^2 - n^2)}{2n^2(n^2 - 2^2 D^2)} \right| \leq 1 + \varepsilon$$

$n \geq N_0$
 $2n \leq 2|z|$
 $2n \leq 2|z|$

$$1 + \frac{z^2(2n^2 D^2 - n^2)}{2n^2(n^2 - 2^2 D^2)} - \frac{1 - \frac{z^2}{2n^2}}{1 - \frac{2^2 D^2}{n^2}}$$

$$\rightarrow 1 - \varepsilon \leq \left| \frac{1 - \frac{z^2}{2n^2}}{1 - \frac{2^2 D^2}{n^2}} \right| \leq 1 + \varepsilon \quad \text{⑦ } n \geq N_0$$

$$\exists C_1 \frac{1}{C_1} \leq \prod_{n \in I} \left| \frac{1 - \frac{z^2}{2n^2}}{1 - \frac{2^2 D^2}{n^2}} \right| \leq C_1 \quad \text{⑧}$$

⑨.⑧ $\# z_n \in I \leq 2|z| D$ grosse $|z|$

$$\rightarrow \frac{1}{C_1} (1 - \varepsilon)^{2|z| D} \prod_{n \in I} \left| 1 - \frac{z^2 D^2}{n^2} \right| \leq \prod_{n \in I} \left| 1 - \frac{z^2}{2n^2} \right| \leq C_1 (1 + \varepsilon)^{2|z| D} \prod_{n \in I} \left| 1 - \frac{z^2 D^2}{n^2} \right|$$

$1+x \leq e^x$ $1-x \geq e^{-x}$ x klein

$$|z_n| \geq 2|z| \quad \text{und} \quad \left| \frac{D^2 z_n^2 - n^2}{2n^2} \right| \leq \frac{1}{8} D^2 \varepsilon^2$$

$$1 - \frac{1}{8} \frac{D^2 \varepsilon^2 |z|^2}{n^2 - 12|z|^2 D^2} \leq \left| 1 + \frac{z^2(2n^2 D^2 - n^2)}{2n^2(n^2 - 2^2 D^2)} \right| \leq 1 + \frac{1}{8} \frac{D^2 \varepsilon^2 |z|^2}{n^2 - 12|z|^2 D^2}$$

$z_n \sim \frac{n}{D}$ $|z|$ gross $|z_n| \geq 2|z|$ immer noch

$$\Rightarrow 1 - \frac{\varepsilon |z|^2 D^2}{n^2} \leq \left| 1 + \frac{z^2(2n^2 D^2 - n^2)}{2n^2(n^2 - 2^2 D^2)} \right| \leq 1 + \frac{\varepsilon |z|^2 D^2}{n^2}$$

damit und mit ⑦

$$\frac{(1 + \varepsilon)^{2|z| D}}{|z_n|^{2|z|}} \prod_{n \in I} \left(1 + \frac{\varepsilon |z|^2 D^2}{n^2} \right) \leq \prod_{n \in I} \left| \frac{1 - \frac{z^2}{2n^2}}{1 - \frac{2^2 D^2}{n^2}} \right|$$

$$\leq (1 + \varepsilon)^{2|z| D} \prod_{|z_n| \geq 2|z|} \left(1 + \frac{\varepsilon |z|^2 D^2}{n^2} \right) \quad \text{⑨}$$

kleine x $1-x \leq e^{-x}$ $1-x \geq e^{-2x}$

$$\prod_{|z_n| \geq 2|z|} \left(1 + \frac{\varepsilon |z|^2 D^2}{n^2} \right) \leq \exp \left(\varepsilon |z|^2 D^2 \sum_{n \geq 2|z|} \frac{1}{n^2} \right) \leq e^{2\varepsilon |z|}$$

weil $\sum_{n \geq 1} \frac{1}{n^2} \leq \frac{1}{1} + \int_1^\infty \frac{1}{x^2} dx = 2$

$$\prod_{|z_n| \geq 2|z|} \left(1 - \frac{\varepsilon |z|^2 D^2}{n^2} \right) \geq \exp \left(-2\varepsilon |z|^2 D^2 \sum_{n \geq 2|z|} \frac{1}{n^2} \right) \geq e^{-4\varepsilon |z|}$$

$$\text{⑩} \rightarrow e^{-8\varepsilon |z|} \leq \prod_{n \in I} \left| \frac{1 - \frac{z^2}{2n^2}}{1 - \frac{2^2 D^2}{n^2}} \right| \leq e^{4\varepsilon |z|} \quad \text{⑩}$$

⑪ ⑫

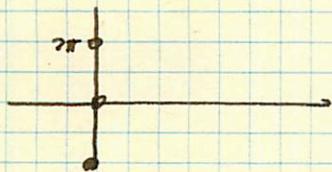
Anwendungen d. Dichtigkeits Theorems:

Betrachten $f(z) = \sum_{n=1}^{\infty} a_n e^{-n^{\lambda} z}$

$$\lim_{n \rightarrow \infty} \frac{n}{\lambda n} = \lim_{n \rightarrow \infty} \frac{n}{n^{\lambda+1}} = 0$$

\Rightarrow DTP \Rightarrow Singularitäten liegen dicht auf konv. Abszisse

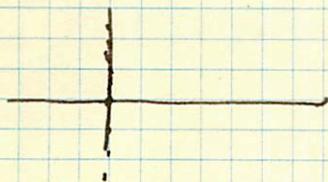
$$f_1(z) = \sum_{n=0}^{\infty} e^{-n^2} = \sum_{n=0}^{\infty} (e^{-z})^n = \frac{1}{1-e^{-z}}$$



\uparrow analytische Fortsetzung von $f_1(z)$ auf ganze Ebene

$$\lim_{n \rightarrow \infty} \frac{n}{\lambda n} = 1 \quad \text{pk} \Rightarrow \text{mind. 1 Singul. in jedem Intervall } > 2\pi 0$$

$$f_2(z) = \sum_{n=0}^{\infty} e^{-n^2 z}$$



$$= -\theta\left(0, \frac{-z}{2}\right) - 1$$

$$\lim_{n \rightarrow \infty} \frac{n}{n^2} = 0 \quad \Rightarrow$$

\rightarrow keine analyt. Fortsetzung möglich

$$f_3(z) = \sum_{n=0}^{\infty} z^{2^n} = \sum_{n=0}^{\infty} e^{2^n \log z} \quad \lim_{n \rightarrow \infty} \frac{n}{\lambda n} = 0$$

keine analyt. Fortsetzung über Einheitskreis möglich

$$f(z) = \sum_{n=1}^{\infty} a_n z^n = \sum_{n=1}^{\infty} a_n e^{n \log z} = \sum a_n e^{-n\omega}$$

$$\log z = -\omega \quad \lambda n = n$$

$$\operatorname{Re} \omega = -\log |z| \geq \lambda_0$$

$$|z| \leq e^{\lambda_0} \quad \text{konv. Kreis}$$

$$2\pi \geq \operatorname{Im} \omega_1 - \operatorname{Im} \omega_2 = \arg |z_1| - \arg |z_2|$$

d.h. mind. eine Singularität auf konv. Kreis

$$f(z) = \sum_{n=0}^{\infty} a_n z^{n^2} \quad \text{Singul. dicht auf konv. Kreis}$$

$$f(z) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{n=1}^{\infty} e^{-\log n^2 z} \quad \text{keine Aussage d. DTP}$$

Das Maximumprinzip

$\mathcal{H}(\mathbb{R})$ Menge aller Holomorphen Funktionen auf \mathbb{R}

Satz

(Maxim. Prinzip)

Ω zust. besch. offene Menge von \mathbb{C}
 $f \in \mathcal{H}(\Omega)$ $M := \sup_{\xi \in \partial \Omega} \lim_{z \rightarrow \xi, z \in \Omega} |f(z)|$

Falls f nicht konst $\Rightarrow |f(z)| < M \quad \forall z \in \Omega$

Lemma 1

(Prinzip d. offen Abb)

Ω offen zust. in \mathbb{C} $f \in \mathcal{H}(\Omega)$
 f nicht konst $U \subset \mathbb{R}$ offen $\Rightarrow f(U)$ offen

Lemma 2

Ω offen $f \in \mathcal{H}(\Omega)$
 U offen $U \subset \mathbb{R}$

$U \cap \mathbb{R} := U \text{ rel. komp. in } \mathbb{R}$

$|f(z)| \leq \sup_{w \in \delta U} |f(w)|$

Bew. Satz 1

$|f(z)| \leq M \quad z \in \mathbb{R} \quad \dashv \dashv \quad$ genügt wegen Lemmas

$\xi \in \partial \Omega \Rightarrow \exists U_\xi$ sodass $|f(z)| < M + \varepsilon \quad z \in U_\xi \cap \Omega$

$V := \bigcup_{\xi \in \partial \Omega} U_\xi \cap \Omega \quad K = \Omega - V \quad K \text{ kompakt}$

$K \subset W \subset \mathbb{R} \Rightarrow \delta W \subset V$

$|f(z)| \leq M \quad z \in \delta W$

Lemma 2
 $\Rightarrow |f(z)| \leq M \quad z \in W$

$\Rightarrow |f(z)| \leq M \quad \forall z \in \Omega \quad \dashv \dashv$

Lemma 1.1

Ω zust. offen in \mathbb{C} $f \in \mathcal{H}(\Omega)$
 $\exists U \neq \emptyset \quad U \subset \mathbb{R}$ offen $f|_U = 0$
 $\Rightarrow f = 0$ auf Ω

Lemma 1.2

Ω zust. offen in \mathbb{C} $f \in \mathcal{H}(\Omega)$
 $Z_f := \{z \in \mathbb{R} \mid f(z) = 0\}$
 $f \neq 0 \Rightarrow Z_f$ diskret (abgesch. isol. Punkte)

Bew. Lemma 1

$a \in \mathbb{R}$
 Annahme $f(a) = 0$ o.F.d. H
 $D(a, r)$ mit $D(a, r) \subset \Omega$ $f(z) \neq 0 \quad z \in D$
 $d := \inf \{ |f(z)| \mid |z - a| = r \}$

zeige: $w \notin |f(\Omega)| \Rightarrow |w| \geq \frac{d}{2}$ d. dann

$\{ |w| < \frac{d}{2} \} \subset f(\Omega)$

$$\phi(z) := \frac{1}{f(z) - \omega} \quad \omega \neq f(z) \quad \phi \in \mathcal{H}(\mathbb{R})$$

Lemma 2 $|\phi(z)| \leq \sup_{|z-a|=r} |f(z)|$

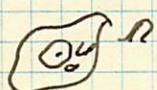
$$|\phi(a)| = \frac{1}{|f(a) - \omega|} \leq \sup_{|z-a|=r} |f(z)| = \frac{1}{\inf_{|z-a|=r} |f(z) - \omega|}$$

1. Fall $|\omega| < d \quad |f(z) - \omega| = |f(z)| - |\omega| \geq d - |\omega|$
auf $|z-a|=r$

$$\frac{1}{|\omega|} \leq \frac{1}{d - |\omega|} \quad |\omega| \geq d - |\omega| \Rightarrow |\omega| \geq \frac{d}{2}$$

2. Fall $|\omega| \geq d \Rightarrow |\omega| \geq \frac{d}{2}$

also insges. $f(\mathbb{R})$ Umgeb. von $f(a)$



Lemma 1.1 - $f|_U$ nicht konst.
 $f|_U$ Umg. von U

Lemma 2.1.2

Sei Ω offen in $\mathbb{C} \quad \forall u \in C^2(\Omega)$ reellwertig

$$\Delta u(x) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \geq 0 \quad z \in \Omega$$

$$(\forall u \in \mathcal{A}) \quad u(z) \leq \sup_{\omega \in \partial \Omega} u(\omega)$$

Bew Lemma 2

$$\omega(z) = |f(z)|^2 \quad f = x + iy$$

$$\Delta \omega = 4|f'(z)|^2$$

$$\Delta \omega = 4|f'(z)|^2 \geq 0$$

$$|f'(z)|^2 \leq \sup_{\omega \in \partial \Omega} |f'(z)|^2 \quad |f(z)| \leq \sup_{\omega \in \partial \Omega} |f(\omega)|$$

Bew Lemma 2.1

Lemma 2.1.1

I offen in \mathbb{R}

$\phi \in C^2(\mathbb{R}) \quad \phi$ reellwertig

$$(\exists t_0 \in I) \quad \phi(t) \leq \phi(t_0) \quad \forall t \in I$$

$$\Rightarrow \phi''(t_0) \leq 0$$

Γ

$$\phi''(t_0) = \lim_{h \rightarrow 0} \frac{1}{h^2} (\phi(t_0+h) + \phi(t_0-h) - 2\phi(t_0))$$

$$= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{\phi(t_0+h) - \phi(t_0)}{h} - \frac{\phi(t_0) - \phi(t_0-h)}{h} \right)$$

$$\leq 0$$

Γ

Koroll. 2.1.1

Ω offen in $\mathbb{R}^2 \quad u \in C^2(\mathbb{R}^2)$ reellwertig

$$(\exists (x_0, y_0) \in \Omega) \quad u(x, y) \leq u(x_0, y_0)$$

$$\Rightarrow \Delta(u(x_0, y_0)) \leq 0$$

Proof Lemma 2.1

$$\Delta u(z) \geq 0 \quad z \in \Omega$$

$$\exists u \in C^2 \quad \exists z_0 \in \bar{\Omega} \quad u(z_0) = \sup_{z \in \bar{\Omega}} u(z)$$

* clear denn würde u nicht ex.

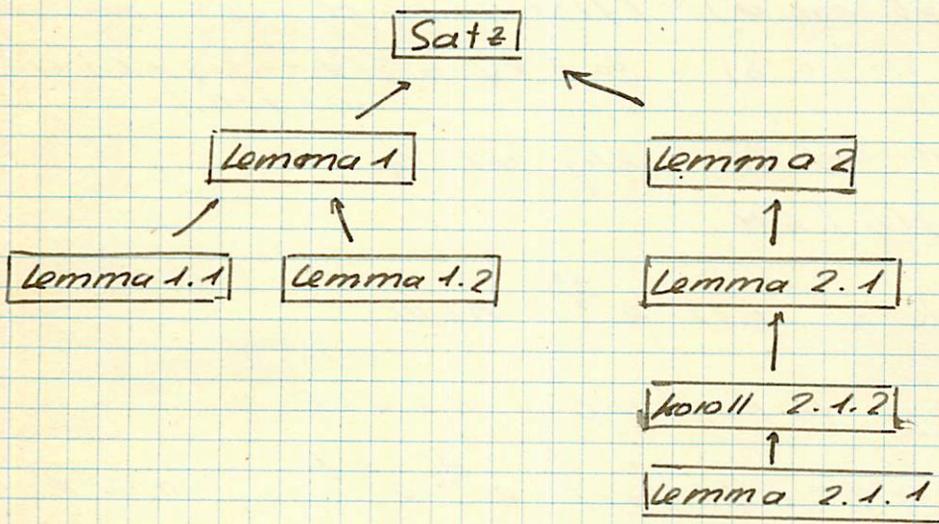
$$\rightarrow \exists \epsilon > 0 \quad u(z) \leq u(z_0) \quad \forall z \in \Omega$$

$$\Delta u(z) \leq 0 \quad \Delta u \geq 0 \quad z \in \Omega$$

$$u_\epsilon(z) < u(z) + \epsilon / |z|^2$$

$$\Delta u_\epsilon(z) = \Delta u(z) + 4\epsilon$$

$$\rightarrow u_\epsilon(z) \leq \sup_{z \in \Omega} u_\epsilon(z) \quad z \in \Omega$$



Phragmen - Lindelöf Prinzip

Satz 0 Max. Prinzip

Satz 1

Phragmen
Lindelöf Prinzip

f analyt. in $D := \{z = x+iy \mid \alpha < x < \beta\}$
 stetig auf ∂D
 $|f(z)| \leq M \quad \forall z \in \partial D$
 f beschränkt auf \bar{D}
Satz $|f(z)| \leq M \quad \forall z \in D$

1. Fall

$$\lim_{y \rightarrow +\infty} f(x+iy) = 0 \quad \alpha < x < \beta \quad \text{glm.}$$

$z_0 \in D$ beliebig.

$\exists M \geq 0$ mit a) $|f(x+iy)| \leq M \quad \forall x \in \mathbb{R}, y \in \mathbb{R}$

b) $z_0 \in R_\eta := \{z = x+iy \in \mathbb{C} \mid \alpha < x < \beta, -\eta < y < \eta\}$

f anal. auf R_η stetig auf \bar{R}_η

$$\Rightarrow |f(z)| \leq M$$

2. Fall

$$f_n(z) := f(z) \cdot e^{-\frac{z^2}{n}} \quad n \in \mathbb{N}$$

1. Schritt

$$\lim_{y \rightarrow +\infty} f_n(x+iy) = 0 \quad \alpha < x < \beta$$

$$|f_n(z)| = |f(z)| \cdot e^{-\frac{y^2}{n}} \quad \text{O.Bd. A} \quad \forall z \in \bar{D}$$

$\varepsilon > 0$

$$y_0^2 := \left(\frac{\beta - \alpha}{\varepsilon} \right)^2$$

da $\exists \delta > 0$ mit $|f(z)| \leq M \quad \forall z \in \bar{D}$

$$y > y_0 \Rightarrow |f_n(z)| \leq M \cdot e^{-\frac{y^2}{n}} < \varepsilon \quad \forall z \in \bar{D}$$

2. Schritt

$$\exists \delta > 0 \quad \forall z \in \bar{D} \quad |f_n(z)| \leq M \cdot e^{-\frac{\delta^2}{n}} < \varepsilon$$

3. Schritt

$$|f(z)| \leq M \cdot e^{\frac{c^2}{n}} \quad \forall z \in \bar{D} \quad \text{da } f \text{ stetig auf } \bar{D}$$

4. Schritt

$$\forall \varepsilon > 0 \quad \exists n_0 \in \mathbb{N} \quad \forall n \geq n_0 \quad |f_n(z) - f(z)| < \varepsilon \quad \forall z \in \bar{D}$$

$\varepsilon > 0$

$z \in \bar{D}$ beliebig

$$|f_n(z) - f(z)| = |f(z)| \cdot |e^{-\frac{z^2}{n}} - 1| < \varepsilon \quad \text{da } |f(z)| \leq M$$

$\frac{\varepsilon}{M} > 0$

5. Schritt

$$|f(z)| \leq M \quad \forall z \in \bar{D}$$

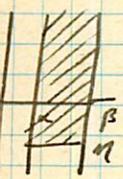
$z \in \bar{D}$ beliebig

$$\forall \varepsilon > 0 \quad \exists n_0 \in \mathbb{N} \quad \forall n \geq n_0 \quad |f_n(z) - f(z)| < \varepsilon \quad \forall z \in \bar{D}$$

$\lim_{n \rightarrow \infty} |f(z)| = \lim_{n \rightarrow \infty} |f(z)| = |f(z)| \leq M$
 sogut strikt

Satz 2

$f \in \mathcal{O}(D_n) \quad D = \{z = x+iy \mid x > \eta, y > \eta\}$
 $\lim_{z \rightarrow \xi} |f(z)| \leq M \quad \forall \xi \in \partial R_n$
 $\exists \epsilon > 0 \exists \theta < 1 \quad |f(z)| \leq \epsilon \exp \left(\frac{\theta |z|}{R-\eta} \right) \quad \forall z \in D_n$
 $\Rightarrow |f(z)| \leq M \quad \forall z \in D_n$



$f(z) = e^{-\sigma z} \quad \alpha = -\frac{\pi}{2} \quad \beta = \frac{\pi}{2} \quad \eta = 0 \quad \theta = 1$
 $|f(\frac{\pi}{2} + iy)| = e^{-\sigma y} \quad |f(x)| = e^{-\sigma x}$

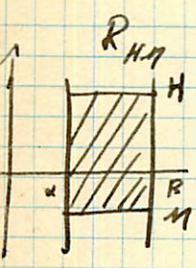
1. Fall $\alpha = -\frac{\pi}{2} \quad \beta = \frac{\pi}{2} \quad \eta = 0 \quad \theta = 1$
 $g(z) := f(z) \exp(-\sigma e^{-ikz})$
 $\sigma > 0$
 $0 < k < 1$

1. Schritt $\exists c > 0 \quad |g(z)| \leq c \cdot \exp \left\{ \sigma |z| - \frac{\sigma}{2} \cos \frac{\pi k}{2} |z|^2 \right\}$
 $\forall z \in D$

$|g(z)| = |f(z)| \exp(\sigma e^{-ikz})$
 $= |f(z)| \exp \left(-\frac{\sigma}{2} \cos \frac{\pi k}{2} |z|^2 \right)$

$\exists c > 0 \quad |f(z)| \leq c \exp(-\theta |z|)$ (aus Satz 2)
 $|g(z)| \leq c \exp \left\{ \sigma |z| - \frac{\sigma}{2} \cos \frac{\pi k}{2} |z|^2 \right\}$

2. Schritt $\lim_{y \rightarrow \infty} |g(z)| = 0 \quad \alpha \leq x \leq \beta \quad g \text{ i.M.}$
 $e^{\theta |x+iy|} - \sigma \cos \frac{\pi k}{2} |z|^2 \leq e^{\theta |x+iy|} - \sigma \cos \frac{\pi k}{2} |z|^2$
 $\leq e^{\theta |x+iy|} - \frac{\sigma}{2} \cos \frac{\pi k}{2} |z|^2 \rightarrow -\infty$



3. Schritt $\exists H > 0 \quad \lim_{D_{2H}} |g(z)| \leq M \quad \forall z \in D_{2H}$

$|z| \leq |f(z)|$
 $\lim |g(z)| \leq \lim |f(z)| \leq M$ (2. Schritt u. Maximumprinzip)

4. Schritt $|f(z)| \leq M e^{-\sigma e^{-ikz}} \quad \forall z \in R_n$
 $z \in D_n$ notwendig

- a) g analyt in D_n
- b) $\exists H > 0$
 - i) $z \in R_{2H}$
 - ii) $\lim_{z \rightarrow \xi} |g(z)| \leq M \quad \forall \xi \in \partial R_{2H}$
- c) notwendig

Beh. wegen Maximumprinzip.

5. Schritt Beh. $\sigma \rightarrow 0$
 $|f(z)| \leq \lim_{\sigma \rightarrow 0} M e^{-\sigma e^{-ikz}} = M$

2. Fall

α, β beliebig

$$y(z) := |f(z)| \exp \left\{ -\sigma \exp \left(-ik \frac{\pi}{\beta - \alpha} \right) \right\}$$

$$(z = \alpha + \frac{\beta - \alpha}{2})$$

Satz 3

$f \in \mathcal{O}(D_M)$ (D_M wie in Satz 2)

$\exists c > 0 \exists \rho > 0 \exists F > 0 \exists \epsilon > 0 \exists \delta > 0$ $|f(z)| \leq c \cdot \exp \left(e^{\frac{\theta |z|}{\beta - \alpha}} \right)$ $\forall z \in D_M$

f stetig auf ∂D_M

$$\lim_{y \rightarrow \infty} |f(x+iy)| \leq M \quad x = \alpha \text{ und } x = \beta$$

$$\Rightarrow \lim_{y \rightarrow \infty} |f(x+iy)| \leq M \quad \alpha \leq x \leq \beta$$

1. Schritt

f beschr. auf D_M

2. Schritt

$$y(z) := |f(z)| \cdot \frac{z}{|z|} \quad n > 0$$

$$|g(z)| < M + \epsilon \quad \forall z \in D_{M+\epsilon}$$

3. Schritt

$$|f(z)| (1 - \epsilon) \leq M + \epsilon$$

Die Γ -Funktion

axiom. Festlegung

- 1) $\Gamma(z)$ merom. Erweiter. von $n!$
 $\rightarrow \Gamma(z+1) = z \Gamma(z) \quad \Gamma(1) = 1$
- 2) logarithm. Konvexität von $\Gamma(x)$ s.h. in $\Gamma(x)$ konk.

Def $\Gamma(z) := e^{-z} z^{-z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^{-1} e^{\frac{z}{n}}$

$\bullet C = \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{k} - \ln n$ wohldefin. \rightarrow An III

$\bullet \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-\frac{z}{n}}$ ist ganz

$$f_n(z) = \prod_{k=1}^n \left(1 + \frac{z}{k}\right) e^{-\frac{z}{k}}$$

$$\log f_n(z) = \sum_{k=1}^n \underbrace{\log\left(1 + \frac{z}{k}\right)}_{\frac{z}{k} + O\left(\frac{1}{k^2}\right)} - \frac{z}{k}$$

$$R > 0 \quad |z| < R \quad \sum_{k=1}^{\infty} O\left(\frac{1}{k^2}\right) < \infty$$

s.h. $\log f_n(z) \xrightarrow{\text{glm}} f(z) \quad |z| \leq R$

$\bullet \Gamma(z+1) = z \cdot \Gamma(z) \quad z \in \mathbb{C} - \{0, -1, \dots\}$

Betrachte $\frac{z \Gamma_n(z)}{\Gamma_n(z+1)} \xrightarrow{n \rightarrow \infty} 1$

$$\Gamma_n(z) = \frac{e^{-cz} z^{\frac{c}{n}} n!}{z \prod_{k=1}^n (z+k)}$$

$$\frac{z \Gamma_n(z+1)}{\Gamma_n(z+1)} = \frac{z e^{-c(z+\frac{z}{n})} (z+\frac{z}{n})!}{z(z+1)\dots(z+n)e^{-c}}$$

$$= \frac{z e^{-c} (z+n+1)}{e^{-c+z\frac{1}{n}}}$$

$$= \frac{z n!}{n} \rightarrow 1$$

weil $\sum \frac{1}{k} = c + \ln n + o(n)$
 $e^{-c + z \frac{1}{n}} = e^{-\ln n + o(n)}$

\bullet Integraldarstellung
 Integral exist. :

$$\int_0^{\infty} e^{-t} t^{z-1} dt \quad \operatorname{Re} z > 0$$

$$\left| \int_0^1 e^{-t} t^{z-1} dt \right| \leq \int_0^1 |e^{-t}| |t^{z-1}| dt$$

$$\leq \int_0^1 e^{-t} t^{\operatorname{Re}(z)-1} dt + \int_0^1 e^{-t} |t^{z-1}| dt$$

$$\leq \int_0^1 e^{-t} \frac{1}{t^{1-\operatorname{Re}z}} dt + \int_0^1 e^{-t} \frac{1}{t^{1-\operatorname{Re}z}} dt$$

$$G_n(z) = \int_0^1 \left(1 - \frac{t}{n}\right)^n t^{z-1} dt \rightarrow G(z) = \int_0^{\infty} e^{-t} t^{z-1} dt \quad \operatorname{Re} z > 0$$

(i) $1 - \frac{t}{n} \leq e^{-\frac{t}{n}} \quad 0 \leq t \leq n$
 (ii) $1 - \frac{t}{n} \leq e^{-\frac{t}{n}} \quad 0 \leq t \leq n$

$G_n = \Gamma_n \quad \frac{t}{n} := z$

Dirichlet reihen

Def

$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n s} \quad \begin{array}{l} \lambda_n \in \mathbb{R} \\ 0 < \lambda_1 < \lambda_2 < \dots \rightarrow \infty \\ a_n \in \mathbb{C} \quad s \in \mathbb{C} \end{array}$$

speziell $\lambda_n = n, a_n = 1 \rightarrow \sum_{n=1}^{\infty} x^n$
 speziell $\lambda_n = \log n, a_n = 1 \rightarrow \sum_{n=1}^{\infty} \frac{1}{n^s}$

Lemma

$$A(x) := \sum_{\lambda_n \leq x} a_n$$

$$x \geq \lambda_1 \quad \phi \in C^1([0, \infty))$$

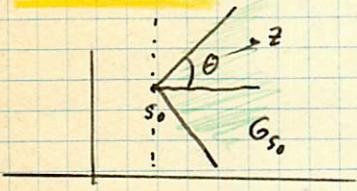
$$\rightarrow \sum_{\lambda_n \leq \omega} a_n \phi(\lambda_n) = - \int_{\lambda_1}^{\omega} A(x) \phi'(x) dx + A(\omega) \phi(\omega)$$

Falls $A(\omega) \phi(\omega) \rightarrow 0 \quad (\omega \rightarrow \infty)$

$$\rightarrow \sum_{n=1}^{\infty} a_n \phi(\lambda_n) = - \int_{\lambda_1}^{\infty} A(x) \phi'(x) dx$$

$$\begin{aligned} & A(\omega) \phi(\omega) - \sum_{\lambda_n \leq \omega} a_n \phi(\lambda_n) = \sum_{\lambda_n \leq \omega} a_n (\phi(\omega) - \phi(\lambda_n)) \\ &= \sum_{\lambda_n \leq \omega} \int_{\lambda_n}^{\omega} a_n \phi'(x) dx = \int_{\lambda_1}^{\omega} A(x) \phi'(x) dx \end{aligned}$$

Theorem 1



Falls $\sum_{n=1}^{\infty} a_n e^{-\lambda_n s}$ für $s = s_0 := \sigma_0 + i\tau_0$ konvergiert dann konvergiert sie in $G_{s_0, \theta} := \{ z \in \mathbb{C} \mid |\arg(z - s_0)| \leq \theta < \frac{\pi}{2} \text{ für } 0 < \theta \leq \frac{\sigma}{2} \}$ gleichmässig

o.B.d.A $s_0 = 0$: $\sum_{n=1}^{\infty} a_n e^{-\lambda_n s} = \sum_{n=1}^{\infty} \frac{a_n e^{-\lambda_n s_0}}{b_n}$

Sei $\gamma \geq \mu + 1, \gamma, \mu \in \mathbb{N}$

$$\begin{aligned} \sum_{n=\mu+1}^{\infty} a_n e^{-\lambda_n s} &= \sum_{n=1}^{\gamma} a_n e^{-\lambda_n s} - \sum_{m=1}^{\mu} a_m e^{-\lambda_m s} \\ &= + \int_{\lambda_n}^{\lambda_{\gamma}} A(x) e^{-xs} (x+s) dx + A(\lambda_{\gamma}) e^{-\lambda_{\gamma} s} + \int_{\lambda_1}^{\lambda_1} e^{-xs} (-s) dx \\ &\quad - A(\lambda_{\mu}) e^{-\lambda_{\mu} s} \\ &= s \int_{\lambda_{\mu}}^{\lambda_{\gamma}} A(x) e^{-xs} dx + A(\lambda_{\gamma}) e^{-\lambda_{\gamma} s} - A(\lambda_{\mu}) e^{-\lambda_{\mu} s} \\ &= s \int_{\lambda_{\mu}}^{\lambda_{\gamma}} (A(x) - A(\lambda_{\mu})) e^{-xs} dx + [A(\lambda_{\gamma}) - A(\lambda_{\mu})] e^{-\lambda_{\mu} s} \end{aligned}$$

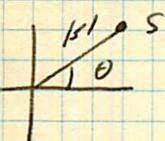
$$\left| \sum_{n=1}^{\infty} a_n e^{-n\sigma} \right| = \left| \sum_{n=1}^{\infty} a_n \right| < \infty$$

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} \forall x \in \mathbb{R} \quad |A(x) - A(N)| < \varepsilon$$

$$\left| \sum_{n=N+1}^{\infty} a_n e^{-n\sigma} \right| \leq \sum_{n=N+1}^{\infty} |a_n| e^{-n\sigma} \leq \sum_{n=N+1}^{\infty} |a_n| e^{-N\sigma} = e^{-N\sigma} \sum_{n=N+1}^{\infty} |a_n|$$

$$\leq |S| \varepsilon \frac{1}{\sigma} (-e^{-N\sigma} + e^{-N\sigma}) + \varepsilon e^{-N\sigma} \leq \varepsilon$$

$$< 2\varepsilon \left(\frac{|S|}{\sigma} + 1 \right) \leq 2\varepsilon \left| \frac{1}{\cos \theta} + 1 \right|$$



Satz 2

Falls $\sum_{n=1}^{\infty} a_n e^{-n\sigma}$ für $\sigma = \sigma_0$ konv.

→ konv. für $\sigma_0(s) > \sigma_0$ u. glm. in jedem beschränkten abgeschl. Gebiet, das in der Halbebene $\sigma > \sigma_0$ enthalten ist

Satz 3

Falls $\sum_{n=1}^{\infty} a_n e^{-n\sigma}$ für $\sigma = \sigma_0$ konv.

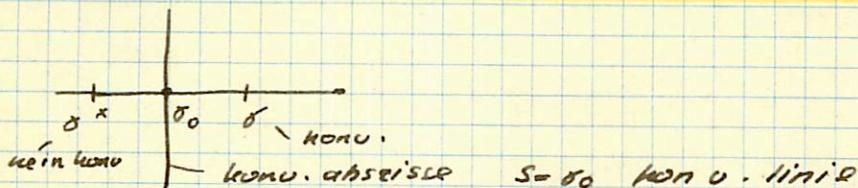
$\sum_{n=1}^{\infty} a_n e^{-n\sigma} = f(\sigma)$ dann $f(\sigma) \rightarrow f(\sigma_0)$ ($\sigma \rightarrow \sigma_0$) entlang dem Weg in σ_0, θ

$$\begin{aligned} \text{Also } \lim_{\sigma \rightarrow \sigma_0} \lim_{m \rightarrow \infty} \sum_{n=1}^m a_n e^{-n\sigma} \\ = \lim_{m \rightarrow \infty} \sum_{n=1}^m \lim_{\sigma \rightarrow \sigma_0} a_n e^{-n\sigma} = \lim_{m \rightarrow \infty} \sum_{n=1}^m a_n e^{-n\sigma_0} \\ = f(\sigma_0) \end{aligned}$$

Satz 4

$f(\sigma) \in A(\sigma_0, \theta)$ falls $\sum_{n=1}^{\infty} a_n e^{-n\sigma}$ in σ_0 konv.

Def



Bsp

$\sum n^{-s}$ $\sigma = 1 + \varepsilon$ glm. konv. Konvabsz. 1
 $\sum n^{-s} (-1)^n$ Konvabsz. 0
 $\sum n! n^{-s}$ mit konv. absz. ∞

Satz 5

$\sum_{n=1}^{\infty} a_n e^{-n\sigma}$ konv. für $\sigma = 0$ und für unendl. viele Werte von σ mit $f(\sigma) = 0$
 $\sigma \in \{ \sigma \in \mathbb{C} \mid \sigma > \varepsilon > 0, |\arg(\sigma)| \leq \theta < \frac{\pi}{2} \}$
 $\Rightarrow a_n = 0 \forall n, f(\sigma) \equiv 0$

Identität \rightarrow kein HP $\Rightarrow s_n \rightarrow \infty$

(Snl neu $\delta_n < \delta_{n+1}$)

$$g(s) := e^{As} f(s) = a_1 + \underbrace{\sum a_n e^{-(\lambda_n - A_1)s}}_{\rightarrow 0 \text{ } s \rightarrow \infty}$$

$$g(s) \rightarrow a_1 \quad s \rightarrow \infty$$

gln konv.

$$g(s_n) = 0 \quad f(s_n) = 0 \quad \forall n \quad \rightarrow a_1 = 0$$

$$\dots \quad \forall n \quad a_n = 0$$

\rightarrow Mittelwertentwicklung eindeutig

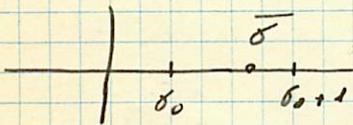
Was passiert für $\sigma = \varepsilon = 0$?

$$g(s) := \prod_{n=1}^{\infty} \left(1 - \frac{s}{s_n}\right) \quad \operatorname{Re}(s) > 0$$

$$N_s \quad s_n = \frac{1}{n} \left(\log \frac{3}{2} + \frac{2\pi i k}{n} \right) \rightarrow 0$$

entspr. absolut konv. abszisse $\bar{\sigma}$

$$\sum a_n n^{-s} \quad \sigma_0 < \sigma \quad \bar{\sigma} - \sigma_0 \leq 1$$



Reihe konv. für $\sigma > \sigma_0 + 1$

$$\text{denn } \frac{1}{n^{\sigma_0 + 1 + \varepsilon}}$$

an reell positiv \Rightarrow in $s = \sigma_0$ Singularität

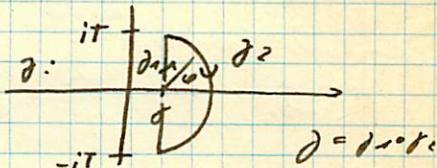
Perron'sche Formel

Lemma 1

$\sigma > 0 \quad u \in \mathbb{R} \rightarrow$

$$\frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{e^{us}}{s} ds = \begin{cases} 1 & u > 0 \\ 0 & u < 0 \\ u = 0 \end{cases}$$

$$\omega_0 \int_{\sigma - i\infty}^{\sigma + i\infty} = \lim_{T \rightarrow \infty} \int_{\sigma - iT}^{\sigma + iT} \quad (T > 0 \in \mathbb{R})$$

1. Fall $u = 0$ $\frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{1}{s} ds$ 

$$\int_{\sigma} \frac{1}{s} ds = 0 \Rightarrow \int_{\sigma_1} = - \int_{\sigma_2} - \int_{-r_2}^{-\sigma_1}$$

$$\sigma_2 : \varphi \mapsto T e^{i\varphi} + \sigma \quad \varphi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$$

$$\int_{-r_2}^{-\sigma_1} \frac{1}{s} ds = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{T e^{i\varphi} d\varphi}{\sigma + T e^{i\varphi}} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{T e^{i\varphi} (i - i)}{\sigma + T e^{i\varphi}} d\varphi + \pi i$$

$$|\dots| \leq \pi \cdot \frac{\sigma}{|T| - \sigma} \xrightarrow{T \rightarrow \infty} 0$$

$$\Rightarrow \int_{\sigma} \frac{1}{s} ds = \pi i, \quad \frac{1}{2\pi i} \int_{\sigma} \frac{1}{s} ds = \frac{1}{2}$$

2. Fall $u > 0 \vee u < 0$

$$\int_{\sigma - iT}^{\sigma + iT} \frac{e^{us}}{s} ds = \underbrace{\frac{e^{us}}{us}}_I \Big|_{\sigma - iT}^{\sigma + iT} + \int_{\sigma - iT}^{\sigma + iT} \frac{e^{us}}{us^2} ds$$

(I) : $|e^{u(\sigma + iT)}| = e^{u\sigma} = |e^{u(\sigma - iT)}| \quad |s| \geq T$

$$\left| \frac{e^{u(\sigma + iT)}}{u(\sigma + iT)} - \frac{e^{u(\sigma - iT)}}{u(\sigma - iT)} \right| \leq \left| \frac{e^{u(\sigma + iT)}}{u(\sigma + iT)} \right| + \left| \frac{e^{u(\sigma - iT)}}{u(\sigma - iT)} \right|$$

$$\leq \frac{2e^{u\sigma}}{|u|T} \xrightarrow{T \rightarrow \infty} 0$$

(II) (a) $u < 0$: $L + CR$
(b) $u > 0$: $L + CL$

$$u < 0 : \left| \frac{e^{us}}{s^2} \right| \leq \frac{e^{u\sigma}}{r^2} \quad \text{für } \operatorname{Re} s > \sigma$$

$$u > 0 : \left| \frac{e^{us}}{s^2} \right| \leq \frac{e^{u\sigma}}{r^2} \quad \operatorname{Re} s < \sigma$$

also $\left| \int_{CR} \frac{e^{us}}{us^2} ds \right| \leq 2\pi \frac{e^{u\sigma}}{|u| \cdot r^2} \xrightarrow{r \rightarrow \infty} 0$

Cauchy : (a) $u < 0$ $\int_{\sigma - i\infty}^{\sigma + i\infty} = - \int_{CR} = 0$ und

also $\int_{\sigma - i\infty}^{\sigma + i\infty} = 0$

$$\leq C_3 N + C_3 (1 + \frac{1}{\epsilon}) e^{-\lambda N \epsilon}$$

② $N \geq n$ $\left| \sum_{r=1}^n \dots \right| \leq C \cdot n \leq C N$ allg. Polynom / $N = N(N) \rightarrow \infty$ Langsame abs.

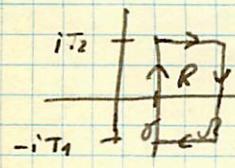
$$\Rightarrow \sum a_r e^{-\lambda r s} = o(1/n)$$

Perron'sche Formel

$\sigma_0 \in \mathbb{R}$ konvergenzabszisse d. DR $\sum a_n e^{-\lambda n s} = f(s)$
 Summenpunkt $f(s)$
 $\rightarrow \sigma > \sigma_0$ $\sigma > 0$ $\sigma + i\infty$
 $\sum_{\lambda n \leq \omega} a_n = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{f(s)}{s} e^{\omega s} ds$

Γ $\lambda n \leq \omega < \lambda(n+1)$
 $g(s) := e^{\omega s} \left\{ f(s) - \sum_{r=1}^n a_r e^{-\lambda r s} \right\}$
 $= \sum_{r=n+1}^{\infty} a_r e^{-(\lambda r - \omega)s} =: \sum_{r \geq 1} b_r e^{-\mu_r s}$ ($b_r = a_{n+r}$, $\mu_r = \lambda(n+r) - \omega$)
 $\Rightarrow 0 < \mu_1 < \mu_2 < \dots$

Lemma ① $\Rightarrow \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{g(s)}{s} ds$
 $= \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{f(s)}{s} e^{\omega s} ds - \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \sum_{r=1}^n \frac{a_r e^{\omega s}}{s} ds$ konstant



Cauchy $\frac{1}{2\pi i} \int_{\partial R} \frac{g(s)}{s} ds = 0$

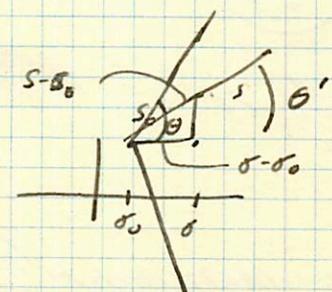
$$\Rightarrow \frac{1}{2\pi i} \int_{\sigma - i\tau_1}^{\sigma + i\tau_2} \dots = \frac{1}{2\pi i} \left[\int_{\sigma - i\tau_1}^{R - i\tau_1} + \int_{R - i\tau_1}^{R + i\tau_2} + \int_{R + i\tau_2}^{\sigma + i\tau_2} \right] = I_1 + I_2 + I_3$$

I_1, I_3 fest $R \rightarrow \infty$ $I_2 \rightarrow 0$ $\therefore \Gamma g(s) = \sum_{r=1}^n \frac{a_r e^{-\mu_r s} e^{-\lambda r s}}{a_r \phi(\mu_r)}$

$$= \sum_{r=1}^n a_r \phi(\mu_r)$$

$$= (s - s_0) \int_{s_0}^{\infty} h(x) e^{-x(s-s_0)} dx$$

v.l. μ



$$|g(s)| = |s - s_0| \int_{s_0}^{\infty} |h(x)| e^{-x(s-s_0)} dx$$

$$= \frac{|s - s_0|}{s - s_0} |h(x)| e^{-x(s-s_0)}$$

$$\leq k$$

$$\leq \frac{|s - s_0|}{s - s_0} \cdot k$$

$$= \frac{1}{\cos \theta} \leq \frac{1}{\cos \theta} = c(\theta)$$

$I_1, I_3 \rightarrow 0$

$$|g(s)| \leq c(\theta) \left(\frac{1}{|s|} \right)$$

$$|I_2| = \left| \frac{1}{2\pi i} \int_{R - i\tau_1}^{R + i\tau_2} \frac{g(s)}{s} ds \right|$$

$$\leq \frac{1}{2\pi} \int_{R - i\tau_1}^{R + i\tau_2} \frac{c}{|s|} ds$$

$$\leq \frac{c}{2\pi} \cdot \frac{I_2 + I_3}{R} \rightarrow 0 \quad R \rightarrow \infty$$

Funktionalgleichung d. Riemannschen Zetafunktion

$$\zeta(s) = \sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^s \quad \operatorname{Re} s > 1$$

$$|\zeta(s)| \leq \sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^{\sigma} \leq \sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^{\sigma} \quad \sigma = \operatorname{Re} s$$

Satz

$$\zeta(s) \cdot \Gamma(s) = \int_0^{\infty} \frac{x^{s-1}}{e^x - 1} dx \quad \operatorname{Re} s > 1$$

$$\Gamma(s) = \int_0^{\infty} x^{s-1} e^{-x} dx \quad \operatorname{Re} s > 0$$

$$= n^s \int_0^{\infty} y^{s-1} e^{-ny} dy$$

$x = ny$
 $dx = ndy$

$$\frac{\Gamma(s)}{n^s} = \int_0^{\infty} y^{s-1} e^{-ny} dy$$

$$\left| \sum_{n=1}^{\infty} \frac{\Gamma(s)}{n^s} \right| = \left| \sum_{n=1}^{\infty} \int_0^{\infty} y^{s-1} e^{-ny} dy \right|$$

$$\leq \sum_{n=1}^{\infty} \int_0^{\infty} |y^{s-1}| e^{-ny} dy \leq \sum_{n=1}^{\infty} \int_0^{\infty} y^{\sigma-1} e^{-ny} dy$$

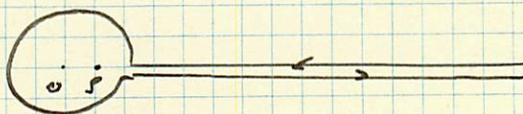
$$= \Gamma(s) \cdot \zeta(s) \quad \text{absolute konv.}$$

$$\sum_{n=1}^{\infty} \int_0^{\infty} y^{s-1} e^{-ny} dy = \int_0^{\infty} \sum_{n=1}^{\infty} e^{-ny} y^{s-1} dy$$

$$= \int_0^{\infty} \frac{e^{-y} - 1}{e^y - 1} y^{s-1} dy = \int_0^{\infty} \frac{y^{s-1}}{e^y - 1} dy$$

$$= \int_0^{\infty} \frac{y^{s-1}}{e^y - 1} dy$$

Betrachte:



$0 < \rho < 2\pi$

$$I(s) = \int_C \frac{z^{s-1}}{e^z - 1} dz$$

Lemma

$\Omega \subseteq \mathbb{C}$ beschränkt ^{Gebiet}, dann ist $I(s) \in H(\Omega)$

$$\int_{|z|=\rho} \frac{z^{s-1}}{e^z - 1} dz \in H(\Omega)$$

$$\left| \int_{\sigma}^{\infty} \frac{x^{s-1}}{e^x - 1} dx \right| \leq \int_{\sigma}^{\infty} \frac{|x^{s-1}|}{|e^x - 1|} dx \approx \int_{\sigma}^{\infty} \frac{x^{\operatorname{Re} s - 1}}{e^x - 1} dx$$

$$\left(\sigma_0 := \sup_{s \in \Omega} \operatorname{Re} s \right) \leq \int_{\sigma_0}^{\infty} \frac{x^{\sigma_0 - 1}}{e^x - 1} dx < \infty$$

7 Theorem

$$\text{Re } s > 1 \Rightarrow \zeta(s) = \frac{I(s) \Gamma(1-s)}{2\pi i \cdot e^{i\pi s}}$$

$$I(s) = \int_C \frac{z^{s-1}}{e^z - 1} dz = - \int_0^\infty \frac{x^{s-1}}{e^x - 1} dx + \int_{|z|=R} \frac{z^{s-1}}{e^z - 1} dz + \int_0^\infty \frac{(x e^{2\pi i})^{s-1}}{e^x - 1} dx$$

$z = \log|z| + i \arg z$

$$\int_{|z|=R} \frac{z^{s-1}}{e^z - 1} dz \cdot \left| \frac{e^z - 1}{z} \right| \xrightarrow{|z|=R} 1$$

$|e^z - 1| > A|z|$ s klein

$$|z^{s-1}| = |e^{(s-1)(\log|z| + i \arg z)}| = e^{(s-1)(\log|z| - \arg z)} \leq |z|^{s-1} e^{-\arg z}$$

$$\left| \int_{|z|=R} \frac{z^{s-1}}{e^z - 1} dz \right| \leq \int_{|z|=R} \frac{|z|^{s-1}}{|e^z - 1|} |dz| \leq \frac{R^{s-1} \cdot 4R\pi}{AR} \xrightarrow{R \rightarrow \infty} 0$$

$$I(s) = - \int_0^\infty \frac{x^{s-1}}{e^x - 1} dx + e^{2\pi i(s-1)} \int_0^\infty \frac{x^{s-1}}{e^x - 1} dx = \Gamma(s) \zeta(s) (e^{2\pi i(s-1)} - 1) = \Gamma(s) \zeta(s) \frac{e^{2\pi i s} - 1}{2i \sin \pi s}$$

$$\Gamma(s) \Gamma(1-s) = \frac{\pi}{\sin \pi s}$$

$$I(s) = \frac{\zeta(s)}{\Gamma(1-s)} \frac{\pi}{\sin \pi s} \cdot 2i \sin \pi s e^{\pi i s} = \frac{\zeta(s)}{\Gamma(1-s)} 2\pi i e^{\pi i s}$$

Schlussl. $\zeta(s) = \frac{I(s) \Gamma(1-s)}{e^{i\pi s} 2\pi i}$ $\text{Re } s > 1$

Pole von $\zeta(s)$

$\Gamma(1-s)$ Pole in $s=1, 2, 3, \dots$

$\zeta(s)$ at

$\zeta(s)$ in $s=2, 3, 4, \dots$ keine Pole

$\Rightarrow I(s)$ hat Pol in $s=1$

$$\Gamma'(z) = \frac{1}{z} + \dots \quad \Gamma'(1-s) = -\frac{1}{s-1} + \dots$$

$$\text{Res } \zeta(s) |_{s=1} = \lim_{s \rightarrow 1} \frac{(s-1) \Gamma(1-s)}{e^{i\pi s} 2\pi i} = 1$$

$$I(1) = \int_{|z|=R} \frac{1}{e^z - 1} dz = 2\pi i \text{Res} \frac{1}{e^z - 1} |_{z=0} = 2\pi i$$

Lemma

$$\zeta(-2m) = \begin{cases} 0 & m > 0 \\ -\frac{1}{2} & m = 0 \end{cases}$$

zu zeigen: $I(1) = 0$

$$\frac{1}{e^z - 1} = \frac{1}{z} - \frac{1}{2} + B_1 z - B_2 \frac{z^3}{4!} \dots$$

$|z|=R$ R klein

$$I(-2m) = \int_{|z|=r} z^{-2m-1} \left(\frac{1}{z} - \frac{1}{z} + \dots \right) dz$$

$$\int_{|z|=r} z^k dz = 0 \text{ falls } k \neq -1 \quad k \in \mathbb{Z}$$

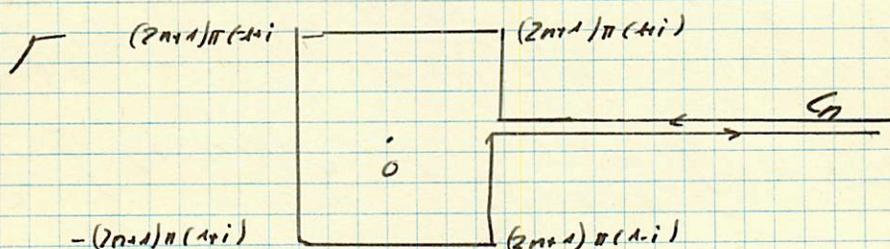
$$I(-2m) = 0 \quad m \neq 0 \quad \text{FT} \quad \int_{|z|=r} \frac{1}{z} dz = -\frac{1}{z} 2\pi i$$

$$|I(0)| = \frac{1}{2}$$

Die Funktionalgleichung

Theorem

$$\begin{aligned} \Gamma(s) &= 4\pi i \sin \frac{\pi s}{2} (2\pi)^{s-1} \zeta(1-s) \text{ Res} \\ &= \Gamma(s) \zeta(s) (e^{2\pi i s} - 1) \end{aligned}$$



$$\int_{C_n} \frac{z^{s-1}}{e^z - 1} dz = 2\pi i \sum_{\substack{m=-n \\ m \neq 0 \\ m \neq 1}}^n \text{Res} \frac{z^{s-1}}{e^z - 1} \Big|_{z=2m\pi i}$$

$$(2m\pi i)^{s-1} = (2m\pi e^{m/i})^{s-1}$$

$$\begin{aligned} &= \sum_{\substack{m=-n \\ m \neq 0 \\ m \neq 1}}^n (2m\pi e^{i\pi/i})^{s-1} = \sum_{k=1}^n (2m\pi)^{s-1} e^{i\pi(s-1)} \left(e^{i\frac{\pi}{2}(s-1)} + e^{-i\frac{\pi}{2}(s-1)} \right) \\ &= 2\pi^{s-1} \sum_{k=1}^n m^{s-1} e^{i\frac{\pi}{2}(s-1)} \cos \frac{\pi}{2}(s-1) \end{aligned}$$

$$I(s) = \underbrace{\int_{C_n} \frac{z^{s-1}}{e^z - 1} dz}_{\xrightarrow{n \rightarrow \infty} 0} + \sum_{k=1}^n 4\pi i \sin \frac{\pi s}{2} e^{\pi i s} (2m\pi)^{s-1}$$

Funktionalgl. $\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{-\frac{(1-s)}{2}} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s)$

Der Primzahlsatz

$$(2) \quad \pi(x) \sim \frac{1}{x} \xrightarrow{x \rightarrow \infty} 1$$

$$\pi(x) = \sum_{p \leq x} 1$$

Bew

$$1 a) \quad (x) \iff \frac{\psi(x)}{x} \xrightarrow{x \rightarrow \infty} 1$$

$$\psi(x) := \sum_{p^m \leq x} \lg p$$

$$b) \quad \psi(x) \leq \tilde{c}x \quad \tilde{c} > 0$$

$$c) \quad \left. \begin{array}{l} i) \quad \zeta(s) = \prod_p (1 - p^{-s})^{-1} \\ \zeta(s) \neq 0 \end{array} \right\} \text{Re } s > 1$$

$$ii) \quad \zeta(s) \sim \frac{1}{s-1} \quad \text{holom. Re } s > 0$$

$$iii) \quad \zeta(s) \neq 0 \quad \text{Re } s \geq 1$$

2 Satz

$$f: [0, \infty[\rightarrow \mathbb{R} \quad (\text{lok. integrierbar})$$

$$|f(x)| < k$$

$$\int_0^{\infty} f(t) e^{-zt} dt =: F(z) \quad (\text{Laplace-Transform})$$

ist holom. Funkl auf $\text{Re } z > 0$

Beh

$$F(z) \text{ holom. } \text{Re } z > 0$$

$$\Rightarrow \int_0^{\infty} f(t) dt \text{ exist und } = F(0)$$

Einsparung

3 korollar

$$g: [1, \infty[\rightarrow \mathbb{R}_0^+$$

monoton steigend

$$g(x) \in \mathcal{O}(x)$$

$$s \int_1^{\infty} g(x) x^{-s-1} dx =: G(s)$$

Wolff
Wetter
Hansf.

holom. auf $\text{Re } s > 1$

Beh

$$G(s) \sim \frac{G(x)}{s-1} \text{ holom. auf } \text{Re } s > 1 \quad \exists c > 0$$

$$\Rightarrow \frac{G(x)}{x} \xrightarrow{x \rightarrow \infty} c$$

4

$$g := \psi \quad c := 1 \xrightarrow{3} \frac{\psi(x)}{x} \xrightarrow{x \rightarrow \infty} 1$$

$$1a) \quad \psi(x) = \sum_{p \leq x} \left[\frac{\log x}{\log p} \right] \log p \leq \sum_{p \leq x} \frac{\log x}{\log p} \log p \\ = \log x \pi(x) \quad (1)$$

$$1cy \leq \pi(x) \leq y + \sum_{y < p \leq x} 1 \leq y + \sum_{y < p \leq x} \frac{\log p}{\log y} \leq y + \frac{\psi(x)}{\log y}$$

$$y := \frac{x}{\log^2 x} \quad \therefore \quad \pi(x) \leq \frac{x}{\log^2 x} + \frac{\psi(x)}{\log x - 2 \log^2 x} \quad (2)$$

$$\frac{\psi(x)}{x} \stackrel{(1)}{\leq} \frac{\log x \pi(x)}{x} \stackrel{(2)}{\leq} \frac{1}{\log x} + \frac{\psi(x)}{x} \frac{\log x}{\log x - 2 \log^2 x}$$

$\xrightarrow{x \rightarrow \infty} 0$

$$1b) \quad p \mid \binom{2n}{n} \quad p \in]n, 2n]$$

$$\prod_{n < p \leq 2n} p \leq \binom{2n}{n} < 2^{2n}$$

$$n := 2^{k-1} \quad \therefore \quad \sum_{p \in]2^{k-1}, 2^k] \log p < 2^k \log 2$$

$$\sum_{p \leq 2^k} \log p < (2^k + 2^{k-1} + \dots + 1) \log 2 < 2^{k+1} \log 2$$

$$2^k \leq x < 2^{k+1} \quad \therefore \quad \sum_{p \leq x} \log p < \sum_{p \leq 2^{k+1}} \log p < 2^{k+2} \log 2 < \\ < \frac{x \cdot 4 \log 2}{c}$$

$$\psi(x) = \left(\sum_{p \leq x} 1 + \sum_{p \leq x} 1 + \dots + \sum_{p \mid \log_2 x \leq x} 1 \right) \log p$$

$$= \sum_{p \leq x} 1 + \sum_{p \leq x} 1/2 + \dots + \sum_{p \leq x} 1/\log_2 x \log p$$

$$< cx + cx^{1/2} + \dots + cx \frac{1}{\log_2 x}$$

$$< cx + \log_2 x \cdot cx^{1/2} < c^2 x$$

$$1c) \quad \zeta(s) = \sum_{n=1}^{\infty} n^{-s}$$

$$(1 - 2^{-s}) \zeta(s) = \sum_{n=1}^{\infty} n^{-s} - \sum_{n=1}^{\infty} (2n)^{-s} = \sum_{\substack{n=1 \\ 2 \nmid n}}^{\infty} n^{-s}$$

$$\prod_{p \in \mathbb{N}} (1 - p^{-s}) \zeta(s) = \sum_{\substack{n=1 \\ p \nmid n \\ p \in \mathbb{N}}} n^{-s} =: \sum$$

$$|\zeta| \geq 1 - \sum_{\substack{n=1 \\ n > N}} |n^{-s}| > 0 \quad \forall \text{ gen\u00fcgend gro\u00df}$$

$\xrightarrow{N \rightarrow \infty}$

$$\zeta(s) \neq 0 \quad \operatorname{Re}(s) > 1$$

$$\lim_{N \rightarrow \infty} \prod_{p \in \mathbb{N}} (1 - p^{-s}) \zeta(s) = \lim_{N \rightarrow \infty} \sum_{\substack{n=1 \\ p \nmid n \\ p \in \mathbb{N}}} n^{-s} = 1$$

iii) Ann. $\zeta(1+iy_0) = 0 \quad y_0 \in \mathbb{R} \setminus \{0\} \quad +21$

$$\begin{aligned} \lg |\zeta(x+iy_0)| &= -\operatorname{Re} \sum_p \lg(1-p^{-s}) \\ &= \operatorname{Re} \sum_p (p^{-s} + \frac{1}{2} p^{-2s} + \frac{1}{3} p^{-3s} + \dots) \\ &= \operatorname{Re} \sum_{n \geq 2} a_n n^{-s} \quad a_n \geq 0 \\ &= \sum_{n \geq 2} \frac{a_n n^{-x}}{n^y} \cos(y_0 \lg n) \end{aligned}$$

$$(3 + 4 \cos \varphi + \cos 2\varphi) = (1 + \cos \varphi)^2 \geq 0 \quad \varphi \in \mathbb{R}$$

$$\varphi(x) := \sum_{n \geq 2} a_n n^{-x} \zeta^3(x) \zeta^4(x+iy_0) \zeta(x+2iy_0)$$

$$\lg |\varphi(x)| = \sum_{n \geq 2} \frac{a_n n^{-x}}{n^y} \underbrace{(3 + 4 \cos(y_0 \lg n) + \cos(2y_0 \lg n))}_{\geq 0} \geq 0$$

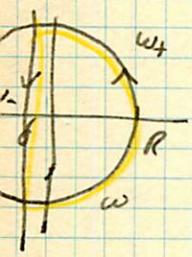
$$\varphi(x) \xrightarrow{x \rightarrow 1} 0$$

$$\zeta(1+iy_0) \neq 0$$

2. $F_\lambda(z) := \int_0^\lambda f(t) e^{-zt} dt \quad \text{ganz} \quad \lambda \geq 0$

$$|F(0) - F_\lambda(0)| \xrightarrow{\lambda \rightarrow \infty} 0$$

$$|F(z) - F_\lambda(z)|_{z=0} = \frac{1}{2\pi} \left| \int_w \frac{F(z) - F_\lambda(z)}{z} dz \right|$$



$0 < d = d(R)$ genug klein:

$F(z)$ holomorph auf $\operatorname{Re} z > -2d$

$w_+ : x > 0$
 $w_- : x < 0$

$$\begin{aligned} w_+ : |F(z) - F_\lambda(z)| &\leq \int_0^\lambda |f(t)| e^{-xt} dt \\ &\leq k \cdot \frac{e^{-x\lambda}}{x} \end{aligned}$$

$$w_- : |F_\lambda(z)| \leq \int_0^\lambda k e^{-xt} dt = k \frac{e^{-x\lambda} - 1}{-x} < \frac{e^{-x\lambda}}{|x|} k$$

$$|F(z) - F_\lambda(z)|_{z=0} = |(F(z) - F_\lambda(z)) e^{\lambda x}|_{z=0}$$

$$\leq \frac{1}{2\pi} \int_w |F(z) - F_\lambda(z)| e^{\lambda x} \left| \frac{1 + \frac{z^2}{R^2}}{z^2} \right| |dz| \quad \left. \frac{z^2}{R^2} \right|_{z=0} |$$

$$\leq \frac{1}{2\pi} \int_{w_+} |F(z) - F_\lambda(z)| e^{\lambda x} \frac{2x}{R^2} |dz| \quad \left. \right\} \leq \frac{1}{2\pi} \cdot \pi R \frac{2k}{R^2} = \frac{1}{R} \cdot k$$

$$+ \frac{1}{2\pi} \int_{w_-} |F_\lambda(z)| e^{\lambda x} \frac{2x}{R^2} |dz| \quad \left. \right\} \leq \frac{k}{R}$$

$$+ \frac{1}{2\pi} \int_{w'} |F(z)| \left| \frac{1 + \frac{z^2}{R^2}}{z^2} \right| e^{\lambda x} |dz| \quad \left. \right\} + \leq B(R) \text{ auf } w' \cap w_x$$

of Kreis
 $\frac{1}{R^2} = \frac{1}{R^2} + \frac{1}{R^2}$
 $= \frac{2}{R^2}$

$\leq B_0(R, d)$ auf

$$3. \text{ Term} \leq \frac{k}{2\pi} \left\{ \underbrace{2\pi \operatorname{arcsin} \frac{J}{R}}_{\delta} B + \frac{2RB_1}{\sigma \Delta d} \right\}$$

$$\varepsilon > 0 : \quad R > \frac{k}{\varepsilon}$$

$$J < \frac{\pi \varepsilon}{k B (R)}$$

$$\forall \lambda > \lambda_0 : \frac{k \cdot R B (R, J)}{\sigma \Delta d \pi} < \varepsilon$$

$$3. \quad g(x), c \quad f(t) := e^{-t} g(e^t) - c$$

$$F(z) := \int_0^{\infty} [e^{-t} f(e^t) - e^{-zt} - c e^{-zt}] dt$$

$$= \int_1^{\infty} x^{-z} g(x) x^{-z} \frac{dx}{x} - \frac{c}{z}$$

$x=e^t \rightarrow dt = \frac{dx}{x}$

$$= \int_1^{\infty} g(x) x^{-(z+1)-1} dx - \frac{c}{z}$$

$$= \frac{1}{z+1} \left[G(z+1) - \frac{c}{z} - c \right]$$

$G(z) = \int_1^{\infty} g(x) x^{-z} dx$ holom. auf $\operatorname{Re} z > 1$

$F(z)$ holom. auf $\operatorname{Re} z > 0$

Satz $\int_0^{\infty} f(t) dt$ exist.

$$= \int_0^{\infty} \frac{g(x) - cx}{x^2} dx$$

g wächst monoton

Ann. $g(y) \geq (c+2d)y$

g steigt

$$g(x) \geq (c+d)x$$

$$x < \frac{(c+2d)y}{c+d}$$

$$\int_y^{\infty} \frac{g(x) - cx}{x^2} dx > \int_y^{\infty} \frac{d}{x} dx = d \ln \infty > 0$$

$\rightarrow \int_0^{\infty} f(t) dt$ würde divergieren. \downarrow

Ann $g(y) < (c-2d)y$

$$\frac{c-2d}{c-d}$$

$\dots \Rightarrow \int_y^{\infty} \frac{d}{x} dx = d \ln \infty < 0$

$$2 < 1$$

$\rightarrow \int_0^{\infty} f(t) dt$ würde divergieren. \downarrow

$$4. \quad \int_1^{\infty} \psi(x) x^{-s-1} dx = s \cdot \sum_{n=2}^{\infty} \int_1^n \psi(x) x^{-s-1} dx$$

$$= s \cdot \sum_{n=2}^{\infty} \psi(n) \int_1^n \left[\frac{1}{n^s} - \frac{1}{(n+1)^s} \right]$$

$$= \sum_{n=2}^{\infty} \left[\psi(n+1) - \psi(n) \right] \frac{1}{n^s} = \sum_p \left[p^{-s} + (p^{-s})^2 + \dots \right]$$

$$= \sum_p \frac{p^{-s}}{1-p^{-s}} \ln p$$

$$- \frac{d}{ds} \ln \zeta(s) = + \sum_p \frac{p^{-s}}{1-p^{-s}} \ln p \quad \text{Re } s > 1$$

↑
Produkt-darst.

$$- \frac{d}{ds} \ln \zeta(s) = - \frac{\zeta'(s)}{\zeta(s)}$$

$$\zeta(s) = \frac{1}{s-1} + a(s) \quad a(s) \text{ analyt.}$$

$$\zeta'(s) = -\frac{1}{(s-1)^2} + a'(s)$$

d.h. $-\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{s-1}$ analytisch in Umgeb. von 1

d.h. nahen situation on Collier

$$\Rightarrow \frac{\psi(x)}{x} \xrightarrow{x \rightarrow \infty} 1$$

Eine Extremaleigenschaft der Riemannschen ζ Funktion

Penellin ~ 1950

$\Lambda := \{ (n_k)_{k \in \mathbb{N}} \mid k \in \mathbb{N} \Rightarrow 0 < n_k \leq n_{k+1} \}$

$\Lambda_1 := \{ \lambda \in \Lambda \mid \inf_{n \in \mathbb{N}} \frac{n_k}{n} > 0 \}$

Für $\lambda \in \Lambda$ definiere $\varphi_\lambda(s) := \sum_{n \in \mathbb{N}} \frac{1}{n^s} = \zeta(s)$

$f(\lambda)(z) = f_\lambda(z) := \prod_{n \in \mathbb{N}} \left(1 + \frac{z^{n_k}}{n^s} \right)$

Für $\lambda \in \Lambda_1$ konvergiert $\varphi(s)$ für $\sigma > 1$ und f_λ ist ganz Funktion vom Exponentialtypus
 $f_\lambda(\lambda) = O(e^{A|\lambda|})$

$\delta := \inf_{n \in \mathbb{N}} \frac{n_k}{n} > 0$

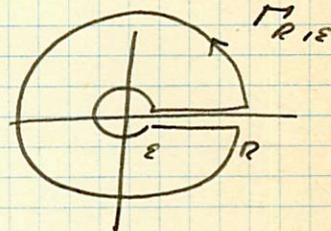
$|f_\lambda(z)| \leq \prod_{n \in \mathbb{N}} \left(1 + \frac{|z|^{n_k}}{n^s} \right) \leq \prod_{n \in \mathbb{N}} \left(1 + \left(\frac{|z|}{\delta} \right)^n \right) = \frac{\sinh \pi \frac{|z|}{\delta}}{\pi \frac{|z|}{\delta}}$
 $\ll e^{A|z|}$

$\lambda \in \Lambda_1$ Für $1 < \sigma < 2$ hat nach $\int_0^\infty \log |f_\lambda| \frac{dx}{x^{1+s}}$

$= \underbrace{-\frac{1}{s} \frac{1}{x^s} \log |f_\lambda| \Big|_0^\infty}_{x \rightarrow \infty} + \frac{1}{s} \int_0^\infty \frac{1}{x^{s+1}} d \log |f_\lambda|$

$x \rightarrow \infty$: f Exponentialtypus $\sigma > 1$
 $x \rightarrow 0$: f gerade $\sigma < 2$

Betrachte folgenden Integrationsweg



$\lim_{\substack{R \rightarrow \infty \\ \epsilon \rightarrow 0}} \int_{\Gamma_{R, \epsilon}} \frac{1}{x^s} d \log |f_\lambda| = \int_0^\infty \frac{1}{x^s} d \log |f_\lambda| + \int_\infty^\epsilon \frac{1}{x^s} d \log |f_\lambda|$
 $= \int_0^\infty \frac{1}{x^s} d \log |f_\lambda| \left[\frac{e^{2\pi i s} - 1}{e^{2\pi i s}} \right]$

andererseits

$\lim_{\substack{R \rightarrow \infty \\ \epsilon \rightarrow 0}} \int_{\Gamma_{R, \epsilon}} \dots = 2\pi i \sum_{n \in \mathbb{N}} \left(\frac{1}{n^s} \frac{\pi i s}{2} + \frac{1}{n^s} \frac{3\pi i s}{2} \right)$
 $= 2\pi i \varphi(s) \frac{e^{2\pi i s} - 1}{e^{2\pi i s}}$

Zusammen: $\int_0^\infty \log |f_\lambda| \frac{dx}{x^{1+s}} = \frac{1}{s} 2\pi i \varphi(s) \frac{e^{\pi i s} - 1}{e^{2\pi i s}} = \frac{\varphi(s) \pi}{s} \frac{2i e^{\pi i s}}{e^{2\pi i s} - 1}$
 $= \frac{\pi \varphi(s)}{s - \sin \pi s}$

$$k > 0$$

$C_k := \{ \lambda_k : \lambda \in \Lambda \quad \sigma > 1 \Rightarrow |\varphi(\lambda)| < \infty; \text{ so class } (A), (B), (C) \}$

(A) $\varphi(\lambda) = \frac{1}{s-1}$ ganz

(B) $\varphi(-22k) = 0$

(C) $|\varphi(\lambda)| \leq \text{const} \frac{|\lambda|}{(2\pi k)^{1/2}}$

$D_k := \bigcap_{\text{OCCUR}} (C_{k-s} - C_{k+s})$

Satz

$$\# D_k = \begin{cases} \infty & \text{falls } 0 < k < \frac{1}{2} \\ 2 & \text{falls } k = 1 \\ 0 & \text{sonst} \end{cases}$$

und $D_k = \{ \varphi_{2k}, \varphi_{2k-\frac{1}{2}} \}$

$\varphi_{2k} = \xi = \xi(\cdot, 1) \quad s \mapsto \xi(s)$

$\varphi_{2k-\frac{1}{2}} = \xi(\cdot, \frac{1}{2}) \quad s \mapsto (2^s-1)\xi(s)$

Lemma

$\lambda \in \Lambda \quad k > 0$

$\varphi_{\lambda} \in D_k \iff \exists$ reelle Zahlen $a > 0$ und p ,

so dass die Funkt. f_{λ} d. Red. $f_{\lambda}(\lambda) = a |\lambda|^p e^{\pi k |\lambda|} + O(e^{\pi(\lambda + \epsilon - 2k)k|\lambda|})$

$|\lambda| \rightarrow \infty \quad x \in \mathbb{R}$

genügt für $\epsilon > 0$ aber nicht für $\epsilon < 0$

Bem

$0 < k < \frac{1}{2} \Rightarrow \# D_k = \infty$

OCC

Bsp. $\cosh \pi k z \quad \cosh \pi(\lambda+k)z \quad (1 + \frac{z^2}{n^2})$

d.h. 1. Teil \checkmark

Prop

f ganze Funkt. von Exponentialtypus mit $f(0) = 1$ und es seien gegeben reelle Zahlen $a > 0, p, \rho > 0$, so dass $f(x) - a|x|^p e^{\pi k|x|} = O(e^{-\rho|x|})$ $|x| \rightarrow \infty$ dann ist entweder

$f(z) = \frac{e^{\pi z} - e^{-\pi z}}{2\pi z}$
 $= \frac{\pi}{\pi^2 z^2} \left(1 + \frac{z^2}{n^2}\right)$

oder $f(z) = \frac{e^{\pi k z} + e^{-\pi k z}}{2} = \frac{\pi}{\pi^2 z^2} \left(1 + \frac{z^2}{(n-\frac{1}{2})^2}\right)$

$\int_{h/k > 2\pi} f(z) = f(z) - a z^p e^{\pi z}$

$\int_{h/k < 2\pi} f(z) = f(z) - a (-z)^p e^{-\pi z}$

Bem

$$n \in \Theta(\mathbb{H}) \cap \mathbb{C}(\overline{\mathbb{H}})$$

$$n(z) = O(e^{A|z|})$$

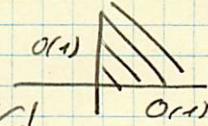
Expon. typus

$$n(z) = O(e^{-d|x|}) \quad (|x| \rightarrow \infty, x \in \mathbb{R})$$

Dann $n = 0$

┌

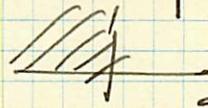
$$n(z) = e^{(d+ia)z}$$



P.L.

$$\Rightarrow |n(z)| \leq C e^{d|x|} \quad \int_{-\infty}^{\infty} e^{-d|x|} dx = \frac{2}{d}$$

$$n(z) = e^{(-d+ia)z}$$



$$0 \leq \theta \leq \frac{\pi}{2}$$

analog $\frac{\pi}{2} \leq \theta \leq \pi$

Zusammen
 $B > 0$

$$|n(re^{i\theta})| \leq C e^{r(-d \cos \theta + A \sin \theta)}$$

$$|n(re^{i\theta})| \leq C e^{r(-d \cos \theta + (A+B) \sin \theta)}$$

wähle θ so dass $\tan \theta = \frac{d}{A+B}$

P.L.

$$\Rightarrow |n(re^{i\theta})| \leq C e^{-r \sin \theta}$$

$$0 \leq \theta \leq \pi$$

$B \rightarrow \infty$

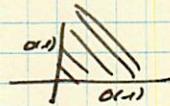
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$$f(x) - f(-x) = h(x) - h(-x) = O(e^{-d|x|}) \quad (|x| \rightarrow \infty, x \in \mathbb{R})$$

Bem

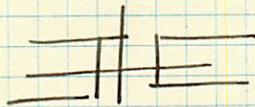
$$f(z) - f(-z) = 0$$

$$h(z) = O(e^{(d+ia)z})$$



$$h(z) = O(e^{Ax - Bx}) \quad \begin{matrix} x \geq 1 \\ y \geq 0 \end{matrix}$$

analog



$$\{x \geq 1\} \cup \{y \leq 1\}$$

$$h(x) = O(e^{-d|x|}) \quad (|x| \rightarrow \infty)$$

wegen $h^{(n)}(x) = \int_{|z-x|=1} \frac{h(z)}{(z-x)^{n+1}} dz = \frac{n!}{2\pi i}$

$$h^{(n)}(x) = O(e^{-d|x|})$$

Betrachte linearen Diff op. $L := x^2 \frac{d^2}{dx^2} - 2x \frac{d}{dx} - (x^2 - p)$

$$(Lf)(x) = Lh(x) = O(e^{-d|x|} x^2) \quad (|x| \rightarrow \infty)$$

da $a/x^p \int e^{-\pi|x|}, a \in \mathbb{R}, p \in \mathbb{N}$ Lösungen von $Lu=0$ sind

Nach obiger Bem: $Lf=0$

$$f(z) = x^p (c_1 e^{\pi x} + c_2 e^{-\pi x})$$

$$f \in \Theta(L) \quad f \neq 0 \Rightarrow p \in \mathbb{Z}$$

f gerade $|c_1| = |c_2| =: c$

entweder $c_1 = -c_2 \Rightarrow p = -1 \quad c = \frac{1}{2\pi}$

oder $c_1 = c_2 \Rightarrow p = 0 \quad c = \frac{1}{2}$

$$\left(\begin{matrix} p = 2\nu(0) \\ a = e^{2\nu(0)} \\ f(0) = -\frac{1}{2} \\ f'(0) = -\frac{1}{2} \log 2 \end{matrix} \right)$$

Poisson Lemma

$\varphi_\lambda \in C^k \xrightarrow{(\lambda)} \lambda \in \Lambda_+$ (Übung $\varphi_\lambda - \xi$)

$\varphi_\lambda \in C^k$ Aufgrund d. Eigenschaft. $(k), (B)$ und mit

$$\int_0^\infty \log f(x) \frac{dx}{x^{1+s}} = \frac{\pi \varphi(s)}{s \sin \frac{\pi s}{2}} =: \varphi(s)$$

folgt. $\varphi \in \mathcal{O}(\sigma < 2, s \in \text{fort. } \mathbb{R})$ mit

Hauptteil in $s=1$ und $s=0$:

$$\boxed{\frac{\pi}{s-1}} \quad \text{und} \quad \lim_{s \rightarrow 0} \frac{\pi \varphi(s) s^k}{s \sin \frac{\pi s}{2}} = \begin{cases} 0 & k \geq 2 \\ \frac{\pi \varphi(0)}{\pi/2} = 2\varphi(0) & k=1 \\ \lim_{s \rightarrow 0} \left(\frac{\pi \varphi(s)}{s \sin \frac{\pi s}{2}} - \frac{2\varphi(0)}{s} \right) & k=0 \end{cases}$$

$$= \lim_{s \rightarrow 0} \frac{\pi \varphi'(s) s - 2\varphi(0) \sin \frac{\pi s}{2}}{s \cdot \sin \frac{\pi s}{2}} \quad \frac{\pi \cos \frac{\pi s}{2}}{2}$$

$$= \lim_{s \rightarrow 0} \frac{\pi \varphi'(s) s + \pi \varphi(s) - 2\varphi(0)}{\sin \frac{\pi s}{2} + \frac{s^2}{2} \cos \frac{\pi s}{2}}$$

$$= \frac{2\pi \varphi'(0)}{2\pi} = 2\varphi'(0)$$

$$\boxed{\frac{2\varphi(0)}{s^2} + \frac{2\varphi'(0)}{s}}$$

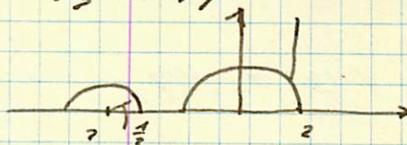
$\varphi_0(s) := \Gamma(2-s) (2\pi k)^s$

Mit dem Lemm. $|\Gamma(\frac{1}{2} - it)| \geq \sqrt{\pi} e^{-\frac{|t|}{2}\pi}$

$|\varphi(\frac{3}{2} + it)| = \mathcal{O}(1) \quad (|t| \rightarrow \infty)$

und $|s|$

$\frac{|\varphi|}{\varphi_0} \Big|_{\frac{3}{2} + it} =$



$s \in \mathbb{R}_+^n (-2n) \quad n \in \mathbb{Z}_+$

$|\varphi(s)| \leq \frac{\varphi(-2n)}{0} + \int_{-2n}^s |\varphi'(s)| |ds|$

$|\varphi(s)| \leq |s+2n| \max_{s \in \mathbb{R}_+^n (-2n)} |\varphi'(s)| \leq |s+2n| \int_{\text{max } s} \frac{|\varphi(s)|}{1-s^2} ds$

$|\varphi(s)| \leq \text{const} |\Gamma(2-s) (2\pi k)^s|$

Bemerkung Lemm.

$\int_{-\infty}^{\infty} |\Gamma(2\sigma - it)| dt \leq \text{const } \sigma^3 \Gamma(\sigma) \quad \sigma \geq 2 \quad (\text{Stirling})$

$\int_{-\infty}^{\infty} |\Gamma(-\sigma + it)| dt \leq \text{const} \frac{\sigma^3 \Gamma(\sigma)}{(2\pi k)^\sigma} \quad \sigma > 2$

$\varphi(\sigma + it) = \int_0^\infty \log f(x) \frac{dx}{x^{1+s+it}} = \int_0^\infty \log f(e^x) \frac{d e^x}{e^{(1+s+it)x}} =$
 $= \int_{-\infty}^\infty \log f(e^x) e^{-\sigma x} e^{-ixt} dx$

$\log f(e^x) e^{-\sigma x} = \frac{1}{2\pi} \int_{-\infty}^\infty \log f(x) e^{ixt} \varphi(\sigma + it) e^{ixt} dt$

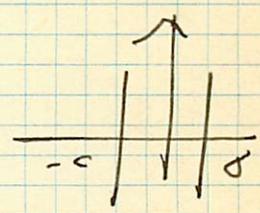
$$\log f(x) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \psi(s) x^s ds = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \psi(s) x^s ds \quad (1.20)$$

$$\psi(s) = \frac{\pi}{s-1} + O(1) \quad (s \rightarrow 1)$$

$$\psi(s) = \frac{2\psi(1)}{s^2} + \frac{2\psi'(1)}{s} + O(1) \quad s \rightarrow 0$$

$$x^s = e^{s \log x} = 1 + s \log x + O(\log^2 x) \quad (s \rightarrow 0)$$

$$x^s = x \cdot x^{s-1} = x + O(|s-1|) \quad (s \rightarrow 1)$$



$$\text{res}_{s=1} (\psi(s) x^s) = \pi x$$

$$\text{res}_{s=0} (\psi(s) x^s) = 2\psi'(1) + 2\psi(1) \log x$$

$$p := 2\psi(1) \\ a := e^{2\psi'(1)}$$

$$\text{also} \quad \oint \log f(x) - \log a x^p e^{\pi x} = \frac{1}{2\pi i} \int_{-c-i\infty}^{c+i\infty} \psi(s) x^s ds \quad c > 0$$

$$|\log f(x) - \log a x^p e^{\pi x}| = O\left(\frac{c^{\sigma} \Gamma(c)}{(2\pi k x)^c}\right) \quad (c \rightarrow \infty)$$

$$c := 2\pi k x \quad x \rightarrow \infty$$

$$|\log f(x) - \log a x^p e^{\pi x}| = O\left(\frac{c^{\sigma} e^{-c}}{c^c}\right) = O(e^{-2\pi k x}) \quad x \rightarrow \infty$$

Benutzen folgende Ungl $1-x \leq e^{-x}$
 $e^x \leq 1+x$ $0 < x < 1 > 0$ $70 k$

$$-\text{const} e^{-2\pi(k-s)x} \leq \log \frac{f(x)}{a x^p e^{\pi x}} \leq \text{const} e^{-2\pi(k-s)x}$$

$$1 - \text{const} e^{-2\pi(k-s)x} \leq \frac{f(x)}{a x^p e^{\pi x}} \leq 1 + \text{const} e^{-2\pi(k-s)x}$$

erhalten $f(x) = a x^p e^{\pi x} + O(e^{\pi(1+s-2k)x}) \quad (x \rightarrow \infty)$
 x gerade
Eine Richtung d. Lemmas

für Umkehrung:

$$f(x) = 1 + \psi(2)x^2 + \frac{\psi(2)^2 - \psi'(2)}{2} x^4 + \dots$$

Methode vom Ende
 für Restriktive Bereiche

Seminar in Analysis u. Zahlentheorie
Chandra

Assistenz E 18 Albert Stadler

Ausblick: Γ Funkt., Dirichletreihen, Perronsche Formel
 Mittelwertsatz d. Dirichletreihen, ζ Funkt., Primzahlansatz
 Dichtigkeitstheorem v. Polya.

Vortrag: 17.6
 24.6 - 1.7 Dichtigkeitstheorem von Polya

Wohin: $\Gamma(1) = 1$
 $n\Gamma(n) = \Gamma(n+1)$ } \rightarrow axiom. eindeut. Entwickl. von $\Gamma(z)$
 $\frac{d^2}{dz^2} \log \Gamma(z) > 0$

wollen solche Entwickl. auch für $\zeta(s)$

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} \text{ Re } s > 1$$

$$\zeta(2) = \frac{\pi^2}{6} \rightarrow \exists \infty \text{ viele Primzahlen}$$

$$\frac{1}{\zeta(s)} = \sum \frac{\mu(n)}{n^s} \quad \mu(n) = (-1)^k \text{ k versch. Primz.} \\ \mu(1) = 1$$

$$\sum_{k=1}^{n-1} \mu(k) = ? \quad \text{Bsp. } 1+2+...+(n-1) = \frac{x^2-x}{2} \\ 1^2+2^2+...+(n-1)^2 = \frac{x^3-x}{3} - \frac{x^2+x}{2} + \frac{x}{6}$$

$$S_n(x) = \frac{x^{n+1}}{n+1} - \frac{x^n}{2} + B_1 \frac{x^{n-1}}{1} - B_2 \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} \frac{x^{n-3}}{6} \\ + \dots + \begin{cases} (-1)^{\frac{n-1}{2}} B_{\frac{n-1}{2}} x & n \text{ gerade} \\ (-1)^{\frac{n-3}{2}} n B_{\frac{n-1}{2}} \frac{x^2}{2} & n \text{ ungerade} \end{cases}$$

$$B_n = \frac{2(2n)! \zeta(2n)}{(2\pi)^{2n}} \quad n=1,2,\dots \quad \text{Bernullizahl.}$$

$$\cot z = \frac{\cos z}{\sin z} = \frac{1}{z} - \frac{2}{\pi^2} \zeta(2) z - \dots - \frac{2}{(2n)^{2n}} \zeta(2n) z^{2n-1}$$

$$\cot z = \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1}{z-n\pi} + \frac{1}{z+n\pi} \right) \\ = \frac{1}{z} - \frac{2}{3} - \frac{2^3}{45} - \frac{2^5}{945} \dots \quad |z| < \pi \\ \Rightarrow \frac{\zeta(2n)}{\pi^{2n}} \text{ rational}$$

Literat: Noiman Lévinson 'gap and density theorems' p 58 ff
 & Polya Unters. über Lücken u. Singul. von Potenzreihen
 Math. Zeitschrift 29 S. 249