Diss. ETH No. 9518

Global Periodic Orbits for Hamiltonian Systems on $T^n \times R^n$

A dissertation submitted to the
SWISS FEDERAL INSTITUTE OF TECHNOLOGY ZURICH

for the degree of
Doctor of Mathematics

presented by
Frank Werner Josellis
Dipl. Math., Ruhr-Universität Bochum
born August 13, 1959
citizen of the Federal Republic of Germany

accepted on the recommendation of
Prof. Dr. Eduard Zehnder, examiner
Prof. Dr. Jürgen Moser, co-examiner

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Abstract

For a large class of periodically time dependent Hamiltonian systems on the cotangent bundle $T^*(\mathsf{T}^n) \cong \mathsf{T}^n \times \mathsf{R}^n$ of the n-dimensional flat torus $\mathsf{T}^n = \mathsf{R}^n/\mathsf{Z}^n$ we prove some existence and multiplicity results for periodic solutions. The Hamiltonian $H(z,t) \in C^{1,0}(T^*(\mathsf{T}^n) \times \mathsf{R}\,,\, \mathsf{R})$ is assumed to be asymptotically quadratic in the fibers. Moreover, the considered Hamiltonian systems are nonresonant at infinity. Note that there are no convexity conditions imposed. On the universal cover $\mathsf{R}^n \times \mathsf{R}^n$ of $T^*(\mathsf{T}^n)$ thus we consider a time dependent Hamiltonian H satisfying $H(x{+}j,y,t) = H(x,y,t) = H(x,y,t{+}1)$ for all $j \in \mathsf{Z}^n$. Moreover, H is required to satisfy asymptotic conditions of the following type: assume there exists a regular matrix $A{=}A^t \in \mathcal{L}(\mathsf{R}^n)$ such that

$$\frac{1}{|y|}|\,\partial_x H(x,y,t)\,| \to 0 \quad \text{and} \quad \frac{1}{|y|}|\,\partial_y H(x,y,t) - Ay\,| \to 0$$

as $|y| \to \infty$ uniformly in x and t. We show that under these hypotheses the Hamiltonian system

$$\dot{z} \;=\; J\nabla H(z,t) \quad , \quad z = (x,y) \in \mathsf{R}^n \times \mathsf{R}^n\,,$$

possesses at least $n+1$ periodic solutions of period 1 in every homotopy class of loops in $\mathsf{T}^n \times \mathsf{R}^n$. Each of these homotopy classes is characterized by the rotation vector of any of its representants. If $z(t) = z(t+1)$ is a loop in $\mathsf{T}^n \times \mathsf{R}^n$ and if $\tilde{z}(t) = (x(t), y(t))$ denotes a lift of z to $\mathsf{R}^n \times \mathsf{R}^n$, then the integer vector $j := x(t{+}1) - x(t) \in \mathsf{Z}^n$ is called the rotation vector of z. We also consider the case of subharmonic solutions z having minimal period $p \in \mathsf{N}$ with $p > 1$. If then correspondingly $j = x(t{+}p) - x(t)$, the solution z has the rational rotation vector $j/p \in \mathsf{Q}^n$. Our result then states the existence of at least $n+1$ solutions having rotation vector j/p, provided j and p are relatively prime. Moreover, if H is C^2 and if all the j/p-solutions of the Hamiltonian equation are nondegenerate, then there exist at least 2^n of them.

The proofs are based on variational methods for indefinite functionals. In order to establish the topological part of the existence proofs, Galerkin approximation is used to obtain a finite-dimensonal reduction. For the general case of $n{+}1$ periodic solutions we present two different proofs. The first proof is done by standard minimax techniques, involving a linking. The second,

more geometric proof uses a Lyusternik-Schnirelman approach for flows, a cohomological version of which is due to C. Conley and E. Zehnder. The geometric version presented here in particular recovers the classical result of L.A. Lyusternik and L.G. Schnirelman concerning the number of critical points for C^1-functions on compact manifolds.
The existence of at least 2^n periodic solutions in the nondegenerate case is obtained via the Conley-Zehnder Morse theory for flows.
As another result we show that the methods introduced to prove the above stated results for Hamiltonian systems on $\mathbb{T}^n \times \mathbb{R}^n$ can be applied to weaken the regularity hypotheses of C. Conley's and E. Zehnder's proof of the Arnold-conjecture for the $2n$-dimensional torus.

Zusammenfassung

Für eine grosse Klasse periodisch zeitabhängiger Hamiltonscher Systeme auf dem Cotangentialbündel $T^*(\mathsf{T}^n) \cong \mathsf{T}^n \times \mathsf{R}^n$ des n-dimensionalen flachen Torus T^n beweisen wir einige Existenz- und Multiplizitätsresultate über periodische Lösungen. Die Hamiltonfunktion $H(z,t) \in C^{1,0}(T^*(\mathsf{T}^n) \times \mathsf{R}, \mathsf{R})$ wird als asymptotisch quadratisch in den Fasern angenommen. Darüber hinaus erfüllen die betrachteten Hamiltonschen Systeme eine Nichtresonanzbedingung im Unendlichen. Jedoch werden keinerlei Konvexitätsbedingungen vorausgesetzt. Auf der universellen Überlagerung $\mathsf{R}^n \times \mathsf{R}^n$ von $T^*(\mathsf{T}^n)$ betrachten wir daher eine zeitabhängige Hamiltonfunktion, welche $H(x+j,y,t) = H(x,y,t) = H(x,y,t+1)$ für alle $j \in \mathsf{Z}^n$ erfüllt. Ausserdem verlangen wir, dass H asymptotische Bedingungen des folgenden Typs erfüllt: es existiere eine reguläre Matrix $A = A^t \in \mathcal{L}(\mathsf{R}^n)$, so dass

$$\frac{1}{|y|}\,|\,\partial_x H(x,y,t)\,| \to 0 \quad \text{und} \quad \frac{1}{|y|}\,|\,\partial_y H(x,y,t) - Ay\,| \to 0$$

gleichmässig in x und t für $|y| \to \infty$. Wir zeigen, dass unter diesen Voraussetzungen das Hamiltonsche System

$$\dot{z} \;=\; J\,\nabla H(z,t) \quad , \quad z = (x,y) \in \mathsf{R}^n \times \mathsf{R}^n\,,$$

mindestens $n+1$ periodische Lösungen der Periode 1 in jeder Homotopieklasse geschlossener Kurven in $\mathsf{T}^n \times \mathsf{R}^n$ besitzt. Jede dieser Homotopieklassen wird durch den Rotationsvektor eines jeden ihrer Repräsentanten charakterisiert. Wenn $z(t) = z(t+1)$ eine geschlossene Kurve in $\mathsf{T}^n \times \mathsf{R}^n$ ist und $\tilde{z}(t) = (x(t), y(t))$ eine Liftung von z nach $\mathsf{R}^n \times \mathsf{R}^n$ bezeichnet, dann heisst der ganzzahlige Vektor $j := x(t+1) - x(t) \in \mathsf{Z}^n$ der Rotationsvektor von z. Wir betrachten ebenso den Fall subharmonischer Lösungen z, die eine minimale Periode $p \in \mathsf{N}$ mit $p > 1$ haben. Wenn dann entsprechend $j = x(t+p) - x(t)$ gilt, so besitzt die Lösung z einen rationalen Rotationsvektor $j/p \in \mathsf{Q}^n$. Unser Resultat gewährleistet dann die Existenz von mindestens $n+1$ Lösungen mit Rotationsvektor j/p, sofern j und p teilerfremd sind. Darüber hinaus, falls $H \in C^2$ und falls alle j/p-Lösungen der Hamiltonschen Gleichung nichtdegeneriert sind, dann existieren mindestens 2^n von ihnen.
Die Beweise basieren auf Variationsmethoden für indefinite Funktionale. Um den topologischen Teil der Existenzbeweise durchzuführen wird eine Galerkin

Approximation zur Reduktion auf endliche Dimension verwendet. Für den allgemeinen Fall von $n + 1$ periodischen Lösungen geben wir zwei unterschiedliche Beweise. Der erste Beweis wird mit üblichen Minimax-Methoden geführt, wobei ein Linking angewendet wird. Der zweite, mehr geometrische Beweis benutzt eine Lyusternik-Schnirelman Methode für Flüsse, die in ihrer cohomologischen Fassung von C. Conley und E. Zehnder stammt. Die geometrische Fassung, die wir präsentieren, enthält insbesondere das klassische Resultat von L.A.Lyusternik und L.G.Schnirelman über die Anzahl kritischer Punkte von C^1-Funktionen auf kompakten Mannigfaltigkeiten.
Die Existenz von mindestens 2^n periodischen Lösungen im nichtdegenerierten Fall wird über die Conley-Zehnder Morse Theorie für Flüsse erhalten.
Als weiteres Resultat zeigen wir, dass die Methoden, die zum Beweis der oben genannten Ergebnisse für Hamiltonsche Systeme auf $T^n \times R^n$ eingeführt werden, auch verwendet werden können um die Differenzierbarkeitsvoraussetzungen des Beweises der Arnold-Vermutung für den $2n$-dimensionalen Torus von C. Conley and E. Zehnder abzuschwächen.

1 Introduction and results

(a) Description of the results

We consider a Hamiltonian equation which depends on time t

$$(1.1) \qquad \dot{z} = J\nabla H(z,t) \quad , \quad z \in \mathbb{R}^{2n}\, , \, t \in \mathbb{R}\, ,$$

where J is the skew-symmetric matrix

$$(1.2) \qquad J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \in \mathcal{L}(\mathbb{R}^n \times \mathbb{R}^n)$$

with I being the identity on $\mathbb{R}^n$. We shall assume that H depends periodically on the time t with period 1, and setting $z = (x,y) \in \mathbb{R}^n \times \mathbb{R}^n$ we shall also assume that H is periodic with respect to $x \in \mathbb{R}^n$:

$$(1.3) \qquad H(x+j,y,t) = H(x,y,t) = H(x,y,t+1) \qquad \text{for all } j \in \mathbb{Z}^n\, .$$

Therefore the Hamiltonian vector field can be considered as a periodically time dependent vector field on the phase space $\mathbb{T}^n \times \mathbb{R}^n$, where $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ denotes the n-dimensional torus.

The interest in Hamiltonian systems arises from the description of frictionless dynamical systems in classical mechanics. The structure of Hamiltonian flows is in general extremely complicated. For an illustration of this complexity we refer to the thesis of C. Genecand [24], where the orbit structure of a low dimensional Hamiltonian system is investigated which is close to an integrable system.
In this connection it is natural to ask for the appearance of periodic phenomena, and we are concerned with the existence and multiplicity of global periodic solutions of special type.

Given $\alpha \in \mathbb{R}^n$ one can ask for solutions $z(t) = (x(t),y(t))$ of (1.1) which satisfy

$$\lim_{|t|\to\infty} \frac{x(t)}{t} = \alpha \, . \tag{1.4}$$

In this case the solution $z(t)$ is said to have the rotation vector α. If in addition we assume this solution to be periodic with integer period p, requiring

$$\begin{aligned} x(t+p) &= x(t) + j \\ y(t+p) &= y(t) \end{aligned} \tag{1.5}$$

for some $j \in \mathbb{Z}^n$, it follows that

$$\alpha p = j \tag{1.6}$$

and

$$x(t) = \alpha t + \xi(t) \quad \text{with} \quad \xi(t+p) = \xi(t) \, . \tag{1.7}$$

Definition

Given $j = (j_1, \ldots, j_n) \in \mathbb{Z}^n$ and an integer $p \geq 1$ such that $j_1, \ldots, j_n, p$ are relatively prime, then we call a solution $z(t) = (x(t), y(t))$ a j/p-solution if $x(t)$ is of the form (1.7) *with $\alpha = j/p$.*

In order to prove the existence of j/p-solutions for all $j \in \mathbb{Z}^n$ and $p \in \mathbb{N}$ we formulate some assumptions on the Hamiltonian H. We shall assume the vector field (1.1) to be asymptotically linear, requiring that

$$\begin{aligned} \frac{1}{|y|} \, | \, \partial_y H(x,y,t) - A(t)y \, | \; &\to \; 0 \\ \frac{1}{|y|} \, | \, \partial_x H(x,y,t) \, | \; &\to \; 0 \end{aligned} \tag{1.8}$$

as $|y| \to \infty$ uniformly in x and t, where $A(t)$ is a symmetric matrix, depending continuously and periodically on t :

$$A(t) = A(t+1) \in \mathcal{L}(\mathbb{R}^n) \, . \tag{1.9}$$

We shall prove

Theorem 1
Assume $H \in C^1$ satisfies (1.3) and (1.8), and assume that, moreover,

$$\det\left[\int_0^1 A(t)\,dt\right] \neq 0\,. \tag{1.10}$$

Then for every given $j \in \mathbb{Z}^n$ and $p \in \mathbb{N}$ relatively prime we have

$$\#\{\, j/p - solutions\,\} \geq n+1\,. \tag{1.11}$$

Here $n+1$ stands for the topological invariant cuplength$(\mathrm{T}^n)+1$ of the torus T^n. The nondegeneracy condition (1.10) is crucial for the existence proof, and as a side remark we observe that it cannot be omitted, as the following example shows :

For $H(x,y,t) = \frac{1}{2}\langle\, A(t)y\,,\,y\,\rangle$ the Hamiltonian equations are

$$\begin{aligned} \dot{x} &= A(t)y \\ \dot{y} &= 0 \end{aligned}$$

so that the solutions are of the form $(x(t),y(t)) = (x(t),y(0))$. If we have a j-solution, i.e. $p=1$, then $x(t) = jt + \xi(t)$ with $\xi(t+1) = \xi(t)$, and integrating the identity

$$j + \dot{\xi}(t) = A(t)\,y(0)$$

from $t=0$ to $t=1$ we arrive at

$$j = \int_0^1 A(t)\,dt\;\,y(0)\,.$$

Consequently j has to be contained in the range of $\int_0^1 A(t)\,dt$.

Setting $p=1$ we conclude from Theorem 1 that there exist at least $n+1$ periodic solutions $(x(t),y(t))$ in every prescribed homotopy class of loops in

$T^n \times R^n$. These homotopy classes are characterized by $j \in Z^n$. We therefore have

Theorem 2
Assume $H \in C^1$ *satisfies* (1.3) , (1.8) *and* (1.10). *Then the Hamiltonian equation* (1.1) *possesses infinitely many 1-periodic solutions.*

The existence statements so far can be viewed as generalizations of the Poincaré-Birkhoff fixed point theorem. This theorem states that an area preserving homeomorphism of an annulus

$$A = S^1 \times I \quad , \quad I = [a, b] \subset R ,$$

rotating the boundaries $S^1 \times \{a\}$ and $S^1 \times \{b\}$ in opposite directions possesses at least $2 = \text{cuplength}(S^1) + 1$ fixed points in the interior. Moreover, it possesses infinitely many periodic orbits, namely for every j/p relatively prime it has a periodic orbit of period p. In our case there are no boundary conditions. Instead the system is assumed to be asymptotically linear and the condition (1.10) plays the role of the twist condition in the Poincaré-Birkhoff theorem.

It should be pointed out that the existence statements above require only conditions at infinity, and no interior conditions are assumed. This is in contrast to recent existence theorems of C. Golé for periodic orbits of monotone symplectomorphisms of $T^n \times R^n$, where the existence of a global generating function is assumed. In [25] C. Golé establishes the existence of j/p-solutions for periodically time dependent Hamiltonians $H \in C^2$ which satisfy

$$H(x, y, t) = \frac{1}{2}\langle Ay , y \rangle + \langle c , y \rangle \quad \text{if } |y| \geq a$$

for some constant $a > 0$, where $A \in \mathcal{L}(R^n)$ is a symmetric matrix with $\det A \neq 0$, and $c \in R^n$. In addition, H is assumed to satisfy the interior condition

$$\det \frac{\partial^2 H}{\partial y^2}(x, y, t) \neq 0 \quad \text{for all } (x, y, t) \in T^n \times R^n \times R .$$

Stronger estimates for the number of periodic orbits are obtained under non-degeneracy assumptions for the orbits. Recall that a j/p-solution is called nondegenerate if it has no Floquet-multiplier equal to 1. Now let $H \in C^2$ and assume, in addition, that there exist constants a, $b \geq 0$ and $1 \leq r < \infty$ such that the Hessian d^2H satisfies

$$|\, d^2H(x,y,t)\, | \leq a + b\,|y|^r \tag{1.12}$$

for all $(x,y,t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$.

Theorem 3

Assume $H \in C^2$ satisfies the assumption of Theorem 1, namely (1.3), (1.8) *and* (1.10), *and assume moreover that* (1.12) *is satisfied. If all the j/p-orbits are nondegenerate then*

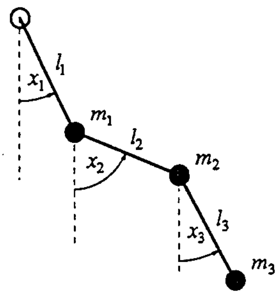
$$\#\{\, j/p - solutions\,\} \geq 2^n\,, \tag{1.13}$$

where the number on the right hand side is the sum of Betti numbers of T^n.

As an application we consider the special Hamiltonian function

$$H(x,y,t) = \frac{1}{2}\,|y|^2 + V(x,t) \tag{1.14}$$

with V depending periodically on x and t. If $V \in C^1$ then H satisfies the assumptions of Theorem 1 with $A = \mathrm{id}_{\mathbb{R}^n}$. An example is the multiple pendulum, see e.g. K.C. Chang, Y. Long and E. Zehnder [9].



In case that $j = 0$ and V is independent of time t the contractible periodic orbits guaranteed by Theorem 1 are of oscillatory type, but may coincide with the $n+1$ critical points of the periodic potential $V(x)$.
If $j \in \mathbb{Z}^n \setminus \{0\}$ then the j/p-orbits given by Theorem 1 are of rotational type. There are infinitely many of them.

We emphasize that Theorems 1-3 hold true also for Hamiltonian systems of the more general form

$$\dot{z} = J[\nabla H(z,t) + f(t)] \tag{1.15}$$

for a continuous map $f(t) = (f_1(t), f_2(t)) \in \mathbb{R}^n \times \mathbb{R}^n$ which satisfies $f(t) = f(t+1)$ and

$$\int_0^1 f_1(t)\, dt = 0 . \tag{1.16}$$

This follows from our proofs.
As a consequence of the Theorems 1 and 3 we point out the following extension of the global Birkhoff-Lewis theorem in [14] :

Theorem 4

Assume that $H \in C^1$ satisfies the assumption (1.3), *and let $A(t)$ satisfy* (1.9) and (1.10). *Moreover assume that there exists $R > 0$ such that*

$$H(x,y,t) = \frac{1}{2}\langle A(t)y, y\rangle + \langle b(t), y\rangle \quad \text{if} \quad |y| \geq R \tag{1.17}$$

where $b : \mathbb{R} \to \mathbb{R}^n$ is a continuous periodic mapping with $b(t) = b(t+1)$. We introduce the following notation for the mean values:

$$[A] := \int_0^1 A(t)\, dt \;\in \mathcal{L}(\mathbb{R}^n) \quad , \quad [b] := \int_0^1 b(t)\, dt \;\in \mathbb{R}^n . \tag{1.18}$$

(i) *Then for every $j \in \mathbb{Z}^n$ satisfying*

$$|j - [b]| < \frac{R}{|[A]^{-1}|} \tag{1.19}$$

there exist at least $n+1$ periodic solutions having rotation vector j which are contained in $\mathbb{T}^n \times D_R$, where $D_R := \{y \in \mathbb{R}^n \mid |y| < R\}$ is the disc of radius R.

(ii) *Assume that $H \in C^2$, and that $j \in \mathbb{Z}^n$ satisfies* (1.19). *If all the periodic solutions having rotation vector j are nondegenerate, then there exist at least 2^n of them.*

For the Birkhoff-Lewis fixed point theorem we refer to J. Moser [43, 44]. In this connection we mention the work of W. Chen [11], where the case of small perturbations of integrable Hamiltonian systems is considered. This is quite in contrast to the above theorem which is global in nature.
For the proof of Theorem 4 consider the orbit $z(t) = (x(t), y(t)) \in \mathbb{R}^n \times \mathbb{R}^n$ of the Hamiltonian flow starting at (x_0, y_0) with $|y_0| \geq R$. Integrating the Hamiltonian equation we obtain

$$y(t) = y_0 \quad , \quad x(t) = x_0 + \int_0^t A(s)\, ds + \int_0^t b(s)\, ds \, .$$

Thus $z(t)$ corresponds to a periodic solution on $\mathbb{T}^n \times \mathbb{R}^n$ with rotation vector j if

$$j = x(1) - x(0) = \int_0^1 A(t)\, dt\, y_0 + \int_0^1 b(t)\, dt = [A]\, y_0 + [b] \ ,$$

hence

$$|\, j - [b]\,| = |\,[A]\, y_0\,| \geq \frac{|y_0|}{|\,[A]^{-1}\,|} \geq \frac{R}{|\,[A]^{-1}\,|} \, .$$

Consequently for every $j \in \mathbb{Z}^n$ satisfying (1.19) the periodic orbits having rotation vector j have to be contained in $\mathbb{T}^n \times D_R$, and there exist at least $n+1$ of them by Theorem 1.
Under the assumption of nondegeneracy of these solutions the number of periodic orbits with rotation vector j is at least 2^n by Theorem 3.

The following observation is related to the conjecture due to V.I. Arnold in the special case of the $2n$-dimensional torus $\mathbb{T}^{2n}$.
Consider a Hamiltonian $H : \mathbb{R}^{2n} \times \mathbb{R} \to \mathbb{R}$ which is periodic in all of its variables :

$$\text{(1.20)} \qquad H(z+j, t) = H(z, t) = H(z, t+1) \quad \text{for all } \ j \in \mathbb{Z}^{2n} \, .$$

We assume that $H \in C^2$. It is a well-known result due to C. Conley and E. Zehnder, see [14], that the Hamiltonian equation (1.1) possesses at least

$2n+1 = \text{cuplength}(\mathrm{T}^{2n})+1$ periodic solutions of period 1 on the torus T^{2n}. If in addition all the periodic solutions are nondegenerate then there exist at least 2^{2n} of them, where 2^{2n} stands for the sum of Betti numbers of T^{2n}.
All these solutions are found in the class of contractible loops on T^{2n}. Indeed, we do not expect to find periodic orbits of rotational type in the case of an exact Hamiltonian vector field, as it is shown by the example $H \equiv 0$.
Recall that a vector field X on a symplectic manifold (M, ω) is called an exact Hamiltonian vector field, if the 1-form $i_X\omega$ is exact. More general, a vector field X on M is said to be Hamiltonian if $\phi_t^*\omega = \omega$, where ϕ_t denotes the flow of X.
Consider the Hamiltonian vector field on T^{2n}

$$\dot{z} \;=\; J\,[\,\nabla H(z,t) + f(t)\,] \;=: X_t(z) \tag{1.21}$$

where $f(t+1) = f(t)$ is a continuous periodic map $f : \mathrm{R} \to \mathrm{R}^{2n}$. This vector field X_t is not exact Hamiltonian on T^{2n} if $f \neq 0$, nevertheless any lift $\tilde{X}_t$ to the covering space R^{2n} is exact Hamiltonian. Considering the flow of $\tilde{X}_t$ we note that it contains a translation by the mean value $[f] = \int_0^1 f(t)dt$ of f, and consequently we cannot hope that the orbits in R^{2n} will project to closed trajectories of X_t on T^{2n} unless this mean value is an integer vector. We restrict ourselves to contractible periodic orbits on T^{2n}.
Weakening the hypothesis on the differentiability of H, the result of C. Conley and E. Zehnder can be generalized as follows :

Theorem 5
(i) *Let* $H \in C^1(\mathrm{R}^{2n} \times \mathrm{R}, \mathrm{R})$ *satisfy* (1.20), *and assume that* $f \in C(\mathrm{R}, \mathrm{R}^{2n})$ *is periodic,* $f(t+1) = f(t)$, *with vanishing mean value :*

$$\int_0^1 f(t)\,dt \;=\; 0\;. \tag{1.22}$$

Then there exist at least $\text{cuplength}(\mathrm{T}^{2n})+1 \;=\; 2n+1$ *contractible periodic solutions of the Hamiltonian equation* (1.21) *on* T^{2n}.
(ii) *If* $H \in C^2$ *and if moreover all the periodic solutions of* (1.21) *are nondegenerate, then there exist at least* 2^{2n} *of them.*

The proof of Theorem 5 is basically done along the same lines as the proofs of the Theorems 1 and 3. Actually, the situation is simpler since the Hamiltonian is assumed to be periodic in all of its variables. In Section 2 we shall say a few words concerning the simplifications of the analytic setup for the variational principle involving a Hamiltonian $H : \mathsf{T}^{2n} \times \mathsf{R} \to \mathsf{R}$.

(b) Related results

The study of asymptotically linear Hamiltonian systems goes back to H. Amann and E. Zehnder [2, 3]. Already for the special case that $j = 0$, i.e. for contractible periodic solutions, Theorem 1 extends the result by C. Conley and E. Zehnder in [14] in several directions : It is not assumed that $H \in C^2$, in particular the Hessian of H is not required to be bounded. In addition, our asymptotic conditions are less restrictive, allowing in particular a time dependent asymptotically linear Hamiltonian system.
For the case of contractible periodic solutions related results have been proved by J.Q. Liu [38], A. Szulkin [57] and P.L. Felmer [20]. Hamiltonian systems on $\mathsf{T}^n \times \mathsf{R}^n$ which are asymptotically linear in the fibres have been considered in particular by A. Szulkin: he assumes the Hamiltonian to be of the form

$$H(x,y,t) = \frac{1}{2}\langle Ay\,,\,y\rangle + G(x,y,t)$$

where G is a C^1-function such that ∇G is bounded, and A is a symmetric matrix. However, A is allowed to have a nontrivial kernel, and in that case G in addition is assumed to satisfy a Landesman-Lazer condition on $\ker(A)$. A Landesman-Lazer condition on H in all those of its variables where it is not periodic, is also assumed by J.Q. Liu in [38]. His work is of special interest to us in view of the methods used .
P. Felmer consideres in [20] a class of Hamiltonians which have superlinear growth and, unlike to the case treated here, are asymptotically convex in the fibers.
All these authors use so-called minimax-principles in their existence proofs. Since we will present two different proofs of Theorem 1, one of which is also based on minimax techniques, it should be pointed out that our setting for the minimax has been influenced by the presentation given by Liu. In this connection we also wish to mention the work of H. Hofer [34], where basically the same strategy was applied to the problem of intersections for Lagrangian embeddings of compact manifolds. We will discuss some of the more technical aspects of the proof in (c) below.

Related results concerning the existence of contractible periodic solutions also have been obtained by K.C. Chang [8] and by G. Fournier and M. Willem [23]. In these works also asymptotically linear conditions on the Hamiltonian are imposed, which are allowed to be periodically time dependent. The assumptions made by G. Fournier and M. Willem are similar to ours, while those of K.C. Chang are slightly more restrictive. However, Chang's assumptions include also the case of resonances.
Both of these works can be viewed as an extension of the results in [14]. As in [14] a Lyapunov-Schmidt reduction to finite dimension is used, which in particular requires that $H \in C^2$ and that the norm of the Hessian of H is bounded. This approach allows to apply the Conley-Zehnder Morse theory directly in case that all the periodic solutions are assumed to be nondegenerate.

(c) Idea of the proof

The proof is based on a classical variational principle for which the critical points are the required periodic solutions. Considering the case $p = 1$ and $j \in \mathbb{Z}^n$ we look for solutions $z(t)$ of the equation

$$\dot{z}(t) = J\nabla H(z,t)$$

of the form $z(t) = (jt+\xi(t),\eta(t)) =: e(t) + \zeta(t)$ with $\zeta(t) = (\xi(t),\eta(t))$ satisfying $\zeta(t) = \zeta(t+1)$, and $e(t) = (jt,0)$. Therefore $\zeta(t)$ satisfies the equation

$$-J\dot{\zeta} = \nabla H(e+\zeta,t) + J\dot{e}\,. \tag{1.23}$$

This is the Euler equation of the action functional Φ defined on the loop space $\Omega = \{\zeta : S^1 \to \mathbb{R}^{2n} \mid \zeta(0) = \zeta(1)\}$ as follows :

$$\Phi(\zeta) = \int_0^1 \left\{\frac{1}{2}\langle -J\dot{\zeta}\,,\,\zeta\,\rangle - H(e+\zeta,t) + \langle -J\dot{e}\,,\,\zeta\,\rangle\right\} dt\,. \tag{1.24}$$

Indeed, taking formally the first variation of Φ we obtain

$$\frac{d}{dt}\bigg|_{\varepsilon=0} \Phi(\zeta+\varepsilon\,\theta) = \int_0^1 \langle -J\dot{\zeta} - \nabla H(e+\zeta,t) - J\dot{e}\,,\,\theta\,\rangle\, dt\,.$$

Hence we have to find critical points of the function $\Phi : \Omega \to \mathbb{R}$ defined on the infinite dimensional loopspace Ω. However, the action functional Φ is indefinite. It is bounded neither from above nor from below, so that the classical techniques of the calculus of variations do not apply directly. Moreover Φ is strongly indefinite, i.e. the restriction of Φ to any closed linear subspace of finite codimension defines an indefinite function on this subspace. For instance, Morse indices of critical points are always infinite, and therefore the critical points of Φ are homotopically invisible.
This is in sharp contrast to a Hamiltonian being of the form

$$H(x,y,t) = \frac{1}{2}|y|^2 + V(x,t),$$

where the existence of critical points is readily established by looking for the minimum of the following quite different variational problem :

Let $L(x,\dot{x},t)$ denote the Lagrangian obtained from $H(x,y,t)$ by Legendre transformation with respect to the variables y and $\dot{x}$. Then we consider

$$\Phi(\xi) := \int_0^1 L(jt+\xi(t), j+\dot{\xi}, t)\, dt$$

and find readily a minimum in the Sobolev space $H^{1,2}(S^1, \mathbb{R}^n)$ by taking a minimizing sequence. In fact, this classical approach applies immediately if $H \in C^2$ since

$$0 < m < L_{\dot{x}\dot{x}} < M$$

so that L satisfies the Legendre condition. The convexity of the Hamiltonian with respect to y is, however, crucial for this approach. Even for a Hamiltonian of the form

$$H(x_1,x_2,y_1,y_2,t) = \frac{y_1^2}{2} - \frac{y_2^2}{2} + V(x_1,x_2,t)$$

where x_i, $y_i \in \mathbb{R}$ and V is a periodic function, this classical approach to the problem already fails.

The breakthrough in the general case of the variational problem for Φ as in (1.24) is due to P. Rabinowitz, who showed that this degenerate principle can be used very effectively for existence proofs if subtle minimax techniques are used. We refer to P. Rabinowitz [52] and M. Struwe [56] for a detailed presentation of this subject and various applications to differential equations. A different approach in the search for critical points of indefinite functionals has been established by C. Conley and E. Zehnder, see [14, 15]. Their ideas are closely related to the dynamics of the gradient flow of Φ on the loopspace Ω. In particular, this approach leads to stronger multiplicity results if the functional Φ is a Morse function on Ω.

We will present in this work two different proofs of Theorem 1, each of them being related to one of the methods just mentioned.
Our first proof is based on minimax methods. The application of such methods to indefinite functionals requires the topological concept of linking in order to guarantee that the values defined by the minimax are finite numbers. This is in contrast to the problem of finding critical points of bounded functionals . For the case of functions on infinite dimensional spaces this construction has been introduced by V. Benci and P. Rabinowitz in [5].
The second difficulty arises from the fact that the action functional is strongly indefinite. This problem will be overcome by means of a Galerkin approximation.
Postponing the details of the variational setup to Section 2, we merely sketch the basic idea of the proof. We shall consider the action functional Φ defined in (1.24) on the Sobolev space $\Omega = W^{\frac{1}{2},2}(S^1, R^{2n})$. By $(\cdot\,,\cdot)$ we denote the inner product on Ω . Thus Ω becomes a Hilbert space, and $\Phi \in C^1(\Omega, R)$ can be represented by

$$(1.25) \qquad \Phi(\xi) = \frac{1}{2}(L\xi, \xi) - \varphi(\xi) + (v, \xi) \ , \ \xi \in \Omega \, ,$$

where $\varphi \in C^1(\Omega, R)$ satisfies $\varphi(\xi) = o(\|\xi\|^2)$ as $\|\xi\| \to \infty$, $v \in \Omega$ is a constant vector depending on the rotation vector j, and where $L \in \mathcal{L}(\Omega)$ is a selfadjoint operator such that there exists an orthogonal splitting $\Omega = \Omega^+ \oplus \Omega^- \oplus \Omega^0$ into the positive, the negative and the zero spectral subspace of L respectively.
In particular, the subspaces Ω^+ and Ω^- are both of infinite dimension, while $\Omega^0 = \ker(L)$ has dimension $n = 1/2 \dim R^{2n}$.
The assumption on the periodicity of the Hamiltonian $H(x, y, t)$ in the variable $x \in R^n$ implies the invariance of Φ under a free Z^n-action on Ω, and by

passing to the quotient with respect to this group action, Φ can be viewed as a R-valued function defined on the trivial vector bundle $\Omega^+ \times \Omega^- \times T^n$. The restricted function $\Phi|_{\Omega^+ \times T^n}$ is bounded from below, and conversely the restriction of Φ to the subbundle $\Omega^- \times T^n$ is bounded from above. Moreover, $\Phi(\xi^-) \to -\infty$ as $\|\xi^-\| \to \infty$, $\xi^- \in \Omega^- \times T^n$.
Consequently, for any disc $D^- \subset \Omega^-$ of sufficiently large radius we have

$$\sup_{\xi \in \partial D^- \times T^n} \Phi(\xi) < \inf_{\xi \in \Omega^+ \times T^n} \Phi(\xi),$$

and since, in addition, Φ satisfies the Palais-Smale compactness condition, the existence of at least one critical point for Φ can be expected to be established similar as in the well-known saddlepoint theorem, see e.g. M. Struwe [56], Theorem 8.4, provided the vector bundle $\Omega^+ \times T^n$ and the sphere bundle $\partial D^- \times T^n$ link, i.e. $h(D^- \times T^n) \cap (\Omega^+ \times T^n) \neq \emptyset$ for all homeomorphisms h of $\Omega^+ \times \Omega^- \times T^n$ which leave $\partial D^- \times T^n$ pointwise fixed.
However, such a topological intersection result cannot be shown, since homology and cohomology of the sphere bundle $\partial D^- \times T^n$ have to be involved. The homotopy-invariants of infinite-dimensional spheres are all trivial, and in order to cope with this difficulty we introduce a Galerkin approximation where Ω^- is approximated by an increasing sequence of finite-dimensional linear subspaces Ω_k^-.
We set $\Omega_k := \Omega^+ \times \Omega_k^-$ for short. With D^- replaced by $D_k^- := D^- \cap \Omega_k^-$ for any homeomorphism h of $\Omega_k \times T^n$ leaving $\partial D_k^- \times T^n$ pointwise fixed, the above intersection is nonempty. Indeed, we shall prove that the intersection contains more topology, which is caused by the torus T^n. In Section 3 we give a detailed proof of the following

Theorem 6
Let $h \in \mathrm{Homeo}(\Omega_k \times T^n)$ be a homeomorphism which leaves the set $D_k^- \times T^n$ pointwise fixed. Then

(1.26) $\quad \mathrm{cat}_{\Omega_k \times T^n}(h(D_k^- \times T^n) \cap (\Omega^+ \times T^n)) \geq \mathrm{cuplength}(T^n) + 1$

where $\mathrm{cat}_{\Omega_k \times T^n}(A)$ denotes the Lyusternik-Schnirelman category of the subset A in $\Omega_k \times T^n$.

This intersection result generalizes the corresponding theorem of J.Q. Liu in [38]. Also see the related result of H. Hofer [34], Proposition 4.

The restriction of the function Φ to the subbundles $\Omega_k \times T^n$ defines a sequence of approximating functionals Φ_k. For each of these functions the existence of critical points can be shown by the means indicated above. Theorem 6 allows to prove a statement stronger than that of the ordinary saddlepoint theorem. For this purpose we shall define an index map defined on the family of subsets of $\Omega^+ \times \Omega_k^- \times T^n$. Such index maps have been introduced by V. Benci [4] and H. Hofer [34] to combine the concept of linking and minimax methods in order to obtain multiplicity of critical points for indefinite functionals in topologically nontrivial situations. Thus having proved the desired result for the approximating functions Φ_k, the claimed statement for the action functional Φ under consideration follows by passing to the limit in the approximation scheme.
The ideas of the proof given by H. Hofer in [34] are related to our approach, although a Galerkin scheme is not used explicitely. Instead H. Hofer introduces a so-called $\mathbb{N}$-family of approximating functionals being defined on the total space and having the same type of approximation properties.

In Section 9 we present an alternative proof of Theorem 1 which does not use minimax methods. The idea of the proof is based on the Lyusternik-Schnirelman theory for flows which is due to C. Conley and E. Zehnder, see [14]. The application of this strategy under the more general hypotheses considered here requires some modifications of the arguments given in [14]. The main difference between the Conley-Zehnder approach and our proof consists in the use of Galerkin approximation instead of Lyapunov-Schmidt reduction, and the use of the geometrical Lyusternik-Schnirelman category instead of the cohomological category. The Galerkin approximation used here is different from that in the first proof. If $\Omega^+ \times \Omega^- \times T^n$ denotes the vector bundle over the torus T^n introduced above, we shall use an approximating sequence $\Omega_k \times T^n := \Omega_k^+ \times \Omega_k^- \times T^n$ where both $\Omega_k^+ \subset \Omega^+$ and $\Omega_k^- \subset \Omega^-$ are linear subspaces of finite dimension for every $k \in \mathbb{N}$.
In order to prove the topological part we have to replace Theorem 6 by a intersection theorem which is suited for the situation considered here, and which corresponds to Theorem 4 in [14]. Let therefore the approximating functions Φ_k be defined by restriction of Φ to $\Omega_k \times T^n$. It will be shown that there always exists a continuous gradient-like flow for Φ_k which admits an isolating block $D_k^+ \times D_k^- \times T^n$ for the invariant set $S^{(k)}$ consisting of the

critical points of Φ_k together with their connecting orbits. Here $D_k^\pm \subset \Omega_k^\pm$ denotes a disc of some radius chosen sufficiently large. Assume, for the sake of simplicity, that the gradient vector field $\nabla\Phi_k$ generates a unique flow on $\Omega_k \times \mathrm{T}^n$. Then we may state our result as

Theorem 7
If $S^{(k)}$ denotes the maximal invariant set of the flow of $\nabla\Phi_k$ which is contained in the isolating block $D_k^+ \times D_k^- \times \mathrm{T}^n$, then

$$\mathrm{cat}_{\Omega_k \times \mathrm{T}^n}(S^{(k)}) \geq \mathrm{cuplength}(\mathrm{T}^n) + 1 \,. \tag{1.27}$$

The existence proof for critical points for Φ_k then is based on the observation that in case that Φ has at most finitely many critical levels $c_1 < \ldots < c_m$, we always can find a corresponding Morse decomposition $(S_1^{(k)}, \ldots, S_m^{(k)})$ of the invariant set $S^{(k)}$, provided $k \in \mathbb{N}$ is sufficiently large.
Beyond its application to the existence of critical points for certain functionals, the Lyusternik-Schnirelman theory for flows prompts the following generalization of the classical Lyusternik-Schnirelman theorem :

Theorem 8
Assume M is a compact absolute neighborhood retract. If there exists a continuous flow on M such that $(M_1, \ldots, M_k)$ is a Morse decomposition of M, then

$$\mathrm{cat}_M(M) \leq \sum_{j=1}^{k} \mathrm{cat}_M(M_j) \,. \tag{1.28}$$

In particular, if the continuous flow is gradient-like, then there exist at least $\mathrm{cat}_M(M)$ rest points.

The corresponding result for the cohomological category, which holds true even if the assumption on M to be an absolute neighborhood retract is dropped, is due to C. Conley and E. Zehnder, see [14], Theorem 5.

In order to establish cuplength-estimates for the number of critical points, G. Fournier and M. Willem have introduced the notion of relative Lyusternik-Schnirelman category, see [22, 23]. A slightly modified definition of relative category has also been given by A. Szulkin in [57]. In particular, the relative category allows to generalize the relation between category and cuplength to the relative case, i.e. for pairs of spaces. However, as for the applications we have in mind it does not seem to be of advantage to make use of this general concept, because the problem can always be reduced to the case where the pair of spaces consists of a manifold M and its boundary ∂M. In that situation, however, the required arguments are easily obtained by Lefschetz duality.

The Galerkin approximation scheme used in Section 9 is already introduced in Section 8 for the proof of Theorem 3. As in the general case it replaces the Lyapunov-Schmidt reduction to finite dimensions which was used by C. Conley and E. Zehnder in [14]. The estimate for the number of critical points by the sum of Betti numbers then is obtained by the use of Conley-Zehnder Morse theory as presented in [15]. It will be shown that for a sufficiently large cut-off-parameter in the Galerkin scheme the critical points of the approximating functionals are in a one-to-one correspondence with those of the unrestricted action functional, and that moreover they are all nondegenerate.
Galerkin methods have already been used earlier in order to establish a Morse theory for asymptotically linear Hamiltonian systems, see Shujie Li, J.Q. Liu [37] and Y. Long, E. Zehnder [39].

The variational methods considered require a certain compactness property for the action functional Φ given in (1.24). It has become quite fashionable to refer to any form of compactness condition as a variant of the well-known Palais-Smale condition, whether it is Palais-Smale or not. For instance, in all the cases we are going to consider this compactness condition for Φ will turn out to be satisfied because its gradient is a proper mapping. The use of an approximation scheme requires, however, an additional compactness argument. This is closely related to the notion of A-proper mappings which is due to F.E. Browder and W.V. Petryshyn. We refer to K. Deimling [17] and W.V. Petryshyn [48] for a detailed survey. A-proper mappings play an important role in the approximation of nondegenerate critical points in Section 8.

(d) The organization of this work

In Section 2 we present the variational principle on which the proofs of Theorem 1 and Theorem 3 are based.
The Sections 3-5 contain the preparations for our first proof of Theorem 1. In Section 3 we give a proof of a generalized version of the topological intersection result stated in Theorem 6. In combination with minimax-methods this result is used in Section 4 to prove the existence of critical points for a special class of indefinite C^1-functionals $f : M \times E \to \mathbb{R}$, where M is a compact smooth manifold without boundary, and E is a Banach space. The case of strongly indefinite functions is not contained herein. In Section 5 it is shown how the existence of critical points for an appropriate strongly indefinite $f : M \times E \to \mathbb{R}$ can be obtained by an approximation argument. This result is applied in Section 6 to the action functional of Section 2, which proves Theorem 1.
In Section 7 we describe the special situation in case that all the periodic solutions are assumed to be non-degenerate. The proof of Theorem 3 is carried out in Section 8 by the use of Morse theory for flows.
In Section 9 an alternative proof of Theorem 1 is presented. The proof is based on an approach to Lyusternik-Schnirelman theory for flows, and we first assume that there exists a gradient flow for the action functional. In general, the gradient vector field has to be replaced by an appropriate Lipschitz-continuous gradient-like vector field, which is constructed at the end of Section 9.

2 The variational formulation

Let $S^1 = R/Z$ denote the circle. Consider the Hilbert space $L^2(S^1, R^{2n})$ together with the inner product

$$(2.1) \qquad (z, w)_2 := \int_0^1 \langle z(t), w(t) \rangle \, dt$$

where $\langle \cdot , \cdot \rangle$ is the Euclidean scalar product in R^{2n}. The corresponding L^2-norm will be denoted by $\|z\|_2 = \sqrt{(z, z)_2}$.

If $\{ e_i \,|\, i = 1, \ldots, 2n \}$ denotes the standard basis of R^{2n}, an orthonormal basis of $L^2(S^1, R^{2n})$ is given by

$$(2.2) \qquad \{ u_{ki}(t) = \exp(2\pi k t J)\, e_i \mid k \in Z \, , \; i = 1, \ldots, 2n \}$$

where J is the skew-symmetric matrix defined in (1.2). Every $z \in L^2(S^1, R^{2n})$ has the Fourier coefficients

$$(2.3) \qquad z_k = \sum_{i=1}^{2n} (u_{ki}, z)_2 \, e_i \;\; \in R^{2n} \quad , \quad k \in Z \, ,$$

and is represented by the Fourier expansion

$$\begin{aligned} z(t) &= \sum_{k \in Z} \sum_{i=1}^{2n} (u_{ki}, z)_2 \, u_{ki}(t) \\ &= \sum_{k \in Z} \exp(2\pi k t J) \, z_k \end{aligned}$$

for almost every $t \in S^1$. The L^2-scalar product defined in (2.1) now can be expressed by

$$(z, w)_2 = \sum_{k \in Z} \langle z_k, w_k \rangle \quad .$$

Using the Fourier expansion we introduce the following Sobolev space

$$(2.4) \qquad W^{\frac{1}{2},2}(S^1, R^{2n}) = \{ z \in L^2(S^1, R^{2n}) \mid \sum_{k\in Z} |k|\,|z_k|^2 < \infty \}$$

which is a Hilbert space with the inner product

$$(2.5) \qquad (z, w) = \langle z_0, w_0 \rangle + 2\pi \sum_{k\in Z} |k| \langle z_k, w_k \rangle$$

and corresponding norm

$$\|z\| = \sqrt{(z, z)}\,.$$

In the following we will abbreviate $W := W^{\frac{1}{2},2}(S^1, R^{2n})$.
There is an orthogonal decomposition $W = W^+ \oplus W^- \oplus W^0$ into closed subspaces according to the subscript $k > 0, k < 0, k = 0$. By P^+, P^-, P^0 we denote the orthogonal projectors on W^+, W^-, W^0.
Now define a differential operator

$$L : \mathrm{dom}(L) \subset L^2(S^1, R^{2n}) \to L^2(S^1, R^{2n})$$

on the domain of definition

$$(2.6) \qquad \mathrm{dom}(L) = \{ z \in L^2(S^1, R^{2n}) \mid \sum_{k\in Z} |k|^2 |z_k|^2 < \infty \}$$

by

$$(2.7) \qquad (Lz)(t) = -J\frac{d}{dt}\, z(t) = \sum_{k\in Z} 2\pi k \, \exp(2\pi k t J) z_k$$

for almost every $t \in S^1$.
Note that $\mathrm{dom}(L)$ coincides with the Sobolev space $W^{1,2}(S^1, R^{2n})$ considered as a linear subspace of $L^2(S^1, R^{2n})$.
We list some properties of L in

Lemma 2.1

(i) *L is densely defined in $L^2(S^1, R^{2n})$, and L is selfadjoint on* $\mathrm{dom}(L)$.

(ii) *The kernel of L consists of the constant loops, hence* $\dim \ker(L) = 2n$.

(iii) *The range of L consists of all those loops which have mean value zero, i.e.*

$$\mathrm{ran}(L) = \{ z \in L^2(S^1, \mathbb{R}^{2n}) \mid z_0 = 0 \} .$$

(iv) *The spectrum of L is $\sigma(L) = \sigma_{pp}(L) = 2\pi\mathbb{Z}$. In particular each u_{ki} is an eigenvector of L corresponding to the eigenvalue $2\pi k$.*

Proof

(i) It is clear from the definition of $\mathrm{dom}(L)$ that L is densely defined.
Let $z, w \in \mathrm{dom}(L)$. Then

$$(Lz \, , \, w)_2 \; = \; \sum_{k \in \mathbb{Z}} 2\pi k \, \langle \, z_k \, , \, w_k \, \rangle \; = \; (\, z \, , \, Lw \,)_2$$

and consequently L is symmetric on $\mathrm{dom}(L)$.
We have to show that $\mathrm{dom}(L^*) \subset \mathrm{dom}(L)$. Let $z \in \mathrm{dom}(L^*)$. Then there exists a constant C depending on z only, such that

$$| (\, z \, , \, Lw \,)_2 \, | \leq C \, \| \, w \, \|_2$$

for all $w \in \mathrm{dom}(L)$. Using Fourier expansion we can write this inequality as

$$\left| \sum_{k \in \mathbb{Z}} 2\pi k \, \langle \, z_k \, , \, w_k \, \rangle \right| \leq C \, \left(\sum_{k \in \mathbb{Z}} | \, w_k \, |^2 \right)^{\frac{1}{2}} . \tag{2.8}$$

For $N \in \mathbb{Z}^+$ we define

$$z_N(t) := \sum_{|k| \leq N} \exp(2\pi k t J) \, \frac{k}{2\pi} z_k \; \in \; \mathrm{dom}(L) \, .$$

With the choice $w = z_N$ the estimate (2.8) becomes

$$\sum_{|k| \leq N} |k|^2 \, |z_k|^2 \leq \frac{C}{2\pi} \left(\sum_{|k| \leq N} |k|^2 \, |z_k|^2 \right)^{\frac{1}{2}} .$$

Consequently

$$\sum_{|k|\le N} |k|^2\,|z_k|^2 \le \frac{C}{2\pi}\,,$$

and the right hand side is independent of N. Therefore we conclude that $z \in \mathrm{dom}(L)$ and thus $\mathrm{dom}(L^*) \subset \mathrm{dom}(L)$, which shows that $L^* = L$.

(ii) By the definition of L we have

$$Lz = 0 \quad \text{if and only if} \quad \| Lz \|_2^2 = \sum_{k\in\mathbb{Z}} (2\pi k)^2\,|z_k|^2 = 0\,,$$

and hence $z \in \ker(L)$ if and only if $z_k = 0$ for all $k \neq 0$, which precisely means that $z(t) = z_0$ is a constant loop.

(iii) By the definition of L we have

$$(\,u_{0i}\,,\,Lz\,)_2 = \sum_{k\in\mathbb{Z}} 2\pi k \left\langle \int_0^1 \exp(2\pi k t J)\,e_i\,dt\,,\,z_k \right\rangle = 0\,.$$

On the other hand, if $z \in L^2(\mathrm{S}^1, \mathrm{R}^{2n})$ with $z_0 = 0$ then consider $w \in \mathrm{dom}(L)$ defined by

$$w_k := \begin{cases} 0 & \text{if} \quad k = 0 \\ \dfrac{1}{2\pi k} z_k & \text{if} \quad k \neq 0\,. \end{cases}$$

Obviously $w \in \mathrm{dom}(L)$ and $Lw = z$.

(iv) From the definitions of u_{ki} and L it is clear that $\sigma_{pp}(L) = 2\pi\mathbb{Z}$. Since the u_{ki} constitute an orthonormal basis of $L^2(\mathrm{S}^1, \mathrm{R}^{2n})$ it follows that $\sigma_{cont}(L) = \emptyset$, and the lemma is proved.

∎

The definitions of $\mathrm{dom}(L)$ and W yield $\mathrm{dom}(L) \subset W$ considered as a linear subspace of $L^2(\mathrm{S}^1, \mathrm{R}^{2n})$, and moreover $\mathrm{dom}(L)$ is dense in W with respect to the norm on W. One verifies by direct computation that we have the identity

$$(2.9) \qquad ((P^+ - P^-)z\,,\,w\,) = (\,Lz\,,\,w\,)_2 \quad \text{if} \quad z,w \in \text{dom}(L)$$

The left hand side is the restriction of a bounded bilinear form on W to $\text{dom}(L)$. Hence the bilinear form on $\text{dom}(L)$ defined by the differential operator L can be extended to a bounded bilinear form on W.
Now recall the Hamiltonian H from the introduction and the definition of $e(t) = (jt, 0) \in \mathbb{R}^n \times \mathbb{R}^n$ for some fixed rotation vector j. Define the functional $\varphi : W \to \mathbb{R}$ by

$$(2.10) \qquad \varphi(z) := \int_0^1 H(z(t)+e(t),t)\,dt\,.$$

Recall from (1.8) that $\nabla H(x,y,t)$ is assumed to be asymptotically linear with respect to y. This hypothesis implies at most linear growth for the gradient of H. Precisely, there exist constants c_1, c_2 such that

$$|\,\nabla H(x,y,t)\,| \leq c_1 + c_2\,|y|$$

uniformly in x and t. Moreover ∇H is continuous. These assumptions are sufficient to prove the following

Lemma 2.2 *The functional* $\varphi : W \to \mathbb{R}$ *defined in* (2.10) *is in* $C^1(W,\mathbb{R})$, *and moreover the derivative of* φ *is represented by*

$$(2.11) \qquad d\varphi(z)\,w = \int_0^1 \langle \nabla H(z+e,t), w\rangle\,dt \quad \text{for} \quad z,w \in W\,.$$

For a proof we refer to Rabinowitz [52], Prop. B37.

The corresponding gradient $\varphi' : W \to W$ defined by

$$(2.12) \qquad (\,\varphi'(z)\,,\,w\,) = d\varphi(z)\,w \quad \text{for all } w \in W$$

is a compact map. This fact is crucial for our purpose, and a proof is given in the appendix; but see also P. Rabinowitz [52], Prop. B37. In order to find a representation for the action functional introduced in (1.24) which is suitable for the estimates in the subsequent paragraphs, the asymptotically quadratic part of the Hamiltonian has to be involved explicitely. Recall the matrix $A(t)$ from (1.9) , (1.10) and define a symmetric matrix

$$(2.13) \qquad Q(t) = \begin{pmatrix} 0 & 0 \\ 0 & A(t) \end{pmatrix} \in M(2n \times 2n, \mathbb{R}) .$$

Define a symmetric bounded linear operator $\hat{Q} \in \mathcal{L}(L^2(S^1, \mathbb{R}^{2n}))$ by

$$(\hat{Q}z)(t) = Q(t)z(t) .$$

There exists a unique symmetric operator $K \in \mathcal{L}(W)$ defined by

$$(2.14) \qquad (Kz, w) := (\hat{Q}z, w)_2 = \int_0^1 \langle Q(t)z(t), w(t) \rangle \, dt .$$

Lemma 2.3 $K \in \mathcal{L}(W)$ *is compact.*

Proof

If $z \in W$ we have

$$\| Kz \|^2 = |(\hat{Q}z, Kz)_2| \leq \| \hat{Q}z \|_2 \, \| Kz \|_2 \leq \| \hat{Q}z \|_2 \, \| Kz \|$$

and hence $\| Kz \| \leq \| \hat{Q}z \|_2 \leq \|\hat{Q}\| \, \|z\|_2$.

Let (z_k) be a bounded sequence in W. Since the embedding $W \rightarrow L^2(S^1, \mathbb{R}^{2n})$ is compact, there exists a subsequence of (z_k) converging in $L^2(S^1, \mathbb{R}^{2n})$. Passing to a subsequence we can assume that (z_k) converges in $L^2(S^1, \mathbb{R}^{2n})$. Hence

$$\| Kz_k - Kz_l \| \leq \|\hat{Q}\| \, \| z_k - z_l \|_2 \rightarrow 0 \quad \text{as} \quad k, l \rightarrow \infty .$$

Thus (Kz_k) is a Cauchy sequence in W, and since W is complete the lemma follows.

∎

Thus the operator $P^+ - P^- - K$ is a bounded symmetric linear operator on W. As K is compact, it is a relatively compact perturbation of $P^+ - P^-$, and therefore the essential spectrum remains unchanged under the perturbation, see Weidmann [58], Satz 9.9. Since $\sigma_{ess}(P^+ - P^-) = \{-1, +1\}$ we conclude

$$\sigma_{ess}(P^+ - P^- - K) = \{-1, +1\} .$$

Consequently, if $\lambda \in \sigma(P^+ - P^- - K) \setminus \{-1, +1\}$ then λ is an isolated eigenvalue of finite multiplicity, and $-1, +1$ are the only possible accumulation points of the spectrum $\sigma(P^+ - P^- - K)$.
Note that we have an identity analogous to (2.9) when the perturbation is involved, i.e.

$$((P^+ - P^- - K)z \,,\, w) = ((L - \hat{Q})z \,,\, w)_2 \quad \text{if} \quad z \in \mathrm{dom}(L) \,,\, w \in W \,.$$

Since $\hat{Q}$ is a bounded symmetric perturbation of L we have by a well-known theorem by Kato-Rellich that $L - \hat{Q}$ is selfadjoint on $\mathrm{dom}(L - \hat{Q}) = \mathrm{dom}(L)$.

We introduce the following notation

$$T := P^+ - P^- - K \,. \tag{2.15}$$

Lemma 2.4 *If*

$$\det \left[\int_0^1 A(t)\, dt \right] \neq 0$$

then

$$\ker(T) = \{ z \in W \mid z(t) = (x(t), y(t)) \in \mathbf{R}^n \times \mathbf{R}^n \ \textit{with}\ x = const. \,,\ y = 0 \} .$$

Moreover, T is a Fredholm operator with $\mathrm{index}(T) = 0$ *and we have*

$$\mathrm{ran}(T) = \ker(T)^{\perp} .$$

Proof
Let $z \in \ker(P^+ - P^- - K)$. Then for any $w \in \mathrm{dom}(L)$:

$$(z \,,\, (L - \hat{Q})w)_2 = (z \,,\, (P^+ - P^- - K)w) = ((P^+ - P^- - K)z \,,\, w) = 0$$

and hence $z \in \mathrm{dom}(L)$ and $(L - \hat{Q})z = 0$.
But $z \in \ker(L - \hat{Q})$ if and only if $z(t) = (x(t), y(t))$ satisfies

$$(2.16) \qquad \begin{cases} -\dot{y} &= 0 \\ \dot{x} &= A(t)y \,. \end{cases}$$

Consequently $y = const.$ and $x(t) = x(0) + \int_0^1 A(s)\,ds\,y$, $0 \le s \le 1$.
Since x is a loop, we find for $t = 1$

$$0 = x(1) - x(0) = \int_0^1 A(t)\,dt\,y \,.$$

From $\det\left[\int_0^1 A(t)\,dt\right] \neq 0$ we obtain $y = 0$ and hence $\dot{x} = 0$ so that $x = const.$
Conversely, any $z = (x, y)$ with $x = const., y = 0$ is in $\ker(L - \hat{Q})$) by (2.16) and hence in $\ker(P^+ - P^- - K)$.

By definition we have $\ker(P^+ - P^-) = W^0$ and $\operatorname{ran}(P^+ - P^-) = W^+ \oplus W^-$.
Consequently $\operatorname{coker}(P^+ - P^-) = W/\operatorname{ran}(P^+ - P^-) \cong \operatorname{ran}(P^+ - P^-)^\perp = W^0$.
Since K is compact it follows that $T = P^+ - P^- - K$ is also Fredholm with $\operatorname{index}(T) = 0$.

It follows from the definitions of P^+ , P^- and K that T is symmetric. Therefore we have

$$(2.17) \qquad \operatorname{ran}(T)^\perp = \ker(T) \,.$$

Since T is Fredholm, its range is closed and therefore we conclude

$$\operatorname{ran}(T) = \operatorname{ran}(T)^{\perp\perp} = \ker(T)^\perp \,,$$

and the lemma is proved.

■

In order to include the case of a not exact Hamiltonian vector field of the form (1.15), where the additional forcing term $f(t)$ satisfies the condition (1.16) we need the following

Lemma 2.5 *Assume that the continuous periodic function* $f(t+1) = f(t) = (f_1(t), f_2(t)) \in \mathbb{R}^n \times \mathbb{R}^n$ *satisfies* (1.16) *:*

$$\int_0^1 f_1(t)\,dt = 0 .$$

Then $f \in \operatorname{ran}(L - \hat{Q})$.

By direct computation we see that $f = (L - \hat{Q})w$ with a continuously differentiable loop $w(t) = (\xi(t), \eta(t)) \in \mathbb{R}^n \times \mathbb{R}^n$ defined by :

$$\begin{aligned} \eta(t) &= \eta_0 - \int_0^t f_1(s)\,ds \\ \xi(t) &= \xi(0) + \int_0^t \left\{ f_2(s) + A(s)\eta_0 - A(s)\int_0^s f_1(\tau)\,d\tau \right\} ds \end{aligned}$$

with

$$\eta_0 = \eta(0) := [A]^{-1} \left(\int_0^1 \left\{ A(s) \int_0^s f_1(\tau)\,d\tau - f_2(s) \right\} ds \right)$$

where $[A] := \int_0^1 A(t)\,dt \in \mathcal{L}(\mathbb{R}^n)$.
Consequently $w \in \operatorname{dom}(L - \hat{Q})$, and for any $z \in W$ we have

$$(2.18) \quad \int_0^1 \langle f(t), z(t) \rangle\,dt = (f, z)_2 = ((L - \hat{Q})w, z)_2 = (Tw, z) .$$

Finally we have to consider the additional term in (1.24) containing the rotation vector. Let $j \in \mathbb{Z}^n$ be fixed and $e(t) = (jt, 0) \in \mathbb{R}^n \times \mathbb{R}^n$. Then we define

$$v_j := -J\dot{e} = (0, j) \in W .$$

Obviously we have the identity

$$(2.19) \quad \int_0^1 \langle v_j, z \rangle\,dt = (v_j, z)_2 = \langle v_j, z_0 \rangle = (v_j, z) \quad \text{for} \quad z \in W .$$

The contributions to the action functional obtained from the forcing term f and from the rotation vector j both consist in additional linear terms. Combining (2.18) and (2.19) we define a continuous linear function $(v, \cdot) : W \to \mathbb{R}$, where

$$(2.20) \qquad (v, z) := (v_j - Tw, z) = \int_0^1 \langle -J\dot{e} - f(t), z(t) \rangle\,dt .$$

Now we define the action functional Φ on W by

$$(2.21) \qquad \Phi(z) = \frac{1}{2}(\,(P^+ - P^-)z\,,\,z\,) - \varphi(z) + (\,v\,,\,z\,)$$

$$(2.22) \qquad = \frac{1}{2}(\,Tz\,,\,z\,) - \hat{\varphi}(z) + (\,v\,,\,z\,)$$

where

$$(2.23) \qquad T := P^+ - P^- - K$$

and

$$(2.24) \quad \hat{\varphi}(z) := \varphi(z) - \frac{1}{2}(\,Kz\,,\,z\,) = \int_0^1 \left\{H(z+e,t) - \frac{1}{2}\langle\,\hat{Q}z\,,\,z\,\rangle\right\}\,dt\,.$$

It follows from the properties of φ and K that $\hat{\varphi} \in C^1(W,\mathbb{R})$ and that moreover $\hat{\varphi}' : W \to W$ is compact. Therefore $\Phi \in C^1(W,\mathbb{R})$ and

$$(2.25) \qquad d\Phi(z)w = (\,Tz\,,\,w\,) - d\hat{\varphi}(z)w + (\,v\,,\,w\,) \quad \text{for} \quad w \in W\,.$$

For the corresponding gradient we have accordingly

$$(2.26) \qquad \begin{aligned} (\,\Phi'(z)\,,\,w\,) &= (\,Tz - \hat{\varphi}'(z) + v\,,\,w\,) \\ &= (\,(P^+ - P^-)z - \varphi'(z) + v\,,\,w\,) \quad \text{for} \quad w \in W\,. \end{aligned}$$

The function $\Phi : W \to \mathbb{R}$ extends the the classical action functional defined on $W^{1,2}$-loops by

$$(2.27) \qquad \Phi(z) = \int_0^1 \left\{\frac{1}{2}\langle -J\dot{z}\,,\,z\,\rangle - H(z+e,t) + \langle\,v_j - f(t)\,,\,z\,\rangle\right\}\,dt$$

for $z \in W^{1,2}(S^1,\mathbb{R}^{2n})$.

The setup on the Sobolev space $W^{\frac{1}{2},2}(S^1,\mathbb{R}^{2n})$ permits a variational characterization of the periodic solutions under consideration. We shall show that the critical points of Φ on W are precisely the classical periodic solutions we are looking for.

Lemma 2.6 *$z \in W$ is a critical point of Φ if and only if $z \in C^1(S^1,\mathbb{R}^{2n})$ and z is a 1-periodic solution of*

$$-J\dot{z} = \nabla H(z+e,t) + f(t) - v_j\,.$$

Proof
Let $z \in W$ be a critical point of Φ, i.e. $\Phi'(z) = 0$. Consequently $(P^+ - P^-)z = \varphi(z) - v$. For $w \in \text{dom}(L)$ we have

$$\begin{aligned}
|(z, Lw)_2| &= |(z, (P^+ - P^-)w)| = |((P^+ - P^-)z, w)| \\
&= |(\varphi(z)' - v, w)| \leq |(\varphi'(z), w)| + |(v, w)| \\
&= \left| \int_0^1 \langle \nabla H(z+e,t), w \rangle \, dt \right| + \left| \int_0^1 \langle v_j - f(t), w \rangle \, dt \right| \\
&= \left(\int_0^1 |\nabla H(z+e,t)|^2 \, dt \right)^{\frac{1}{2}} \|w\|_2 + \|v_j - f\|_2 \, \|w\|_2 \\
&\leq C \, \|w\|_2
\end{aligned}$$

with a constant C independent of w. Hence $z \in \text{dom}(L)$. So we have

$$(\Phi(z), w) = 0 \qquad \text{if and only if}$$

(2.28) $$\int_0^1 \langle -J\dot{z} - \nabla H(z+e,t) + v_j - f, w \rangle \, dt = 0 \quad \text{for all} \quad w \in W$$

and therefore

(2.29) $$-J\dot{z}(t) = \nabla H(z(t)+e(t),t) + f(t) - v_j$$

for almost every $t \in S^1$.

By the Sobolev embedding theorem, $z \in \text{dom}(L)$ implies the continuity of z, so the right hand side of (2.29) is continuous, and this gives $z \in C^1(S^1, R^{2n})$. Conversely, if z is a classical solution of (2.29), i.e. $z \in C^1(S^1, R^{2n})$, then $z \in W^{1,2}(S^1, R^{2n}) = \text{dom}(L) \subset W$. Hence by (2.28) we have $(\Phi'(z), w) = 0$ for all $w \in W$, and thus $\Phi'(z) = 0$.

■

The periodicity of the Hamiltonian H with respect to x implies the invariance

of Φ and Φ' under a free action of the group $\mathbb{Z}^n$ on W which is defined as follows:
Identify $g = (g_1, \ldots, g_n) \in \mathbb{Z}^n$ with $(g_1, \ldots, g_n, 0, \ldots, 0) \in \mathbb{R}^{2n}$, which can be considered as a constant loop, i.e. an element of W. Now define the group action by

(2.30) $$g \cdot z := z + g \quad \text{for} \quad z \in W\,.$$

Then $\Phi(g \cdot z) = \Phi(z)$ and $\Phi'(g \cdot z) = \Phi'(z)$.
Passing to the quotient we obtain

(2.31) $$W/\mathbb{Z}^n \cong E \times \mathbb{T}^n$$

where

(2.32) $$E = \ker(T)^{\perp}\,.$$

By Lemma 2.4 we have

(2.33) $$E \cong W/\ker(T)\,.$$

We will consider E equipped with the scalar product induced from the inner product on W. In particular, because of the $\mathbb{Z}^n$-invariance we can consider Φ as a an element of $C^1(E \times \mathbb{T}^n, \mathbb{R})$.
In the subsequent sections we shall make crucial use of the fact that there exists a splitting

$$E = E^+ \oplus E^-$$

into closed orthogonal subspaces E^+, E^- according to the positive and negative eigenvalues of the operator T.

The variational setup described until now is suited for periodic solutions having the period $p = 1$, i.e. they have the same periodicity as the Hamiltonian H. These solutions are usually called forced oscillations; they correspond to the j-solutions of Theorem 1. In order to establish the existence of j/p-solutions with $j \in \mathbb{Z}^n$ and $p \in \mathbb{N}$ relatively prime we define $e_p(t) := (p^{-1}j\,t, 0) \in \mathbb{R}^n \times \mathbb{R}^n$, and instead of (2.27) we have to consider the action functional

$$\Phi_p(z) := \frac{1}{p}\int_0^p \left\{\frac{1}{2}\langle -J\dot{z}, z\rangle - H(z+e_p,t) + \langle -J\dot{e}_p - f(t), z\rangle\right\} dt$$

on the loop space $\Omega = W^{1,2}(\mathbb{R}/p\mathbb{Z}, \mathbb{R}^{2n})$. The variational formulation on the Sobolev space $W^{\frac{1}{2},2}(\mathbb{R}/p\mathbb{Z}, \mathbb{R}^{2n})$ then can be carried out analogous to the case $p = 1$.

In the case of Theorem 5 the claimed periodic solutions are characterized as the critical points of the functional

$$\Phi(z) := \int_0^1 \left\{\frac{1}{2}\langle -J\dot{z}, z\rangle - H(x,t) + \langle -f(t), z\rangle\right\} dt$$

defined on the loop space $W^{1,2}(S^1, \mathbb{R}^{2n})$. The periodicity of the Hamiltonian H in all its variables leads to a simpler situation. On the Sobolev space $W^{\frac{1}{2},2}(S^1, \mathbb{R}^{2n})$ the representation of the action functional

$$\Phi(z) = \frac{1}{2}((P^+ - P^-)z, z) - \varphi(z) + (v, z),$$

where v is defined by

$$(v, z) := (-(P^+ - P^-)w, z) := \int_0^1 \langle -f(t), z(t)\rangle\, dt \ , \ z \in W,$$

is already well suited for the existence proof, which can be done using the same strategy as for the proof of Theorem 1.

3 An intersection result

We consider the manifold $E \times M$ where E is a normed vector space, and where M is an oriented compact manifold without boundary. Moreover, we shall assume that E splits into two closed linear subspaces

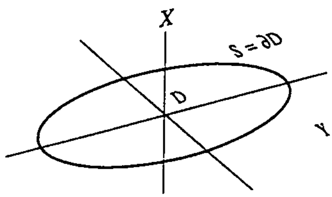
$$E = X \oplus Y \quad \text{and} \quad \dim Y < \infty \, .$$

For a fixed $R > 0$ we define the embedded compact disc of radius R in Y by

$$D = \{ (x, y) \in X \oplus Y \mid x = 0 \, , \, \|y\| \leq R \} \tag{3.1}$$

and its boundary in Y by

$$S = \partial D = \{ (0, y) \in X \oplus Y \mid \|y\| = R \} \, . \tag{3.2}$$



Next we define a family of homeomorphisms

$$\mathcal{H} := \{ h \in \text{Homeo}(E \times M) \mid h(z) = z \;\text{ if }\; z \in S \times M \} \, . \tag{3.3}$$

Our aim is to show that

$$h(D \times M) \cap (X \times M) \neq \emptyset \quad \text{for all} \quad h \in \mathcal{H} \, . \tag{3.4}$$

From the definition of S we obtain $S \times M = \partial(D \times M)$ and

$$\partial(D \times M) \cap (X \times M) = \emptyset \, . \tag{3.5}$$

We say that $S \times M$ and $X \times M$ link with respect to $\mathcal{H}$ when (3.4) and (3.5) hold true, see M. Struwe [56], Chapter 8.
Actually we will prove a stronger result, namely that

$$\text{(3.6)} \qquad \mathrm{cat}_{E\times M}(\,h(D \times M) \cap (X \times M)) \geq \mathrm{cuplength}(M) + 1$$

for all $h \in \mathcal{H}$.
Here cat stands for the Lyusternik-Schnirelman category. Recall that for a nonempty subset $A \subset E \times M$ the category $\mathrm{cat}_{E\times M}(A)$ of A in $E \times M$ is defined to be the smallest positive integer k such that there exist k closed subsets $A_j \subset E \times M$, $j = 1, \ldots, k$, which satisfy

1. A_j is contractible in $E \times M$ for $j = 1, \ldots, k$
2. $A \subset A_1 \cup \ldots \cup A_k$

If such k does not exist we define $\mathrm{cat}_{E\times M}(A) = +\infty$. Moreover, we set $\mathrm{cat}_{E\times M}(\emptyset) := 0$.
Also recall the definition of the cuplength of a topological space :
cuplength(M) is defined to be the largest integer k such that there exist k cohomology classes $\omega_j \in H^{k_j}(M)$, $k_j \geq 1$, $j = 1, \ldots, k$, with nonvanishing cup-product $\omega_1 \cup \ldots \cup \omega_k$. Here $H^*(M)$ denotes the singular cohomology of M.
The number cuplength$(M) + 1$ can be considered as a kind of cohomological category, see H. Hofer [34]. One has the well known relation $\mathrm{cat}_M(M) \geq \mathrm{cuplength}(M) + 1$, see e.g. J.T. Schwartz [54].
In the appendix we recollect some of the properties of cat. However, we point out the following result which we shall need below :

Lemma 3.1 *Let W be a topological space, and assume $V \subset W$ is a closed retract of W, i.e. V is closed and there exists a continuous map $r : W \to V$ such that $r(v) = v$ for all $v \in V$.*
Then for $A \subset V$ we have

$$\mathrm{cat}_V(A) = \mathrm{cat}_W(A) \ .$$

Proof

(a) Since V is closed, we have trivially

$$\text{cat}_W(A) \le \text{cat}_V(A)\,.$$

(b) First assume $A \neq \emptyset$ and $\text{cat}_W(A) = k < \infty$. Then there exist k subsets $B_j \subset W$, $j = 1,\ldots,k$, B_j closed and contractible in W, such that $A \subset B_1 \cup \ldots \cup B_k$. Consider $A_j := B_j \cap V \subset V$. Then A_j is closed and $A \subset A_1 \cup \ldots \cup A_k$ and we have to show that A_j is contractible in V.
If $r_j : [0,1] \times B_j \to W$ is the contraction of B_j in W, then

$$r \circ r_j : [0,1] \times (B_j \cap V) \to V$$

is a contraction of A_j in V. Consequently

$$\text{cat}_V(A) \le \text{cat}_W(A)\,.$$

If $\text{cat}_W(A) = \infty$ the claim is trivial. If $A = \emptyset$ then $\text{cat}_W(A) = \text{cat}_V(A) = 0$.

∎

We are now ready to prove the main result of this section. It extends the corresponding Intersection Theorem in J.Q. Liu [38]; the proof is different from that of Liu.

Proposition 3.1 (Intersection Lemma) *Let E be a normed linear space, splitting into closed subspaces $E = X \oplus Y$ with $\dim Y < \infty$. Let D be the closed disc of radius R in Y with boundary $S = \partial D$ as defined in* (3.1), (3.2). *Let M be an oriented compact manifold without boundary. Then*

$$\text{cat}_{E\times M}(h(D \times M) \cap (X \times M)) \;\ge\; \text{cuplength}(M) + 1 \tag{3.7}$$

for every homeomorphism h of $E \times M$ which leaves the boundary $S \times M$ of $D \times M$ pointwise fixed.

Proof

Let $h \in \mathcal{H}$ be fixed and define

$$I \;:=\; h(D \times M) \cap (X \times M)\,.$$

Define $m :=$ cuplength(M) and assume that $m \geq \text{cat}_{E\times M}(I)$. We set $l := \text{cat}_{E\times M}(I)$.

(a) We first consider the case $l > 0$. Then in particular $I \neq \emptyset$. Since I is compact there exists, by the properties of cat, an open neighborhood V_1 of I in $E \times M$ such that $\text{cat}_{E\times M}(V_1) = l$.
Moreover, since I does not intersect $S\times M$, there exists an open neighborhood V_2 of I in $E \times M$ such that $\overline{V_2} \cap (S \times M) = \emptyset$.
Define $V := V_1 \cap V_2$. Then $\text{cat}_{E\times M}(V) = l > 0$, and by definition of cat there exist l closed subsets $A_j \subset E \times M$ for $j = 1, \ldots, l$, such that each A_j is contractible in $E \times M$ and $V \subset A_1 \cup \ldots \cup A_l$. Define

$$(3.8) \qquad A'_j := A_j \cap h(D \times M) \quad , \quad j = 1, \ldots, l\,.$$

Then $A'_j \subset h(D \times M)$ is a compact subset of $E \times M$ which is contractible in $E \times M$. Moreover,

$$(3.9) \qquad V \cap h(D \times M) \subset A'_1 \cup \ldots \cup A'_l\,.$$

We claim that A'_j is contractible in $h(D \times M)$.
Indeed, let $\tilde{r} : E \times M \to D \times M$ be a retraction of $E \times M$ onto $D \times M$ so that $\tilde{r}(z) = z$ for $z \in D \times M$. Since, by assumption, $h \in \mathcal{H}$ is a homeomorphism, the map

$$r := h \circ \tilde{r} \circ h^{-1} : E \times M \to h(D \times M)$$

is a retraction of $E \times M$ to $h(D \times M)$. As in the proof of Lemma 3.1 we conclude

$$\text{cat}_{h(D\times M)}(A'_j) = \text{cat}_{E\times M}(A'_j)$$

and hence A'_j is contractible in $h(D \times M)$, which proves our claim.
Define the sets

$$(3.10) \qquad B_j := h^{-1}(A'_j) \subset D \times M \quad , \quad j = 1, \ldots, l\,.$$

The sets B_j are compact and contractible in $D \times M$.

Lemma 3.2 *There exists a closed set B satisfying*

$$(3.11) \qquad S \times M \subset B \subset D \times M$$

such that $S \times M$ is a retract of B, and such that moreover

$$B \cup B_1 \cup \ldots \cup B_l = D \times M . \tag{3.12}$$

Proof

We set

$$U := V \cap h(D \times M) .$$

Observe that $h(z) = z$ for $z \in S \times M$ by assumption on h. Consequently

$$S \times M \subset h(D \times M) \setminus U \subset h(D \times M) ,$$

and we claim that $S \times M$ is a retract of $h(D \times M) \setminus U$. Indeed, with the identification $X = X \oplus \{0\} \subset X \oplus Y$,

$$S \times M \subset (E \setminus X) \times M$$

is a deformation retract, given by the homotopy

$$\rho : [0,1] \times (E \setminus X) \times M \to (E \setminus X) \times M$$

which is defined by

$$\rho(t, x + y, p) = \left((1-t)(x+y) + t\frac{y}{\|y\|} , p \right)$$

for $z = x + y \in X \oplus Y$ with $y \neq 0$, and $p \in M$.
By construction $h(D \times M) \setminus U \subset (E \setminus X) \times M$. Therefore, with the retraction $r : E \times M \to h(D \times M)$ defined above, the composition

$$r \circ \rho_1 : h(D \times M) \setminus U \to S \times M ,$$

where $\rho_1 = \rho(1, \cdot)$, is the required retraction, as claimed.
Finally, the set $B \subset D \times M$, defined by

$$B := h^{-1}(h(D \times M) \setminus U) , \tag{3.13}$$

satisfies, in view of the hypotheses on h, the claimed properties of the lemma.

■

Lemma 3.3

(i) *The injections* $\iota : (D \times M, \emptyset) \to (D \times M, B_j)$ *for* $j = 1, \ldots, l$ *induce homomorphisms*

$$\iota^* : H^*(D \times M, B_j) \to H^*(D \times M)$$

which are onto for $* \geq 1$.

(ii) *The injection map* $g : (D \times M, S \times M) \to (D \times M, B)$ *induces homomorphisms in relative cohomology*

$$g^* : H^*(D \times M, B) \to H^*(D \times M, S \times M)$$

which are onto.

Proof
(i) Let $i : B_j \to D \times M$ denote the inclusion map. Since B_j is contractible in $D \times M$, the inclusion i is homotopic to a constant map

$$i \simeq i_0 : B_j \to D \times M \ , \quad i_0(z) = z_0 \ \text{ for all } z \in B_j \ ,$$

and since homotopic maps induce the same homomorphism in cohomology we have $i^* = i_0^* : H^*(D \times M) \to H^*(B_j)$.
We denote $P = \{z_0\}$ and $\gamma : B_j \to P$. Let $p : P \to D \times M$ be the inclusion. We have a commutative diagram :

$$\begin{array}{ccc} H^*(D \times M) & \xrightarrow{\ p^*\ } & H^*(P) \\ & \overset{i^*}{\searrow} & \big\downarrow \gamma^* \\ & & H^*(B_j) \end{array}$$

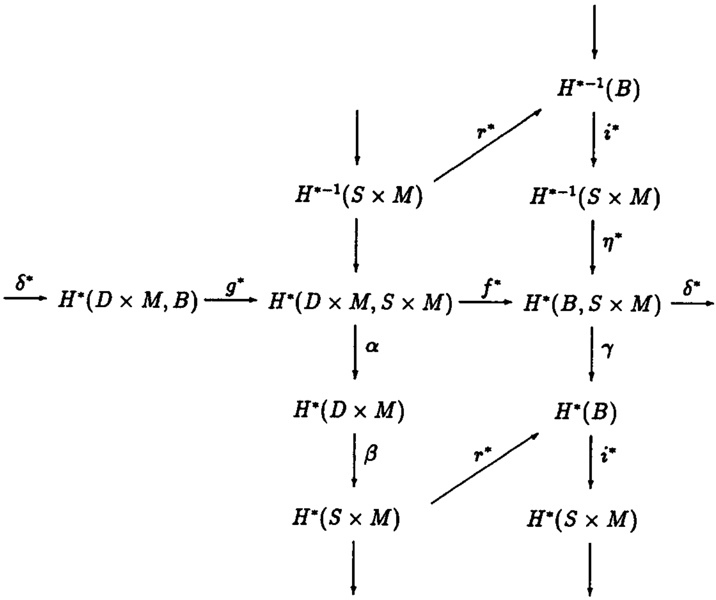
P consists of a single point. Hence $H^*(P) = 0$ for $* > 0$ and thus $\gamma^* = 0$. By the commutativity of the diagram $i^* = 0$.

Consider the exact cohomology sequence

$$\cdots \longrightarrow H^{*-1}(B_j) \xrightarrow{\delta^*} H^*(D \times M, B_j) \xrightarrow{\iota^*} H^*(D \times M) \xrightarrow{i^*} H^*(B_j) \longrightarrow \cdots$$

The statement of (i) now follows from $i^* = 0$ and the exactness of the sequence.

(ii) Consider the commutative diagram



Observe that the horizontal line is the exact cohomology sequence of the triple $(D \times M, B, S \times M)$; δ^* is the connecting homomorphism. The vertical lines are the exact cohomology sequences of the pairs $(D \times M, S \times M)$ and $(B, S \times M)$ respectively.

Let now $r : B \to S \times M$ be the retraction of Lemma 3.2, and consider

$$S \times M \xrightarrow{i} B \xrightarrow{r} S \times M$$

where i is the inclusion map. Then $r \circ i = id$ and therefore $i^* \circ r^* = id^* = id$. Consequently r^* is injective and i^* is onto. From the exactness of the sequence of $(B, S \times M)$ we conclude that the connecting homomorphism is $\eta^* = 0$, and hence γ is one-to-one.
We claim that $f^* = 0$, so that, by exactness of the horizontal line, g^* is onto, as claimed. Indeed, if $\omega \in H^*(D \times M, S \times M)$ we conclude from the commutativity of the diagram and the exactness of the sequence of the triple $(D \times M, S \times M)$ that

$$(\gamma \circ f^*)(\omega) = (r^* \circ \beta \circ \alpha)(\omega) = r^*(0) = 0 ,$$

so $f^*(\omega) = 0$, since γ is injective.
This proves (ii) of the lemma.

∎

Using Lemma 3.2 and Lemma 3.3 we finally arrive at a contradiction.
Since cuplength$(M) = m$, and since $H^*(M) \cong H^*(D \times M)$ by the Künneth formula, there exist cohomology classes

$$\omega_j \in H^{k_j}(D \times M) , \quad j = 1, \ldots, m , \ k_j \geq 1 ,$$

such that $\omega_1 \cup \ldots \cup \omega_m \neq 0$. In particular then for $0 < l \leq m$:

$$\omega_1 \cup \ldots \cup \omega_l \neq 0 .$$

Let $k := k_1 + \ldots + k_l$. Then there is a homology class $\alpha \in H_k(D \times M)$ such that

$$< \omega_1 \cup \ldots \cup \omega_l \, , \, \alpha > \neq 0 \, , \tag{3.14}$$

where $< \cdot \, , \, \cdot >$ denotes the duality pairing

$$H^*(D \times M) \times H_*(D \times M) \to \mathbb{Z} .$$

Denote by $d = \dim Y$, and $n = \dim M$, and let $\xi \in H_{n+d}(D \times M, S \times M)$ be the fundamental class. Thus by Lefschetz duality, there is an isomorphism given by the cap-product

$$H^{n+d-k}(D \times M, S \times M) \xrightarrow{\cap \xi} H_k(D \times M) \ .$$

Here we have used that $S \times M = \partial(D \times M)$ and that $D \times M$ is oriented, since M is oriented. Consequently there exists a class $\omega_0 \in H^{n+d-k}(D \times M, S \times M)$ satisfying $\omega_0 \cap \xi = \alpha$, and we conclude

$$\begin{aligned} < \omega_1 \cup \ldots \cup \omega_l \ , \ \alpha > \ &= \ < \omega_1 \cup \ldots \cup \omega_l \ , \ \omega_0 \cap \xi > \\ &= \ < \omega_0 \cup \omega_1 \cup \ldots \cup \omega_l \ , \ \xi > \ . \end{aligned}$$

We claim that $\omega_0 \cup \omega_1 \cup \ldots \cup \omega_l = 0$, hence contradicting (3.14).
Indeed, by (i) of Lemma 3.3 we can choose cohomology classes $\tilde{\omega}_j \ , \ j = 1, \ldots, l$ such that

$$\omega_j = \iota^*(\tilde{\omega}_j) \quad , \quad \tilde{\omega}_j \in H^{k_j}(D \times M, B_j) \ . \tag{3.15}$$

Consequently, if $a : (D \times M, \emptyset) \to (D \times M, B_1 \cup \ldots \cup B_l)$ denotes the injection map, we have that

$$a^* : H^k(D \times M, B_1 \cup \ldots \cup B_l) \to H^k(D \times M)$$

contains $\omega_1 \cup \ldots \cup \omega_l$ in its image.
Consider the diagram

$$\begin{array}{ccc} H^{n+d-k}(D \times M, B) & \xrightarrow{\cup \tilde{\omega}_1 \cup \ldots \cup \tilde{\omega}_l} & H^{n+d}(D \times M, B \cup B_1 \cup \ldots \cup B_l) \\ \Big\downarrow g^* & & \Big\downarrow \\ H^{n+d-k}(D \times M, S \times M) & \xrightarrow{\cup \omega_1 \cup \ldots \cup \omega_l} & H^{n+d}(D \times M, S \times M) \end{array}$$

g^* is surjective by (ii) of Lemma 3.3. Moreover $B \cup B_1 \cup \ldots \cup B_l = D \times M$ by Lemma 3.2, and hence $H^{n+d}(D \times M, B \cup B_1 \cup \ldots \cup B_l) = 0$.
Therefore $\omega_0 \cup \omega_1 \cup \ldots \cup \omega_l$ factors through 0, and consequently vanishes, as claimed.

(b) Now assume $l = 0$. Hence $h(D \times M) \cap (X \times M) = \emptyset$ and therefore $h(D \times M) \subset (E \setminus X) \times M$.
Thus in the proof of Lemma 3.2 we can replace V by the empty set $\emptyset$, and consequently we have a retraction of $B := D \times M$ to $S \times M$.
Now we apply Lemma 3.3 (ii), which states that

$$g^* : H^*(D \times M, B) \to H^*(D \times M, S \times M)$$

is onto. But since $B = D \times M$, we conclude $H^*(D \times M, S \times M) = 0$, and therefore, by Lefschetz duality, the homology groups $H_*(D \times M)$ are all trivial. So we obtain a contradiction.
This finishes the proof of the proposition.

■

We end this section with some remarks on the role of the hypotheses we have introduced.
Note that the continuity property of the Lyusternik-Schnirelman category is crucial for the proof of Proposition 3.1. In order to have this property satisfied it is sufficient for the considered topological space to be an absolute neighborhood retract (ANR). For a proof see K. Deimling [17], Proposition 27.3.
Recall that a space X is called a (metric) ANR if given any metric space Y, a closed $A \subset Y$ and a continuous map $f : A \to X$, then there always exists a continuous extension of f to some neighborhood of A. If f always can be extended to Y, then X is called an absolute retract (AR).
More general we may require Y to be a normal space in this definition. For our purposes it suffices to consider the case of metric ANR's only.
Observe that every compact manifold is an ANR; for a proof see e.g. M. Greenberg, J. Harper [31], Theorem (26.17.4).
Every normed linear space also is an AR, which is easily obtained from the Arens-Eells embedding theorem, see e.g. E. Michael [42], together with

Dugundji's extension theorem, see K. Deimling [17], Theorem 7.2. Consequently the space $E \times M$ considered in Proposition 3.1 is an ANR.

The hypothesis of M to be oriented means that M is $\mathbb{Z}$-orientable, and therefore M is orientable for every coefficient ring R. The assertion of Proposition 3.1 still holds true for non-orientable M if homology and cohomology are taken with $\mathbb{Z}/2\mathbb{Z}$-coefficients.

4 Minimax and critical points

It is well-known that a differentiable function f on a compact manifold M has at least as many critical points as $\mathrm{cat}_M(M)$, the category of M.
In order to prove this, Lyusternik and Schnirelman introduced the following minimax-principle. If Γ_j denotes the family of subsets of M defined by

$$\Gamma_j = \{ A \subset M \mid \mathrm{cat}_M(A) \geq j \}$$

then the real numbers

$$c_j = \inf_{A \in \Gamma_j} \sup_{x \in A} f(x) \ , \quad j = 1, \ldots, \mathrm{cat}_M(M)$$

are critical values of the function f. The proof follows immediately from the observation that f is a Lyapunov function for the negative gradient flow ϕ^t, defined by

$$\begin{aligned} \frac{d}{dt}\phi^t(x) &= -\nabla f(\phi^t(x)) \quad , \quad x \in M \ , \\ \phi^0(x) &= x \end{aligned}$$

i.e. f is strictly decreasing along nonconstant orbits. Similarly one finds critical points for functions defined on noncompact and even not locally compact manifolds, such as infinite dimensional Banach manifolds, provided the considered function is bounded from below and satisfies in addition appropriate compactness conditions, see e.g. R. Palais [47], J.T. Schwartz [53].
In the case we are interested in, the function f is, however, neither bounded from below nor from above, and the ideas just described do not apply directly. Indeed, the existence of critical points is more subtle. It requires in particular more information about the behaviour of the function.
We are interested in the case of a function f defined on a trivial bundle $E \times M$:

(4.1) $$f : E \times M \to \mathbb{R} \ ,$$

where E is a Banach space and M is a compact connected oriented C^1-manifold without boundary. We can assume M to be smooth, since a C^1-manifold always can be equipped with a compatible C^∞-differentiable structure, see M. Hirsch [32], Chapter 2, Theorem 2.9. In addition, on every differentiable manifold there exists a Riemannian metric, see e.g. W. Klingenberg [36], Theorem 1.8.5. Since M is compact, the Riemannian metric is complete.

Note that the norm on E and the Riemannian structure on M together induce a smooth Finsler structure on $E \times M$ which is complete with respect to the corresponding Finsler metric. We refer to K. Deimling [17], R. Palais [47], M. Struwe [56] for more details.

We shall assume that E decomposes into closed linear subspaces

$$E = X \oplus Y \quad , \quad \dim Y < \infty .$$

The compact disc of radius $R > 0$ in $\{0\} \oplus Y$ will be denoted by

$$D = \{ (0,y) \in X \oplus Y \mid \|y\| \leq R \} \tag{4.2}$$

and its boundary in Y by

$$S = \partial D = \{ (0,y) \in X \oplus Y \mid \|y\| = R \} . \tag{4.3}$$

We introduce the family $\mathcal{H}$ of homeomorphisms of $E \times M$ by

$$\mathcal{H} = \{ h \in \mathrm{Homeo}(E \times M) \mid h(z) = z \text{ if } z \in S \times M \} . \tag{4.4}$$

For a subset $A \subset E \times M$ we define

$$\mathrm{cat}^*(A) := \inf_{h \in \mathcal{H}} \mathrm{cat}_{E \times M}(h(A) \cap (X \times M)) \tag{4.5}$$

which is a nonnegative integer or $+\infty$.

This construction has been used by J.Q. Liu [38]; index maps like cat* have been introduced by H. Hofer [34] and V. Benci [4].

We state some properties of cat* in

Lemma 4.1 *If $A, B \subset E \times M$ and $h \in \mathcal{H}$, then*

(i) $\mathrm{cat}^*(A) \leq \mathrm{cat}_{E \times M}(A)$.

(ii) *If* $A \subset B$ *then* $\mathrm{cat}^*(A) \leq \mathrm{cat}^*(B)$.

(iii) $\mathrm{cat}^*(A \setminus B) \geq \mathrm{cat}^*(A) - \mathrm{cat}_{E\times M}(B)$.

(iv) $\mathrm{cat}^*(\, h(A)\,) = \mathrm{cat}^*(A)$.

Proof
(i) $\mathrm{cat}^*(A) \leq \mathrm{cat}_{E\times M}(\, A \cap (X \times M)) \leq \mathrm{cat}_{E\times M}(A)$.

(ii) $A \subset B$ implies $h(A) \cap (X \times M) \subset h(B) \cap (X \times M)$, and therefore $\mathrm{cat}^*(A) \leq \mathrm{cat}_{E\times M}(\, h(B) \cap (X \times M))$ for every $h \in \mathcal{H}$, from which the assertion follows.

(iii) From $A = (A \setminus B) \cup (A \cap B)$ we conclude $h(A) = h(A \setminus B) \cup h(A \cap B)$ for every $h \in \mathcal{H}$, and consequently

$$\begin{aligned} h(A) \cap (X \times M)) &= [h(A \setminus B) \cup h(A \cap B)] \cap (X \times M) \\ &= [h(A \setminus B) \cap (X \times M)] \cup [h(A \cap B) \cap (X \times M)] \, . \end{aligned}$$

Hence it follows from the subadditivity of cat and the definition of cat* that $\mathrm{cat}^*(A) \leq \mathrm{cat}^*(A \setminus B) + \mathrm{cat}^*(A \cap B)$. Applying now (ii) and (i) we conclude $\mathrm{cat}^*(A \cap B) \leq \mathrm{cat}^*(B) \leq \mathrm{cat}_{E\times M}(B)$.
This proves (iii).

(iv) follows immediately from the definition of cat*.

■

For $k = 1, 2, \ldots$ we now define families Γ_k of subsets of $E \times M$ by

(4.6) $$\Gamma_k := \{\, A \subset E \times M \mid \mathrm{cat}^*(A) \geq k \,\} \ , \ \ k = 1, 2, \ldots$$

Note that the Γ_k form a decreasing sequence, since $\Gamma_k \subset \Gamma_j$ if $k \geq j$.
The statement of intersection, Proposition 3.1, now can be reformulated as

(4.7) $$\mathrm{cat}^*(D \times M) \geq \mathrm{cuplength}(M) + 1 \, .$$

Define $m := \mathrm{cuplength}(M)$. Then

$$\Gamma_k \neq \emptyset \quad \text{for} \quad k = 1, \dots, m+1 . \tag{4.8}$$

Assume now that the function $f : E \times M \to \mathbb{R}$ satisfies

$$\sup_{z \in S \times M} f(z) < \inf_{z \in X \times M} f(z) ,$$

and, in addition, assume that the so-called Palais-Smale compactness condition holds for f. Then the real numbers c_k defined by the following minimax-principle

$$c_k := \inf_{A \in \Gamma_k} \sup_{z \in A} f(z) \quad , \quad k = 1, \dots, m+1 , \tag{4.9}$$

are critical values of f. The proof of this fact will be given in Proposition 4.1 below.
Recall the following

Definition 4.1 (Palais-Smale condition) *A C^1-function $f : E \times M \to \mathbb{R}$ is said to satisfy the condition* (PS), *if every sequence (z_j) in $E \times M$ which satisfies*

$$f(z_j) \to c \quad \text{and} \quad df(z_j) \to 0 \quad \text{as} \quad j \to \infty$$

for some $c \in \mathbb{R}$, contains a convergent subsequence.

The main result of this section is

Proposition 4.1 *Let M be a compact connected oriented C^∞-manifold without boundary, and let E be a Banach space which decomposes into closed linear subspaces $E = X \oplus Y$ with* $\dim Y < \infty$.
Let $f \in C^1(E \times M, \mathbb{R})$ satisfy the Palais-Smale condition (PS), *and assume there exist constants $a < b$ and $R > 0$ such that*

$$f(z) \geq b \quad \text{if } z \in X \times M \tag{4.10}$$

$$f(z) \leq a \quad \text{if } z \in S \times M \tag{4.11}$$

where $S = \{ z = (0, y) \in X \oplus Y \mid \|z\| = R \} \subset \{0\} \oplus Y$ is the sphere of radius R in Y.
Then f has at least $\text{cuplength}(M) + 1$ *critical points.*

Proof
The proof uses the standard minimax arguments described in detail for example in R. Palais [47], P. Rabinowitz [52], M. Struwe [56].
Let $m = \text{cuplength}(M)$ and let D be the compact disc

$$D = \{ z = (0,y) \in X \oplus Y \mid \|z\| \leq R \} \subset \{0\} \oplus Y .$$

Thus $S = \partial D$ is the boundary of D in Y. We recall from (4.6) :

$$\Gamma_k = \{ A \subset E \times M \mid \text{cat}^*(A) \geq k \} \quad , \quad k = 1, 2, \ldots$$

By the Proposition 3.1 we have $D \times M \in \Gamma_k$ for $k = 1, \ldots, m+1$, so that $\Gamma_k \neq \emptyset$ for these subscripts k. Corresponding to these families of sets we define

$$c_k := \inf_{A \in \Gamma_k} \sup_{z \in A} f(z) \quad , \quad k = 1, \ldots, m+1 . \tag{4.12}$$

Since $D \times M$ is compact we have

$$c_k \leq \sup_{z \in D \times M} f(z) < +\infty \quad , \quad k = 1, \ldots, m+1 . \tag{4.13}$$

Moreover, $\text{cat}^*(A) \geq 1$ implies $\text{cat}_{E \times M}(A \cap (X \times M)) \geq 1$, so that in particular $A \cap (X \times M) \neq \emptyset$.
Consequently, in view of the assumptions of the proposition

$$-\infty < b \leq \inf_{z \in X \times M} f(z) \leq \sup_{z \in A} f(z) . \tag{4.14}$$

This inequality still holds true if we take the infimum over $A \in \Gamma_k$. Therefore, by definition of Γ_k :

$$-\infty < b \leq c_1 \leq c_2 \leq \ldots \leq c_{m+1} < \infty . \tag{4.15}$$

We show next that the c_k are critical values of f for $k = 1, \ldots, m+1$.

Lemma 4.2 *Assume*

$$c := c_k = c_{k+1} = \ldots = c_{k+j} \tag{4.16}$$

for $k \geq 1$ and $k + j \leq m+1$. Denote by

$$K_c := \{ z \in E \times M \mid f(z) = c \text{ and } df(z) = 0 \}$$

the set of critical points at the level c. Then

$$\text{cat}_{E\times M}(K_c) \geq j+1 . \tag{4.17}$$

Postponing the proof we first observe that the Proposition 4.1 follows from the lemma. Indeed, if $k = 1, \ldots, m+1$ then $\text{cat}_{E\times M}(K_{c_k}) \geq 1$ and hence $K_{c_k} \neq \emptyset$. Consequently every c_k is a critical value of f. As a side remark we observe that if at least two of these critical levels are equal, say equal to c, then $\text{cat}_{E\times M} \geq 2$ and consequently, since M and therefore $E \times M$ is arcwise connected, K_c containes infinitely many points.
We shall use the following notation:

Definition *For $s \in \mathbb{R}$ define the sublevel sets*

$$A_s := \{ z \in E \times M \mid f(z) \leq s \} . \tag{4.18}$$

The proof of Lemma 4.2 is based on the following well-known Deformation Theorem.

Proposition 4.2 (Deformation Theorem) *Assume $f \in C^1(E \times M, \mathbb{R})$ and let f satisfy the condition* (PS). *If $c \in \mathbb{R}$, $\bar{\varepsilon} > 0$, and U is any neighborhood of K_c, then there exist $0 < \varepsilon < \bar{\varepsilon}$ and $\eta \in C([0,1] \times E \times M, E \times M)$ such that*

(1) $\eta(0,z) = z$ *for all* $z \in E \times M$.

(2) $\eta(t,z) = z$ *for all* $0 \leq t \leq 1$ *if* $f(z) \notin [c-\bar{\varepsilon}, c+\bar{\varepsilon}]$.

(3) $\eta(t,\cdot)$ *is a homeomorphism of* $E \times M$ *for all* $0 \leq t \leq 1$.

(4) $f(\eta(t,z)) \leq f(z)$ *for all* $0 \leq t \leq 1$ *and* $z \in E \times M$.

(5) $\eta(1, A_{c+\varepsilon} \setminus U) \subset A_{c-\varepsilon}$.

(6) *If* $K_c = \emptyset$ *then* $\eta(1, A_{c+\varepsilon}) \subset A_{c-\varepsilon}$.

The proof of the Deformation Theorem in the case of a Banach space is given in [52]. The case of a Finsler manifold requires only minor modifications; we refer to R. Palais [47], and M. Struwe [56] for the technical details.

Proof of Lemma 4.2 Assume now, by contradiction, that $\text{cat}_{E\times M}(K_c) \leq j$. In view of the defintiton of c we can find for every $\varepsilon > 0$ a set $A \in \Gamma_{k+j}$ such that $A \subset A_{c+\varepsilon}$. Since f satisfies (PS), the set K_c is compact, and consequently there exists a neighborhood N of K_c satisfying $\text{cat}_{E\times M}(N) = \text{cat}_{E\times M}(K_c)$, and moreover $N \subset A$. Observe now that by our assumptions (4.10) , (4.11) on f and by (4.15) we have

$$S \times M \subset A_a \quad \text{and} \quad a < b \leq c\,.$$

Provided that $0 < \bar{\varepsilon} < b - a$ we conclude from the Deformation Theorem that there exist $\varepsilon > 0$ sufficiently small and a homeomorphism $h = \eta(1,\cdot)$ of $E \times M$ which leaves $S \times M$ pointwise fixed, and consequently belongs to $\mathcal{H}$, such that

$$h(A \setminus N) \subset A_{c-\varepsilon}\,.$$

Moreover, by the properties of cat* we have

$$\begin{aligned} \text{cat}^*(\,h(A \setminus N)\,) &= \text{cat}^*(A \setminus N) \\ &\geq \text{cat}^*(A) - \text{cat}_{E\times M}(N) \\ &\geq (k + j) - j \;=\; k\,. \end{aligned}$$

Consequently $h(A \setminus N) \in \Gamma_k$, and we arrive at a contradiction since

$$c = c_k \leq \sup_{z\in h(A\setminus N)} f(z) \leq c - \varepsilon < c\,. \tag{4.19}$$

This proves Lemma 4.2, and the proof of Proposition 4.1 is finished.

∎

5 Galerkin approximation for strongly indefinite functionals

The existence theorem Proposition 4.1 requires for the decomposition $E = X \oplus Y$ that $\dim Y < \infty$. It is therefore not directly applicable to strongly indefinite functionals f, which do not admit such a decomposition of E with finite dimensional Y such that the assumptions of Proposition 4.1 are satisfied. In order to cope with this difficulty we introduce a Galerkin type approximation scheme as follows:

Definition 5.1 *Assume the Banach space E decomposes into closed linear subspaces $E = X \oplus Y$ such that $\dim X = \dim Y = \infty$. We assume moreover that there exists an increasing sequence (Y_k) of finite dimensional linear subspaces of Y, $Y_k \subset Y_{k+1}$, with continuous projections $\Pi_k : Y \to Y_k$, such that*

$$\Pi_k y \to y \quad as \quad k \to \infty \tag{5.1}$$

for every $y \in Y$.
Then we define the linear subspaces

$$E_k := X \oplus Y \subset E \tag{5.2}$$

and the corresponding bounded linear projections

$$P_k : E \to E_k \quad , \quad P_k z = P_k(x, y) := (x, \Pi_k y) \tag{5.3}$$

for every $z = (x, y) \in X \oplus Y = E$.
We shall call (E_k, P_k) an approximation scheme for E.

Note that $P_k \to \mathrm{id}_E$ pointwise, i.e. the projection scheme (E_k, P_k) is projectionally complete. We do not require that the projectors P_k are uniformly bounded in the norm of $\mathcal{L}(E)$. The above definition of an approximation scheme is very special adapted to our purposes.

Now let M be a compact connected smooth manifold without boundary as in the preceding section, and let $f : E \times M \to \mathrm{R}$ be a C^1-function. Define a sequence f_k of C^1-functions

$$f_k : E_k \times M \to \mathbb{R} \tag{5.4}$$

by restriction of f to $E_k \times M$:

$$f_k(z) := f(z) \quad \text{for} \quad z \in E_k \times M \ . \tag{5.5}$$

In order to find critical points of f we require that f and each f_k satisfies the Palais-Smale condition (PS). However, this will not be sufficient, and we introduce an additional compactness condition which is related to the notion of A-properness. In the application of the of the results of the present section to the action functional Φ we actually will show that its gradient Φ' is proper and A-proper with respect to the approximation scheme (E_k, P_k). We also emphasize that there is a relation between the N-families introduced by H. Hofer [34] and the follwing compactness condition :

Definition 5.2 *A C^1-function $f : E \times M \to \mathbb{R}$ is said to satisfy the condition* $(\mathrm{PS})_k$ *with respect to the approximation scheme (E_k, P_k), if every sequence (z_{k_i}) with $z_{k_i} \in E_{k_i} \times M$, satisfying*

$$f_{k_i}(z_{k_i}) \to c \quad \textit{and} \quad df_{k_i}(z_{k_i}) \to \infty \quad \textit{as} \quad i \to \infty$$

for some $c \in \mathbb{R}$, containes a convergent subsequence (z_{k_i}) such that the limit $z := \lim z_{k_i} \in E \times M$ is a critical point of f at the level c :

$$f(z) = c \quad \textit{and} \quad df(z) = 0 \ .$$

After these preparations we now can state the following critical point theorem for strongly indefinite functionals.

Proposition 5.1 *Let M be a compact connected oriented smooth manifold without boundary, and let E be a Banach space which decomposes into closed linear subspaces $E = X \oplus Y$. Let (E_k, P_k) be a approximation scheme for E. Let $f \in C^1(E \times M, \mathbb{R})$ satisfy the compactness conditions* (PS) *and* $(\mathrm{PS})_k$ *with respect to (E_k, P_k) and assume $f_k = f|_{E_k \times M}$ satisfies* (PS) *for every k. Assume there are constants $\alpha < \beta \le \gamma$ and $R > 0$ such that*

$$\begin{aligned} f(z) &\ge \beta \quad \textit{for} \quad z \in X \times M \\ f(z) &\le \alpha \quad \textit{for} \quad z \in S \times M \\ f(z) &\le \gamma \quad \textit{for} \quad z \in D \times M \ , \end{aligned}$$

where $D = \{ z = (0, y) \in X \oplus Y \mid \|z\| \leq R \} \subset \{0\} \oplus Y$ *is the disc of radius* R *in* Y*, and*
$S = \partial D = \{ z = (0, y) \in X \oplus Y \mid \|z\| = R \} \subset \{0\} \oplus Y$ *is the boundary of* D *in* Y*.*
Then f *has at least* $\text{cuplength}(M) + 1$ *critical points.*

Proof
Denote by $D_k := D \cap E_k$, and $S_k := S \cap E_k$. Then by the assumptions for $k = 1, 2, \ldots$ we have

$$f_k(z) \;\geq\; \alpha \quad \text{for} \quad z \in X \times M \tag{5.6}$$

$$f_k(z) \;\leq\; \beta \quad \text{for} \quad z \in S_k \times M \tag{5.7}$$

and each f_k satisfies (PS). Hence $f_k \in C^1(E_k \times M, \mathbb{R})$ satisfies the assumptions of Proposition 4.1.

Replacing E by E_k and f by f_k, the notions $\mathcal{H}_k$, cat_k^*, $\Gamma_j^{(k)}$ are defined as in Section 4. Set $m = \text{cuplength}(M)$. Then we have

$$\Gamma_j^{(k)} \neq \emptyset \quad \text{if} \quad j = 1, \ldots, m+1 \tag{5.8}$$

since $\text{cat}_k^*(D_k \times M) \geq m + 1$ by the Intersection Lemma, and we obtain critical values of f_k by the minimax-principle for every fixed k

$$c_j^{(k)} = \inf_{A \in \Gamma_j^{(k)}} \sup_{z \in A} f_k(z) \quad , \quad j = 1, \ldots, m+1 \,. \tag{5.9}$$

From the assumptions on f we deduce

$$-\infty < \beta \leq c_1^{(k)} \leq \ldots \leq c_{m+1}^{(k)} \leq \gamma < +\infty \,. \tag{5.10}$$

Hence for a fixed $j = 1, \ldots, m+1$ the sequence $(c_j^{(k)})_k$ has a convergent subsequence. So eventually choosing a subsequence we may assume

$$c_j^{(k)} \to c_j \quad \text{as} \quad k \to \infty \,, \tag{5.11}$$

and these numbers c_j satisfy

$$-\infty < \beta \le c_1 \le \ldots \le c_{m+1} \le \gamma < +\infty . \tag{5.12}$$

As a consequence of the condition $(\mathrm{PS})_k$ the c_j for $j = 1, \ldots, m+1$ are critical values of f.

It remains to show that we do not lose the multiplicity in the limit.

Lemma 5.1 *Assume*

$$c = c_j = \ldots = c_{j+r} \tag{5.13}$$

for $j \ge 1$ *and* $j + r \le m + 1$. *Let*

$$K_c = \{ \, z \in E \times M \mid f(z) = c \ \text{ and } \ df(z) = 0 \, \}$$

denote the set of critical points of f *at the level* c. *Then*

$$\mathrm{cat}_{E\times M}(K_c) \ \ge \ r + 1 \, . \tag{5.14}$$

Proof

We claim that for every $\delta' > 0$ there exist positive constants $b > 0$, $\hat{\varepsilon} > 0$ and $k_0 \in \mathbb{N}$ depending on δ' such that

$$\| \, df_k(z_k) \, \| \ge b \quad \text{for } \ k \ge k_0 \, , \ f_k(z_k) \in [c - \hat{\varepsilon}, c + \hat{\varepsilon}] \ \text{ and } \ z_k \notin N_{\delta'}(K_c) \cap (E \times M) \tag{5.15}$$

where $N_{\delta'}(K_c)$ is the δ'-neighborhood of K_c in $E \times M$.

We define $A_s^{(k)} := A_s \cap (E_k \times M)$. Arguing by contradiction we now assume that there is a $\delta' > 0$ for which such numbers do not exist. Consequently we can find sequences $b_k \to 0$, $\varepsilon_k \to 0$ and $z_k \in A_{c+\varepsilon_k}^{(k)} \setminus (A_{c-\varepsilon_k}^{(k)} \cup N_{\delta'})$ such that $\| \, df_k(z_k) \, \| \le b_k$. By the $(\mathrm{PS})_k$-condition we may assume that z_k converges to some $z \in E \times M$ such that $f(z) = c$ and $df(z) = 0$.
On the other hand $z_k \notin N_{\delta'}$ for all k and consequently $\mathrm{dist}(z, K_c) \ge \delta'$, contradicting the definition of K_c.

Since f satisfies (PS) the set K_c is compact, and by the properties of cat there exists a δ-neighborhood $N_\delta = N_\delta(K_c)$ of K_c such that $\mathrm{cat}_{E\times M}(N_\delta) =$

$\mathrm{cat}_{E\times M}(K_c)$. As in the proof of the Deformation Theorem, see P. Rabinowitz [52], p 82, we can choose $\delta' := \delta/8$ in the proof of (5.15) above.
Given $\bar{\varepsilon} > 0$ there exists a constant $\varepsilon \in (0, \bar{\varepsilon})$, depending on b, δ and $\hat{\varepsilon}$, and there exists $h_k \in \mathcal{H}_k$ for any $k \geq k_0$ such that

$$(5.16) \qquad f_k(A^{(k)}_{c+\varepsilon} \setminus (N_\delta \cap (E_k \times M))) \subset A^{(k)}_{c-\varepsilon} .$$

Note that $N_\delta \cap (E \times M)$ is a neighborhood of the set of critical points of f_k at a level $c_k \in [c-\varepsilon, c+\varepsilon]$. In particular ε is independent of k.
Since $c_i^{(k)} \to c_i$ as $k \to \infty$ for each i there exists an integer k_1 such that for $k \geq k_1$:

$$(5.17) \qquad c-\varepsilon \leq c_j^{(k)} \leq \ldots \leq c_{j+r}^{(k)} \leq c+\varepsilon .$$

Consequently for $k \geq \max\{k_0, k_1\}$ we conclude as in the proof of Lemma 4.2 that

$$(5.18) \qquad \mathrm{cat}_{E_k \times M}(\, N_\delta \cap (E_k \times M)\,) \geq r+1 .$$

We have a retraction $E \times M \to E_k \times M$, given by $P_k \times \mathrm{id}_M$. It therefore follows from Lemma 3.3 that

$$(5.19) \qquad \mathrm{cat}_{E\times M}(\, N_\delta \cap (E_k \times M)\,) = \mathrm{cat}_{E_k\times M}(\, N_\delta \cap (E_k \times M)\,) .$$

By the monotonicity of cat we have

$$\mathrm{cat}_{E\times M}(K_c) = \mathrm{cat}_{E\times M}(N_\delta) \geq \mathrm{cat}_{E\times M}(\, N_\delta \cap (E_k \times M)) \geq r+1 ,$$

thus proving Lemma 5.1 and Proposition 5.1.

∎

6 A first proof of Theorem 1

We are ready to prove Theorem 1: we shall verify that the action functional $\Phi : T^n \times E \to R$ introduced in Section 2 satisfies the hypotheses of Proposition 5.1 of the previous section which then guarantees the claimed number of critical points.

First we recall that on the Hilbert space $W = W^{\frac{1}{2},2}(S^1, R^{2n})$ the functional Φ is defined by

$$\Phi(u) = \frac{1}{2}(\, Tu\,,\, u\,) - \hat{\varphi}(u) + (\,v\,,\, u\,)$$

where $v = v_j - Tw$ is defined in (2.20). v_j contains the rotation vector j :

$$v_j \;=\; (0, j) \in R^n \times R^n$$

with $j \in Z^n$ fixed. Recall the definition of $\hat{\varphi}$ given in (2.24) :

$$\hat{\varphi}(u) = \int_0^1 \left\{ H(u + e, t) - \frac{1}{2}\langle \hat{Q}u\,,\, u\rangle \right\} dt$$

for $u \in W$, with $e(t) = (jt, 0) \in R^n \times R^n$.

Moreover $H(z,t) = H(x,y,t)$, with $z = (x,y) \in R^{2n}$, is periodic in $x \in R^n$ and periodic in $t \in R$. In addition $H \in C^1(R^{2n} \times R, R)$ is assumed to satisfy the following asymptotic condition as $|y| \to \infty$:

$$\frac{1}{|y|}\, |\partial_y H(x,y,t) - A(t)y| \;\to\; 0$$

$$\frac{1}{|y|}\, |\partial_x H(x,y,t)| \;\to\; 0$$

uniformly in x and t.

Recall moreover the orthogonal splitting

$$W = E^+ \oplus E^- \oplus E^0$$

where E^0 consists of the special constant loops

$$E^0 = \ker(T) = \{ (x, 0) \mid x \in \mathbb{R}^n \} .$$

Correspondingly the bounded and selfadjoint operator T splits; if

$$u = u^+ + u^- + u^0$$

then

$$Tu = T^+u^+ + T^-u^- \tag{6.1}$$

and there exists $\lambda > 0$ such that

$$\Big(T^+u^+ \, , \, u^+ \Big) \;\geq\; \lambda \|u^+\|^2 \tag{6.2}$$

$$\Big(T^-u^- \, , \, u^- \Big) \;\leq\; -\lambda \|u^-\|^2 . \tag{6.3}$$

From the asymptotic conditions on H we conclude

Lemma 6.1 *For every $\varepsilon > 0$ there exist constants $c_1 = c_1(\varepsilon)$ and $c_2 = c_2(\varepsilon)$ such that*

$$\left| \, H(x, y, t) - \frac{1}{2} \langle \hat{Q} z \, , \, z \rangle \, \right| \;\leq\; \varepsilon |y|^2 + c_1 |y| + c_2$$

for every $z = (x, y)$ and t.

Proof

Since $H(x + j, y, t) = H(x, y, t)$, $j \in \mathbb{Z}^n$, for every $x \in \mathbb{R}^n$ we can choose $\xi = \xi(x)$ in the unit cube $[0, 1]^n = \{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n \mid 0 \leq x \leq 1 \}$, such that $|\xi| \leq \sqrt{n}$, satisfying $H(x, y, t) = H(\xi(x), y, t)$. Therefore, by the Taylor formula :

$$H(z, t) - \frac{1}{2} \langle \hat{Q} z, z \rangle = H(0, t) + \langle \nabla H(\xi, t) - (\hat{Q} \bar{z})(t), z \rangle$$

for some $\bar{z} = \mu z$ with $0 < \mu < 1$.

Hence

$$H(x,y,t) - \frac{1}{2}\langle \hat{Q}z, z\rangle = H(\xi,y,t) - \frac{1}{2}\langle A(t)y\,,\,y\rangle$$
$$= H(0,t) + \langle \frac{\partial}{\partial x}H(\bar{\xi},\bar{\eta},t)\,,\,\xi\,\rangle + \langle \frac{\partial}{\partial y}H(\bar{\xi},\bar{\eta},t) - A(t)\bar{\eta}\,,\,y\rangle$$

where $(\bar{\xi},\bar{\eta}) = \mu(\xi,y)$ for some μ with $0 < \mu < 1$, and the lemma follows immediately from the asymptotic conditions.

∎

Lemma 6.2 *Φ is bounded below on $E^+ \oplus E^0$, i.e. there is a constant c such that*

$$\Phi(u) \geq c \quad \text{if} \quad u = u^+ + u^0\,.$$

Proof
Observe that $W \to L^2(S^1, R^{2n})$ is a continuous embedding, and by the definition of the L^2- and the Sobolev-norm given in Section 2 we have $\|u\|_2 \leq \|u\|$ for every $u \in W$. In view of the properties (6.1) , (6.2) of T and Lemma 6.1 we find for $u = u^+ + u^0$

$$\begin{aligned}
\Phi(u) &\geq \lambda\|u^+\|^2 - |\hat{\varphi}(u)| - |(\,v\,,\,u\,)| \\
&\geq \lambda\|u^+\|^2 - \int_0^1 \left|H(u+e,t) - \frac{1}{2}\langle\,\hat{Q}u\,,\,u\,\rangle\right| dt - \|v\|\;\|u^+\| \\
&\geq \lambda\|u^+\|^2 - c_1 - c_2\|u^+\|_2 - \varepsilon\|u^+\|^2 - \|v\|\;\|u^+\| \\
&\geq (\lambda - \varepsilon)\,\|u^+\|^2 - c_3\|u^+\| - c_1\,.
\end{aligned}$$

Choosing $\varepsilon > 0$ sufficiently small such that $\varepsilon \leq \lambda$, the lemma follows.

∎

Lemma 6.3 *Φ is bounded from above on $E^- \oplus E^0$. Moreover there exists $R^* > 0$ such that for every $R \geq R^*$*

$$\sup_{u \in S \times E^0} \Phi(u) < \inf_{u \in E^+ \times E^0} \Phi(u) \tag{6.4}$$

where S is the sphere of radius R in E^- :

$$S = \left\{ u^- \in E^- \mid \|u^-\| = R \right\} .$$

Proof

Assume $u = u^- + u^0 \in E^- \oplus E^0$. Choosing $0 < \varepsilon \leq \lambda/2$ we find similarly as in the proof of Lemma 6.2 the estimate :

$$\Phi(u) \leq -\frac{\lambda}{2}\|u^-\|^2 + c_3\|u^-\| + c_1$$

from which the first statement follows.
Clearly $\Phi(u) \to -\infty$ as $\|u^-\| \to \infty$ uniformly on E^0, and the second statement follows, since, by Lemma 6.2, the function Φ is bounded from below on $E^+ \oplus E^0$.

∎

Summarizing we have proved

Lemma 6.4 *There exist constants $\alpha < \beta \leq \gamma$ and $R > 0$ such that*

$$\begin{aligned} \Phi(u) &\geq \beta \quad \text{if} \quad u \in E^+ \oplus E^0 \\ \Phi(u) &\leq \alpha \quad \text{if} \quad u \in S \times E^0 \\ \Phi(u) &\leq \gamma \quad \text{if} \quad u \in D \times E^0 \end{aligned}$$

where $D = \{ u^- \in E^- \mid \|u^-\| \leq R \}$ and $S = \partial D \subset E^-$.

It remains to show that the function Φ satisfies the desired compactness conditions. We shall first prove that $\Phi' : E \times T^n \to E \times T^n$ is a proper map, i.e. the preimage of a compact set is compact. It should be emphasized that, of course, this compactness property does not hold true for Φ' considered as a function on $W = E \oplus E^0$. However, since $E^0/\mathbb{Z}^n = \mathbb{T}^n$ all we have to verify is the following

Lemma 6.5 *Every sequence $u_k \in W = E \oplus E^0$ satisfying*

$$\Phi'(u_k) \to u^* \quad as \quad k \to \infty$$

for some $u^* \in W$ containes a subsequence which is convergent in $E \times E^0/\mathbb{Z}^n$.

Proof
Assume the sequence

$$u_k \;=\; u_k^+ + u_k^- + u_k^0 \;\in\; E^+ \oplus E^- \oplus E^0$$

satisfies

$$(6.5) \qquad \Phi'(u_k) \;=\; Tu_k - \hat{\varphi}(u_k) + v \;\to\; u^* \quad \text{in } W \quad \text{as} \quad k \to \infty\,.$$

Since $\Phi'(u_k + g) = \Phi'(u_k)$ for every $g \in \mathbb{Z}^n \subset E^0$, we can and do assume that the sequence u_k^0 is contained in the n-dimensional unit cube $[0,1]^n \subset E^0$. Hence in particular u_k^0 is bounded.

Assume now that the sequence u_k is bounded in W. In view of the compactness of $\hat{\varphi}' : W \to W$ we can choose a subsequence u_{k_i} such that $\hat{\varphi}'(u_{k_i}) \to w \in W$ along this subsequence. Consequently

$$Tu_{k_i} \;=\; T(u_{k_i}^+ + u_{k_i}^-) \;\to\; u^* + w - v \;\in\; W \quad \text{as} \quad i \to \infty\,.$$

Recall from Section 2 that T is Fredholm with $\operatorname{index} T = 0$ and $\ker(T) = \operatorname{ran}(T)^\perp = E^0$. In particular the range of T is closed and it follows $u^* + w - v \in \operatorname{ran}(T)$. Moreover, $T|_{E^+ \oplus E^-}$ is a linear isomorphism of $E^+ \oplus E^-$. Thus we have

$$u_{k_i}^+ + u_{k_i}^- \;\to\; T^{-1}(u^* + w - v) \;\in\; E^+ \oplus E^- \quad \text{as} \quad i \to \infty$$

and consequently $u_k^+ \to u^+ \in E^+$ and $u_k^- \to u^- \in E^-$.

Since u_k^0 is bounded we can assume $u_k^0 \to u^0 \in E^0$ along a subsequence, and so we finally have singled out a convergent subsequence of the bounded sequence u_k.

To finish the proof of the lemma it therefore remains to show that every sequence satisfying (6.5) is bounded in W. We argue by contradiction and assume that there is a sequence $u_k \in W$ which satisfies (6.5) and $\|u_k\| \to \infty$. We therefore may assume $u_k \neq 0$ for all k and define

$$v_k := \frac{u_k}{\|u_k\|} .$$

From (6.5) we conclude that

$$Tv_k - \frac{1}{\|u_k\|}\hat{\varphi}'(u_k) \to 0 \quad \text{as} \quad k \to \infty . \tag{6.6}$$

Since $\|v_k\| = 1$ for all subscripts k and since T is a bounded operator, the sequence $w_k \in W$ defined by

$$w_k := \frac{1}{\|u_k\|}\hat{\varphi}'(u_k)$$

is bounded in W.

We claim that

$$\frac{1}{\|u_k\|}\hat{\varphi}'(u_k) \to 0 \quad \text{as} \quad k \to \infty . \tag{6.7}$$

It then follows that $v_k \to 0$ in W from (6.6) together with the fact that $v_k^0 = \|u_k\|^{-1}u_k^0 \to 0$, thus contradicting $\|v_k\| \equiv 1$.
In order to prove (6.7) recall

$$\begin{aligned} \left\| \frac{\hat{\varphi}'(u_k)}{\|u_k\|} \right\|^2 &= \left(\frac{\hat{\varphi}'(u_k)}{\|u_k\|} \,,\, w_k \right) \\ &= \frac{1}{\|u_k\|} \left| \int_0^1 \langle \nabla H(u_k + e, t) - \hat{Q}u_k \,,\, w_k \rangle dt \right| . \end{aligned}$$

The right hand side is less or equal

$$
\begin{aligned}
& \frac{1}{\|u_k\|}\int_0^1 |\partial_x H(u_k+e,t)|\,|w_k(t)|\,dt \\
(6.8) \qquad & +\frac{1}{\|u_k\|}\int_0^1 |\partial_y H(u_k+e,t)-A(t)y_k(t)|\,|w_k(t)|\,dt \\
& =: I_k^{(1)}+I_k^{(2)}
\end{aligned}
$$

where $u_k(t)=(x_k(t),y_k(t))\in \mathbb{R}^n\times\mathbb{R}^n$.

We estimate $I_k^{(1)}$. Let $\varepsilon>0$ be given. Then from the asymptotic estimates of the Hamiltonian we obtain constants $M=M(\varepsilon)$ and $C=C(\varepsilon)$ such that

$$
\begin{aligned}
|\,\partial_x H(x_k(t)+e(t),y_k(t),t)\,| &\leq \varepsilon|y_k(t)| \quad \text{if} \quad |y_k(t)|\geq M \\
|\,\partial_x H(x_k(t)+e(t),y_k(t),t)\,| &\leq C \qquad\quad \text{if} \quad |y_k(t)|\leq M
\end{aligned}
$$

Since by assumption $\|u_k\|\to\infty$ we can choose $k^*=k^*(\varepsilon)$ such that for all $k\geq k^*$

$$\|u_k\|\geq\frac{C}{\varepsilon}$$

and we define for $k\geq k^*$ the subsets of $[0,1]$

$$
\begin{aligned}
\Omega_{1,k} &:= \{\, t\in[0,1] \mid |y_k(t)|\geq M \,\} \\
\Omega_{2,k} &:= [0,1]\setminus\Omega_{1,k}\,.
\end{aligned}
$$

Then

$$
\begin{aligned}
I_k^{(1)} &= \frac{1}{\|u_k\|}\int_{\Omega_{1,k}} |\,\partial_x H(x_k+e,y_k,t)\,|\,|\,w_k(t)\,|\,dt \\
&+ \frac{1}{\|u_k\|}\int_{\Omega_{2,k}} |\,\partial_x H(x_k+e,y_k,t)\,|\,|\,w_k(t)\,|\,dt
\end{aligned}
$$

$$
\begin{aligned}
&\le \frac{\varepsilon}{\|u_k\|}\int_0^1 |y_k|\,|w_k|\,dt + \frac{C}{\|u_k\|}\int_0^1 |w_k|\,dt \\
&\le \left(\varepsilon\frac{\|u_k\|_2}{\|u_k\|} + \frac{C}{\|u_k\|}\right)\|w_k\|_2 \\
&\le 2\varepsilon\,\|w_k\| = 2\varepsilon\left\|\frac{\hat{\varphi}'(u_k)}{\|u_k\|}\right\| .
\end{aligned}
$$

Similarly we obtain

$$
I_k^{(2)} \le 2\varepsilon\left\|\frac{\hat{\varphi}'(u_k)}{\|u_k\|}\right\| .
$$

This proves the claim (6.7), and therefore the proof of the lemma is finished. ∎

We now have to introduce the Galerkin type approximation scheme for the present situation :
Recall that $W = E^+ \oplus E^- \oplus E^0$ and that $Tu = Tu^+ + Tu^- \in E^+ \oplus E^-$ if $u = u^+ + u^-$. Take an orthonormal basis $\{u_i\}$ of E^- consisting of eigenvectors of the operator T and define

$$
\begin{aligned}
E_k^- &:= \operatorname{span}\{\, u_i \mid i = 1, \ldots, k \,\} \\
W_k &:= E^+ \oplus E_k^- \oplus E^0
\end{aligned}
$$

By P_k we denote the corresponding orthogonal projectors

$$
P_k : W \to W_k .
$$

Then $\|P_k\| = 1$ and $P_k u \to u$ as $k \to \infty$ for every $u \in W$. The corresponding restricted functionals are defined by

$$\Phi_k : W_k \to \mathbb{R}$$

$$\Phi_k(u) = \Phi(u) \quad \text{if} \quad u \in W_k \ .$$

The gradient of Φ_k is given by

$$\Phi_k'(u) = P_k\Phi'(u) = Tu - P_k\hat{\varphi}'(u) + v \quad \text{for} \quad u \in W_k \ .$$

Lemma 6.6 *Every sequence $u_{k_i} \in W_{k_i}$ satisfying*

$$\Phi_{k_i}'(u_{k_i}) \to u^*$$

for some $u^* \in W$ as $i \to \infty$ contains a subsequence which is convergent in $E \times E^0/\mathbb{Z}^n$.
Moreover if $u_{k_i} \in W_{k_i}$ satisfies $\Phi_{k_i}'(u_{k_i}) \to u^* \in W$ and $u_{k_i} \to u \in W$ as $i \to \infty$, then $\Phi'(u) = u^*$.

Proof
The proof of the first statement is almost literally the same as the proof of Lemma 6.5.

Suppose that $u_{k_i} \in W_{k_i}$ is a bounded in W and satisfies $\Phi_{k_i}'(u_{k_i}) \to u^*$, i.e.

$$Tu_{k_i} - P_{k_i}\hat{\varphi}'(u_{k_i}) + v \to u^* \quad \text{as} \quad i \to \infty \tag{6.9}$$

Since $\hat{\varphi}'$ is a compact map we may assume that $\hat{\varphi}'(u_{k_I}) \to w \in W$. Then

$$\begin{aligned} \| P_{k_i}\hat{\varphi}'(u_{k_i}) - w \| &\leq \| P_{k_i}\hat{\varphi}'(u_{k_i}) - P_{k_i}w \| + \| P_{k_i}w - w \| \\ &\leq \| P_{k_i} \| \, \| \hat{\varphi}'(u_{k_i}) - w \| + \| P_{k_i}w - w \| \end{aligned}$$

and the right hand side tends to zero as $i \to \infty$. As in the proof of Lemma 6.5 we now conclude that the bounded sequence u_{k_i} containes a convergent subsequence.

The boundedness of a sequence u_{k_i} which satisfies the hypothesis of the lemma is obtained the same way as it was done in the proof of Lemma 6.5, we only have to replace $\|u_k\|^{-1}\hat{\varphi}'(u_k)$ by $\|u_{k_i}\|^{-1}P_k\hat{\varphi}'(u_{k_i})$.

It remaines to prove the second statement of Lemma 6.6. Let $w \in W$ be arbitrarily chosen. Assume $u_{k_i} \to u \in W$, $u_{k_i} \in W_k$, $\Phi'_{k_i}(u_{k_i}) \to u^*$. Then we have

$$\begin{aligned}(\Phi'(u), w) &= \lim_{i\to\infty} (\Phi'(u_{k_i}), w) \\ &= \lim_{i\to\infty} \{(\Phi'(u_{k_i}), w - P_{k_i}w) + (P_{k_i}\Phi'(u_{k_i}), P_{k_i}w)\}\end{aligned}$$

Since

$$|(\Phi'(u_{k_i}), w - P_{k_i}w)| \leq \|\Phi'(u_{k_i})\| \, \|w - P_{k_i}w\| \to 0 \quad \text{as} \quad i \to \infty$$

and

$$\lim_{i\to\infty}(P_{k_i}\Phi'(u_{k_i}), P_{k_i}w) = \lim_{i\to\infty}(\Phi'_{k_i}(u_{k_i}), w) = (u^*, w)$$

we consequently have

$$(\Phi'(u), w) = (u^*, w) \quad \text{for all} \quad w \in W,$$

and therefore $\Phi'(u) = u^*$, as claimed. This finishes the proof of Lemma 6.6.

■

Lemma 6.7 *For every k the functional $\Phi'_k : (E^+ \oplus E^-_k) \times E^0/Z^n$ is proper.*

Proof

Fix a positive $k_0 \in Z$. Then the proof of Lemma 6.7 is the same as the proof of Lemma 6.5 with the sequence $u_k \in W_{k_0}$ and $\|u_k\|^{-1}\hat{\varphi}'(u_k)$ replaced by $\|u_k\|^{-1}P_{k_0}\hat{\varphi}'(u_k)$.

■

Proof of Theorem 1

From Section 2 we know the special solutions of the Hamiltonian equation we are looking for are precisely the critical points of the action functional $\Phi : W \to \mathbf{R}$ on the Hilbert space $W = E^+ \oplus E^- \oplus E^0$. We therefore can write $\Phi(u) = \Phi(u^+, u^-, u^0)$, and in view of the periodicity of the Hamiltonian H in the x-variable we know that $\Phi(u^+, u^-, u^0) = \Phi(u^+, u^-, u^0 + g)$ for every $g \in \mathbf{Z}^n$. Therefore Φ is a function

$$\Phi : E \times \mathrm{T}^n \to \mathbf{R}$$

with $E = E^+ \oplus E^-$ and $\mathrm{T}^n = E^0/\mathbf{Z}^n$. In view of the Lemmata 6.4, 6.5, 6.6, and 6.7 this function satisfies all the assumptions of Proposition 5.2 with the manifold M being the torus T^n. Since $\text{cuplength}(\mathrm{T}^n) = n$ we conclude that Φ possesses at least $n + 1$ critical points, and the proof of Theorem 1 is finished.

7 The nondegenerate case

We have used a variant of Lyusternik-Schnirelman theory in the preceding sections to prove existence and multiplicity of critical points of the action functional Φ . From the hypothesis in Theorem 2 we obtain a more detailed information about the the critical points of Φ , which, by use of the generalized Morse theory developed by C. Conley and E. Zehnder, makes it possible to estimate the number of critical points of Φ by the sum of the Betti numbers of $\mathbf{T}^n$. Of course, this requires a reduction to finite dimension, and this will be achieved using a Galerkin approximation.

We briefly recall the classical Morse theory. Consider a smooth compact closed manifold M and a smooth function $f : M \to \mathbf{R}$ such that all critical points of f are nondegenerate. Such a function f is called a Morse function. Recall that a critical point x of f is called nondegenerate if the Hessian form $d^2f(x)$ is a nondegenerate bilinear form on the tangent space T_xM. For a nondegenerate critical point x the Morse index $m(x)$ is defined to be the dimension of the maximal linear subspace V of T_xM such that $d^2f(x)$ is negative definite on V. Hence $m(x)$ is an integer $0 \leq m(x) \leq \dim M$. The local structure of the gradient flow

$$\frac{d}{dt}\phi^t = \nabla f \circ \phi^t$$

on M near a nondegenerate critical point x is completely determined by the Morse index $m(x)$, in particular one concludes that the nondegenerate critical points of f are isolated. Consequently, since M is compact, a Morse function f possesses only finitely many critical points.
If a_k denotes the number of critical points of f having Morse index $m(x) = k$, and if b_k is the k-th Betti number of M, then the Morse inequalities give an estimate

$$a_k \geq b_k \ , \ \ k = 0, \dots, \dim M \ ,$$

and consequently the total number of critical points of f has to be greater or equal to the sum of the Betti numbers of M.

Conversely, the homotopy type of a compact manifold M as a CW-complex is completely determined by the Morse indices $m(x)$ of the critical points x of a Morse function $f : M \to \mathbb{R}$, since passing a critical level from below means one has to attach cells. To be precise, let $A_s = \{x \in M | f(x) \leq s\}$, $s \in \mathbb{R}$, and let c be a critical level for f. Let $x_1, \ldots, x_n$ be the critical points of f at the level c with Morse indices $m(x_1), \ldots, m(x_n)$. Then, for $\varepsilon > 0$ sufficiently small, $A_{c+\varepsilon}$ is homotopically equivalent to $A_{c-\varepsilon}$ with n cells $e^{m(x_1)}, \ldots, e^{m(x_n)}$ disjointly attached to its boundary. Here e^k denotes the k-dimensional standard cell. We refer to J. Milnor [45] for a detailed survey. The classical Morse theory has been generalized by R. Palais and S. Smale for functions $f : M \to \mathbb{R}$ where M is a Hilbert manifold and f is bounded from below, see e.g. R. Palais [46]. However, this approach is not appropriate to prove the existence of critical points of strongly indefinite Morse functions. Considering the example of the action functional $\Phi : \mathbb{T}^n \times E \to \mathbb{R}$, we point out the following observations.

First, the Morse index of each critical point Φ is infinite. Therefore, passing a critical level means one has to attach infinite dimensional cells, which are homotopically invisible. This difficulty of infinite Morse indices is overcome by means of a Galerkin approximation which reduces the problem to a finite dimensional one, as will be seen below.

Secondly, the function Φ and also the functions Φ_k used in the finite dimensional reduction are unbounded from below and above. Even if we assume the functions under consideration to be at least C^3, the Morse inequalities proved by R. Palais [46], Theorem 7, are not sufficient to obtain the required estimate for the total number of critical points of Φ.

This difficulty will not arise in the Conley - Zehnder Morse theory for flows as in [15]. The topological obstructions for the gradient flow of Φ_k on $E_k \times \mathbb{T}^n$ are reflected in the structure of the invariant set S_k, which consists of the critical points of Φ_k together with their connecting orbits. It will be shown that for k sufficiently large the critical points of Φ_k are all nondegenerate and correspond uniquely to the critical points of Φ. The properties of the gradient flow of Φ_k then will be used to prove the existence of a very special isolating block for S_k, which allows to relate the Conley index of S_k to the Betti numbers of $\mathbb{T}^n$. Via the Morse equation, this relation finally yields the desired estimate for the number of critical points.

In the following we assume that the Hamiltonian H is in C^2. This is quite in contrast to the previous sections, where only $H \in C^1$ was required. Moreover, we still assume the asymptotic estimates, which we recall:

$$\frac{1}{|y|}|\,\partial_y H(x,y,t) - A(t)y\,| \to 0 \quad \text{and} \quad \frac{1}{|y|}|\,\partial_x H(x,y,t)\,| \to 0$$

uniformly in x and t as $|y| \to \infty$.

We start with defining the condition of nondegeneracy for periodic solutions of the Hamiltonian equation. For this purpose some basic facts on ordinary differential equations are needed, which can be found e.g. in Amann [1].

Fix a rotation vector $j \in \mathbb{Z}^n$ and set

$$(7.1) \qquad v_j = -J\frac{d}{dt}\,e(t) = (0,j) \in \mathbb{R}^n \times \mathbb{R}^n$$

where $e(t) = (jt, 0) \in \mathbb{R}^n \times \mathbb{R}^n$.

A 1-periodic solution $\hat{u}$ with rotation vector j of

$$(7.2) \qquad \frac{d}{dt}\,\hat{u} = J\,[\,\nabla H(\hat{u},t) + f(t)\,] \quad \text{for} \quad \hat{u} \in \mathbb{T}^n \times \mathbb{R}^n\,,$$

where $f(t)$ satisfies (1.15) , (1.16), is represented in the covering space $\mathbb{R}^{2n}$ by $\hat{u}(t) = u(t) + e(t)$. Then u is a 1-periodic solution of

$$(7.3) \qquad \frac{d}{dt}\,u = J\nabla H(u{+}e,t) + J\,\{f(t){-}v_j\} \quad , \quad u(t) \in \mathbb{R}^{2n}\,.$$

By $\alpha^t(z)$ we denote the flow of the vector field in (7.3):

$$(7.4) \qquad \begin{cases} \dfrac{d}{dt}\alpha^t(z) &= J\,\nabla H(\alpha^t(z){+}e(t),t) + J\{f(t){-}v_j\} \\ \alpha^0(z) &= z \end{cases}$$

for $z \in \mathbb{R}^{2n}$.

Since the Hamiltonian $H(z,t)$ is twice differentiable with respect to z, the flow $\alpha^t(z)$ is differentiable with respect to z, and the derivative $d\alpha^t(z)$ satisfies the linearized equation

$$(7.5) \qquad \begin{cases} \dfrac{d}{dt}\,d\alpha^t(z) &= J\,d^2H(\alpha^t(z){+}e(t),t)\;d\alpha^t(z) \\ d\alpha^0(z) &= id_{\mathbb{R}^{2n}} \end{cases}$$

along the solution $\alpha^t(z)$. Here d denotes the derivative with respect to z. Suppose now that $\alpha^t(z)$ is a 1-periodic solution of (7.4), i.e. $\alpha^1(z) = \alpha^0(z) = z$. Then the Hessian $d^2H(\alpha^t(z) + e(t), t)$ is periodic in t with period 1, which follows from the periodicity

$$H(x+j, y, t+1) = H(x, y, t)$$

of the Hamiltonian, and

$$U(t) := d\alpha^t(z) \tag{7.6}$$

is a fundamental system of the periodically t-dependent linear equation

$$\begin{cases} \frac{d}{dt}U(t) &= J\, d^2H(\alpha^t(z)+e(t), t)\, U(t) \\ U(0) &= id_{\mathbb{R}^{2n}} \end{cases} \tag{7.7}$$

The eigenvalues of $U(1)$ are called the Floquet-multipliers of the 1-periodic solution $\alpha^t(z)$ of the Hamiltonian equation (7.3).

Definition 7.1 *A 1-periodic solution $u(t) = \alpha^t(z)$ of the Hamiltonian equation (7.3) is called nondegenerate if it has no Floquet-multiplier equal to 1.*

Lemma 7.1 *The linearized equation*

$$\frac{d}{dt}u = J\, d^2H(\alpha^t(z)+e(t), t)\, u \tag{7.8}$$

has a nontrivial 1-periodic solution if and only if 1 is a Floquet-multiplier of $\alpha^t(z)$.

Proof

If $u(t)$ is a nontrivial 1-periodic solution, then there exists $t_0 \in \mathbb{R}$ such that $u(t_0) = z_0 \neq 0$, and therefore $u(t + t_0) = U(t)z_0$. Hence

$$U(1)z_0 = U(0)z_0 = z_0$$

and consequently 1 is an eigenvalue of $U(1)$.

Conversely, if 1 is an eigenvalue of $U(1)$ and $z_0 \neq 0$ is a corresponding eigenvector, then $U(t)z_0$ is a nontrivial 1-periodic solution of the linearized equation.

∎

Recall from Section 2 the action functional Φ on $W = W^{\frac{1}{2},2}(S^1, \mathbb{R}^{2n})$

$$\begin{aligned}\Phi(u) &= \frac{1}{2}(Tu, u) - \hat{\varphi}(u) + (v, u) \\ &= \frac{1}{2}((P^+ - P^-)u, u) - \varphi(u) + (v, u).\end{aligned}$$

We shall assume now that $H \in C^2$ and

$$|d^2H(x,y,t)| \leq a_1 + a_2|y|^r \tag{7.9}$$

for some constants $a_1, a_2 \in \mathbb{R}$ and $1 \leq r < \infty$, uniformly in x and t. Consequently, $\varphi : W \to \mathbb{R}$ defined by

$$\varphi(u) = \int_0^1 H(u+e,t)\, dt \tag{7.10}$$

is in C^2, and moreover

$$d^2\varphi(u)(\xi,\eta) = \int_0^1 \langle d^2H(u+e,t)\xi(t), \eta(t)\rangle\, dt \tag{7.11}$$

for $\xi, \eta \in W$.
$d^2H(\cdot)$ is a symmetric matrix, so that the Hessian form $d^2\varphi(u)$ is symmetric. Moreover, with $u(t) = (x(t), y(t)) \in \mathbb{R}^n \times \mathbb{R}^n$,

$$|d^2\varphi(u)(\xi,\eta)| \leq \int_0^1 |d^2H(u+e,t)|\,|\xi|\,|\eta|\, dt \tag{7.12}$$

$$\leq a_1\int_0^1 |\xi|\,|\eta|\, dt + a_2\int_0^1 |y|^r\,|\xi|\,|\eta|\, dt. \tag{7.13}$$

We apply the generalized Hölder inequality with $p_1 = (r+1)/r$ and $p_2 = p_3 = 2r+2$, to estimate the second integral. Then $p_1^{-1} + p_2^{-1} + p_3^{-1} = 1$, and

$$\int_0^1 |y|^r\,|\xi|\,|\eta|\,dt \;\leq\; \left(\int_0^1 |y|^{r+1}\,dt\right)^{r/(r+1)} \|\xi\|_{2r+2}\,\|\eta\|_{2r+2}$$

$$= \|y\|_{r+1}^r\,\|\xi\|_{2r+2}\,\|\eta\|_{2r+2}$$

where $\|\cdot\|_q$ denotes the L^q-norm.
We have compact embeddings

$$W = W^{\frac{1}{2},2}(S^1, R^{2n}) \to L^q(S^1, R^{2n}) \;,\; 1 \leq q < \infty\,. \tag{7.14}$$

In particular there exist constants c_q such that for every $\xi \in W$

$$\|\xi\|_q \;\leq\; c_q\,\|\xi\| \;,\; 1 \leq q < \infty\,. \tag{7.15}$$

Finally applying the Cauchy-Schwarz inequality to estimate the first integral on the right hand side of (7.13), we therefore conclude that there exists a constant $c = c(u)$ depending only on u and the constants a_1 , a_2 and r given in (7.9), such that

$$|\,d^2\varphi(u)\,(\xi,\eta)\,| \;\leq\; c(u)\,\|\xi\|\,\|\eta\|\,. \tag{7.16}$$

Therefore the Hessian form $d^2\varphi(u)$ is a bounded symmetric bilinear form on W, and by Riesz' theorem there exists a uniquely determined symmetric bounded linear operator $\varphi''(u) \in \mathcal{L}(W)$ defined by

$$(\,\varphi''(u)\xi\,,\,\eta\,) \;:=\; d^2\varphi(u)\,(\xi,\eta) \quad , \quad \xi,\eta \in W\,. \tag{7.17}$$

Lemma 7.2 *The linear operator $\varphi''(u) \in \mathcal{L}(W)$ defined in* (7.17) *is compact for every $u \in W$.*

Proof
By means of (7.16) and (7.17) we have for $\xi \in W$

$$\|\,\varphi''(u)\xi\,\|^2 \;=\; (\,\varphi''(u)\xi\,,\,\varphi''(u)\xi\,) \;=\; |\,d^2\varphi(u)\,(\xi,\varphi''(u)\xi)\,|$$

$$
\begin{aligned}
&\leq \; a_1 \,\|\xi\|_2 \,\| \varphi''(u)\xi \,\|_2 \;+\; a_2 \,\|y\|_{r+1}^r \,\|\xi\|_{2r+2} \,\| \varphi''(u)\xi \,\|_{2r+2} \\
&\leq \; \{a_1 + a_2 \,\|y\|_{r+1}^r\} \,\|\xi\|_{2r+2} \,\| \varphi''(u)\xi \,\|_{2r+2} \\
&=: \; C'(u) \,\|\xi\|_{2r+2} \,\| \varphi''(u)\xi \,\|_{2r+2}
\end{aligned}
$$

where $u(t) = (x(t), y(t)) \in \mathrm{R}^n \times \mathrm{R}^n$. Note that we have used $\|\xi\|_2 \leq \|\xi\|_{2r+2}$. By use of (7.15) we have with $C(u) := c_{2r+2}\, C'(u)$

$$
\| \varphi''(u)\xi \,\| \;\leq\; C(u)\,\|\xi\|_{2r+2} \tag{7.18}
$$

for all $\xi \in W$.
Since the embedding $W \to L^{2r+2}(\mathrm{S}^1, \mathrm{R}^{2n})$ is compact, the lemma follows.

■

Recall the definition of $\hat{\varphi}$ from (2.24) :

$$
\hat{\varphi} \;=\; \varphi(u) - \frac{1}{2}(\,Ku\,,\,u\,) \tag{7.19}
$$

where $K \in \mathcal{L}(W)$ is compact. From the above considerations concerning φ we deduce that $\hat{\varphi} \in C^2(W, \mathrm{R})$ and $\hat{\varphi}''(u) = \varphi''(u) - K$. Moreover, $\hat{\varphi}''(u) \in \mathcal{L}(W)$ is a compact linear operator for every $u \in W$.
Consequently the action functional Φ is in $C^2(W, \mathrm{R})$, and its Hessian form is given by

$$
d^2\Phi(u)\,(\xi,\eta) \;=\; (\,T\xi\,,\,\eta\,) \;-\; d^2\hat{\varphi}(u)\,(\xi,\eta) \tag{7.20}
$$

$$
\;=\; (\,[T - \hat{\varphi}''(u)]\,\xi\,,\,\eta\,)\,. \tag{7.21}
$$

For $u \in W$ we define the bounded linear operator $\Phi''(u)$ by

$$
\Phi''(u) := T - \hat{\varphi}''(u) \;\in\; \mathcal{L}(W)\,. \tag{7.22}
$$

Definition 7.2 *A critical point u of Φ is called nondegenerate if the Hessian form $d^2\Phi(u)$ is nondegenerate.*

Lemma 7.3 *Suppose $u \in W$ is a nondegenerate critical point of Φ. Then $\Phi''(u) \in \mathcal{L}(W)$ is an isomorphism of W.*

Proof

We have proved in Lemma 2.4 that $T \in \mathcal{L}(W)$ is a Fredholm operator with $\text{index}(T) = 0$ and $\dim\ker(T) = n$. Since $\hat{\varphi}''(u)$ is compact, $\Phi''(u)$ is Fredholm with $\text{index}(T) = 0$. Consequently $\text{ran}(\Phi''(u))$ is closed and $\dim\ker(\Phi''(u)) = \dim\text{coker}(\Phi''(u))$.

If $d^2\Phi(u)$ is nondegenerate, then $\Phi''(u) \in \mathcal{L}(W)$ is one-to-one, and consequently $\dim\ker(\Phi''(u)) = 0$. Hence $\Phi''(u)$ is onto, and by the open-mapping principle $\Phi''(u)$ has a bounded inverse on W.

∎

The next lemma shows that the Definitions 7.1 and 7.2 of nondegeneracy are equivalent.

Lemma 7.4 *A 1-periodic solution u of the Hamiltonian equation* (7.3) *is nondegenerate if and only if u is a nondegenerate critical point of the action functional Φ.*

Proof

We have already proved in Section 2 that the 1-periodic solutions of (7.3) are in one-to-one correspondence with the critical points of Φ.

Let u be a nondegenerate periodic solution of the Hamiltonian equation, and let

$$d^2\Phi(u)(\xi,\eta) = 0 \quad \text{for all} \quad \eta \in W\,. \tag{7.23}$$

In particular then (7.23) holds for $\eta \in \text{dom}(L) = W^{1,2}(S^1, R^{2n})$, and we have

$$\begin{aligned}
|(\xi, L\eta)_2| &= |(\xi, [P^+ - P^-]\eta)| \\
&\le |(\xi, [P^+ - P^- - \varphi''(u)]\eta)| + |(\xi, \varphi''(u)\eta)| \\
&= |d^2\Phi(u)(\xi,\eta)| + \left| \int_0^1 \langle d^2H(u+e,t)\xi, \eta\rangle\, dt \right|
\end{aligned}$$

$$\begin{aligned}
&\le \int_0^1 |d^2H(u+e,t)|\,|\xi|\,|\eta|\,dt \\
&\le \int_0^1 \{a_1 + a_2|y|^r\}\,|\xi|\,|\eta|\,dt \\
&\le a_1\,\|\xi\|_2\,\|\eta\|_2 + a_2\int_0^1 |y|^r\,|\xi|\,|\eta|\,dt\,.
\end{aligned}$$

The remaining integral can be estimated using the generalized Hölder inequality:

$$\begin{aligned}
\int_0^1 |y|^r\,|\xi|\,|\eta|\,dt \;&\le\; \Big(\int_0^1 |y|^{4r}\,dt\Big)^{1/4}\Big(\int_0^1 |\xi|^4\,dt\Big)^{1/4}\Big(\int_0^1 |\eta|^2\,dt\Big)^{1/2} \\
&= \|y\|_{4r}^r\,\|\xi\|_4\,\|\eta\|_2\,.
\end{aligned}$$

Consequently

$$|(\,\xi\,,\,L\eta\,)_2\,| \le \{a_1\,\|\xi\|_2 + a_2\,\|y\|_{4r}^r\,\|\xi\|_4\}\,\|\eta\|_2\,, \tag{7.24}$$

and therefore we conclude $\xi \in \mathrm{dom}(L^*) = \mathrm{dom}(L)$. Then we have for every $\eta \in W$:

$$\begin{aligned}
0 \;&=\; d^2\Phi(u)\,(\xi,\eta) = (\,[T-\hat{\varphi}''(u)]\,\xi\,,\,\eta\,) \\
&=\; (\,[P^+-P^- - \varphi''(u)]\,\xi\,,\;,\eta\,) \\
&=\; (\,L\xi\,,\,\eta\,)_2 - \int_0^1 \langle\, d^2H(u+e,t)\,\xi\,,\,\eta\,\rangle\,dt \\
&=\; \int_0^1 \langle\, -J\dot{\xi} - d^2H(u+e,t)\,\xi\,,\,\eta\,\rangle\,dt\,.
\end{aligned}$$

We conclude that ξ is a 1-periodic solution of the linearized Hamiltonian equation

$$\dot{\xi} \;=\; J\,d^2H(u+e,t)\,\xi \tag{7.25}$$

and, since u is a nondegenerate solution of (7.3), we have $\xi = 0$ by Lemma 7.2.

Thus we have shown: if the 1-periodic solution u of (7.3) has no Floquet-multiplier equal to 1, then u is a nondegenerate critical point of Φ.
Conversely, if 1 is Floquet-multiplier of u, then, by Lemma 7.2, there exists a nontrivial solution of the linearized Hamilton equation. In particular then $\xi \in C^1(S^1, \mathbb{R}^{2n}) \subset W$, and consequently

$$\int_0^1 \langle -J\dot{\xi} - d^2H(u+e,t)\,\xi\,,\,\eta\,\rangle\,dt \;=\; (\,\Phi''(u)\xi\,,\,\eta\,) \;=\; 0 \tag{7.26}$$

for all $\xi \in W$. Hence $\xi \neq 0$ and $\xi \in \ker \Phi''(u)$. Then, by Lemma 7.5, u has to be degenerate. This finishes the proof of the lemma.

∎

Recall the orthogonal decomposition of W introduced in Section 2

$$W = E^+ \oplus E^- \oplus E^0\,.$$

Also recall that Φ is invariant under the $\mathbb{Z}^n$-action defined in (2.30), and passing to the quotient , we can consider the action functional as a map

$$\Phi : E^+ \times E^- \times E^0/\mathbb{Z}^n \to \mathbb{R}$$

Lemma 7.5

(i) *The nondegenerate critical points of Φ are isolated.*

(ii) *If all critical points of $\Phi : E \times E^0/\mathbb{Z}^n \to \mathbb{R}$ are nondegenerate then there exist only finitely many of them.*

Proof

(i) Let u be a nondegenerate critical point of Φ. Then $\Phi''(u)$ is an isomorphism of W by Lemma 7.3, and the claim follows from

$$\Phi'(u+\xi) \;=\; \Phi''(u)\,\xi + o(\,|\xi|) \tag{7.27}$$

(ii) We have proved in Lemma 6.5 that $\Phi' : E \times E^0/\mathbb{Z}^n \to E \times E^0/\mathbb{Z}^n$ is proper, where $E = E^+ \oplus E^-$. Consequently the set of critical points of Φ is compact. If all the critical points of Φ are nondegenerate, this set consists of isolated points and therefore has to be finite.

∎

8 Morse theory and proof of Theorem 3

Recall the action functional $\Phi : W = W^{\frac{1}{2},2}(S^1, R^{2n}) \to R$, defined by

$$\Phi(u) = \frac{1}{2}(Tu, u) - \hat{\varphi}(u) + (v, u) \tag{8.1}$$

where $v = v_j - Tw$ defined in (2.20), with $v_j = -J\dot{e}(t) = (0, j) \in R^n \times R^n$ depending on the fixed rotation vector $j \in Z^n$, and where

$$\begin{aligned} \hat{\varphi}(u) &= \varphi(u) - \frac{1}{2}(Ku, u) \\ &= \int_0^1 H(u+e, t)\, dt - \frac{1}{2}\int_0^1 \langle \hat{Q}u, u \rangle\, dt\,. \end{aligned}$$

We assume now that $H(x, y, t)$ is a C^2-function, periodic in x and t, satisfying the asymptotic condition as $|y| \to \infty$:

$$\begin{aligned} \frac{1}{|y|}| \partial_y H(x, y, t) - A(t)y | &\to 0 \\ \frac{1}{|y|}| \partial_x H(x, y, t) | &\to 0 \end{aligned} \tag{8.2}$$

uniformly in x and t. Moreover we shall assume that the Hessian of H satisfies

$$| d^2H(x, y, t) | \leq a_1 + a_2|y|^r \tag{8.3}$$

uniformly in x and t with some constants $a_1, a_2 \geq 0$ and $r \in [1, \infty)$. This condition on the Hessian implies that $\varphi \in C^2(W, R)$, see e.g. P. Rabinowitz [52]. Moreover, a representation of the Hessian form of φ at $u \in W$ is given by

$$(\varphi''(u)\xi, \eta) = \int_0^1 \langle d^2H(u+e, t)\xi, \eta \rangle\, dt\,. \tag{8.4}$$

We shall make crucial use of the fact that φ'' is a compact map.

Lemma 8.1 $\quad \varphi'' : W \to \mathcal{L}(W)$ *is compact.*

Proof

Let u_m be a bounded sequence in W. Then u_m has a weakly convergent subsequence, and thus we may assume $u_{m_i} \rightharpoonup u \in W$ weakly. Let $\xi \in W$ with $\|\xi\| \leq 1$. We have

$$\begin{aligned} |(\,\varphi''(u_{m_i}) - \varphi''(u))\,\xi\,,\,\xi\,)| &= \left| \int_0^1 \langle\, d^2H(u_{m_i}+e,t)\,\xi\,,\,\xi\,\rangle\; dt \right| \\ &\leq \int_0^1 |d^2H(u_{m_i}+e,t) - d^2H(u+e,t)|\; |\xi|^2\; dt\,. \end{aligned}$$

We write $w(t) = (x(t), y(t)) \in \mathbb{R}^n \times \mathbb{R}^n$, $w \in W$. Choose $\alpha > 1$. Then the assumption on the norm of the Hessian gives

$$|\,d^2H(w(t)+e(t),t)\,| \;\leq\; a_1 + a_2|y(t)|^{\frac{\alpha r}{\alpha}}\,. \tag{8.5}$$

As in [52], Prop. B1, we conclude that the map $w \mapsto d^2H(w+e,\cdot)$ is in $C(L^{\alpha r}(\mathrm{S}^1, \mathbb{R}^{2n}), L^{\alpha}(\mathrm{S}^1, \mathcal{L}(\mathbb{R}^{2n}))\,)$. Consequently,

$$\begin{aligned} &\int_0^1 |\,d^2H(u_{m_i}+e,t) - d^2H(u+e,t)\,|\; |\xi|^2\; dt \\ &\leq \left(\int_0^1 |\,d^2H(u_{m_i}+e,t) - d^2H(u+e,t)\,|^{\alpha}\; dt \right)^{1/\alpha} \left(\int_0^1 |\xi|^{\frac{2\alpha}{\alpha-1}}\; dt \right)^{(\alpha-1)/\alpha} \\ &= \|\,d^2H(u_{m_i}+e,\cdot) - d^2H(u+e,\cdot)\,\|_{L^{\alpha}(\mathrm{S}^1,\mathcal{L}(\mathbb{R}^{2n}))}\;\; \|\xi\|^2_{\frac{2\alpha}{\alpha-1}}\;\;. \end{aligned}$$

By the continuous embedding $W \to L^{\frac{2\alpha}{\alpha-1}}(\mathrm{S}^1, \mathbb{R}^{2n})$ we conclude

$$\begin{aligned} \|\,\varphi''(u_{m_i}) - \varphi''(u)\,\| &= \sup_{\|\xi\| \leq 1} |(\,(\varphi''(u_{m_i}) - \varphi''(u)\,)\,\xi\,,\,\xi\,)\,| \\ &\leq a_3\,\|\,d^2H(u_{m_i}+e,\cdot) - d^2H(u+e,\cdot)\,\|_{L^{\alpha}(\mathrm{S}^1,\mathcal{L}(\mathbb{R}^{2n}))} \end{aligned}$$

with some constant a_3 depending on α. Note we have used that $\varphi''(w)$ is a symmetric operator.

Since $u_m \rightharpoonup u$ in W we have $u_m \to u$ in $L^{\alpha r}(S^1, R^{2n})$ by the compactness of the embedding $W \to L^{\alpha r}(S^1, R^{2n})$. By the above continuity the right hand side tends to zero as $i \to \infty$, which proves the lemma.

■

We now introduce the Galerkin approximation scheme:
Recall we have an orthogonal splitting $W = E^+ \oplus E^- \oplus E^0$ corresponding to the positive, negative and zero eigenspaces of the operator $T = P^+ - P^- - K$. In E^+ and E^- we choose orthonormal bases consisting of eigenvectors of T:

$$(8.6) \qquad \{u_i \mid i \in Z^+\} \subset E^+$$

$$(8.7) \qquad \{u_i \mid i \in Z^-\} \subset E^- .$$

Define the finite dimensional linear subspaces E_k of $E = E^+ \oplus E^-$ by

$$(8.8) \qquad E_k := \text{span}\{u_i \mid 0 < |i| \leq k\} \ , \ k = 1, 2, \ldots$$

and denote by

$$(8.9) \qquad P_k : E \oplus E^0 \to E_k \oplus E^0$$

the corresponding orthogonal projections. We also occasionally shall use the notation

$$(8.10) \qquad W = E \oplus E^0 \quad , \quad W_k = E_k \oplus E^0 .$$

Note that we have $\|P_k\| = 1$, and the projection scheme (W_k, P_k) is projectionally complete, i.e. $P_k u \to u$ as $k \to \infty$ for every $u \in W$.
Define the restricted functionals

$$(8.11) \qquad \Phi_k : W_k \to R \ , \ k = 1, 2, \ldots$$

by

$$(8.12) \qquad \Phi_k(u) := \Phi(u) \quad \text{if} \quad u \in W_k .$$

We note that the statement of Lemma 6.6 holds true for the projection scheme just defined; the proof here is literally the same.

The subsequent lemmata will show that the situation considered here is well suited for Galerkin approximation. In fact, we shall prove that for sufficiently large k there is a one-to-one correspondence between the critical points of Φ_k and those of Φ, provided all the critical points of Φ are nondegenerate. It then turns out that the critical points of Φ_k are also nondegenerate.
We recall the definition of an A-proper map.

Definition 8.1 *A map $F : W \to W$ is called A-proper with respect to the approximation scheme (W_k, P_k) if P_kF is continuous and, if given any infinite subscheme (W_{k_i}, P_{k_i}) and a bounded subsequence $z_i \in W_{k_i}$, satisfying*

$$P_{k_i}F(z_i) \to w \in W \quad \textit{as} \quad i \to \infty\,,$$

then there exists an element $z \in W$ and a convergent subsequence z_{i_l} such that $z_{i_l} \to z$ as $l \to \infty$ and $F(z) = w$.

Lemma 8.2 *Let $T = P^+ - P^- - K$, and let $F : W \to W$ be a compact map. Then $T - F$ is A-proper with respect to (W_k, P_k).*

Proof
Let (u_{k_i}) be a bounded sequence in W such that $u_{k_i} \in W_{k_i}$, and

$$(8.13) \qquad P_{k_i}(Tu_{k_i} - F(u_{k_i})) \to u^* \in W \quad \text{as} \quad i \to \infty\,.$$

By definition of W_k the operators P_k and T commute. Hence

$$(8.14) \qquad Tu_{k_i} - P_{k_i}F(u_{k_i}) \to u^* \quad \text{as} \quad i \to \infty\,.$$

Since F is compact and (u_{k_i}) is bounded, the sequence $F(u_{k_i})$ contains a convergent subsequence. We assume $F(u_{k_i}) \to w \in W$. Consequently

$$\begin{aligned} \| P_{k_i}F(u_{k_i}) - w \| &\leq \| P_{k_i}F(u_{k_i}) - P_{k_i}w \| + \| P_{k_i}w - w \| \\ &\leq \| F(u_{k_i}) - w \| + \| P_{k_i}w - w \| \to 0 \quad \text{as} \quad i \to \infty\,. \end{aligned}$$

We therefore conclude $Tu_{k_i} \to u^* + w$ as $i \to \infty$.
Recall the orthogonal decomposition

$$W = E^+ \oplus E^- \oplus E^0$$

corresponding to

$$u_{k_i} = u_{k_i}^+ + u_{k_i}^- + u_{k_i}^0 \, .$$

Then $u_{k_i}^0$ is a bounded sequence in $E^0 = \ker(T)$. Since dim $E^0 = n < \infty$, we may assume $u_{k_i}^0 \to u^0 \in E^0$. Thus consider

$$Tu_{k_i} \; = \; T(u_{k_i}^+ + u_{k_i}^-) \to u^* + w \, .$$

Since T is Fredholm, ran(T) is closed, and consequently $u^* + w \in \text{ran}(T)$. Moreover, $T|_{E^+ \oplus E^-}$ is continuously invertible, and therefore

$$u_{k_i}^+ + u_{k_i}^- \to T^{-1}(u^* + w) \, .$$

Thus we have found a convergent subsequence

$$u_{k_i} = u_{k_i}^+ + u_{k_i}^- + u_{k_i}^0 \to T^{-1}(u^* + w) + u^0 \; =: \; u \, .$$

Of course we have

$$(8.15) \qquad Tu - F(u) = u^* + w - \lim_{i \to \infty} F(u_{k_i}) = u^* \, ,$$

and the proof of the lemma is finished.

■

We apply this result to the action functional Φ.

Lemma 8.3

(i) *For every* $u \in W$ *the operator* $\Phi''(u) \in \mathcal{L}(W)$ *is A-proper with respect to* (W_k, P_k).

(ii) $\Phi' : W \to W$ *is A-proper with respect to* (W_k, P_k).

Proof

(i) The statement is obtained from Lemmata 8.2 and 7.2 with $F = \hat{\varphi}''(u)$.

(ii) The statement follows with $F = \hat{\varphi}' - v$.

■

Lemma 8.4 *If $u_0 \in W$ is a nondegenerate critical point of Φ, then there exist $\varepsilon > 0$ and $k_0 \in \mathbb{Z}^+$ such that for every $k \geq k_0$ the function Φ_k possesses exactly one critical point $u_k \in B(u_0, \varepsilon) \cap W_k$.*
Moreover, this critical point u_k of Φ_k is nondegenerate.

Proof

(a) Recall from Lemma 7.3 that $\Phi''(u_0) \in \mathcal{L}(W)$ is an isomorphism, and we have for all $\xi \in W$

$$\| \Phi''(u_0)\xi\| \geq \frac{1}{\| \Phi''(u_0)^{-1} \|} \|\xi\| . \tag{8.16}$$

Since $\Phi'' : W \to \mathcal{L}(W)$ is continuous, there exists $\varepsilon_1 > 0$ such that

$$\frac{\| \Phi''(u_0)^{-1} \|}{\| \Phi''(u)^{-1} \|} \geq \frac{1}{2} \tag{8.17}$$

if $u \in B(u_0, \varepsilon_1)$. Consequently for these u we have an estimate

$$\| \Phi''(u)\xi \| \geq \frac{1}{\| \Phi''(u)^{-1} \|} \|\xi\| \geq \frac{1}{2\| \Phi''(u_0)^{-1} \|} \|\xi\| . \tag{8.18}$$

(b) We claim there exist $c > 0$ and $k_1 \in \mathbb{Z}^+$ such that for $\xi \in W_k$, $k \geq k_1$ and $u \in B(u_0, \varepsilon_1)$ the estimate

$$\| P_k\Phi''(u)\xi \| \geq c\|\xi\| \tag{8.19}$$

holds true.

Assuming, in contradiction, that such numbers c and k_1 do not exist, we can find sequences u_{k_i} in $B(u_0, \varepsilon_1)$ and $\xi_{k_i} \in W_{k_i}$, $\|\xi_{k_i}\| = 1$, such that

$$P_{k_i}\Phi''(u_{k_i})\xi_{k_i} \to 0 \quad \text{as} \quad i \to \infty . \tag{8.20}$$

Recall $\Phi''(u_{k_i}) = T - \hat{\varphi}''(u_{k_i})$, and $\hat{\varphi}''(u_{k_i}) = \varphi''(u_{k_i}) - K$. By Lemma 8.1 the map φ'' is compact, and therefore $\hat{\varphi}'' : W \to \mathcal{L}(W)$ is compact. Since (u_{k_i}) is a bounded sequence we can choose a subsequence, again denoted by (u_{k_i}), such that $\hat{\varphi}''(u_{k_i})$ converges to $B \in \mathcal{L}(W)$ in $\mathcal{L}(W)$. The set of compact linear operators on W is closed in $\mathcal{L}(W)$, and hence B is compact. Consequently

$$\Phi''(u_{k_i}) \to T - B =: S \in \mathcal{L}(W) \quad \text{as} \quad i \to \infty . \tag{8.21}$$

Note that

$$\begin{aligned} \| P_{k_i} S \xi \| &\leq \| P_{k_i} \Phi''(u_{k_i}) \xi_{k_i} \| + \| P_{k_i} S \xi_{k_i} - P_{k_i} \Phi''(u_{k_i}) \xi_{k_i} \| \\ &\leq \| P_{k_i} \Phi''(u_{k_i}) \xi_{k_i} \| + \| S - \Phi''(u_{k_i}) \| \to 0 \quad \text{as} \quad i \to \infty . \end{aligned}$$

By Lemma 8.2 the operator $S = T - B$ is A-proper. Hence (ξ_{k_i}) contains a convergent subsequence. Thus we may assume $\xi_{k_i} \to \xi \in W$, and from $\|\xi_{k_i}\| = 1$ we conclude $\|\xi\| = 1$. By the A-properness, ξ satisfies $S\xi = 0$. Consequently

$$\|\Phi''(u_{k_i}) \xi \| = \| \Phi''(u_{k_i}) \xi - S\xi \| \leq \| \Phi''(u_{k_i}) - S \| \to 0 \quad \text{as} \quad i \to \infty .$$

On the other hand, (8.18) yields

$$\| \Phi''(u_{k_i}) \xi \| \geq \frac{1}{2 \| \Phi''(u_0)^{-1} \|} \|\xi\| = \frac{1}{2 \| \Phi''(u_0)^{-1} \|} > 0 .$$

Thus we arrived at a contradiction, and the claim is proved.

(c) In particular we have with c, k_1 as in (b)

$$\| P_k \Phi''(u_0) \xi \| \geq c \|\xi\| \quad \text{if} \quad k \geq k_1 . \tag{8.22}$$

Hence $P_k \Phi''(u_0)|_{W_k} \in \mathcal{L}(W_k)$ is an isomorphism of W_k. Moreover, we have

$$\Phi'(u) = \Phi''(u_0)(u - u_0) + o(\|u - u_0\|) . \tag{8.23}$$

Consequently we can choose $0 < \varepsilon_2 \leq \varepsilon_1$ such that $\Phi'(u) \neq 0$ whenever $u \in \overline{B(u_0, \varepsilon_2)} \setminus \{u_0\}$. By (ii) of Lemma 8.3 we can find $k_2 \geq k_1$ such that $P_k \Phi'(u) \neq 0$ if $u \in \partial B(u_0, \varepsilon_2)$ and $k \geq k_2$.

The Brouwer mapping degree is therefore defined and, by the homotopy given by

$$P_k\Phi''(u_0)\,(u-u_0)+t\,o(\,\|u-u_0\|\,)\quad ,\quad 0\le t\le 1$$

we find

$$\deg\,(P_k\Phi'\,,B(u_0,\varepsilon_2)\cap W_k,\,0)=\deg\,(P_k\Phi''(u_0)\,(\cdot-u_0)\,,B(u_0,\varepsilon_2)\cap W_k,\,0)\,.$$

Note that $P_k\Phi''(u_0)$ is an isomorphism for all $k\ge k_1$. Hence the equation

$$P_k\Phi''(u_0)\,u \;=\; P_k\Phi''(u_0)\,u_0 \tag{8.24}$$

has exactly one solution $u=\tilde u_k\in W_k\,,\,k\ge k_1$. By the convergence $P_k\Phi''(u_0)\,u_0\to\Phi''(u_0)\,u_0$ as $k\to\infty$ and by the A-properness of $\Phi''(u_0)$ it follows that $\tilde u_k\to u_0$ as $k\to\infty$. Consequently there exists $k_3\ge k_2$ such that $\tilde u_k\in B(u_0,\varepsilon_2)\cap W_k$ for $k\ge k_3$. Computing the Brouwer degree we obtain

$$\begin{aligned}&\deg\,(\,P_k\Phi''(u_0)\,(\cdot-u_0)\,,\,B(u_0,\varepsilon_2)\cap W_k\,,\,0\,)\\ =\;&\deg\,(\,P_k\Phi''(u_0)\,,\,B(u_0,\varepsilon_2)\cap W_k)\,,\,P_k\Phi''(u_0)\,u_0\,)\\ =\;&\operatorname{sign}\,\Big(\det\,[\,P_k\Phi''(u_0)\,]\Big)\;=\;\pm1\end{aligned}$$

and therefore

$$\deg\,(\,P_k\Phi'\,,\,B(u_0,\varepsilon_2)\cap W_k\,,\,0\,)\;=\;\pm1\,. \tag{8.25}$$

Consequently there exists a critical point $u_k\in B(u_0,\varepsilon_2)\cap W_k$ of Φ_k, if $k\ge k_3$.

(d) We finally claim that there exists $k_4\ge k_3$ such that there is only one single critical point u_k of Φ_k in $B(u_0,\varepsilon_2)\cap W_k$.
By contradiction we assume there exist $u_k,u_k'\in B(u_0,\varepsilon_2)\cap W_k\,,u_k\ne u_k'$, such that $P_k\Phi'(u_k')=P_k\Phi'(u_k)=0$ for infinitely many $k\ge k_3$. The identity

$$\Phi'(u_k')\;=\;\Phi'(u_k)+\Phi''(u_k)\,(u_k'-u_k)+o(\,\|u_k'-u_k\|\,) \tag{8.26}$$

implies

$$0\;=\;P_k\Phi''(u_k)\,(u_k'-u_k)+o(\,\|u_k'-u_k\|\,)\,. \tag{8.27}$$

Since Φ' is A-proper and since $u_k, u'_k \in B(u_0, \varepsilon_2)$ there exist subsequences u_{k_i}, u'_{k_i} which converge to u_0 as $i \to \infty$. Hence $\|u'_{k_i} - u_{k_i}\| \to 0$, and combining (8.19) and (8.27) leads to

$$(8.28) \qquad c < \left\| P_{k_i} \Phi''(u_{k_i}) \frac{u'_{k_i} - u_{k_i}}{\|u'_{k_i} - u_{k_i}\|} \right\| \leq \frac{o(\|u'_{k_i} - u_{k_i}\|)}{\|u'_{k_i} - u_{k_i}\|} \to 0$$

as $i \to \infty$, in contradiction to $c > 0$, and the claim is proved.

We define $k_0 := k_4$, $\varepsilon := \varepsilon_2$ in the statement of the lemma, and note that the equation (8.19) shows that the critical points u_k of Φ_k are nondegenerate. This finishes the proof of the lemma.

■

Under the hypothesis that all critical points of $\Phi : E \times \mathbb{T}^n \to \mathbb{R}$ are non-degenerate, we have shown in Lemma 7.5 that the set of critical points of Φ contains at most finitely many elements. By Theorem 1 the number of critical points is at least $n+1$. Now let $u^1, \dots, u^m$ denote the critical points of Φ. In view of Lemma 8.4 we can find $\varepsilon > 0$ and $k_0 \in \mathbb{Z}^+$ such that

1. $B(u^i, \varepsilon) \cap B(u^j, \varepsilon) = \emptyset \qquad \text{if} \quad i \neq j,\ 1 \leq i, j \leq m,$
2. There exists at least one critical point u^i_k of Φ_k in each $B(u^i, \varepsilon) \cap (E \times \mathbb{T}^n)$ if $k \geq k_0$. Moreover, the u^i_k are are nondegenerate and $u^i_k \to u^i$ as $k \to \infty$, since u^i is the only critical point of Φ contained in $B(u^i, \varepsilon)$.

Assume we already know that there exists a bounded set $B \subset E \times \mathbb{T}^n$ such that the critical points of Φ_k on $E_k \times \mathrm{T}^n$ are entirely contained in B. Then for k sufficiently large there are no critical points of Φ_k contained in $(B \setminus (B(u^1, \varepsilon) \cup \dots \cup B(u^m, \varepsilon))) \cap (E_k \times \mathrm{T}^n)$. Assuming we can find an infinite sequence $u_k \in B \cap (E_k \times \mathrm{T}^n)$, $\Phi'_k(u_k) = 0$, $u_k \notin B(u^1, \varepsilon) \cup \dots \cup B(u^m, \varepsilon)$, we conclude by the A-properness of Φ that this sequence must have a convergent subsequence whose limit is a critical point of Φ , different from the u^i, and thus we have a contradiction.
Consequently there exists k_0 such that for $k \geq k_0$ the critical points of Φ_k are in a one-to-one correspondence with the critical points of Φ, and therefore a

lower bound for the number of critical points of Φ_k is also a lower bound for the critical set of Φ.

To carry out this strategy we first prove the required a priori estimate for the critical points of Φ on $E \times T^n$ and Φ_k on $E_k \times T^n$. From now on let $\Phi_{(k)}$ and $E_{(k)}$ denote either Φ or Φ_k and E or E_k respectively. Consider the negative gradient flow $u{\cdot}s$, $s \in \mathbb{R}$, of the vector field $-\Phi'_{(k)}$ on $E_{(k)} \times T^n$, i.e.

$$\frac{d}{ds}(u{\cdot}s) = -\Phi'_{(k)}(u{\cdot}s) .$$

Lemma 8.5

(i) *There exists $R > 0$ such that the restpoints of the flow of $-\Phi'$ are contained in*

$$B = D^+ \times D^- \times T^n$$

where $D^\pm = \{ u^\pm \in E^\pm \mid \|u^\pm\| \le R \}$.

(ii) *The restpoints of the flow $-\Phi'_k$ are all contained in*

$$B_k = D_k^+ \times D_k^- \times T^n$$

where $D_k^\pm = D^\pm \cap E_k$.

Proof

The proof of (i) and (ii) can be done simultaneously using the notational convention that the symbol $A_{(k)}$ means either A or A_k.

Recall from (6.1) that there exists $\lambda > 0$ such that $(Tu^+, u^+) \ge \lambda \|u^+\|^2$ for all $u^+ \in E^+_{(k)}$, and moreover $(Tu^-, u^-) \ge -\lambda \|u^-\|^2$ for $u^- \in E^-_{(k)}$. We consider $u = u^+ + u^- + u^0 \in E^+_{(k)} \oplus E^-_{(k)} \oplus E^0$ such that $\|u^-\| \le \|u^+\| + r$ with a fixed constant $r \ge 0$. For these u we have

$$\begin{aligned}
\left(\Phi'_{(k)}(u), u^+ \right) &= (Tu^+, u^+) - (\hat{\varphi}'(u), u^+) + (v, u^+) \\
&\ge \lambda \|u^+\|^2 - \int_0^1 \langle \nabla H(u+e,t) - \hat{Q}(t)u, u^+ \rangle \, dt + (v, u^+) \\
&\ge \lambda \|u^+\|^2 - \int_0^1 |\nabla H(u+e,t) - \hat{Q}(t)u| \, |u^+| \, dt - \|v\| \, \|u^+\|
\end{aligned}$$

$$\geq \lambda \|u^+\|^2 - \{c_1(\varepsilon) + \varepsilon\|u^+ + u^-\|\}\,\|u^+\| - \|v\|\,\|u^+\|$$

$$\geq \lambda \|u^+\|^2 - \varepsilon\{2\,\|u^+\| + r\}\,\|u^+\| - c_1(\varepsilon)\,\|u^+\| - \|v\|\,\|u^+\|$$

$$\geq (\lambda - 2\varepsilon)\,\|u^+\|^2 - c_2(\varepsilon, r)\,\|u^+\|$$

where we have used the asymptotic condition

$$|\,\partial_x H(x,y,t)\,| + |\,\partial_y H(x,y,t) - A(t)y\,| \leq c_1(\varepsilon) + \varepsilon|y|\,.$$

We choose $0 < \varepsilon < \lambda/2$. Consequently there exists $R > 0$ such that with this choice of ε :

$$\text{(8.29)} \qquad \begin{aligned} &(\,-\Phi'_{(k)}(u)\,,\,u^+\,) \;<\; 0 \quad \text{where} \quad u = u^+ + u^- + u^0 \\ &\text{satisfying} \quad \|u^-\| \leq \|u^+\| + r \quad \text{and} \quad \|u^+\| \geq R\,. \end{aligned}$$

By a similar argument we can find $R > 0$ such that

$$\text{(8.30)} \qquad \begin{aligned} &(\,-\Phi'_{(k)}(u)\,,\,u^-\,) \;>\; 0 \quad \text{where} \quad u = u^+ + u^- + u^0 \\ &\text{satisfying} \quad \|u^+\| \leq \|u^-\| + r \quad \text{and} \quad \|u^-\| \geq R\,. \end{aligned}$$

Note that R can be chosen the same constant in (8.29) and (8.30), and moreover R is independent of k. Thus we have proved that all the possible restpoints of the flow of $-\Phi'_{(k)}$ on $E^+_{(k)} \oplus E^-_{(k)} \oplus E^0$ have to be contained in $D^+_{(k)} \times D^-_{(k)} \times E^0$.
We point out the fact that it is sufficient for our purposes in this section to choose $r = 0$. However, the case $r > 0$ turns out to be important in the applications considered in Section 9.

Observe that the above estimates are uniform in $u^0 \in E^0$. Passing now to the quotient $E_{(k)} \times \mathrm{T}^n$ we see the following :
If $\|u^+\| = R$ and $\|u^-\| < R$ the inequality (8.29) shows that the vector $\Phi'_{(k)}(u)$ points into the exterior of $B_{(k)}$, and, vice versa, if $\|u^-\| = R$ and

$\|u^+\| < R$ the vector $-\Phi'_{(k)}(u)$ points into the exterior of $B_{(k)}$ by (8.30). Consequently, the set

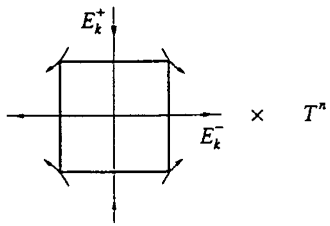
$$B^+_{(k)} := \partial D^+_{(k)} \times D^-_{(k)} \times \mathrm{T}^n \tag{8.31}$$

is the entrance set of $B_{(k)}$ for the flow of $-\Phi'_{(k)}$, and

$$B^-_{(k)} := D_{(k)} \times \partial D^-_{(k)} \times \mathrm{T}^n \tag{8.32}$$

is the corresponding exit set.

■



Note that B_k is an isolating block in the sense of Conley and Zehnder, see [14, 15].
We are ready now to apply the Conley-Zehnder Morse theory to the flow of $-\Phi'_k$ on $E_k \times \mathrm{T}^n$ near the isolating block found in Lemma 8.5. Here we make use of the fact that $E_k \times \mathrm{T}^n$ is a finite dimensional manifold.

Definition 8.2 *The Morse index $m(x)$ of a nondegenerate critical point $u \in E_k \times \mathrm{T}^n$ of Φ'_k is the number of the negative eigenvalues of the Hessian matrix $P_k\Phi''(u)$.*

Proposition 8.1 (Morse theory for Φ_k) *Assume $k \in \mathbb{Z}^+$ is sufficiently large. Then all critical points of Φ_k on $E_k \times \mathrm{T}^n$ are nondegenerate and there are only finitely many of them. Denote by $m(x)$ the Morse index of a critical point u, then*

$$\sum_{\{u_i | \Phi'_i(u_i)=0\}} t^{m(u_i)} = p(t, \mathrm{T}^n)t^a + (1+t)\,Q(t) \tag{8.33}$$

where $p(t, T^n)$ is the Poincaré polynomial of the n-dimensional torus T^n, a is a positive integer depending on k, and $Q(t)$ is a polynomial having nonnegative integer coefficients.

Proof

We merely describe the main ideas and refer to [14, 15] for the details.
Let S_k denote the maximal invariant set of the gradient flow for Φ_k which is contained in the isolating set $D_k^+ \times D_k^- \times T^n$ having $D_k^+ \times \partial D_k^- \times T^n$ as exit set. The set S_k is compact and consists of finitely many critical points together with all their connecting orbits. It has a Conley index which is the homotopy type of a pointed compact space, denoted by $h(S_k)$. From the above isolating block and the exit set one computes readily for the Poincaré polynomial of $h(S_k)$ that

$$(8.34) \qquad p(t, h(S_k)) = p(t, T^n)\, p(t, \dot{S}^a) = p(t, T^n)\, t^a$$

for some integer a. Denote the Conley index of a critical point u by $h(u)$. Then, since the critical points constitute a Morse decomposition of S_k, we have the following Morse equation relating the global index of S_k with the local invariants of the critical points by

$$(8.35) \qquad \sum_{i=1}^{N} p(t, h(u_i)) = p(t, h(S_k)) + (1 + t)\, Q(t)$$

with a polynomial Q having nonnegative integer coefficients. Observe now that if a critical point is nondegenerate, then its Conley index is a pointed sphere of dimension equal to the Morse index $m(u)$ of u, so that

$$(8.36) \qquad p(t, h(u_i)) = p(t, \dot{S}^{m(u_i)}) = t^{m(u_i)}$$

and hence the proposition is proved.

■

From the Morse equation it is now easy to derive a lower bound for the number of critical points . Observe that the Poincaré polynomial is given by

$$p(t, \mathrm{T}^n) = \sum_{i=0}^{n} \binom{n}{i} t^i . \tag{8.37}$$

Setting $t = 1$ we obtain

$$p(1, \mathrm{T}^n) = 2^n \tag{8.38}$$

which is equal to the sum of Betti numbers of T^n. Consequently, setting $t = 1$ in the Morse equation, we obtain the

Corollary 8.1

$$\#\{\, \textit{critical points of } \Phi_k \,\} \geq 2^n . \tag{8.39}$$

This proves Theorem 2.

9 Lyusternik-Schnirelman theory for flows and a second proof of Theorem 1

In the previous section we have exploited the structure of the gradient flow of the action functional Φ to construct an isolating block for the invariant set S which consists of the critical points of Φ together with their connecting orbits. By use of the A-properness of the gradient Φ' with respect to the Galerkin approximation scheme we have introduced a reduction to finite dimension in order to establish the topological arguments which finally led to the claimed multiplicity statement for the number of critical points of Φ. Basically the setting which was introduced in Section 8 can still be used even if the assumption of nondegeneracy is dropped. Replacing the Morse equation by some appropriate cuplength-estimate, we shall prove the existence of at least cuplength$(\mathrm{T}^n) + 1$ critical points. This will provide an alternative proof of Theorem 1 which does not involve minimax-methods. In contrast to our former proof of Theorem 1 the dynamical properties of the gradient flow will be used explicitly.

For the sake of simplicity we shall assume in the sequel that the gradient vector field of Φ generates a unique flow on $E \times \mathrm{T}^n$. This is obviously the case if the Hamiltonian H satisfies the hypotheses stated in Section 8. Note that under the more general assumptions of Theorem 1 a unique gradient flow may not exist. However, it is possible to replace the gradient vector field in the proof below by some appropriate Lipschitz-continuous almost gradient vector field, see Proposition 9.3.

For the notation used in the following we refer to Section 8.
We state precisely the main result of this section

Proposition 9.1 *Let $\Phi \in C^1(E \times \mathrm{T}^n, \mathrm{R})$ be the action functional defined by*

$$\Phi(u) = \frac{1}{2}(Tu\,,\,u) - \hat{\varphi}(u) + (v\,,\,u)\,.$$

Assume that the gradient Φ' generates a unique flow on $E \times \mathrm{T}^n$. Then Φ has at least $n+1$ critical points.

In particular, if Φ has only finitely many critical values $c_1 < \ldots < c_m$ and if

$$(9.1) \quad M_j := \{u \in E \times \mathrm{T}^n \mid \Phi(u) = c_j\,,\ \Phi'(u) = 0\,\}\quad,\quad j = 1, \ldots, m$$

denotes the critical set at the level c_j, then

$$(9.2) \qquad \sum_{j=1}^{m} \mathrm{cat}_{E\times \mathrm{T}^n}(M_j) \ \geq\ \mathrm{cuplength}(\mathrm{T}^n) + 1\,.$$

Proof

(i) We first assume there exist finitely many critical levels of Φ denoted by $c_1 < \ldots < c_m$. In this case the existence of at least $n+1 = \mathrm{cuplength}(\mathrm{T}^n)+1$ critical points of Φ follows from the estimate (9.2). Note that by our assumption each of the critical sets M_j at the level c_j is nonempty and consequently $\mathrm{cat}_{E\times\mathrm{T}^n}(M_j) \geq 1$. If there exists a subscript j such that $\mathrm{cat}_{E\times\mathrm{T}^n}(M_j) \geq 2$ then M_j has infinitely many elements since $E \times \mathrm{T}^n$ is arcwise connected. If such subscript j does not exist we conclude that $m \geq n+1$, and the claim follows.

In Lemma 6.5 we have proved that the mapping $\Phi' : E \times \mathrm{T}^n \to E \times \mathrm{T}^n$ is proper, and consequently the critical set of Φ is compact. As a consequence each of the components M_j, $j = 1, \ldots, m$, defined in (9.1), is compact. By the continuity property of cat there exists $\delta > 0$ such that

$$(9.3) \qquad \mathrm{cat}_{E\times\mathrm{T}^n}(N_{\delta,j}) = \mathrm{cat}_{E\times\mathrm{T}^n}(M_j)$$

where for $j = 1, \ldots, m$

$$N_{\delta,j} := N_\delta(M_j) = \{\, u \in E \times \mathrm{T}^n \mid \mathrm{dist}(u, M_j) \leq \delta\,\}$$

denotes the δ-neighborhood of M_j in $E \times \mathrm{T}^n$.

Observe that if the number m of critical values of Φ is greater than 1 we can choose $\bar{\varepsilon} > 0$ such that $c_j + \bar{\varepsilon} < c_{j+1} - \bar{\varepsilon}$ for $j = 1, \ldots, m-1$.

If $0 < \varepsilon \leq \bar{\varepsilon}$ is given, then by the continuity of Φ there exists $\delta = \delta(\varepsilon) > 0$ such that for $j = 1, \ldots, m$:

$$(9.4) \qquad c_j - \varepsilon \ <\ \Phi(u)\ <\ c_j + \varepsilon \quad \text{if} \quad u \in N_{\delta,j}\,.$$

The choice of $\bar{\varepsilon}$ implies that for $0 < \varepsilon \le \bar{\varepsilon}$ and $\delta = \delta(\varepsilon)$:

$$N_{\delta,i} \cap N_{\delta,j} = \emptyset \quad \text{if} \quad i \neq j \,. \tag{9.5}$$

Recall from Section 8 the Galerkin approximation scheme, and also recall the definition of the restricted functionals Φ_k. In the subsequent Lemma 9.1 we present an a priori-estimate concerning the approximation of critical points of the action functional Φ. Considering Φ resp. Φ_k on the covering space $W = E \oplus E^0$ resp. $W_k = E_k \oplus E^0$ we have the following

Lemma 9.1 *Let*

$$K_{c_j} = \{\, u \in W \mid \Phi(u) = c_j \ \textit{and}\ \Phi'(u) = 0 \,\}$$

and let $\delta > 0$. *Define*

$$N := N_{\delta/4}(K_{c_1}) \cup \ldots \cup N_{\delta/4}(K_{c_m}) \subset W \,. \tag{9.6}$$

Then there exist $b > 0$ *and* $k_1(\delta) \in \mathbb{Z}^+$ *depending on* δ *such that*

$$\|\Phi_k'(u_k)\| \ge b \quad \textit{for} \quad k \ge k_1(\delta) \quad \textit{and} \quad u_k \notin N \cap W_k \,.$$

Proof

Assuming such constants b and $k_1(\delta)$ do not exist we can find a sequence $u_{k_i} \in W_{k_i} \setminus N$ such that $\Phi_{k_i}'(u_{k_i}) \to 0$ as $i \to \infty$. Note that Lemma 6.6 holds true for the approximation scheme considered here, and therefore u_{k_i} contains a convergent subsequence in $E \times T^n \cong E \times E^0/\mathbb{Z}^n$. By use of the $\mathbb{Z}^n$-invariance of Φ_k, we may assume that this subsequence is also convergent in $E \times E^0$, and again by Lemma 6.6 we conclude that the limit $u \in W$ is a critical point of Φ. On the other hand $u_{k_i} \notin N$ implies $\mathrm{dist}(u, K_{c_j}) \ge \delta/2$ for $j = 1, \ldots, m$, and therefore u is not contained in the critical set of Φ, which gives a contradiction.

∎

From Lemma 9.1 we conclude immediately

Corollary 9.1 *Suppose that $\delta > 0$ is chosen such that* (9.3) *and* (9.4) *hold with some $\varepsilon \leq \bar{\varepsilon}$, and let $k_1(\delta)$ be the integer determined in Lemma 9.1. If $k \geq k_1(\delta)$ and if u is a critical point of $\Phi_k : E_k \times \mathrm{T}^n \to \mathbf{R}$, then*

$$u \in (N_{\delta,1} \cup \ldots \cup N_{\delta,m}) \cap (E_k \times \mathrm{T}^n) \ .$$

In this case there exists $j \in \{1, \ldots, m\}$ such that $u \in N_{\delta,j}$, and consequently

$$\Phi_k(u) \ \in \ (c_j - \varepsilon, c_j + \varepsilon) \,. \tag{9.7}$$

For $k \geq k_1(\delta)$ and $j = 1, \ldots, m$ we define the following subsets of $E_k \times \mathrm{T}^n$:

$$S_j^{(k)} := \left\{ u \in E_k \times \mathrm{T}^n \mid \lim_{|\tau| \to \infty} u{\cdot}\tau \in N_{\delta,j} \right\} , \tag{9.8}$$

where $u{\cdot}\tau$ denotes the flow of $-\Phi_k'$ on $E_k \times \mathrm{T}^n$, i.e.

$$\frac{d}{d\tau}(u{\cdot}\tau) = -\Phi_k'(u{\cdot}\tau) \,.$$

Thus $S_j^{(k)}$ consists precisely of the critical points of Φ_k which are contained in $N_{\delta,j}$ together with the connecting orbits of these critical points. Quite in contrast to the nondegenerate case it is not sufficient in the present situation to consider the critical points only. We have to take into account an estimate concerning the behavior of the invariant sets $S_j^{(k)}$ as $k \to \infty$:

Lemma 9.2 *There exist $\varepsilon > 0$, $\delta > 0$ sufficiently small, and $k_0 \in \mathbb{Z}$ depending on δ, such that*

$$S_j^{(k)} \ \subset \ N_{\delta,j} \quad , \quad j = 1, \ldots, m \,, \tag{9.9}$$

for all $k \geq k_0$.

Proof
We will consider the functionals Φ_k on the covering space $W_k = E_k \times E^0$ of $E_k \times \mathrm{T}^n$. Our notation will not distinguish between the flow of $-\Phi_k'$ on $E_k \times \mathrm{T}^n$ and its lift on W_k.

We choose $\bar{\varepsilon} > 0$ so small that $c_j + \bar{\varepsilon} < c_{j+1} - \bar{\varepsilon}$. Let $\delta > 0$ be chosen such that (9.3) and (9.4) are satisfied, i.e. we have $\mathrm{cat}_{E\times T^n}(N_{\delta,j}) = \mathrm{cat}_{E\times T^n}(M_j)$ and $\Phi(u) \in (c_j - \bar{\varepsilon}, c_j + \bar{\varepsilon})$ for $u \in N_{\delta,j}$.
Let $b > 0$ and $k_1(\delta)$ denote the constants depending on δ which are given by Lemma 9.1.
In the following we choose a fixed ε satisfying

$$0 < \varepsilon < \min\left\{\bar{\varepsilon}, \frac{b\delta}{4}\right\} . \tag{9.10}$$

Corresponding to this ε we can find $0 < \delta_0 \leq \delta$ such that (9.4) holds with ε , δ_0. Let $k_1(\delta_0)$ be the integer assigned to δ_0 as in Lemma 9.1.
We define $k_0 := \max\{k_1(\delta), k_1(\delta_0)\}$.

Fix $k \geq k_0$ and consider $u \in W_k$ such that

$$\Phi_k(u) \leq c_j + \varepsilon \quad \text{and} \quad u \notin N_\delta(K_{c_j}) .$$

We claim that the orbit $\{u\cdot\tau \,|\, \tau \in \mathbb{R}\}$ of u under the flow of $-\Phi_k'$ will not approach a critical point of Φ_k on a level greater or equal $c_j - \varepsilon$ in foreward direction. Note that for $k \geq k_0$ any critical point u_0 of Φ_k satisfying $c_j - \varepsilon \leq \Phi_k(u_0) \leq \Phi_k(u)$ has to be contained in $N_{\delta_0/4}(K_{c_j}) \subset N_{\delta/4}(K_{c_j})$. Consequently we only have to consider the case where the orbit of u enters the set $N_{\delta/2}(K_{c_j})$.

In this case we can find $\tau_1 > \tau_0 \geq 0$ such that

$$u\cdot\tau_0 \notin N_\delta(K_{c_j}) \cap W_k \tag{9.11}$$

$$u\cdot\tau_1 \in N_{\delta/2}(K_{c_j}) \cap W_k \tag{9.12}$$

$$u\cdot\tau \notin N \cap W_k \quad \text{for} \quad \tau_0 \leq \tau \leq \tau_1 \tag{9.13}$$

where N is defined in (9.6).
Then we have

$$\int_{\tau_0}^{\tau_1} \|\frac{d}{ds}(u\cdot s)\|\, ds \geq \mathrm{dist}(u\cdot\tau_0, u\cdot\tau_1) \geq \frac{\delta}{2} . \tag{9.14}$$

Therefore we conclude

$$\begin{aligned}
\Phi_k(u\cdot\tau_1) &= \Phi_k(u\cdot\tau_0) + \int_{\tau_0}^{\tau_1} \frac{d}{ds}\Phi_k(u\cdot s)\, ds \\
&= \Phi_k(u\cdot\tau_0) + \int_{\tau_0}^{\tau_1} \left(\Phi_k'(u\cdot s)\, ,\, \frac{d}{ds}(u\cdot s) \right) ds \\
&= \Phi_k(u\cdot\tau_0) - \int_{\tau_0}^{\tau_1} \| \Phi_k'(u\cdot s) \|^2\, ds \\
&\leq c_j + \varepsilon - b\int_{\tau_0}^{\tau_1} \| \Phi_k'(u\cdot s) \|\, ds \\
&\leq c_j + \varepsilon - \frac{b\delta}{2} \\
&< c_j - \varepsilon\, .
\end{aligned}$$

The claim now follows since Φ_k is a Lyapunov function for the flow of $-\Phi_k'$.

Thus we have proved that u is not contained in the stable manifold of some critical point of Φ_k in $N_{\delta_0/4}(K_{c_j})$. In particular u does not belong to any connecting orbit of such critical points. Finally passing from W_k to the quotient $E_k \times \mathrm{T}^n$ completes the proof of the lemma.

■

Recall Lemma 8.5 from the previous section. There we have proved the existence of an isolating block

(9.15) $$B_k \;=\; D_k^+ \times D_k^- \times \mathrm{T}^n \;\subset\; E_k^+ \times E_k^- \times \mathrm{T}^n$$

where $D_k^\pm = \{\, u^\pm \in E_k^\pm \mid \|u^\pm\| \leq R \,\}$ for some $R > 0$ independent of k.

We denote by $S^{(k)}$ the maximal invariant set of the flow of $-\Phi_k'$ which is contained in the isolating block B_k :

(9.16) $$S^{(k)} := \{\, u \in E_k \times \mathrm{T}^n \mid u\cdot\tau \in B_k \;\text{ for all }\; \tau \in \mathrm{R} \,\}$$

Note that $S^{(k)}$ is compact and invariant under the flow of $-\Phi_k'$. As an easy consequence of Lemma 9.2 we have

Corollary 9.2 *For $k \geq k_0$ there is an ordered Morse decomposition of $S^{(k)}$ given by $(S_1^{(k)}, \ldots, S_m^{(k)})$.*

For sufficiently large k the Lyusternik-Schnirelman category of $S^{(k)}$ is related to the category of the components $S_j^{(k)}$ of the Morse decomposition :

Lemma 9.3 *For $k \geq k_0$ we have*

$$\text{cat}_{E_k \times \mathrm{T}^n}(S^{(k)}) \leq \sum_{j=1}^{m} \text{cat}_{E_k \times \mathrm{T}^n}(S_j^{(k)}) . \tag{9.17}$$

Proof

(a) Note that the assumption $S_j^{(k)} = \emptyset$ for all $j = 1, \ldots, m$ implies $S^{(k)} = \emptyset$, and then (9.17) holds trivially.

(b) Dealing with the case $S^{(k)} \neq \emptyset$ we can assume that $S_j^{(k)} \neq \emptyset$ for $j = 1, \ldots, m$. Observe that the nonempty $S_j^{(k)}$ constitute an admissibly ordered Morse decomposition of $S^{(k)}$ and therefore only the contribution of the nonempty components has to be taken into consideration. Up to renumbering of these components the proof remains unchanged.

If $m = 1$ then obviously $S^{(k)} = S_1^{(k)}$ and we are already done.
Let $m > 1$. Using the continuity property of cat we can choose $\delta > 0$ such that

$$\text{cat}_{E_k \times \mathrm{T}^n}(N_\delta(S_j^{(k)})\,) = \text{cat}_{E_k \times \mathrm{T}^n}(S_j^{(k)})$$

where $N_\delta(S_j^{(k)})$ denotes the closed δ-neighborhood of $S_j^{(k)}$ in $E_k \times \mathrm{T}^n$. We define for $j = 1, \ldots, m$:

$$p_j := \text{cat}_{E_k \times \mathrm{T}^n}(N_\delta(S_j^{(k)})\,) \tag{9.18}$$

Since the sets $S_j^{(k)}$ are compact, we have $p_j < \infty$.
Moreover we consider all ordered pairs $(S_i^{(k)}, S_j^{(k)})$ where $1 \leq i < j \leq m$. Given any such pair we define the invariant set of connecting orbits from $S_j^{(k)}$ to $S_i^{(k)}$:

$$(9.19)\quad S_{i,j} := \left\{ u \in S^{(k)} \mid \lim_{\tau\to+\infty} u\cdot\tau \in S_i^{(k)} \text{ and } \lim_{\tau\to-\infty} u\cdot\tau \in S_j^{(k)} \right\} .$$

Let $1 \le i < m$ be fixed, and let $U_i^1, \ldots, U_i^{p_i} \subset E_k \times \mathrm{T}^n$ be a collection of closed subsets such that

$$U_i^r \text{ is contractible in } E_k \times \mathrm{T}^n$$
$$N_\delta(S_i^{(k)}) \subset U_i^1 \cup \ldots \cup U_i^{p_i}$$

Then for every $i < j \le m$ there exists $\tau_{i,j} \in \mathrm{R}^+$ such that for $t \ge \tau_{i,j}$

$$(9.20)\qquad S_{i,j} \setminus N_\delta(S_j^{(k)}) \subset \bigcup_{r=1}^{p_i} U_i^r\cdot(-t) .$$

In order to prove this claim we choose $j > i$ and define for $u \in S_{i,j}$

$$(9.21)\qquad \tau(u) := \inf\left\{ \tau \in \mathrm{R} \mid u\cdot s \in N_\delta(S_i^{(k)}) \text{ for all } s \ge \tau \right\} .$$

Note that $\tau(u) < \infty$ for all $u \in S_{i,j}$. We consider

$$(9.22)\qquad \tau_{i,j} := \sup\left\{ \tau(u) \mid u \in S_{i,j} \setminus N_\delta(S_j^{(k)}) \right\} .$$

By definition of $\tau_{i,j}$ there exists a sequence $u_l \in S_{i,j} \setminus N_\delta(S_j^{(k)})$ such that $\tau(u_l) \to \tau_{i,j}$ as $l \to \infty$. Since $S_{i,j} \setminus N_\delta(S_j^{(k)})$ is relatively compact, u_l contains a convergent subsequence whose limit $\bar{u}$ is contained in the closure of $S_{i,j} \setminus N_\delta(S_j^{(k)})$. Hence $\operatorname{dist}(\bar{u}, S_j^{(k)}) \ge \delta$ and $\tau(\bar{u}) = \tau_{i,j} < +\infty$, and (9.20) is proved.

We define

$$(9.23)\qquad \tau_i := \begin{cases} \max\{ \tau_{i,j} \mid i < j \le m \} & \text{if } 1 \le i \le m-1 \\ 0 & \text{if } i = m \end{cases}$$

From (9.20) we now conclude that for $t \ge \tau_i$ and every $j = i+1, \ldots, m$

$$(9.24)\qquad S_{i,j} \subset \left(\bigcup_{r=1}^{p_i} U_i^r\cdot(-t) \right) \cup N_\delta(S_j^{(k)}) .$$

Let $U_j^1, \ldots, U_j^{p_j} \subset E_k \times T^n$ denote a collection of closed contractible subsets such that $N_\delta(S_j^{(k)})$ is contained in the union of the U_j^r. Then (9.22) holds true with $N_\delta(S_j^{(k)})$ replaced by $U_j^1 \cup \ldots \cup U_j^{p_j}$. Moreover, since $S_{i,j}$ is invariant under the flow of $-\Phi_k'$, we conclude that

$$S_{i,j} \subset \left(\bigcup_{r_i=1}^{p_i} U_i^{r_i} \cdot (-t-s) \right) \cup \left(\bigcup_{r_j=1}^{p_j} U_j^{r_j} \cdot (-s) \right) \tag{9.25}$$

for all $s \in \mathbb{R}$ and $t \geq \tau_i$.
We define the subset $U \subset E_k \times T^n$ by

$$\begin{aligned} U \ := \ & \left(\bigcup_{r_1=1}^{p_1} U_1^{r_1} \cdot (-\tau_1 - \ldots - \tau_m) \right) \cup \left(\bigcup_{r_2=1}^{p_2} U_2^{r_2} \cdot (-\tau_2 - \ldots - \tau_m) \right) \cup \ldots \\ & \ldots \cup \left(\bigcup_{r_m=1}^{p_m} U_m^{r_m} \cdot (-\tau_m) \right) \end{aligned}$$

where $U_i^{r_i} \subset E_k \times T^n$ is contractible in $E_k \times T^n$ and $N_\delta(S_i^{(k)}) \subset U_i^1 \cup \ldots \cup U_i^{p_i}$.
We claim that

$$S^{(k)} \subset U\,. \tag{9.26}$$

For this purpose we note that we have a decomposition of $S^{(k)}$ into disjoint invariant subsets

$$S^{(k)} = \left(\bigcup_{i=1}^{m} S_i^{(k)} \right) \cup \left(\bigcup_{1 \leq i < j \leq m} S_{i,j} \right). \tag{9.27}$$

By the invariance of $S_i^{(k)}$ under the gradient flow we have for all $t \in \mathbb{R}$

$$S_i^{(k)} \subset \bigcup_{r_i=1}^{p_i} U_i^{r_i} \cdot (-t)\,. \tag{9.28}$$

Choosing $t = \tau_i + \ldots + \tau_m$ we conclude $S_i^{(k)} \subset U$ for every $i = 1, \ldots, m$.
Let $1 \leq i < j \leq m$ be given. With the choice $t = \tau_i + \ldots + \tau_{j-1}$ and $s = \tau_j + \ldots + \tau_m$ the inclusion $S_{i,j} \subset U$ follows from (9.25), and the claim is proved.
Note that the flow of $-\Phi_k'$ on $E_k \times T^n$ is a 1-parameter family of homeomorphisms of $E_k \times T^n$. Consequently each of the sets $U_i^{r_i} \cdot \tau$ introduced above is

closed and contractible for any $\tau \in \mathbb{R}$. Thus we have constructed a covering of the set $S^{(k)}$ by $p_1 + \ldots + p_m$ closed and contractible subsets of $E_k \times \mathrm{T}^n$. Therefore by the definition of cat

$$\mathrm{cat}_{E_k \times \mathrm{T}^n}(S^{(k)}) \leq p_1 + \ldots + p_m$$

and the lemma is proved.

∎

Summarizing the results obtained until now we have the following

Corollary 9.3 *Suppose $\Phi : E \times \mathrm{T}^n \to \mathbb{R}$ has only finitely many critical values $c_1 < \ldots < c_m$, corresponding to the critical sets $M_1, \ldots, M_m$. Then for $k \geq k_0$ we have*

$$\sum_{j=1}^{m} \mathrm{cat}_{E \times \mathrm{T}^n}(M_j) \geq \mathrm{cat}_{E_k \times \mathrm{T}^n}(S^{(k)}) . \tag{9.29}$$

Proof

Note that $S_j^{(k)} \subset E_k \times \mathrm{T}^n$ and $E_k \times \mathrm{T}^n$ is a retract of $E \times \mathrm{T}^n$ for every $k \in \mathbb{Z}^+$. Consequently by Lemma 3.1 we have

$$\mathrm{cat}_{E_k \times \mathrm{T}^n}(S_j^{(k)}) = \mathrm{cat}_{E \times \mathrm{T}^n}(S_j^{(k)}) . \tag{9.30}$$

Moreover by the monotonicity of cat we get from (9.9)

$$\mathrm{cat}_{E \times \mathrm{T}^n}(S_j^{(k)}) \leq \mathrm{cat}_{E \times \mathrm{T}^n}(N_{\delta,j}) . \tag{9.31}$$

Using (9.3), (9.17), (9.30) and (9.31) we have for $k \geq k_0$

$$\begin{aligned} \sum_{j=1}^{m} \mathrm{cat}_{E \times \mathrm{T}^n}(M_j) &\overset{(9.3)}{=} \sum_{j=1}^{m} \mathrm{cat}_{E \times \mathrm{T}^n}(N_{\delta,j}) \overset{(9.31)}{\geq} \sum_{j=1}^{m} \mathrm{cat}_{E \times \mathrm{T}^n}(S_j^{(k)}) \\ &\overset{(9.30)}{=} \sum_{j=1}^{m} \mathrm{cat}_{E_k \times \mathrm{T}^n}(S_j^{(k)}) \overset{(9.17)}{\geq} \mathrm{cat}_{E_k \times \mathrm{T}^n}(S^{(k)}) . \end{aligned}$$

∎

The remaining step in the proof of the estimate (9.2) consists in relating the category of the isolated invariant set $S^{(k)}$ to the cuplength of the torus T^n. This result corresponds to the Intersection Lemma in Section 3.

Proposition 9.2 *For any $k \in \mathbb{Z}^+$ let $S^{(k)} \subset E \times T^n$ denote the maximal invariant set of the flow of $-\Phi'_k$ which is contained in the isolating block B_k determined in Lemma 8.5 :*

$$B_k = D_k^+ \times D_k^- \times T^n \subset E_k^+ \times E_k^- \times T^n$$

where $D_k^\pm = \{ u^\pm \in E_k^\pm \mid \|u^\pm\| \leq R \}$ for some $R > 0$ sufficiently large. Then

$$\text{cat}_{E_k \times T^n}(S^{(k)}) \geq \text{cuplength}(T^n) + 1 . \tag{9.32}$$

Proof

We assume, by contradiction, that $\text{cat}_{E_k\times T^n}(S^{(k)}) \leq \text{cuplength}(T^n) = n$. In the following we shall abbreviate $l := \text{cat}_{E_k \times T^n}(S^{(k)})$.
Observe that by definition of the isolating block B_k, the compact invariant set $S^{(k)}$ is contained in the interior $\text{int}B_k$. Moreover note that B_k is a retract of $E_k \times T^n$, and therefore by Lemma 3.1

$$\text{cat}_{B_k}(S^{(k)}) = \text{cat}_{E_k \times T^n}(S^{(k)}) = l . \tag{9.33}$$

Using the continuity property of cat we can find an open neighborhood U of $S^{(k)}$ in B_k such that

$$\begin{aligned} \text{cat}_{B_k}(U) &= \text{cat}_{B_k}(S^{(k)}) \\ \text{and} \qquad & \overline{U} \subset \text{int}B_k . \end{aligned} \tag{9.34}$$

(a) First we consider the case $l \geq 1$, which in particular implies $S^{(k)} \neq \emptyset$.
By definition of cat there exist $l \leq n$ closed subsets $A'_j \subset B_k$, $j = 1, \ldots, l$ such that

$$A'_j \text{ is contractible in } B_k$$

$$\overline{U} \subset A'_1 \cup \ldots \cup A'_l$$

We can assume that $A'_j \subset \text{int}B_k$, since otherwise A'_j can be replaced by $A'_j \cap \overline{U}$. We point out that $A'_j \cap S^{(k)} \neq \emptyset$ for all $j = 1, \ldots, l$. Using the flow $u \cdot t$ of $-\Phi'_k$ we define the following compact sets

$$\begin{aligned} S^+ &:= \{ u \in B_k \mid u\cdot t \in B_k \text{ for all } t \geq 0 \} \\ S^- &:= \{ u \in B_k \mid u\cdot t \in B_k \text{ for all } t \leq 0 \} \end{aligned}$$

In particular we have $S^{(k)} = S^+ \cap S^-$.
The entrance set B_k^+ and the exit set B_k^- of the isolating block B_k are given by

$$\begin{aligned} B_k^+ &= \partial D_k^+ \times D_k^- \times \mathrm{T}^n \subset \partial B_k \\ B_k^- &= D_k^+ \times \partial D_k^- \times \mathrm{T}^n \subset \partial B_k \end{aligned}$$

Observe that $\partial B_k = B_k^+ \cup B_k^-$.
By means of the flow we can use the sets A'_j to construct a covering of S^+ by l contractible closed subsets A_j which do not intersect the exit set B_k^-: For $u \in S^+$ we define the real number $\tau^+(u)$ by

$$\tau^+(u) := \inf \left\{ \tau \in \mathrm{R} \mid u \cdot s \in \overline{U} \text{ for all } s \geq \tau \right\} . \tag{9.35}$$

We set

$$\tau^+ := \sup \left\{ \tau^+(u) \mid u \in S^+ \right\} . \tag{9.36}$$

Using the compactness of $S^+ \subset B_k$ we find that $\tau^+ < \infty$, and we conclude

$$S^+ \subset \overline{U} \cdot (-\tau^+) \subset \bigcup_{j=1}^{l} A'_j \cdot (-\tau^+) . \tag{9.37}$$

Note that $A'_j \cdot (-\tau^+) \subset E_k \times \mathrm{T}^n$ is a closed subset which is contractible in $E_k \times \mathrm{T}^n$, and that $A'_j \cdot (-\tau^+) \cap B_k^- = \emptyset$. Now we define

$$A_j := A'_j \cdot (-\tau^+) \cap B_k \quad \text{for} \quad j = 1, \ldots, l . \tag{9.38}$$

Then $A_j \subset B_k$ is closed and contractible in B_k, and

$$S^+ \subset A_1 \cup \ldots \cup A_l .$$

Note that $A_j \cap S^{(k)} \neq \emptyset$ and consequently $A_j \neq \emptyset$ for all j, which follows from the fact that $A'_j \cap S^{(k)} \neq \emptyset$.

Considering now $B_k \setminus S^+$ we have the following

Lemma 9.4 (Wazewski's principle) *The entrance set B_k^+ is a strong deformation retract of $B_k \setminus S^-$, and the exit set B_k^- is a strong deformation retract of $B_k \setminus S^+$.*

Proof
Define the continuous function $t^+ : B_k \setminus S^+ \to \mathbf{R}$ by

$$t^+(u) = \sup \{ \, t \geq 0 \mid u \cdot [0,t] \subset B_k \, \} \ .$$

Then $t^+(u) = 0$ if and only if $u \in B_k^-$. The deformation retraction

$$F : (B_k \setminus S^+) \times [0,1] \to B_k \setminus S^+$$

which retracts $B_k \setminus S^+$ to B_k^- is given by $F(u,s) := u \cdot \{s\, t^+(u)\}$. This proves that B_k^- is a deformation retract of $B_k \setminus S^+$. The second part of the lemma is proved similarly.

∎

As a consequence of this lemma we have

Corollary 9.4 *There exists a closed set A satisfying*

$$B_k^- \ \subset \ A \ \subset \ B_k$$

such that B_k^- is a retract of A, and such that moreover

$$A \cup A_1 \cup \ldots \cup A_l \ = \ B_k \ .$$

Proof
Let U be the open neighborhood of $S^{(k)}$ chosen in (9.34). Then we define

$$A := B_k \setminus \Big(U \cdot (-\tau^+) \Big) \ . \tag{9.39}$$

Observe that $A \subset B_k$ is closed. Since $S^+ \subset B_k \setminus A$ it follows immediately from Wazewski's principle that B_k^- is a strong deformation retract of A.

∎

In the following H^* and H_* will denote singular cohomology and homology respectively. Corresponding to Lemma 3.3 we have

Lemma 9.5

(i) *The injections* $\iota : (B_k, \emptyset) \to (B_k, A_j)$ *for* $j = 1, \dots, l$ *induce homomorphisms*

$$\iota^* : H^*(B_k, A_j) \to H^*(B_k)$$

which are onto for $* \geq 1$.

(ii) *The injection* $g : (B_k, B_k^-) \to (B_k, A)$ *induces isomorphisms in relative cohomology*

$$g^* : H^*(B_k, A) \to H^*(B_k, B_k^-) \, .$$

The proof of (i) is the same as the proof of the corresponding claim in Lemma 3.3. The statement (ii) follows immediately from the observation that B_k^- is a deformation retract of A.

After these preparations we are ready to establish the argument which contradicts the assumption $l \leq n$.
Observe that $\{0\} \times D_k^- \times \mathrm{T}^n$ is a strong deformation retract of B_k, and similarly $\{0\} \times \{0\} \times \mathrm{T}^n$ is a strong deformation retract of B_k. Consequently we have isomorphisms which commute with the cup-product

$$H^*(B_k) \cong H^*(D_k^- \times \mathrm{T}^n) \cong H^*(\mathrm{T}^n)$$

and

$$H^*(B_k, B_k^-) \cong H^*(D_k^- \times \mathrm{T}^n, \partial D_k^- \times \mathrm{T}^n) \, .$$

Since cuplength$(\mathrm{T}^n) =$ cuplength$(D_k^- \times \mathrm{T}^n)$ there exist cohomology classes

$$\omega_j \in H^{q_j}(D_k^- \times \mathrm{T}^n) \ , \ j = 1, \dots, n \ , \ q_j \geq 1 \ ,$$

such that $\omega_1 \cup \dots \cup \omega_n \neq 0$. In particular, if $l < n$ we consider the first l factors of this cup-product only. Of course the cup-product $\omega_1 \cup \dots \cup \omega_l \neq 0$. Let $q := q_1 + \dots + q_l$. Consequently there exists a homology class $\alpha \in H_q(D_k^- \times \mathrm{T}^n)$ such that

(9.40) $$< \omega_1 \cup \dots \cup \omega_l \ , \ \alpha > \neq 0$$

where $< \cdot, \cdot >$ denotes the duality pairing

$$H^*(D_k^- \times \mathrm{T}^n) \times H_*(D_k^- \times \mathrm{T}^n) \to \mathbb{Z} \, .$$

Recall that $\dim E_k^{\pm} = k$. Let $\xi \in H_{n+k}(D_k^- \times \mathrm{T}^n, \partial D_k^- \times \mathrm{T}^n)$ denote the fundamental class. By Lefschetz duality there is an isomorphism given by the cap-product

$$H^{n+k-q}(D_k^- \times \mathrm{T}^n, \partial D_k^- \times \mathrm{T}^n) \xrightarrow{\cap \xi} H_q(D_k^- \times \mathrm{T}^n)\,.$$

Hence there exists $\omega_0 \in H^{n+k-q}(D_k^- \times \mathrm{T}^n, \partial D_k^- \times \mathrm{T}^n)$ such that $\alpha = \omega_0 \cap \xi$, and we have

$$(9.41)\qquad \begin{aligned} <\omega_1 \cup \ldots \cup \omega_l\,,\, \alpha> &= <\omega_1 \cup \ldots \cup \omega_l\,,\, \omega_0 \cap \xi> \\ &= <\omega_0 \cup \omega_1 \cup \ldots \cup \omega_l\,,\, \xi> \,. \end{aligned}$$

We claim that $\omega_0 \cup \omega_1 \ldots \cup \omega_l = 0$.
Consider the commutative diagram

$$\begin{array}{ccccc}
H^{n+k-q}(B_k, A) & \xrightarrow{g^*} & H^{n+k-q}(B_k, B_k^-) & \xrightarrow{\rho^{*-1}} & H^{n+k-q}(D_k^- \times \mathrm{T}^n, \partial D_k^- \times \mathrm{T}^n) \\
\otimes & & \otimes & & \otimes \\
H^{q_1}(B_k, A_1) & \xrightarrow{\iota^*} & H^{q_1}(B_k) & \xrightarrow{r^{*-1}} & H^{q_1}(D_k^- \times \mathrm{T}^n) \\
\otimes & & \otimes & & \otimes \\
\vdots & & \vdots & & \vdots \\
\otimes & & \otimes & & \otimes \\
H^{q_l}(B_k, A_l) & \xrightarrow{\iota^*} & H^{q_l}(B_k) & \xrightarrow{r^{*-1}} & H^{q_l}(D_k^- \times \mathrm{T}^n) \\
\Big\downarrow \cup & & \Big\downarrow \cup & & \Big\downarrow \cup \\
H^{n+k}(B_k, B_k) & \longrightarrow & H^{n+k}(B_k, B_k^-) & \xrightarrow{\rho^{*-1}} & H^{n+k}(D_k^- \times \mathrm{T}^n, \partial D_k^- \times \mathrm{T}^n)
\end{array}$$

Here $\iota^* : H^*(B_k, A_j) \to H^*(B_k)$ denotes the surjective homomorphism from Lemma 9.6 (i), and $g^* : H^*(B_k, A) \to H^*(B_k, B_k^-)$ is the isomorphism from Lemma 9.6 (ii). Let $r : [0,1] \times B_k \to B_k$ denote the deformation retraction from B_k to $D_k^- \times \mathbf{T}^n$, and let r^* and ρ^* be the isomorphisms

$$r^* : H^*(D_k^- \times \mathbf{T}^n) \rightarrow H^*(B_k)$$
$$\rho^* : H^*(D_k^- \times \mathbf{T}^n, \partial D_k^- \times \mathbf{T}^n) \rightarrow H^*(B_k, B_k^-)$$

induced by $r(1,\cdot) : B_k \rightarrow D_k^- \times \mathbf{T}^n$.
For $j = 1, \ldots, l$ we can find cohomology classes $\tilde{\omega}_j \in H^{q_j}(B_k, A_j)$ such that

$$\omega_j = (r^{*^{-1}} \circ \iota^*)(\tilde{\omega}_j) \in H^{q_j}(D_k^- \times \mathbf{T}^n) . \tag{9.42}$$

Similarly we define

$$\tilde{\omega}_0 := (g^{*^{-1}} \circ \rho^*)(\omega_0) \in H^{n+k-q}(B_k, A) . \tag{9.43}$$

Consequently we have

$$\tilde{\omega}_0 \cup \tilde{\omega}_1 \cup \ldots \cup \tilde{\omega}_l \in H^{n+k}(B_k, A \cup A_1 \cup \ldots \cup A_l) = H^{n+k}(B_k, B_k) = \{0\}$$

and therefore the cup-product vanishes. By the commutativity of the above diagram then also $\omega_0 \cup \omega_1 \cup \ldots \cup \omega_l = 0$ contradicting (9.40).

(b) We still have to consider the case $l = 0$. Then $S^{(k)} = \emptyset$ and consequently $S^+ = \emptyset$, which by Wazewski's principle implies that B_k^- is a strong deformation retract of B_k. Consequently we can choose $A = B_k$ in the above proof, and therefore we have

$$H^*(B_k, B_k) \cong H^*(B_k, B_k^-) \cong H^*(D_k^- \times \mathbf{T}^n, \partial D_k \times \mathbf{T}^n)$$

which obviously is a contradiction.
This finishes the proof of Proposition 9.2.

∎

We have shown that (9.2) holds, and the first case of Proposition 9.1 is proved.

(ii) If the action functional Φ possesses infinitely many critical values then there exist correspondingly infinitely many critical points, and we are already done.

(iii) Assuming that Φ has no critical values at all, we conclude by the A-properness of Φ with respect to the Galerkin approximation scheme that there has to be a $k_0 \in \mathbb{Z}^+$ such that for $k \geq k_0$ the functions Φ_k have no critical points. This, however, contradicts Proposition 9.2, and the proof is complete.

■

To complete this section we consider the case where the action functional $\Phi : E_k \times T^n \to \mathbb{R}$ is only assumed to be continuously differentiable. We shall construct a Lipschitz-continuous vector field $V_k : E_k \times T^n \to E_k \times T^n$ close to the gradient vector field Φ_k' such that in the proof of Proposition 9.1 the negative gradient flow can be replaced by the flow of $-V_k$. The construction of V_k is similar to well-known constructions of pseudo-gradient vector fields, which originally have been introduced by R. Palais [47] in connection with minimax methods. However, the intended application considered here requires a somewhat more sophisticated determination of what is meant by saying V_k is close to the gradient vector field :
First note that in order to have the flow of $-\Phi_k'$ replaced by that of $-V_k$, the vector field V_k has to be defined on all of $E_k \times T^n$; the set of critical points is not an exceptional set. Secondly, it is essential to guarantee that the flow of $-V_k$ admits a very special isolating block $B_k = D_k^+ \times D_k^- \times T^n$.

We shall carry out the construction of V_k in the covering space $W_k = E_k^+ \oplus E_k^- \oplus E^0$. In detail, the vector field $V_k : W_k \to W_k$ has to satisfy the following assumptions :

(i) $V_k : W_k \to W_k$ is Lipschitz-continuous.

(ii) For any $u \in W_k$ we have

$$\| V_k(u) \| \leq 2 \| \Phi_k'(u) \| . \tag{9.44}$$

(iii) Let η denote the flow of $-V_k$, i.e.

$$\begin{cases} \dfrac{d}{dt}\eta^t(u) &= -V_k(\eta^t(u)) \\ \eta^0(u) &= u\,. \end{cases}$$

Then Φ_k is a Lyapunov function for the flow η, i.e.

$$\frac{d}{dt}\bigg|_{t=0} \Phi_k(\eta^t(u)) \le 0\,. \tag{9.45}$$

(iv) By $K = \{\, u \in W \mid \Phi'(u) = 0 \,\}$ we denote the critical set of Φ. Let $\delta > 0$ be a given parameter and let $N_{\delta/4}(K)$ denote the $\delta/4$-neighborhood of K. Let $k_1(\delta)$ be the integer determined in Lemma 9.1. Then for $k \ge k_1(\delta)$ there are no critical points of Φ_k contained in $W_k \setminus N_{\delta/4}(K)$, and

$$(\,\Phi_k'(u)\,,\, V_k(u)\,) \ge \|\,\Phi_k'(u)\,\|^2 \quad \text{if} \quad u \in W_k \setminus N_{3\delta/8}(K)\,. \tag{9.46}$$

(v) Let $u = u^+ + u^- + u^0 \in E_k^+ \oplus E_k^- \oplus E^0$. There exists $R' > 0$ such that

$$\Big(-V_k(u)\,,\, u^+\Big) < 0 \tag{9.47}$$

if $\|u^+\| \ge R'$ and $\|u^+\| \ge \|u^-\|$, and

$$\Big(-V_k(u)\,,\, u^-\Big) > 0 \tag{9.48}$$

if $\|u^-\| \ge R'$ and $\|u^-\| \ge \|u^+\|$.

(vi) V_k is $\mathbb{Z}^n$-invariant and thus projects to a vector field on $E_k \times T^n$ under the canonical mapping $W_k \to E_k \times T^n$.

Note that V_k is a pseudo-gradient vector field on $W_k \setminus N_{3\delta/8}(K)$ by the conditions (9.44) and (9.46). The choice of the neighborhoods in (iv) is done with respect to the application in Lemma 9.2. We note that the flow η^t exists for all $t \in \mathbb{R}$, which follows from (9.44) and the fact that Φ_k' is linearly bounded. The conditions (9.47) and (9.48) are sufficient to show that η admits the desired isolating block.

Proposition 9.3 *There exists a $\mathbb{Z}^n$-invariant Lipschitz-continuous vector field $V_k : W_k \to W_k$ satisfying the conditions* (9.44)-(9.48).

Proof

The proof will use a partition of the unity. We begin with defining the sets needed for the covering.

Consider for $u_0 \in W_k$ the vector $\frac{3}{2}\Phi_k'(u_0)$. Then there exists an open neighborhood $U^1_{u_0}$ of u_0 such that

$$(9.49) \qquad \| \frac{3}{2}\Phi_k'(u_0) \| \le 2 \| \Phi_k'(u) \| \quad \text{if} \quad u \in U^1_{u_0}$$

If $\Phi_k'(u_0) \neq 0$ then there exists an open neighborhood $U^2_{u_0}$ of u_o such that

$$(9.50) \qquad (\frac{3}{2}\Phi_k'(u_0) \, , \, \Phi_k'(u)) \ge \| \Phi_k'(u) \|^2 \quad \text{if} \quad u \in U^2_{u_0}$$

Let $u_0 = u_0^+ + u_0^- + u_0^0 \in E_k^+ \oplus E_k^- \oplus E^0$. If $(\Phi_k'(u_0) \, , \, u_0^+) \neq 0$ then there exists an open neighborhood $U^+_{u_0}$ of u_0 such that

$$(9.51) \qquad \left| (\frac{3}{2}\Phi_k'(u_0) \, , \, u^+) - (\Phi_k'(u) \, , \, u^+) \right| \le \frac{3}{4} \left| (\Phi_k'(u) \, , \, u^+) \right|$$

for $u = u^+ + u^- + u^0 \in U^+_{u_0}$. If $(\Phi_k(u_0) \, , \, u_0^-) \neq 0$ then we have correspondingly an open neighborhood $U^-_{u_0}$ of u_0 such that

$$(9.52) \qquad \left| (\frac{3}{2}\Phi_k'(u_0) \, , \, u^-) - (\Phi_k'(u) \, , \, u^-) \right| \le \frac{3}{4} \left| (\Phi_k'(u) \, , \, u^-) \right|$$

for $u \in U^-_{u_0}$. We briefly write $U^3_{u_0}$ meaning one of the neighborhoods $U^+_{u_0}$ or $U^-_{u_0}$ or $U^+_{u_0} \cap U^-_{u_0}$. Note that all the neighborhoods $U^i_{u_0}$ can be chosen $\mathbb{Z}^n$-equivariant, i.e. if $g \in E^0 \cong \mathbb{R}^n$ denotes an integer vector, then we will assume that $U^i_{u_0+g} = U^i_{u_0} + g$.

Finally we consider the open balls of radius $\delta/8$ centered at u_0 which will be denoted by $B(u_0, \delta/8)$. We now define open neighborhoods of u_0 by

$$(9.53) \qquad U_{u_0} := \begin{cases} U^1_{u_0} \cap U^2_{u_0} \cap U^3_{u_0} \cap B(u_0, \delta/8) & \text{if} \quad (\Phi_k'(u_0) \, , \, u_0^\pm) \neq 0 \\ U^1_{u_0} \cap U^2_{u_0} \cap B(u_0, \delta/8) & \text{if} \quad \Phi_k'(u_0) \neq 0 \\ B(u_0, \delta/8) & \text{if} \quad \Phi_k'(u_0) = 0 \end{cases}$$

The collection of open sets $\{ U_{u_0} \mid u_0 \in W_k \}$ constitutes an open covering of W_k. Since W_k is paracompact this open covering contains a locally finite refinement, which will be denoted by $\mathcal{U} = \{ U_i \mid i \in I \}$, where $U_i := U_{u_i}$. This refinement can be chosen $\mathbb{Z}^n$-equivariant, i.e.

$$U_{u_i} \in \mathcal{U} \quad \text{if and only if} \quad U_{u_i+g} \in \mathcal{U}$$

for every integer vector $g \in E^0$.
Let $\Psi = \{ \psi_i \mid i \in I \}$ be a Lipschitz-continuous partition of the unity subordinated to the locally finite open covering $\mathcal{U}$. In view nof the $\mathbb{Z}^n$-equivariant choice of the sets U_i the functions $\psi_i \in \Psi$ can be chosen $\mathbb{Z}^n$-invariant. Then we define the vector field V_k by

$$V_k(u) := \sum_{i \in I} \psi_i(u) \left(\frac{3}{2} \Phi_k'(u_i) \right) . \tag{9.54}$$

The verification of the properties (9.44) and (9.45) is standard. Considering (9.46) we note that the critical points of Φ_k are all contained in $N_{\delta/4}(K)$ which by the definition of the open sets U_i implies that there are no rest points of the flow η^t of $-V_k$ in $W_k \setminus N_{3\delta/8}(K)$.
We emphasize that (9.51) and (9.52) are only needed to show that the conditions (9.47) and (9.48) are satisfied. Let $c > 0$ be an arbitrarily chosen constant. The first estimate in the proof of Lemma 8.5 allows to obtain a stronger statement than (8.29), (8.30), namely that there exists $R > 0$ such that for a given constant $r \geq 0$:

$$\begin{aligned} &(\Phi_k'(u), u^+) > c \quad \text{if} \quad \|u^+\| \geq R \\ &\text{where} \quad u = u^+ + u^- + u^0 \quad \text{with} \quad \|u^-\| \leq \|u^+\| + r . \end{aligned} \tag{9.55}$$

In addition we have with the same constant $R > 0$:

$$\begin{aligned} &-(\Phi_k'(u), u^-) > c \quad \text{if} \quad \|u^-\| \geq R \\ &\text{where} \quad u = u^+ + u^- + u^0 \quad \text{with} \quad \|u^+\| \leq \|u^-\| + r . \end{aligned} \tag{9.56}$$

We shall prove (9.47). Let δ be the constant used in (9.53), and choose $r := \delta/4$. Fix an arbitrary constant $c > 0$, and let $R > 0$ such that (9.55) holds for this choice of c and r.

Now consider $u = u^+ + u^- + u^0 \in E_k^+ \oplus E_k^- \oplus E^0$ which satisfies $\|u^+\| \geq R + \delta/8$ and $\|u^-\| \leq \|u^+\|$. Then

$$V_k(u) = \sum_{i \in I_u} \psi_i(u) \left(\frac{3}{2} \Phi_k'(u_i)\right)$$

where $I_u \subset I$ is a finite subset. Consequently $u \in U_i$ for $i \in I_u$, and therefore $\| u - u_i \| < \delta/8$ by the definition of the neighborhoods U_i. Hence it follows that $\|u_i^+\| \geq R$ and $\|u_i^-\| \leq \|u_i^+\| + \delta/4$, and consequently by (9.55) :

$$(\Phi_k'(u_i), u_i^+) > c > 0 \quad \text{for} \quad i \in I_u . \tag{9.57}$$

From (9.51) we now conclude

$$\left| (\frac{3}{2} \Phi_k'(u_i), u^+) - (\Phi_k'(u), u^+) \right| \leq \frac{3}{4} \left| (\Phi_k'(u), u^+) \right| \tag{9.58}$$

for all $i \in I_u$. Summarizing we finally have

$$\begin{aligned} (-V_k(u), u^+) &\leq -(\Phi_k'(u), u^+) + \left| (V_k(u) - \Phi_k'(u), u^+) \right| \\ &\leq -(\Phi_k'(u), u^+) + \sum_{i \in I_u} \psi_i(u) \left| (\frac{3}{2} \Phi_k'(u_i) - \Phi_k'(u), u^+) \right| \\ &\leq -(\Phi_k'(u), u^+) + \sum_{i \in I_u} \psi_i(u) \frac{3}{4} \left| (\Phi_k'(u), u^+) \right| \\ &= \left(\frac{3}{4} - 1\right) (\Phi_k'(u), u^+) < -\frac{c}{4} < 0 . \end{aligned}$$

Hence (9.47) is proved with $R' = R + \delta/8$. The condition (9.48) is proved similarly, which finishes the proof of Proposition 9.3.

■

Finally we want to point out the abstract content of Lemma 9.3. For this purpose we consider the following situation :

Let M be a compact space and consider a continuous flow defined on M. Assume there exists a Morse decomposition $\{M_1, \ldots, M_k\}$ of M with respect

to this flow. Recall that a Morse decomposition of M is a finite collection of disjoint compact invariant subsets which can be ordered, say $(M_1, \ldots, M_k)$, such that the following condition is satisfied :
For any $x \in M \setminus (M_1 \cup \ldots \cup M_k)$ there exist indices $1 \leq i < j \leq k$ such that

$$\omega^+(x) \subset M_i \quad \text{and} \quad \omega^-(x) \subset M_j$$

where $\omega^{\pm}(x)$ denote the positive and negative limit set of x respectively. Then the ordering $(M_1, \ldots, M_k)$ is called an admissible ordering of the Morse decomposition.

Lyusternik-Schnirelman theory for this situation has been established by C. Conley and E. Zehnder. In [14], Theorem 5, they give a proof of the estimate

$$l(M) \leq \sum_{j=1}^{k} l(M_j) \tag{9.59}$$

where $l(A)$:= cuplength(A)+1 denotes the cohomological category of the set A, defined in the Alexander cohomology. Suppose now in addition that M is an absolute neighborhood retract. Then the construction used in the proof of Lemma 9.3 provides the corresponding inequality with the cohomological category replaced by the geometrical Lyusternik-Schnirelman category :

$$\text{cat}_M(M) \leq \sum_{j=1}^{k} \text{cat}_M(M_j) \ . \tag{9.60}$$

Note that the continuity property of the category is crucial. In case of the cohomological category it is guaranteed by the use of Alexander cohomology, and for the Lyusternik-Schnirelman category the assumption on M to be an ANR is sufficient.
In particular the estimate (9.60) allows to recover the original result of L.A. Lyusternik and L.G. Schnirelman, which states that the number of critical points of a C^1-function $f : M \to \mathbb{R}$ on a compact closed differentiable manifold M is at least $\text{cat}_M(M)$. Recall that the manifold M can be endowed with a Riemannian structure, and the continuous flow can be defined as the flow of a gradient-like vector field which satisfies the conditions (i)-(iv) stated before Proposition 9.3.

Appendix

(a) Proof of the second part of Lemma 2.2

We denote $W = W^{\frac{1}{2},2}(S^1, R^{2n})$ for short. Recall from Section 2 the nonlinear part $\varphi \in C^1(W, R)$ of the action functional which depends on the Hamiltonian H:

$$\text{(A.1)} \qquad \varphi(u) = \int_0^1 H(u(t)+e(t), t)\, dt \quad , \quad u \in W\,,$$

where $e(t) = (jt, 0) \in R^n \times R^n$ with a fixed rotation vector j.
Its gradient $\varphi' : W \to W$ is represented by

$$\text{(A.2)} \qquad (\varphi'(u), w) = \int_0^1 \langle \nabla H(u+e, t), w \rangle\, dt \quad \text{for} \quad u, w \in W\,.$$

We claim that φ' is a compact map $W \to W$.
First we note that if $z \in W$ is a loop $z(t) = (x(t), y(t)) \in R^n \times R^n$ then the asymptotic conditions (1.8) on $\partial_x H$ and $\partial_y H$ imply that

$$|\nabla H(z(t)+e(t), t)| \leq a_1 + a_2\, |y(t)|$$

with some constants a_1 and a_2. It then follows that the map $z \mapsto \nabla H(z+e, \cdot)$ is in $C(L^p(S^1, R^{2n}), L^p(S^1, R^{2n}))$ for every $p > 1$, see e.g. Proposition B1 in [52]. In particular $\nabla H(z+e, \cdot) \in L^2(S^1, R^{2n})$ for every $z \in L^2(S^1, R^{2n})$.

Consider a weakly convergent sequence $u_k \rightharpoonup u$ in W. We have to show that $\varphi'(u_k) \to \varphi'(u)$ in the norm of W. Let w_k be any sequence in W. Then

$$\begin{aligned} |(\varphi'(u_k) - \varphi'(u), w_k)| &= \left| \int_0^1 \langle \nabla H(u_k+e, t) - \nabla H(u+e, t), w_k(t) \rangle\, dt \right| \\ &\leq \int_0^1 |\nabla H(u_k+e, t) - \nabla H(u+e, t)|\, |w_k(t)|\, dt\,. \end{aligned}$$

Applying the Cauchy-Schwarz inequality we obtain

$$\int_0^1 |\nabla H(u_k+e,t) - \nabla H(u+e,t)|\, |w_k(t)|\, dt$$

$$\leq \left(\int_0^1 |\nabla H(u_k+e,t) - \nabla H(u+e,t)|^2 dt\right)^{1/2} \|w_k\|_2$$

$$\leq \|\nabla H(u_k+e,\cdot) - \nabla H(u+e,\cdot)\|_2\, \|w_k\|\,.$$

Since the embedding $W \to L^2(S^1, \mathbb{R}^{2n})$ is compact, $u_k \rightharpoonup u$ weakly in W implies $u_k \to u$ in $L^2(S^1, \mathbb{R}^{2n})$. Choosing $w_k := \varphi'(u_k) - \varphi'(u)$ we conclude by the above continuity that

$$0 \leq \|\varphi'(u_k) - \varphi'(u)\| \leq \|\nabla H(u_k+e,\cdot) - \nabla H(u+e,\cdot)\|_2 \to 0$$

as $k \to \infty$, which proves our claim.

(b) Compact embeddings of $W^{\frac{1}{2},2}(S^1, \mathbb{R}^{2n})$

We have repeatedly used the following

Theorem A.1 *The embedding $W^{\frac{1}{2},2}(S^1, \mathbb{R}^{2n}) \to L^q(S^1, \mathbb{R}^{2n})$ is compact for $1 \leq q < \infty$.*

Proof
Recall the notation from Section 2, (2.1)-(2.5). We consider three different cases :

(i) The case $q = 2$
It follows from the definition of the Sobolev-norm and Plancherel's formula that

$$\text{(A.3)} \qquad \|z\|_2^2 = \sum_{k\in\mathbb{Z}} |z_k|^2 \leq |z_0|^2 + 2\pi \sum_{k\in\mathbb{Z}} |k|\, |z_k|^2 = \|z\|^2$$

for $z \in W^{\frac{1}{2},2}(S^1, \mathbb{R}^{2n})$. This shows that the embedding $I : W^{\frac{1}{2},2}(S^1, \mathbb{R}^{2n}) \to L^2(S^1, \mathbb{R}^{2n})$ is continuous. In order to prove the compactness of I we show the existence of a sequence $I_m \in \mathcal{L}(W^{\frac{1}{2},2}(S^1,R^{2n}), L^2(S^1,R^{2n}))$ of operators of finite rank satisfying $I_m \to I$ in $\mathcal{L}(W^{\frac{1}{2},2}(S^1,R^{2n}), L^2(S^1,R^{2n}))$ as $m \to \infty$. For $z \in W^{\frac{1}{2},2}(S^1,R^{2n})$ with Fourier coefficients $(z_k)_{k\in\mathbb{Z}}$, we define

$$(A.4) \qquad (I_m z)(t) := \sum_{|k|<m} \exp(2\pi k t J)\, z_k \quad , \quad m = 1,2,\dots$$

If $z = (z_k)_{k\in\mathbb{Z}} \in W^{\frac{1}{2},2}(S^1,R^{2n})$, we have

$$\|(I - I_m)z\|_2^2 = \sum_{|k|\geq m} |z_k|^2 \leq \frac{1}{2\pi m} \sum_{|k|\geq m} 2\pi|k|\,|z_k|^2 \leq \frac{1}{2\pi m}\|z\|^2 ,$$

and consequently $\|I - I_m\| \to 0$ as $m \to \infty$. This proves the compactness of I, and the proof for $q = 2$ is finished.

(ii) The case $1 \leq q < 2$
Recall that $L^2(S^1,R^{2n})$ embeds continuously into $L^q(S^1,R^{2n})$ for $1 \leq q \leq 2$; the estimate

$$\left(\int_0^1 |z(t)|^q\, dt\right)^{1/q} \leq \left(\int_0^1 |z(t)|^2\, dt\right)^{1/2}$$

for $z \in L^2(S^1,R^{2n})$ is an easy consequence of Hölder's inequality. Combining this continuous embedding with the result from (i) we conclude that the considered embedding

$$(A.5) \qquad W^{\frac{1}{2},2}(S^1,R^{2n}) \to L^2(S^1,R^{2n}) \to L^q(S^1,R^{2n})$$

is compact for $1 \leq q < 2$.

(iii) The case $2 < q < \infty$
We introduce the following notation: if $z = (z_k)_{k\in\mathbb{Z}} \in W^{\frac{1}{2},2}(S^1,R^{2n})$ then we define $z^{(\frac{1}{2})} \in L^2(S^1,R^{2n})$ by its Fourier coefficients as follows

$$(A.6) \qquad (z^{(\frac{1}{2})})_k := \begin{cases} \sqrt{2\pi|k|}\, z_k & \text{if } k \neq 0 \\ z_0 & \text{if } k = 0 . \end{cases}$$

By this definition we have $\|z\| = \|z^{(\frac{1}{2})}\|_2$ for all $z \in W^{\frac{1}{2},2}(S^1,R^{2n})$. We restrict ourselves for a moment to the special case $z_0 = 0$. Then we apply Corollary (9.22) in Zygmund [61], p. 142, which, adapted to our situation, can be formulated as

Lemma A.6 *Assume* $z = (z_k)_{k\in \mathbb{Z}} \in W^{\frac{1}{2},2}(S^1, \mathbb{R}^{2n})$ *satisfies* $z_0 = 0$, *and suppose that* $1 < r < q < \infty$ *with* $1/r - 1/q = 1/2$. *Then* $z^{(\frac{1}{2})} \in L^r(S^1, \mathbb{R}^{2n})$ *implies* $z \in L^q(S^1, \mathbb{R}^{2n})$, *and moreover there exists a constant* $A_{r,q}$ *depending on* r *and* q, *such that*

$$\|z\|_q \leq A_{r,q} \|z^{(\frac{1}{2})}\|_r \tag{A.7}$$

where $\|\cdot\|_p$ *denotes the* L^p*-norm.*

If $2 < q < \infty$ is given, we choose $r = 2q/(q+2) < 2$. Then (A.7) and (A.5) yield

$$\|z\|_q \leq A_{r,q} \|z^{(\frac{1}{2})}\|_r \leq A_{r,q} \|z^{(\frac{1}{2})}\|_2 = A_{r,q} \|z\|$$

for every $z \in W^{\frac{1}{2},2}(S^1, \mathbb{R}^{2n})$ with $z_0 = 0$.
If $z_0 \in \mathbb{R}^{2n}$ is arbitrary, we write $\zeta = z - z_0 \in W^{\frac{1}{2},2}(S^1, \mathbb{R}^{2n})$. Then it follows that

$$\|z\|_q \leq |z_0| + \|\zeta\|_q \leq |z_0| + A_{r,q} \|\zeta\| \leq A'_{r,q} \|z\| \tag{A.8}$$

with some appropriate constant $A'_{r,q}$. Thus we have shown that the embedding is continuous for $2 < q < \infty$. To prove the compactness we observe that for every $z \in W^{\frac{1}{2},2}(S^1, \mathbb{R}^{2n})$ we have

$$\|z\|_q \leq \|z\|_{2q-2}^{(q-1)/q} \, \|z\|_2^{1/2q} , \tag{A.9}$$

which follows from the Cauchy-Schwarz inequality. Then by the continuity of the embedding $W^{\frac{1}{2},2}(S^1, \mathbb{R}^{2n}) \to L^{2q-2}(S^1, \mathbb{R}^{2n})$ and by the compactness of $W^{\frac{1}{2},2}(S^1, \mathbb{R}^{2n}) \to L^2(S^1, \mathbb{R}^{2n})$ the assertion of the theorem follows.

■

(c) Some properties of the Lyusternik-Schnirelman category

We first recall the following

Definition A.1 *Let* M *be a topological space and* $A \subset M$. *If* $A \neq \emptyset$, *the Lyusternik-Schnirelman category* $\mathrm{cat}_M(A)$ *of* A *in* M *is defined to be the least positive integer* k *such that there exist closed subsets* $B_j \subset M$, $j = 1, \ldots, k$, *being contractible in M, which satisfy* $A \subset B_1 \cup \ldots \cup B_k$.
If no such integer exists, we define $\mathrm{cat}_M(A) := \infty$.
Moreover we set $\mathrm{cat}_M(\emptyset) := 0$.

Recall that a subset $B \subset M$ is called contractible in M if the inclusion map $B \to M$ is homotopic to a constant map.

We list some properties of cat :

(i) $\mathrm{cat}_M(A) = 1$ if and only if the closure $\bar{A}$ is contractible in M.

(ii) $\mathrm{cat}_M(A) = \mathrm{cat}_M(\bar{A})$.

(iii) If A is closed in M then $\mathrm{cat}_M(A) \leq k$ if and only if A is the union of k closed sets each contractible in M.

(iv) cat is monotone, i.e. if $A \subset B \subset M$ then $\mathrm{cat}_M(A) \leq \mathrm{cat}_M(B)$.

(v) cat is subadditive, i.e. $\mathrm{cat}_M(A \cup B) \leq \mathrm{cat}_M(A) + \mathrm{cat}_M(B)$.

(vi) If A is closed and $h_t : A \to M$, $0 \leq t \leq 1$, is a homotopy such that h_0 is the inclusion map of A into M, then $\mathrm{cat}_M(h_1(A)) \geq \mathrm{cat}_M(A)$.

(vii) If $h_t : M \to M$, $0 \leq t \leq 1$ is a homotopy of M such that $h_0 = \mathrm{id}_M$, then $\mathrm{cat}_M(h_1(A)) \geq \mathrm{cat}_M(A)$ for every $A \subset M$. A closed in Palais but see (ii)

(viii) If $h : M \to M$ is a homeomorphism of M, then $\mathrm{cat}_M(h(A)) = \mathrm{cat}_M(A)$ for every $A \subset M$.

(ix) Let M be arcwise connected. Then $\mathrm{cat}_M(A) = 1$ if $A \neq \emptyset$ is a finite set.

(x) Let M be an absolute neighborhood retract (ANR). If $A \neq \emptyset$ then $\mathrm{cat}_M(A) = \mathrm{cat}_M(U)$ for some neighborhood U of A in M.

(xi) Let M be a metric ANR. If $K \neq \emptyset$ is compact then there exists $\delta > 0$ such that $\mathrm{cat}_M(K) = \mathrm{cat}_M(U_\delta(K))$, where $U_\delta(K)$ denotes the δ-neighborhood of K in M.

For the proofs we refer to R. S. Palais [47] and K. Deimling [17].

$\mathrm{cat}_M(M) = \mathrm{cat}(M)$

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