LYUSTERNIK-SCHNIRELMAN THEORY FOR FLOWS AND PERIODIC ORBITS FOR HAMILTONIAN SYSTEMS ON $\mathbb{T}^n \times \mathbb{R}^n$

FRANK W. JOSELLIS

[Received 7 May 1992—Revised 11 December 1992]

ABSTRACT

We present a Lyusternik–Schnirelman approach for flows which generalizes the classical result concerning the number of critical points for a differentiable function on a compact manifold. This method can also be extended to special gradient-like flows on non-compact manifolds. Our main goal is the application to the existence and multiplicity of critical points for certain strongly indefinite functions of the form $f\colon M \times E \to \mathbb{R}$, where M is a compact manifold and E is a Hilbert space. The case $M = \mathbb{T}^n$ of the n-dimensional torus arises in the study of periodic solutions of Hamiltonian systems which are global perturbations of completely integrable systems. For a large class of Hamiltonian systems on $T^*\mathbb{T}^n$ we prove the existence of at least $n+1$ forced oscillations in every homotopy class of loops in $T^*\mathbb{T}^n$. Moreover, there exist at least $n+1$ periodic solutions having an arbitrarily prescribed rational rotation vector.

1. *Lyusternik–Schnirelman theory for flows*

The idea of generalizing Lyusternik–Schnirelman theory to topological flows goes back to C. Conley and E. Zehnder. In **[6, 8]** this concept is developed and successfully applied to various problems concerning the existence of periodic solutions of Hamiltonian systems. Further applications of this method to related problems can be found in the papers of A. Floer **[10, 11]**, J.-C. Sikorav **[33]**, and C. Golé **[17, 18]**. In the first part of this paper we describe a new approach to Lyusternik–Schnirelman theory for flows, which differs from that of Conley and Zehnder by the use of the geometrical Lyusternik–Schnirelman category instead of the cohomological category. The second part contains an application to the existence of periodic orbits for Hamiltonian systems on $\mathbb{T}^n \times \mathbb{R}^n$, assuming hypotheses on the Hamiltonian which are not suited for the direct use of the cohomological category.

We begin by recalling the following definitions and notation.

DEFINITION 1. Let M be a topological space.

(i) For a non-empty subset $A \subset M$ the *Lyusternik–Schnirelman category* of A in M is defined by

$$\begin{aligned}\operatorname{cat}_M(A) := \inf\{k \in \mathbb{N} \mid\ &\text{there exist closed subsets } A_1, \ldots, A_k,\\ &A_j \text{ contractible in } M, \text{ such that } A \subset A_1 \cup \ldots \cup A_k\}.\end{aligned}$$

If no such integer k exists, we define $\operatorname{cat}_M(A) := \infty$. Moreover, $\operatorname{cat}_M(\emptyset) := 0$. In particular, we denote the category of M by $\operatorname{cat}(M) := \operatorname{cat}_M(M)$.

(ii) The *cuplength* of M is defined by

$$\begin{aligned}\operatorname{cuplength}(M) := \sup\{k \in \mathbb{N} \mid\ &\text{there exist cohomology classes } \omega_1, \ldots, \omega_k,\\ &\omega_j \in H^{q_j}(M) \text{ with } q_j \geq 1, \text{ such that } \omega_1 \cup \ldots \cup \omega_k \neq 0\}.\end{aligned}$$

1991 *Mathematics Subject Classification*: 58E05.

Proc. London Math. Soc. (3) 68 (1994) 641–672.

If no such classes exist, we define cuplength$(M) := 0$. We shall call the number

$$l(M) := \text{cuplength}(M) + 1$$

the *cohomological category* of M.

Throughout this work M will be a Hausdorff space, and $H^*(M) := H^*(M; R)$ denotes either the Alexander cohomology or the singular cohomology of M with coefficients in some principal ideal domain R.

Between cat(M) and $l(M)$ there holds the well-known relation $\text{cat}(M) \geq l(M)$; see, for example, [**32**]. For convenience, some more properties of cat are collected in the appendix. It is crucial for our purposes that cat_M satisfies the continuity property, that is, every non-empty $A \subset M$ possesses a neighbourhood N in M such that $\text{cat}_M(N) = \text{cat}_M(A)$. This is guaranteed if the space M is an absolute neighbourhood retract; see for example, [**9**].

Recall that a topological space M is called an *absolute neighbourhood retract* (ANR) if the following condition holds true: given any normal or metric space X, a closed subset $A \subset X$ and a continuous map $f: A \to M$, then f can be extended to a continuous function defined on some neighbourhood of A in X.

Occasionally we distinguish between an ANR(normal) and an ANR(metric), according to whether X is a normal or a metric space in the above condition. For example, every compact manifold is an ANR(normal), see [**20**], and every metric manifold is an ANR(metric), see [**9**].

We also recall the definition of a Morse decomposition. Consider a compact space M together with a continuous flow $M \times \mathbb{R} \to M$. A *Morse decomposition* of M with respect to this flow is a finite collection of disjoint compact invariant subsets $M_1, \ldots, M_k$, which can be ordered, say $(M_1, \ldots, M_k)$, such that the following condition is satisfied: for every $x \in M \backslash \{M_1 \cup \ldots \cup M_k\}$ there exist indices $1 \leq i < j \leq k$ such that

$$\omega^+(x) \subset M_i \quad \text{and} \quad \omega^-(x) \subset M_j,$$

where $\omega^{\pm}(x)$ denote the positive and the negative limit set of x respectively. Then the ordering $(M_1, \ldots, M_k)$ is called an *admissible ordering* of the Morse decomposition.

THEOREM 1. *Let M be a compact ANR. If there exists a continuous flow on M which admits a Morse decomposition $(M_1, \ldots, M_k)$ of M, then*

$$\text{cat}(M) \leq \sum_{j=1}^{k} \text{cat}_M(M_j).$$

In particular, if M is pathwise connected and if the flow is gradient-like, then there exist at least cat(M) *rest points.*

Recall that a flow is said to be *gradient-like* if there exists a continuous function $f: M \to \mathbb{R}$ which is strictly decreasing along non-constant trajectories.

The result corresponding to Theorem 1 for the cohomological category is due to C. Conley and E. Zehnder; see [**6**, Theorem 5]. They showed that

$$l(M) \leq \sum_{j=1}^{k} l(M_j),$$

where $l(\cdot)$ is defined for Alexander cohomology with real coefficients. The use of Alexander cohomology guarantees that $l(\cdot)$ possesses the continuity property, similar to cat_M in Theorem 1. However, the space M is not required to be an ANR. So the cohomological category can be applied for more general spaces, while the geometrical category gives possibly stronger estimates in the case that M is an ANR.

In particular, let M be a compact connected differentiable manifold without boundary. We can assume that M is smooth, since every C^r-manifold with $r \geq 1$ can be equipped with a compatible C^∞-differentiable structure; see for example, **[21]**. Moreover, M can be endowed with a Riemannian structure; see for example, **[26]**. By a theorem of Lyusternik and Schnirelman, a C^1-function $f\colon M \to \mathbb{R}$ has at least $\mathrm{cat}(M)$ critical points. This result is also a consequence of Theorem 1. Note that if f possesses infinitely many critical values, then there exist correspondingly infinitely many distinct critical points. So we only have to consider the case that f has at most finitely many critical values, say $c_1 < \ldots < c_m$. By $K_j \subset M$, for $j = 1, \ldots, m$, we denote the critical set at the level c_j. In view of the Riemannian structure, the gradient vector field ∇f is defined on M. Suppose first that the vector field $-\nabla f$ generates a flow. Then $(K_1, \ldots, K_m)$ is an admissibly ordered Morse decomposition of M with respect to the flow of $-\nabla f$, and consequently f has at least $\mathrm{cat}(M)$ critical points by Theorem 1. However, a gradient flow for a C^1-function in general does not exist, and therefore we have to make use of the following theorem.

THEOREM 2. *Let M be a compact Riemannian manifold without boundary, and let $f \in C^1(M, \mathbb{R})$ have at most finitely many critical values $c_1 < \ldots < c_m$. Then there exists a Lipschitz-continuous vector field V on M such that the following hold.*

(i) *The map f is strictly decreasing along non-constant orbits of the flow of $-V$.*

(ii) *For $j = 1, \ldots, m$, let $K_j := \{x \in M \mid f(x) = c_j, \nabla f(x) = 0\}$ denote the critical set of f at the level c_j. Then there exists a Morse decomposition $(M_1, \ldots, M_m)$ of M with respect to the flow of $-V$ such that*

$$\mathrm{cat}_M(M_j) = \mathrm{cat}_M(K_j) \quad \text{for } j = 1, \ldots, m.$$

Moreover, if $\delta > 0$ is given, then V can be chosen such that for $j = 1, \ldots, m$, the Morse set M_j is contained in the δ-neighbourhood $N_\delta(K_j)$ of K_j.

Clearly, we then have $\mathrm{cat}(M) \leq \sum_{j=1}^m \mathrm{cat}_M(M_j) = \sum_{j=1}^m \mathrm{cat}_M(K_j)$, and hence the critical set of f contains at least $\mathrm{cat}(M)$ points if M is connected.

The proof of Theorem 2 will use a suitable modification of the so-called pseudo gradient vector fields introduced by R. Palais **[28]**. To be more precise we introduce the following definition.

DEFINITION 2. Let $(M, \langle\cdot,\cdot\rangle_x)$ be a Riemannian manifold, and let $f \in C^1(M, \mathbb{R})$. A Lipschitz-continuous vector field V on M will be called a *gradient-like vector field* for f if V satisfies:

$$\|V(x)\|_x \leq 2\,\|\nabla f(x)\|_x \quad \text{for all } x \in M, \tag{1}$$

$$\langle V(x), \nabla f(x)\rangle_x > 0 \quad \text{if } V(x) \neq 0. \tag{2}$$

Moreover, if K denotes the critical set of f, then we say that the gradient-like vector field V for f is a *δ-pseudo gradient vector field* for f if, in addition,

$$\langle V(x), \nabla f(x)\rangle_x \geq \|\nabla f(x)\|_x^2 \quad \text{for } x \in M\backslash N_\delta(K). \tag{3}$$

Here $\|v\|_x = \sqrt{\langle v, v\rangle_x}$ denotes the norm on T_xM derived from the Riemannian metric.

If V is a gradient-like vector field for f, then f is strictly decreasing along non-constant orbits of the flow of $-V$, and the critical points of f are rest points of this flow. Also note that a δ-pseudo gradient vector field for f is a pseudo gradient field in the sense of Palais on $M\backslash N_\delta(K)$.

Under the hypotheses of Theorem 2 we shall construct for every given $\delta > 0$ a corresponding δ-pseudo gradient vector field for f.

In contrast to the compact case, the set of rest points for a continuous flow on a non-compact space might be empty. However, we point out the following result for a non-compact space M, which is important for our application to Hamiltonian systems below.

PROPOSITION 1. *Let M be an ANR together with a continuous flow $M \times \mathbb{R} \to M$. Suppose there exists a compact invariant subset $S \subset M$ which admits a Morse decomposition $(S_1, \ldots, S_k)$. Then*

$$\operatorname{cat}_M(S) \leq \sum_{j=1}^{k} \operatorname{cat}_M(S_j).$$

If M is pathwise connected and if the flow is gradient-like, then there exist at least $\operatorname{cat}_M(S)$ rest points in S.

Of course, this result does not give the existence of a single rest point unless more information about $\operatorname{cat}_M(S)$ is available. In order to relate the quantity $\operatorname{cat}_M(S)$ to global topological invariants of the space M, some additional structure has to be imposed. The subsequent application may serve as an example where the compact invariant set S admits a special isolating neighbourhood.

In the study of gradient-like flows on a not locally compact space M this strategy cannot be applied. However, it is still possible to obtain results for special gradient-like flows on not locally compact manifolds. For the sake of simplicity, in the subsequent considerations we restrict ourselves to the case of manifolds modelled over a separable Hilbert space.

Let M be a Hilbert manifold, and let $f \in C^1(M, \mathbb{R})$ be a function which satisfies the Palais–Smale compactness condition (PS). Recall that f is said to satisfy (PS) if the following condition holds true: if $(x_k)_k$ is a sequence in M such that $f(x_k)$ is bounded and $\nabla f(x_k) \to 0$ as $k \to \infty$, then $(x_k)_k$ possesses a convergent subsequence.

If in addition the set of critical values of f is finite, then for every $\delta > 0$ there always exists a δ-pseudo gradient vector field for f. Up to some obvious modifications, this follows from the proof of Theorem 2.

Corresponding to Proposition 1 we have the following theorem.

THEOREM 3. *Let M be a smooth Hilbert manifold without boundary, and assume $f \in C^1(M, \mathbb{R})$ satisfies* (PS). *Suppose that f has at most finitely many critical values $c_1 < \ldots < c_m$, corresponding to the critical sets $K_1, \ldots, K_m$. Let V be a*

δ-pseudo gradient vector field for f, where δ is chosen such that $\operatorname{cat}_M(K_j) = \operatorname{cat}_M(N_\delta(K_j))$ *for $j = 1, \ldots, m$. Denoting by $(x, t) \mapsto x \cdot t$ the maximal local flow of $-V$, we define the positively invariant set*

$$S^+ := \left\{ x \in M \mid \inf_{t \geq 0} f(x \cdot t) > -\infty \right\}.$$

Then

$$\operatorname{cat}_M(S^+) \leq \sum_{j=1}^{m} \operatorname{cat}_M(K_j).$$

Note that the definition of S^+ is acceptable, even if the flow of $-V$ is not globally defined on $M \times \mathbb{R}$. Clearly, a corresponding result holds true for the negatively invariant set S^- consisting of those trajectories along which f remains bounded from above under the backward flow.

The statement of Theorem 3 is related to a result due to G. Fournier and M. Willem. In **[15, 16]** they introduced a relative Lyusternik–Schnirelman category as follows. The *relative category* $\operatorname{cat}_M(A, B)$ of a non-empty subset $A \subset M$ in M relative to $B \subset M$ is defined to be the least integer k such that there exist $k+1$ closed subsets $A_0, A_1, \ldots, A_k \subset M$, satisfying:

(a) $A \subset A_0 \cup A_1 \cup \ldots \cup A_k$,

(b) $A_1, \ldots, A_k$ are contractible in M,

(c) $B \cap A_0$ is a strong deformation retract of A_0.

Denoting by $M^a := \{x \in M \mid f(x) \leq a\}$ the sub-level set of $a \in \mathbb{R}$, we formulate the following consequence of Theorem 3.

THEOREM 4. *Let $f \in C^1(M, \mathbb{R})$ satisfy the assumptions of Theorem* 3. *Consider $-\infty \leq a < b \leq +\infty$, and assume that a, b are no critical values of f. Define*

$$S_a^+ := \{x \in S^+ \mid f(x \cdot t) > a \text{ for all } t \geq 0\}.$$

Then we have

$$\operatorname{cat}_M(M^b, M^a) \leq \operatorname{cat}_M(S_a^+ \cap M^b).$$

In particular, if M is connected then there exist at least $\operatorname{cat}_M(M^b, M^a)$ *critical points in $M^b \backslash M^a$.*

A corollary of Theorem 4 is the following general saddle point theorem, involving a linking.

THEOREM 5. *Let $f \in C^1(M, \mathbb{R})$ satisfy the assumptions of Theorem* 3. *Assume that $Q \subset M$ is an embedded compact manifold with relative boundary ∂Q, such that $\partial Q \subset M^a$ for some regular value $a \in \mathbb{R}$, and moreover assume that ∂Q is a strong deformation retract of M^a. Suppose there exists a closed subset $B \subset M$ such that*

(i) $\inf\{f(x) \mid x \in B\} > a$,

(ii) *B and ∂Q link, that is, $B \cap (Q \cdot t) \neq \emptyset$ for all $t \geq 0$.*

Then

$$\operatorname{cat}_M(Q, \partial Q) \leq \operatorname{cat}_M(S_a^+).$$

In particular, if M is connected then there exist at least $\operatorname{cat}_M(Q, \partial Q)$ *critical points of f on some level greater than a.*

Of course, more general cases can be included by the use of a modified relative category. For instance, instead of (c) it is possible to require the condition

(c′) $B \cap A_0$ is a retract of A_0.

2. *Periodic orbits for Hamiltonian systems on* $\mathbb{T}^n \times \mathbb{R}^n$

We consider a Hamiltonian equation which depends on time t,

$$\dot{z} = J\nabla H(z, t), \quad z \in \mathbb{R}^{2n}, t \in \mathbb{R}, \tag{4}$$

where J is the skew-symmetric matrix

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \in \mathscr{L}(\mathbb{R}^n \times \mathbb{R}^n) \tag{5}$$

with I being the identity on $\mathbb{R}^n$. We shall assume that H depends periodically on the time t with period 1. Writing $z = (x, y) \in \mathbb{R}^n \times \mathbb{R}^n$, we shall moreover assume that H is periodic with respect to $x \in \mathbb{R}^n$:

$$H(x + j, y, t) = H(x, y, t) = H(x, y, t + 1) \quad \text{for all } j \in \mathbb{Z}^n. \tag{6}$$

Therefore the Hamiltonian vector field can be considered as a periodically time-dependent vector field on the phase space $\mathbb{T}^n \times \mathbb{R}^n$, where $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ denotes the n-dimensional flat torus.

Given $\alpha \in \mathbb{R}^n$ one can ask for solutions $z(t) = (x(t), y(t)) \in \mathbb{R}^n \times \mathbb{R}^n$ of (4) which satisfy

$$\lim_{|t| \to \infty} \frac{x(t)}{t} = \alpha.$$

In this case the solution $z(t)$ is said to have the *rotation vector* α. If in addition we assume this solution to be periodic with integer period p, requiring

$$x(t + p) = x(t) + j \quad \text{and} \quad y(t + p) = y(t)$$

for some $j \in \mathbb{Z}^n$, it follows that $\alpha p = j$ and

$$x(t) = \alpha t + \xi(t) \quad \text{with } \xi(t + p) = \xi(t). \tag{7}$$

Definition 3. If $j = (j_1, \dots, j_n) \in \mathbb{Z}^n$, and if $p \in \mathbb{N}$ such that $j_1, \dots, j_n$, and p are relatively prime, then we call a solution $z(t) = (x(t), y(t))$ a *j/p-solution* if $x(t)$ is of the form (7) with $\alpha = j/p$.

Observe that a j/p-solution considered in the universal cover $\mathbb{R}^n \times \mathbb{R}^n$ of the phase space $\mathbb{T}^n \times \mathbb{R}^n$ is not periodic unless $j = 0$. Its projection to $\mathbb{T}^n \times \mathbb{R}^n$ is always periodic, and solutions having the same period but different rotation vectors belong to different homotopy classes of loops in $\mathbb{T}^n \times \mathbb{R}^n$. Recall that $\pi_1(\mathbb{T}^n \times \mathbb{R}^n) \cong \pi_1(\mathbb{T}^n) \cong \mathbb{Z}^n$. Therefore a j/p-solution belongs to the homotopy class characterized by the integer vector j.

The assumption on j and p to be relatively prime implies that a j/p-solution has p as its minimal period. In particular, a j/p-solution with $p > 1$ is not the iterate of a 1-solution.

In order to prove the existence of j/p-solutions for all $j \in \mathbb{Z}^n$ and $p \in \mathbb{N}$ we formulate some assumptions on the Hamiltonian H. We shall assume the vector field (4) to be asymptotically linear with respect to y, requiring that

$$\text{(8)} \qquad \frac{1}{|y|}|\partial_y H(x, y, t) - A(t)y| \to 0 \quad \text{and} \quad \frac{1}{|y|}|\partial_x H(x, y, t)| \to 0$$

as $|y| \to \infty$ uniformly in x and t, where $A(t)$ is a symmetric matrix, depending continuously and periodically on t:

$$\text{(9)} \qquad A(t) = A(t+1) \in \mathscr{L}(\mathbb{R}^n).$$

We state our main result, which generalizes Theorem 3 in [**6**]:

THEOREM 6. *Assume $H \in C^1$ satisfies* (6) *and* (8), *and assume that moreover*

$$\text{(10)} \qquad \det\left[\int_0^1 A(t)\,dt\right] \neq 0.$$

Then for every given $j \in \mathbb{Z}^n$ and $p \in \mathbb{N}$ relatively prime we have

$$\#\{j/p\text{-}solutions\} \geq n + 1.$$

In particular, there exist at least $n+1$ periodic solutions with period 1 *in every homotopy class of loops on $\mathbb{T}^n \times \mathbb{R}^n$.*

Here $n+1$ stands for $l(\mathbb{T}^n) := \text{cuplength}(\mathbb{T}^n) + 1$. The non-resonance condition (10) is crucial for the existence proof, and as a side remark we observe that it cannot be omitted, as the following example shows. For $H(x, y, t) = \frac{1}{2}\langle A(t)y, y\rangle$ the Hamiltonian equations are

$$\dot{x} = A(t)y \quad \text{and} \quad \dot{y} = 0,$$

so that the solutions are of the form $(x(t), y(t)) = (x(t), y(0))$. If we have a j-solution, that is, $p = 1$, then $x(t) = jt + \xi(t)$ with $\xi(t+1) = \xi(t)$, and integrating the identity $j + \dot{\xi}(t) = A(t)y(0)$ from $t = 0$ to $t = 1$ we arrive at

$$j = \int_0^1 A(t)\,dt\,y(0).$$

Consequently j has to be contained in the range of $\int_0^1 A(t)\,dt$.

The existence statements so far can be viewed as generalizations of the Poincaré-Birkhoff fixed point theorem. This theorem states that an area-preserving homeomorphism of an annulus $A = \mathbb{S}^1 \times [a, b]$ rotating the boundaries $\mathbb{S}^1 \times \{a\}$ and $\mathbb{S}^1 \times \{b\}$ in opposite directions, possesses at least $2 = \text{cuplength}(\mathbb{S}^1) + 1$ fixed points in the interior. Moreover, it possesses infinitely many periodic orbits; namely for every j and p relatively prime it has a periodic orbit of period p. In our case there are no boundary conditions. Instead the system is assumed to be asymptotically linear and the condition (10) plays the role of the twist condition in the Poincaré–Birkhoff theorem.

We point out the relation of Theorem 6 to the corresponding result of C. Golé for periodic orbits of monotone symplectomorphisms of $\mathbb{T}^n \times \mathbb{R}^n$; see [**17, 18**]. In contrast to our situation, where only asymptotic conditions are imposed, the existence of a global generating function is assumed in [**17**]. Golé establishes the existence of j/p-solutions for periodically time-dependent Hamiltonians $H \in C^2$, satisfying

$$H(x, y, t) = \tfrac{1}{2}\langle Ay, y\rangle + \langle c, y\rangle \quad \text{if } |y| \geq a$$

for some constant $a > 0$, where $A \in \mathscr{L}(\mathbb{R}^n)$ is a symmetric matrix with $\det A \neq 0$, and where $c \in \mathbb{R}^n$ is a constant vector. In addition, H is assumed to satisfy the interior condition

$$\det \frac{\partial^2 H}{\partial y^2}(x, y, t) \neq 0 \quad \text{for all } (x, y, t) \in \mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R},$$

which generalizes the monotone twist condition.

By the use of Conley–Zehnder Morse theory, the number of periodic orbits can be estimated by the sum of Betti numbers of $\mathbb{T}^n$, provided the orbits are non-degenerate. This is carried out for our case in **[25]** under suitably stronger hypotheses on the Hamiltonian H, and the assumptions that $H \in C^2$ and the norm of the Hessian of H has at most polynomial growth with respect to y.

Observe that Theorem 6 includes the case of the special Hamiltonian function

$$H(x, y, t) = \tfrac{1}{2}|y|^2 + V(x, t)$$

with V depending periodically on x and t. If $V \in C^1$ then H satisfies the assumptions of Theorem 1 with $A = \mathrm{id}_{\mathbb{R}^n}$. An example is the multiple pendulum; see, for example, a paper by K. C. Chang, Y. Long and E. Zehnder **[3]**.

In the case that $j = 0$ and V is independent of time t, the contractible periodic orbits guaranteed by Theorem 1 are of oscillatory type, but may coincide with the $n+1$ critical points of the periodic potential $V(x)$. If $j \in \mathbb{Z}^n \setminus \{0\}$ then the j/p-orbits given by Theorem 1 are of rotational type. There are infinitely many of them.

It follows immediately from our proofs that the assertion of Theorem 6 also holds true for non-exact Hamiltonian systems of the form

$$\dot{z} = J[\nabla H(z, t) + f(t)] \tag{11}$$

for a continuous 1-periodic map $f(t) = (f_1(t), f_2(t)) \in \mathbb{R}^n \times \mathbb{R}^n$ which satisfies $\int_0^1 f_1(t)\, dt = 0$.

As a consequence of Theorem 6 we point out the following extension of the global Birkhoff–Lewis theorem in **[6]**.

THEOREM 7. *Assume that $H \in C^1$ satisfies the assumption* (6), *and let $A(t)$ satisfy* (9) *and* (10). *Moreover assume that there exists $R > 0$ such that*

$$H(x, y, t) = \tfrac{1}{2}\langle A(t)y, y\rangle + \langle b(t), y\rangle \quad \text{if } |y| \geq R,$$

where $b\colon \mathbb{R} \to \mathbb{R}^n$ is a continuous periodic mapping with $b(t) = b(t+1)$. We introduce the following notation for the mean values:

$$[A] := \int_0^1 A(t)\, dt \in \mathscr{L}(\mathbb{R}^n), \quad [b] := \int_0^1 b(t)\, dt \in \mathbb{R}^n.$$

Then for every $j \in \mathbb{Z}^n$ and $p \in \mathbb{N}$ relatively prime, which satisfy

$$|j/p - [b]| < R/|[A]^{-1}|, \tag{12}$$

there exist at least $n+1$ periodic solutions of period p having rotation vector j/p which are contained in $\mathbb{T}^n \times D_R$, where $D_R := \{y \in \mathbb{R}^n \mid |y| < R\}$ is the disc of radius R.

The proof of Theorem 7 is readily derived from Theorem 6: consider the orbit $z(t) = (x(t), y(t)) \in \mathbb{R}^n \times \mathbb{R}^n$ of the Hamiltonian flow starting at (x_0, y_0) with $|y_0| \geqslant R$. Integrating the Hamiltonian equation we obtain

$$y(t) = y_0, \quad x(t) = x_0 + \int_0^t A(s)\, ds + \int_0^t b(s)\, ds.$$

Thus $z(t)$ corresponds to a p-periodic solution on $\mathbb{T}^n \times \mathbb{R}^n$ with rotation vector j/p if

$$j = x(p) - x(0) = \int_0^p A(t)\, dt\, y_0 + \int_0^p b(t)\, dt = p[A]y_0 + p[b],$$

whence

$$|j/p - [b]| = |[A]y_0| \geqslant \frac{|y_0|}{|[A]^{-1}|} \geqslant \frac{R}{|[A]^{-1}|}.$$

Consequently, for every $j \in \mathbb{Z}^n$ and $p \in \mathbb{N}$ relatively prime which satisfy (12), the periodic orbits having rotation vector j/p have to be contained in $\mathbb{T}^n \times D_R$, and there exist at least $n+1$ of them by Theorem 6.

The proof of Theorem 6 is organized as follows. In § 6 we describe the variational formulation for the problem. For fixed $p \in \mathbb{N}$, the claimed j/p-solutions are characterized as the critical points of the action functional Φ of the form

$$\Phi(u) = \tfrac{1}{2}(Tu, u) - \hat{\varphi}(u) + (v, u),$$

defined on a suitable Hilbert space of loops u: $\mathbb{R}/p\mathbb{Z} \to \mathbb{R}^{2n}$. Here T is a bounded Fredholm operator of index 0, and $\dim \ker(T) = n$. The non-linearity $\hat{\varphi}$ and the linear part $(v, \cdot)$ depend on the particular integer vector j.

The action functional is invariant under a free $\mathbb{Z}^n$-action on $\ker(T)$, such that mod $\mathbb{Z}^n$ we have Φ: $\ker(T)^\perp \times \mathbb{T}^n \to \mathbb{R}$.

In § 7 we introduce a Galerkin approximation by finite-dimensional subspaces of $\ker(T)^\perp$, and the gradient of Φ is shown to be A-proper with respect to this projection scheme. This replaces the Lyapunov–Schmidt reduction in **[6]**.

Section 8 contains *a priori* bounds for the critical points of the action functional Φ and of the approximating functions Φ_k, respectively. In case that Φ_k' generates a gradient flow, this allows us to define an isolating block for the invariant set of bounded trajectories. The existence of at least $n+1$ critical points of Φ is proved in § 9, assuming first that there exists a gradient flow for Φ_k. At the end of § 9 we show that for sufficiently large k there always exists a Lipschitz-continuous gradient-like vector field for Φ_k, admitting an isolating block, which can take the part of the gradient field in the preceding existence proof.

Related results for Hamiltonian systems on $\mathbb{T}^n \times \mathbb{R}^n$ concerning the existence of contractible periodic solutions have also been obtained by K. C. Chang **[2]** and by G. Fournier and M. Willem **[16]**. In particular, as in **[6]**, a Lyapunov–Schmidt reduction is used, which requires that the Hamiltonian H is in C^2 and the norm of its Hessian is bounded. Also for the case of contractible periodic solutions, we point out the results obtained by J. Q. Liu **[27]**, by A. Szulkin **[34]**, and by P. L.

Felmer [13], where minimax methods for critical values have been used. See also P. Rabinowitz [29] for the case of Lagrangian systems. We also mention the work of W. Chen [4], who considers the case of small perturbations of completely integrable Hamiltonian systems.

3. *Proof of Theorem* 1

Let M satisfy the hypothesis of Theorem 1, and assume that $(M_1, \ldots, M_k)$ is an admissible ordering of the Morse decomposition. We shall assume that $M_j \neq \emptyset$ for $j = 1, \ldots, k$. For $1 \leqslant i \leqslant j \leqslant k$ we define

$$M_{i,j} := \{x \in M \mid \omega^+(x) \cup \omega^-(x) \subset M_i \cup \ldots \cup M_j\}.$$

Note in particular that we have $M_{j,j} = M_j$ for $j = 1, \ldots, k$, and $M_{1,k} = M$. Assume that $k \geqslant 2$, and let $1 \leqslant l < k$ be fixed. Then $(M_{1,l}, M_{l+1,k})$ is also an admissibly ordered Morse decomposition of M.

We claim that $\mathrm{cat}_M(M) \leqslant \mathrm{cat}_M(M_{1,l}) + \mathrm{cat}_M(M_{l+1,k})$. Observe that $(M_1, \ldots, M_l)$ is a Morse decomposition of $M_{1,l}$ and $(M_{l+1}, \ldots, M_k)$ is a Morse decomposition of $M_{l+1,k}$. Unless $M_{i,j}$ is not equal to one of the given Morse sets M_j, we can define a Morse decomposition $(M_{i,i'}, M_{i'+1,j})$ of $M_{i,j}$ for some $i < i' < j$ just as before, and the above argument applied to $M_{i,j}$ gives $\mathrm{cat}_M(M_{i,j}) \leqslant \mathrm{cat}_M(M_{i,i'}) + \mathrm{cat}_M(M_{i'+1,j})$. This procedure can be repeated until the required decomposition is reached, and this finally proves that $\mathrm{cat}_M(M) \leqslant \mathrm{cat}_M(M_1) + \ldots + \mathrm{cat}_M(M_k)$.

In order to prove the above claim it remains to consider the case of a Morse decomposition (M_1, M_2) of M consisting of two components. By the continuity property of cat_M we can find neighbourhoods N_i of M_i, with $i = 1, 2$, such that $\mathrm{cat}_M(N_i) = \mathrm{cat}_M(M_i)$. We denote the flow $M \times \mathbb{R} \to M$ by $(x, t) \mapsto x \cdot t$. Using the flow we define

$$t(x) := \inf\{t \in \mathbb{R}_+ \mid x \cdot s \in N_2 \text{ for all } s \geqslant t\}.$$

Note that $0 \leqslant t(x) < +\infty$ for $x \in \overline{M \backslash N_1}$. Moreover we set

$$\tau := \sup\{t(x) \mid x \in M \backslash N_1\}.$$

It follows that $\tau < +\infty$, since otherwise we could find a sequence $x_n \in M \backslash N_1$ such that $t(x_n) \to +\infty$ monotonically as $n \to \infty$. In view of the compactness of M this sequence contains a convergent subsequence $x_{n_j} \to \bar{x} \in \overline{M \backslash N_1}$, and consequently $\sup\{t(x_{n_j}) \mid j \in \mathbb{N}\} \leqslant t(\bar{x}) < \infty$, which gives a contradiction.

Therefore we conclude that $M \subset N_2 \cdot (-t) \cup N_1$ for $t \geqslant \tau$. We set $m_i := \mathrm{cat}_M(M_i) = \mathrm{cat}_M(N_i)$, for $i = 1, 2$. Consequently there exist subsets $B_1^1, \ldots, B_{m_1}^1$, $B_1^2, \ldots, B_{m_2}^2$ of M such that each B_j^i is closed and contractible in M, and $\bar{N}_i = B_1^i \cup \ldots \cup B_{m_i}^i$. For $t \geqslant \tau$ it follows that

$$M \subset \bigcup_{j=1}^{m_1} B_j^1 \cup \bigcup_{j=1}^{m_2} B_j^2 \cdot (-t).$$

Since the flow is a family of homeomorphisms of M, each of the sets $B_j^2 \cdot (-t)$ is closed and contractible in M. Thus M can be covered by $m_1 + m_2$ closed contractible subsets, and hence $\mathrm{cat}(M) \leqslant m_1 + m_2$, which proves our claim.

Assume now that M is arcwise connected and that the flow is gradient-like. Let $f \in C(M, \mathbb{R})$ be a function which is strictly decreasing along non-constant orbits. Observe that for a gradient-like flow on a compact space the set of rest points consists precisely of the positive and the negative limit sets $\omega^+(x)$, $\omega^-(x)$ for $x \in M$. A value $c \in f(M)$ is called *critical* if there exists $x \in M$ such that $f(x \cdot t) = c$ for all $t \in \mathbb{R}$. Since f is gradient-like, it follows that $x = x \cdot t$ for all t.

If f has infinitely many critical values then there also exist infinitely many rest points of the flow. If there are only finitely many critical values $c_1 < \dots < c_k$, then the critical sets $M_j := \{x \in M | \ f(x \cdot t) = c_j\}$ constitute a Morse decomposition of M. Consequently $M_j \neq \emptyset$, and hence $\mathrm{cat}_M(M_j) \geq 1$. If $\mathrm{cat}_M(M_j) = 1$ for $j = 1, \dots, k$, we conclude that $k \geq \mathrm{cat}(M)$. If $\mathrm{cat}_M(M_j) > 1$ for at least one j, then M_j already contains infinitely many points since M is arcwise connected. This completes the proof of Theorem 1.

The proof of Proposition 1 is based on precisely the same arguments, and is therefore omitted.

4. *Proof of Theorem* 2

Let $(M, \langle \cdot, \cdot \rangle_x)$ be a compact Riemannian manifold. Suppose $f \in C^1(M, \mathbb{R})$ has only finitely many critical values $c_1 < \dots < c_m$, and denote by K_j the critical set at the level c_j. We shall abbreviate $K := K_1 \cup \dots \cup K_m$. In the following, $N_\delta(K_j) = \{x \in M | \ \mathrm{dist}(x, K_j) \leq \delta\}$ denotes the closed δ-neighbourhood of K_j in M.

Choose $\bar{\varepsilon} > 0$ such that $c_j + \bar{\varepsilon} < c_{j+1} - \bar{\varepsilon}$ for $j = 1, \dots, m-1$. Hence we can find $\delta > 0$ satisfying $|f(x) - c_j| < \bar{\varepsilon}$ if $x \in N_\delta(K_j)$. This assumption implies, in particular, that the sets $N_\delta(K_j)$ are mutually disjoint.

Moreover, by the continuity property of cat_M we can assume that δ is chosen such that $\mathrm{cat}_M(N_\delta(K_j)) = \mathrm{cat}_M(K_j)$ for $j = 1, \dots, m$. In view of the compactness of M there exists a positive constant $\beta \in \mathbb{R}$, depending on δ, such that $\|\nabla f(x)\|_x \geq \beta$ for all $x \in M \backslash N_{\delta/4}(K)$.

Now fix $\varepsilon > 0$ such that $\varepsilon \leq \min\{\bar{\varepsilon}, \frac{1}{8}\beta\delta\}$. According to this choice of ε we find some positive $\delta_0 = \delta_0(\varepsilon) \leq \delta$ such that $|f(x) - c_j| < \varepsilon$ if $x \in N_{\delta_0}(K_j)$.

For each $x_0 \in M$, let $B(x_0, r_{x_0}) = \{x \in M | \ \mathrm{dist}(x, x_0) < r_{x_0}\}$ denote the open r_{x_0}-ball centred at x_0, where the radius r_{x_0} is sufficiently small such that $B(x_0, r_{x_0})$ is contained in the domain of normal coordinates at x_0. Then on $B(x_0, r_{x_0})$ there exists a canonical smooth vector field $V_{x_0}(x) \in T_xM$ satisfying $V_{x_0}(x_0) = \nabla f(x_0)$, defined simply by the parallel translation of the Levi–Cività connection along geodesic arcs. The following two conditions are needed to guarantee that the vector field we are going to define is sufficiently close to the gradient field for our purposes:

(i) there exists $r_{x_0}^{(1)} \leq r_{x_0}$ such that

$$\|\tfrac{3}{2} V_{x_0}(x)\|_x \leq 2\,\|\nabla f(x)\|_x \quad \text{if } x \in B(x_0, r_{x_0}^{(1)});$$

(ii) if $\nabla f(x_0) \neq 0$ then there exists $r_{x_0}^{(2)} \leq r_{x_0}$ such that

$$\langle \tfrac{3}{2} V_{x_0}(x), \nabla f(x) \rangle_x > \|\nabla f(x)\|_x^2 \quad \text{if } x \in B(x_0, r_{x_0}^{(2)}),$$

and if $\nabla f(x_0) = 0$, we set $r_{x_0}^{(2)} := r_{x_0}$.

Now we define $\varrho_{x_0} := \min\{\frac{1}{4}\delta_0, r_{x_0}^{(1)}, r_{x_0}^{(2)}\}$ for $x_0 \in M$. The collection of open balls $\{B(x, \varrho_x) | \ x \in M\}$ covers M, and in view of the compactness of M we can choose a finite subcover $\{B(x_i, \varrho_{x_i}) | \ i \in I\}$. Let $\{\psi_i | \ i \in I\}$ be a Lipschitz-continuous partition of the unity subordinated to this subcover, such that, in particular, $\psi_i(x) \neq 0$ for $x \in B(x_i, \varrho_{x_i})$. Following R. Palais [28], we can do this by setting

$$\varphi_i(x) := \inf\{\text{dist}(x', x) \mid x' \notin B(x_i, \varrho_{x_i})\}$$

and

$$\psi_i(x) := \varphi_i(x) \Big/ \sum_{j \in I} \varphi_j(x).$$

Then we define the Lipschitz-continuous vector field V on M by

$$V(x) := \tfrac{3}{2} \sum_{i \in I} \psi_i(x) V_{x_i}(x) \in T_x M.$$

It follows immediately from our definitions that

$$\|V(x)\|_x \leqslant 2\, \|\nabla f(x)\|_x \quad \text{for all } x \in M.$$

By $(x, t) \mapsto x \cdot t$ we denote the flow of $-V$. Note that f is strictly decreasing along non-constant orbits. To see this, consider $x \in M$ such that $V(x) \neq 0$. Set $I_x := \{i \in I \mid x \in B(x_i, \varrho_{x_i}), \nabla f(x_i) \neq 0\}$. Then $V(x) \neq 0$ implies $I_x \neq \varnothing$, and we find that

$$\left.\frac{d}{dt}\right|_{t=0} f(x \cdot t) = -\sum_{i \in I_x} \psi_i(x) \langle \nabla f(x), \tfrac{3}{2} V_{x_i}(x) \rangle_x < 0.$$

Hence V is a gradient-like vector field for f. In particular, V is a $\frac{1}{4}\delta_0$-pseudo-gradient vector field for f, since

$$\langle V(x), \nabla f(x) \rangle_x \geqslant \|\nabla f(x)\|_x^2 \quad \text{if } \text{dist}(x, K) > \tfrac{1}{4}\delta_0.$$

Therefore the positive and the negative limits $\omega^{\pm}(x)$ for every $x \in M$ have to be contained in $N_{\delta_0/4}(K)$. Now we define for $j = 1, \ldots, m$, the invariant subsets

$$M_j := \{x \in M \mid \omega^+(x) \cup \omega^-(x) \subset N_{\delta_0/4}(K_j)\}.$$

In particular, $K_j \subset M_j$, and hence $\text{cat}_M(K_j) \leqslant \text{cat}_M(M_j)$. Observe that $x \in M_j$ implies $|f(x \cdot t) - c_j| < \varepsilon$ for all $t \in \mathbb{R}$. Consequently, $(M_1, \ldots, M_m)$ is an admissibly ordered Morse decomposition of M.

In order to show that $\text{cat}_M(M_j) \leqslant \text{cat}_M(K_j)$, we claim that $M_j \subset N_\delta(K_j)$. Therefore consider $x \in M \backslash N_\delta(K_j)$ such that $|f(x) - c_j| < \varepsilon$. We are going to show that $f(x \cdot t) \leqslant c_j - \varepsilon$ for some sufficiently large $t > 0$, which proves that $x \notin M_j$.

Assume that the positive half-orbit of x enters $N_{\delta/4}(K_j) \supset N_{\delta_0/4}(K_j)$. Consequently there exists $\tau > 0$ such that

$$x \cdot \tau \in N_{\delta/2}(K_j) \quad \text{and} \quad x \cdot [0, \tau] \cap N_{\delta/4}(K) = \varnothing.$$

Then we have

$$\int_0^\tau \|\nabla f(x \cdot s)\|_{x \cdot s}\, ds \geqslant \frac{1}{2} \int_0^\tau \|V(x \cdot s)\|_{x \cdot s}\, ds = \frac{1}{2} \int_0^\tau \left\| \frac{d}{ds}(x \cdot s) \right\|_{x \cdot s} ds$$
$$\geqslant \tfrac{1}{2}\, \text{dist}(x, x \cdot \tau) \geqslant \tfrac{1}{4}\delta.$$

Therefore we conclude that

$$\begin{aligned} f(x \cdot \tau) &= f(x) - \int_0^\tau \langle \nabla f(x \cdot s), V(x \cdot s) \rangle_{x \cdot s} \, ds \\ &\leqslant f(x) - \int_0^\tau \|\nabla f(x \cdot s)\|_{x \cdot s}^2 \, ds \\ &\leqslant f(x) - \beta \int_0^\tau \|\nabla f(x \cdot s)\|_{x \cdot s} \, ds \\ &\leqslant c_j + \varepsilon - \tfrac{1}{4}\beta\delta. \end{aligned}$$

In view of the above choice of $\varepsilon \leqslant \frac{1}{8}\beta\delta$, it follows that $f(x \cdot \tau) \leqslant c_j - \varepsilon$. This proves the claim, and the proof of Theorem 2 is complete.

5. *Proof of Theorem* 3

We have already pointed out that under the assumptions of Theorem 3 a δ-pseudo gradient vector field V for f always exists. The flow of V is in general not defined globally on $M \times \mathbb{R}$, and, in order to avoid the difficulties arising from this fact, it is convenient to choose a suitable normalization of V as follows. Let μ: $\mathbb{R}_+ \to (0, 1]$ be a smooth function such that $\mu(s) = 1$ if $0 \leqslant s \leqslant 1$, such that $s\mu(s)$ is non-decreasing for all $s \geqslant 0$, and such that $s\mu(s) = 2$ if $s \geqslant 2$. Then we define

$$V_\mu(x) := \mu(\|V(x)\|_x)V(x).$$

The orbits of the vector field V_μ are thus geometrically the same as the orbits of V, but have a different parametrization. Since V_μ is uniformly bounded, the corresponding flow is defined on $M \times \mathbb{R}$.

(a) If c_j is one of the critical values of f, then consider the interval $(c_j - \bar{\varepsilon}, c_j + \bar{\varepsilon})$, which can be assumed to obtain no critical values other than c_j. Since f satisfies (PS), the critical set K_j is compact.

We proceed as in the proof of Theorem 2. Choose $\delta > 0$ such that $\mathrm{cat}_M(N_\delta(K_j)) = \mathrm{cat}_M(K_j)$. Again by (PS), it follows that there exists $\beta > 0$ such that $\|\nabla f(x)\|_x \geqslant \beta$ for all those $x \in M \backslash N_{\delta/4}(K_j)$ which satisfy $|f(x) - c_j| < \bar{\varepsilon}$. Now fix $\varepsilon > 0$ such that $\varepsilon < \min\{\bar{\varepsilon}, \frac{1}{8}\beta\delta\}$. Then we choose $0 < \delta_0 \leqslant \delta$ such that $|f(x) - c_j| < \varepsilon$ for all $x \in N_{\delta_0}(K_j)$.

Now let V be a $\frac{1}{4}\delta_0$-pseudo gradient vector field for f constructed as in the proof of Theorem 3, and let V_μ be a normalization as described above. Let $(x, t) \mapsto x \cdot t$ denote the flow of $-V_\mu$. Consider $x \in M \backslash N_\delta(K_j)$ such that $|f(x) - c_j| \leqslant \varepsilon$. As in the proof of Theorem 2, we conclude that the orbit of x cannot enter $N_{\delta/4}(K_j)$ in a positive direction without before having passed the level $c_j - \varepsilon$. Hence for every such x there exists a unique time $\tau_x^+ \geqslant 0$ such that $f(x \cdot \tau_x^+) = c_j - \varepsilon$. Now observe that there exists a constant $\mu_0 > 0$ such that $\|V_\mu(x)\| \geqslant \mu_0$ for all those $x \in M \backslash N_{\delta/4}(K_j)$ which satisfy $|f(x) - c_j| \leqslant \bar{\varepsilon}$. Otherwise we could find a sequence x_i with $\|V_\mu(x_i)\|_{x_i} \to 0$, which implies that $\|V(x_i)\|_{x_i} \to 0$ and hence $\|\nabla f(x_i)\|_{x_i} \to 0$, in contradiction to $\|\nabla f(x_i)\|_{x_i} \geqslant \beta > 0$ for all i. Consequently,

$$f(x \cdot \tau_x^+) \leqslant f(x) - \frac{1}{2} \int_0^{\tau_x^+} \|V_\mu(x \cdot s)\|_{x \cdot s} \|\nabla f(x \cdot s)\|_{x \cdot s} \, ds \leqslant c_j + \varepsilon - \tfrac{1}{2}\beta\mu_0\tau_x^+.$$

Therefore we conclude that $\tau_x^+ \leq 4\varepsilon/\mu_0\beta =: \tau^+$, and hence

$$([M^{c_j+\varepsilon}\backslash N_\delta(K_j)] \cap S^+) \cdot \tau^+ \subset M^{c_j-\varepsilon} \cap S^+.$$

It follows that

$$\begin{aligned}\operatorname{cat}_M(M^{c_j+\varepsilon} \cap S^+) &\leq \operatorname{cat}_M([M^{c_j+\varepsilon}\backslash N_\delta(K_j)] \cap S^+) + \operatorname{cat}_M(N_\delta(K_j))\\ &\leq \operatorname{cat}_M(M^{c_j-\varepsilon} \cap S^+) + \operatorname{cat}_M(K_j).\end{aligned}$$

(b) Assume now that the interval (a, b) does not contain a critical value of f. Then we find that $\operatorname{cat}_M(S^+ \cap M^b) = \operatorname{cat}_M(S^+ \cap M^a)$, as a trivial case of (a).

Suppose $a < \inf_{x\in S} f(x) \leq \sup_{x\in S} f(x) < b$. Combining the steps (a) and (b), we obtain

$$\begin{aligned}\operatorname{cat}_M(S^+) = \operatorname{cat}_M(M^b \cap S^+) &\leq \operatorname{cat}_M(M^a \cap S^+) + \sum_{j=1}^m \operatorname{cat}_M(K_j)\\ &= \sum_{j=1}^m \operatorname{cat}_M(K_j).\end{aligned}$$

The proof of the inequality in Theorem 4 follows immediately from the definition of the relative category by the use of Wazewski's principle, which implies that M^a is a strong deformation retract of $M^b\backslash S_a^+$. For a proof of Wazewski's principle we refer to Conley [**5**, p. 24ff].

The number of critical points in $M^b\backslash M^a$ can be estimated from below in a way similar to the proof of Theorem 3. Under the assumption of there being at most finitely many critical levels $c_1 < \ldots < c_m$ of f in the interval (a, b), we obtain

$$\operatorname{cat}_M(S_a^+ \cap M^b) \leq \sum_{j=1}^m \operatorname{cat}_M(K_j \cap M^b).$$

For the proof of Theorem 5 we observe that the linking condition $B \cap Q \cdot t \neq \varnothing$ implies that Q has a non-empty intersection with S_a^+. In view of the compactness of Q we can find a neighbourhood N of $Q \cap S_a^+$ in M such that $\operatorname{cat}_M(N) = \operatorname{cat}_M(Q \cap S_a^+)$, and such that $\bar{N} \subset M^b\backslash M^a$. Consequently, $Q\backslash N$ can be retracted to M^a by Wazewski's principle, and M^a can be retracted to ∂Q by assumption. In view of the definition of the relative category the assertion of Theorem 5 follows.

6. *The variational setup for Theorem* 6

By $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ we denote the circle. Consider the Hilbert space $L^2(\mathbb{S}^1, \mathbb{R}^{2n})$ together with the inner product

$$(z, w)_2 := \int_0^1 \langle z(t), w(t)\rangle\, dt,$$

where $\langle \cdot, \cdot \rangle$ is the Euclidean scalar product in $\mathbb{R}^{2n}$. The corresponding L^2-norm will be denoted by $\|z\|_2 = \sqrt{(z, z)_2}$.

If $\{e_i |\ i = 1, \ldots, 2n\}$ denotes the standard basis of $\mathbb{R}^{2n}$, an orthonormal basis of $L^2(\mathbb{S}^1, \mathbb{R}^{2n})$ is given by

$$\{u_{ki}(t) = \exp(2\pi k t J)e_i |\ k \in \mathbb{Z}, i = 1, \ldots, 2n\},$$

where J is the skew-symmetric matrix defined in (5). Every $z \in L^2(\mathbb{S}^1, \mathbb{R}^{2n})$ has the Fourier coefficients

$$z_k = \sum_{i=1}^{2n} (u_{ki}, z)_2 e_i \in \mathbb{R}^{2n}, \quad \text{for } k \in \mathbb{Z},$$

and is represented by the Fourier expansion

$$z(t) = \sum_{k \in \mathbb{Z}} \sum_{i=1}^{2n} (u_{ki}, z)_2 u_{ki}(t) = \sum_{k \in \mathbb{Z}} \exp(2\pi k t J) z_k$$

for almost every $t \in \mathbb{S}^1$. The L^2-scalar product can be expressed by

$$(z, w)_2 = \sum_{k \in \mathbb{Z}} \langle z_k, w_k \rangle.$$

Using the Fourier expansion we introduce the following Sobolev space:

$$W^{\frac{1}{2},2}(\mathbb{S}^1, \mathbb{R}^{2n}) = \left\{ z \in L^2(\mathbb{S}^1, \mathbb{R}^{2n}) \,\middle|\, \sum_{k \in \mathbb{Z}} |k|\, |z_k|^2 < \infty \right\},$$

which is a Hilbert space with the inner product

$$(z, w) = \langle z_0, w_0 \rangle + 2\pi \sum_{k \in \mathbb{Z}} |k| \langle z_k, w_k \rangle$$

and corresponding norm $\|z\| = \sqrt{(z, z)}$.

In the following we will abbreviate $W := W^{\frac{1}{2},2}(\mathbb{S}^1, \mathbb{R}^{2n})$. There is an orthogonal decomposition $W = W^+ \oplus W^- \oplus W^0$ into closed subspaces according to the subscript $k > 0$, $k < 0$, $k = 0$. By P^+, P^-, P^0 we denote the orthogonal projectors on W^+, W^-, W^0.

Now define a differential operator L: $\operatorname{dom}(L) \subset L^2(\mathbb{S}^1, \mathbb{R}^{2n}) \to L^2(\mathbb{S}^1, \mathbb{R}^{2n})$ on the domain of definition

$$\operatorname{dom}(L) = \left\{ z \in L^2(\mathbb{S}^1, \mathbb{R}^{2n}) \,\middle|\, \sum_{k \in \mathbb{Z}} |k|^2 \, |z_k|^2 < \infty \right\}$$

by

$$(Lz)(t) := -J \frac{d}{dt} z(t) = \sum_{k \in \mathbb{Z}} 2\pi k \exp(2\pi k t J) z_k$$

for almost every $t \in \mathbb{S}^1$. Note that $\operatorname{dom}(L)$ coincides with the Sobolev space $W^{1,2}(\mathbb{S}^1, \mathbb{R}^{2n})$ considered as a linear subspace of $L^2(\mathbb{S}^1, \mathbb{R}^{2n})$. We list some properties of L, which are easily verified.

LEMMA 1. (i) *The operator L is densely defined in $L^2(\mathbb{S}^1, \mathbb{R}^{2n})$, and L is self-adjoint on* $\operatorname{dom}(L)$.

(ii) *The kernel of L consists of the constant loops, and hence* $\dim \ker(L) = 2n$.

(iii) *The range of L consists of all those loops which have mean value zero, that is,*

$$\operatorname{ran}(L) = \{ z \in L^2(\mathbb{S}^1, \mathbb{R}^{2n}) \mid z_0 = 0 \}.$$

(iv) *The spectrum of L is $\sigma(L) = \sigma_{pp}(L) = 2\pi\mathbb{Z}$. In particular, each u_{ki} is an eigenvector of L corresponding to the eigenvalue $2\pi k$.*

Note that $\operatorname{dom}(L)$ is dense in W with respect to the norm on W. In fact, W is the form-domain of the operator L, and it is readily verified by direct computation that we have the identity

$$((P^+ - P^-)z, w) = (Lz, w)_2 \quad \text{if } z, w \in \operatorname{dom}(L).$$

From now on let $j \in \mathbb{Z}^n$ be a fixed rotation vector. We define

$$e(t) := (jt, 0) \in \mathbb{R}^n \times \mathbb{R}^n.$$

Let H be the Hamiltonian as introduced above. Then we define the functional $\varphi\colon W \to \mathbb{R}$ by

$$\varphi(z) := \int_0^1 H(z(t) + e(t), t)\, dt. \tag{13}$$

From our assumptions (8) it is seen that $\nabla H(x, y, t)$ is asymptotically linear with respect to y. This hypothesis implies at most linear growth for the gradient ∇H in each fibre, that is, there exist constants $c_1, c_2 > 0$ such that $|\nabla H(x, y, t)| \leqslant c_1 + c_2 |y|$ uniformly in x and t. Moreover ∇H is continuous. These assumptions are sufficient to prove the following lemma.

LEMMA 2. *The functional $\varphi\colon W \to \mathbb{R}$ defined in* (13) *is in $C^1(W, \mathbb{R})$, and moreover the derivative of φ is represented by*

$$d\varphi(z)w = \int_0^1 \langle \nabla H(z + e, t), w\rangle\, dt \quad \textit{for } z, w \in W.$$

For a proof we refer to Rabinowitz [**30**, Proposition B 37]. The corresponding gradient $\varphi'\colon W \to W$, defined by

$$(\varphi'(z), w) = d\varphi(z)w \quad \text{for all } w \in W,$$

is a compact map. This fact is crucial for our purpose; for a proof see Rabinowitz [**30**, Proposition B 37].

Recall the matrix $A(t)$ from (9), (10) and define a symmetric matrix

$$Q(t) := \begin{pmatrix} 0 & 0 \\ 0 & A(t) \end{pmatrix} \in M(2n \times 2n, \mathbb{R}).$$

Define a symmetric bounded linear operator $\hat{Q} \in \mathscr{L}(L^2(\mathbb{S}^1, \mathbb{R}^{2n}))$ by

$$(\hat{Q}z)(t) := Q(t)z(t).$$

Then there exists a unique symmetric operator $K \in \mathscr{L}(W)$, defined by

$$(Kz, w) := (\hat{Q}z, w)_2 = \int_0^1 \langle Q(t)z(t), w(t)\rangle\, dt.$$

Using the compactness of the embedding $W^{\frac{1}{2},2}(\mathbb{S}^1, \mathbb{R}^{2n}) \to L^2(\mathbb{S}^1, \mathbb{R}^{2n})$, one shows the following lemma.

LEMMA 3. *The operator $K \in \mathscr{L}(W)$ is compact.*

Hence the operator $P^+ - P^- - K$ is a bounded symmetric linear operator on W. As K is compact, it is a relatively compact perturbation of $P^+ - P^-$, and therefore the essential spectrum remains unchanged under the perturbation; see Weidmann [**35**, Satz 9.9]. Therefore we have

$$\sigma_{\mathrm{ess}}(P^+ - P^- - K) = \sigma_{\mathrm{ess}}(P^+ - P^-) = \{-1, +1\}.$$

Consequently, if $\lambda \in \sigma(P^+ - P^- - K)\backslash\{-1, +1\}$ then λ is an isolated eigenvalue of finite multiplicity, and $-1, +1$ are the only possible accumulation points of the spectrum $\sigma(P^+ - P^- - K)$. Having the perturbation involved, we find the identity

$$((P^+ - P^- - K)z, w) = ((L - \hat{Q})z, w)_2 \quad \text{if } z, w \in \mathrm{dom}(L).$$

Since $\hat{Q}$ is a bounded symmetric perturbation of L, a well-known theorem by Kato and Rellich states that $L - \hat{Q}$ is self-adjoint on $\mathrm{dom}(L - \hat{Q}) = \mathrm{dom}(L)$.

In the following we shall write

$$T := P^+ - P^- - K.$$

We shall make crucial use of the following property of T.

LEMMA 4. *We have*

$$\ker(T) = \{z \in W \mid z(t) = (x(t), y(t)) \in \mathbb{R}^n \times \mathbb{R}^n \text{ with } x = \text{const.}, y = 0\}.$$

Moreover, T is Fredholm with $\mathrm{index}(T) = 0$, *and we have* $\mathrm{ran}(T) = \ker(T)^\perp$.

In order to include the case of a non-exact Hamiltonian vector field of the form (11), we note that the vanishing mean value of f_1 implies that $f \in \mathrm{ran}(L - \hat{Q})$, which is seen by elementary calculations. In particular, we then have

$$\int_0^1 \langle f(t), z(t)\rangle\, dt = (f, z)_2 = ((L - \hat{Q})w, z)_2 = (Tw, z). \tag{14}$$

Finally we have to consider the contribution inherited from the rotation vector. Let $j \in \mathbb{Z}^n$ be fixed and $e(t) = (jt, 0) \in \mathbb{R}^n \times \mathbb{R}^n$. Then we define

$$v_j := -J\dot{e} = (0, j) \in W.$$

Obviously, we have the identity

$$\int_0^1 \langle v_j, z\rangle\, dt = (v_j, z)_2 = \langle v_j, z_0\rangle = (v_j, z) \quad \text{for } z \in W. \tag{15}$$

Combining the contributions from (14) and (15), we define a continuous linear function $(v, \cdot)$: $W \to \mathbb{R}$ by

$$(v, z) := (v_j - Tw, z) = \int_0^1 \langle -J\dot{e} - f(t), z(t)\rangle\, dt.$$

Now we define the action functional Φ on W by

$$\Phi(z) = \tfrac{1}{2}(Tz, z) - \hat{\varphi}(z) + (v, z),$$

where

$$\hat{\varphi}(z) := \varphi(z) - \tfrac{1}{2}(Kz, z) = \int_0^1 \{H(z + e, t) - \tfrac{1}{2}\langle \hat{Q}z, z\rangle\}\, dt.$$

It follows from the properties of φ and K that $\hat{\varphi} \in C^1(W, \mathbb{R})$ and that moreover $\hat{\varphi}'$: $W \to W$ is compact. Therefore $\Phi \in C^1(W, \mathbb{R})$ and

$$d\Phi(z)w = (Tz, w) - d\hat{\varphi}(z)w + (v, w) \quad \text{for } w \in W.$$

For the corresponding gradient we have accordingly

$$(\Phi'(z), w) = (Tz - \hat{\varphi}'(z) + v, w) \quad \text{for } w \in W.$$

The function Φ: $W \to \mathbb{R}$ extends the classical action functional defined on $W^{1,2}$-loops by

$$\Phi(z) = \int_0^1 \{\tfrac{1}{2}\langle -J\dot{z}, z\rangle - H(z+e, t) + \langle v_j - f(t), z\rangle\}\, dt$$

for $z \in W^{1,2}(\mathbb{S}^1, \mathbb{R}^{2n})$.

The setup on the Sobolev space $W^{\frac{1}{2},2}(\mathbb{S}^1, \mathbb{R}^{2n})$ permits a variational characterization of the required periodic solutions. Using standard regularity arguments, one sees that the critical points of Φ on W are precisely the classical periodic solutions we are looking for:

LEMMA 5. *The point $z \in W$ is a critical point of Φ if and only if $z \in C^1(\mathbb{S}^1, \mathbb{R}^{2n})$ and z is a* 1-*periodic solution of*

$$-J\dot{z} = \nabla H(z+e, t) + f(t) - v_j.$$

The periodicity of the Hamiltonian H with respect to x implies the invariance of Φ and Φ' under a free action of the group $\mathbb{Z}^n$ on W which is defined as follows. Identify $g = (g_1, \dots, g_n) \in \mathbb{Z}^n$ with $(g_1, \dots, g_n, 0, \dots, 0) \in \mathbb{R}^{2n}$, which can be considered as a constant loop, that is, an element of W. Now define the group action by

$$g \cdot z := z + g \quad \text{for } z \in W.$$

Then $\Phi(g \cdot z) = \Phi(z)$ and $\Phi'(g \cdot z) = \Phi'(z)$. Passing to the quotient we obtain

$$W/\mathbb{Z}^n \cong E \times \mathbb{T}^n$$

where $E = \ker(T)^\perp \cong W/\ker(T)$. We will consider E equipped with the scalar product induced from the inner product on W. In particular, because of the $\mathbb{Z}^n$-invariance, we can consider Φ as an element of $C^1(E \times \mathbb{T}^n, \mathbb{R})$.

Subsequently we shall make crucial use of the fact that there exists a splitting

$$W = E^+ \oplus E^- \oplus E^0 =: E \oplus E^0$$

into closed orthogonal subspaces E^+, E^-, E^0 according to the positive, the negative, and the zero eigenspaces of the operator T.

The variational setup described up to now is suited for periodic solutions having the period $p = 1$, that is, they have the same periodicity as the Hamiltonian H. These solutions are usually called forced oscillations; they correspond to the j-solutions of Theorem 1. In order to establish the existence of j/p-solutions with $j \in \mathbb{Z}^n$ and $p \in \mathbb{N}$ relatively prime we define $e_p(t) := (p^{-1}jt, 0) \in \mathbb{R}^n \times \mathbb{R}^n$, and we have to consider the action functional

$$\Phi_p(z) := \frac{1}{p}\int_0^p \{\tfrac{1}{2}\langle -J\dot{z}, z\rangle - H(z+e_p, t) + \langle -J\dot{e}_p - f(t), z\rangle\}\, dt$$

on the loop space $W^{1,2}(\mathbb{R}/p\mathbb{Z}, \mathbb{R}^{2n})$. The variational formulation on the Sobolev space $W^{\frac{1}{2},2}(\mathbb{R}/p\mathbb{Z}, \mathbb{R}^{2n})$ can then be carried out in a way analogous to the case $p = 1$.

7. *Finite-dimensional approximation and A-properness*

We introduce a Galerkin approximation scheme to obtain a finite-dimensional reduction. Recall the orthogonal splitting $W = E^+ \oplus E^- \oplus E^0$ according to the positive, negative, and zero eigenspaces of the operator $T = P^+ - P^- - K$. In E^+ and in E^- we choose orthonormal bases consisting of eigenvectors of T:

$$\{u_i |\ i \in \mathbb{Z}^+\} \subset E^+ \quad \text{and} \quad \{u_i |\ i \in \mathbb{Z}^-\} \subset E^-.$$

Define the finite-dimensional subspaces E_k of $E = E^+ \oplus E^-$ by

$$E_k := \operatorname{span}\{u_i |\ 0 < |i| \leq k\}, \quad \text{for } k = 1, 2, \ldots .$$

By P_k we denote the corresponding orthogonal projectors $P_k\colon\ E \oplus E^0 \to E_k \oplus E^0$. Note that the projection scheme $(E_k \oplus E^0, P_k)$ is projectionally complete, that is, $P_k u \to u$ as $k \to \infty$ for every $u \in E \oplus E^0$. We define the functionals $\Phi_k\colon\ E_k \oplus E^0 \to \mathbb{R}$ by restriction:

$$\Phi_k(u) = \Phi(u) \quad \text{for } u \in E_k \oplus E^0.$$

The gradient of Φ_k is given by

$$\Phi_k'(u) = P_k \Phi'(u) = Tu - P_k \hat{\varphi}'(u) + v \quad \text{for } u \in E_k \oplus E^0.$$

The action functional Φ turns out to be sufficiently well suited for approximation with respect to the projection scheme $(E_k \oplus E^0, P_k)$. The next lemma shows that the gradient Φ' considered on $E \times \mathbb{T}^n$ is an A-proper map in the sense of F. Browder and V. I. Petryshin; see for example, [**9**].

LEMMA 6. *Every bounded sequence $u_{k_i} \in E_{k_i} \oplus E^0$ which satisfies*

$$\Phi_{k_i}'(u_{k_i}) \to u^* \quad \textit{as } i \to \infty$$

for some $u^* \in E \oplus E^0$, contains a subsequence which is convergent in $E \times (E^0/\mathbb{Z}^n)$. Moreover, if $u_{k_i} \in E_{k_i} \oplus E^0$ satisfies $\Phi_{k_i}'(u_{k_i}) \to u^* \in E \oplus E^0$ and $u_{k_i} \to u \in E \oplus E^0$, then $\Phi'(u) = u^*$.

Proof. Let $u_{k_i} \in E_{k_i} \oplus E^0$ be a bounded sequence in $E \oplus E^0$ satisfying $\Phi_{k_i}'(u_{k_i}) \to u^*$, that is,

$$Tu_{k_i} - P_{k_i} \hat{\varphi}'(u_{k_i}) + v \to u^* \quad \text{as } i \to \infty.$$

Since $\hat{\varphi}'$ is a compact map, we may assume that $\hat{\varphi}'(u_{k_i}) \to w \in E \oplus E^0$. Then

$$\|P_{k_i} \hat{\varphi}'(u_{k_i}) - w\| \leq \|P_{k_i}\| \, \|\hat{\varphi}'(u_{k_i}) - w\| + \|P_{k_i} w - w\|$$

and the right-hand side tends to zero as $i \to \infty$. Consequently,

$$Tu_{k_i} = T(u_{k_i}^+ + u_{k_i}^-) \to u^* + w - v \in E \oplus E^0 \quad \text{as } i \to \infty.$$

Recall that T is Fredholm with $\operatorname{index}(T) = 0$ and $\ker(T) = \operatorname{ran}(T)^\perp = E^0$. In particular, the range of T is closed and it follows that $u^* + w - v \in \operatorname{ran}(T)$. Moreover, $T|_{E^+ \oplus E^-}$ is a linear isomorphism of $E^+ \oplus E^-$. Thus we have

$$u_{k_i}^+ + u_{k_i}^- \to T^{-1}(u^* + w - v) \in E^+ \oplus E^- \quad \text{as } i \to \infty,$$

and consequently $u_{k_i}^+ \to u^+ \in E^+$ and $u_{k_i}^- \to u^- \in E^-$. In view of the $\mathbb{Z}^n$-invariance of Φ_k we can assume that $u_{k_i}^0 \in E^0$ is bounded, hence $u_{k_i}^0 \to u^0 \in E^0$ along a subsequence, and so we have finally singled out a convergent subsequence of the bounded sequence u_{k_i}.

It remains to prove the second statement of Lemma 6. Let $w \in E \oplus E^0$ be arbitrarily chosen. Assume that $u_{k_i} \to u \in E \oplus E^0$, $u_{k_i} \in E_{k_i} \oplus E^0$, $\Phi'_{k_i}(u_{k_i}) \to u^*$ as $i \to \infty$. Then we have

$$(\Phi'(u), w) = \lim_{i \to \infty} \{(\Phi'(u_{k_i}), w - P_{k_i} w) + (P_{k_i}\Phi'(u_{k_i}), P_{k_i} w)\}$$

$$= \lim_{i \to \infty} (\Phi'_{k_i}(u_{k_i}), w_{k_i}) = (u^*, w).$$

Therefore $\Phi'(u) = u^*$, as claimed. This finishes the proof of Lemma 6.

We shall write $\Phi'_{(k)}$ and $E_{(k)}$ meaning either Φ' or Φ_k and E or E_k respectively. By the same arguments as those used in the proof of Lemma 6 we obtain the following lemma.

LEMMA 7. *Every bounded sequence $u_i \in E_{(k)} \oplus E^0$ which satisfies*

$$\Phi'_{(k)}(u_i) \to u^* \quad \textit{as } i \to \infty$$

for some $u^* \in E \oplus E^0_{(k)}$, contains a subsequence which converges in $E_{(k)} \times (E^0/\mathbb{Z}^n)$.

As a side remark we point out that the assertions of the above Lemmata 6 and 7 still hold true if the hypothesis of boundedness on the considered sequences is omitted. This means that Φ satisfies a strong form of A-properness, and in particular the gradients $\Phi'_{(k)}$: $E_{(k)} \times \mathbb{T}^n \to E_{(k)} \oplus E^0$ are proper mappings.

However, the subsequent *a priori* estimates for critical points will give us the desired boundedness.

8. *A priori estimates for critical points and isolating neighbourhoods*

In the following let $\Phi_{(k)}$ and $E_{(k)}$ denote either Φ or Φ_k and E or E_k respectively.

LEMMA 8. (i) *There exists $R > 0$ such that the critical points of Φ are contained in*

$$B = D^+ \times D^- \times \mathbb{T}^n$$

where $D^\pm = \{u^\pm \in E^\pm \mid \|u^\pm\| \leq R\}$.

(ii) *The critical points of Φ_k are contained in*

$$B_k = D_k^+ \times D_k^- \times \mathbb{T}^n$$

where $D_k^\pm = D^\pm \cap E_k$.

(iii) If $b > 0$ *is given, then in addition $R > 0$ can be chosen such that*

$$\|\Phi'_{(k)}(u)\| \geq b$$

for all $u \in (E_{(k)} \times \mathbb{T}^n) \backslash B_{(k)}$.

Proof. Recall that the restriction of the operator T to $E^+ \oplus E^-$ is a linear automorphism. Hence there exists $\lambda > 0$ such that $(Tu^+, u^+) \geq \lambda \|u^+\|^2$ for all $u^+ \in E^+_{(k)}$, and such that $(Tu^-, u^-) \leq -\lambda \|u^-\|^2$ for $u^- \in E^-_{(k)}$.

We consider $u = u^+ + u^- + u^0 \in E^+_{(k)} \oplus E^-_{(k)} \oplus E^0$ such that $\|u^-\| \leq \|u^+\| + r$ with a fixed constant $r \geq 0$. For these u we have

$$\begin{aligned}
(\Phi'_{(k)}(u), u^+) &\geq \lambda \|u^+\|^2 - \int_0^1 \langle \nabla H(u+e, t) - \hat{Q}(t)u, u^+ \rangle \, dt + (v, u^+) \\
&\geq \lambda \|u^+\|^2 - \int_0^1 |\nabla H(u+e, t) - \hat{Q}(t)u| \, |u^+| \, dt - \|v\| \, \|u^+\| \\
&\geq \lambda \|u^+\|^2 - \{c_1(\varepsilon) + \varepsilon \|u^+ + u^-\|\} \|u^+\| - \|v\| \, \|u^+\| \\
&\geq \lambda \|u^+\|^2 - \varepsilon\{2\|u^+\| + r\} \|u^+\| - c_1(\varepsilon) \|u^+\| - \|v\| \, \|u^+\| \\
&\geq (\lambda - 2\varepsilon) \|u^+\|^2 - c_2(\varepsilon, r) \|u^+\|,
\end{aligned}$$

where we have used the asymptotic condition

$$|\partial_x H(x, y, t)| + |\partial_y H(x, y, t) - A(t)y| \leq c_1(\varepsilon) + \varepsilon |y|.$$

We choose $0 < \varepsilon < \frac{1}{2}\lambda$. Then for $u^+ \neq 0$ we find that

$$\|\Phi'_{(k)}(u)\| \geq (\lambda - 2\varepsilon) \|u^+\| - c_2(\varepsilon, r) \quad \text{if } \|u^-\| \leq \|u^+\| + r. \tag{16}$$

By a similar estimate it follows, in the case where $u^- \neq 0$, that

$$\|\Phi'_{(k)}(u)\| \geq (\lambda - 2\varepsilon) \|u^-\| - c_2(\varepsilon, r) \quad \text{if } \|u^+\| \leq \|u^-\| + r. \tag{17}$$

Now choose some large $R > 0$ such that the right-hand sides of the inequalities (16) and (17) are positive for $\|u^+\| \geq R$ and $\|u^-\| \geq R$ respectively.

Consider now the case that the gradient Φ' is Lipschitz-continuous. Then it follows that Φ'_k is also Lipschitz for every k, and we can consider the corresponding negative gradient flow $(u, t) \mapsto u \cdot t$ on $E_{(k)} \times \mathbb{T}^n$:

$$\frac{d}{dt}(u \cdot t) = -\Phi'_{(k)}(u \cdot t).$$

Note that this flow is globally defined for all $t \in \mathbb{R}$, since $\|\Phi'\|$ is linearly bounded in the fibres. For this situation we conclude from Lemma 8 that the following lemma holds.

LEMMA 9. *Let $S^{(k)} \subset E_{(k)} \times \mathbb{T}^n$ denote the invariant set consisting of the critical points of $\Phi_{(k)}$ together with their connecting orbits under the flow of $-\Phi'_{(k)}$. If $R > 0$ in Lemma 8 is chosen sufficiently large, then*

$$S^{(k)} \subset D^+_{(k)} \times D^-_{(k)} \times \mathbb{T}^n = B_{(k)}.$$

Moreover, the compact set B_k is an isolating neighbourhood for the compact invariant set S^k, having the entrance set

$$B^+_k := \partial D^+_k \times D^-_k \times \mathbb{T}^n, \tag{18}$$

and the corresponding exit set

$$B^-_k := D^+_k \times \partial D^-_k \times \mathbb{T}^n. \tag{19}$$

Proof. Let $c \geq 0$ be fixed. Choosing $0 < \varepsilon < \frac{1}{2}\lambda$, we conclude from the proof of Lemma 8 that, for sufficiently large $R > 0$,

$$(\Phi'_{(k)}(u), u^+) > c \quad \text{where } u = u^+ + u^- + u^0, \text{ satisfying } \|u^-\| \leq \|u^+\| + r \text{ and } \|u^+\| \geq R. \tag{20}$$

By a similar argument we find that

$$(21)\qquad (-\Phi'_{(k)}(u), u^-) > c \quad \text{where } u = u^+ + u^- + u^0, \quad \text{satisfying } \|u^+\| \leqslant \|u^-\| + r \text{ and } \|u^-\| \geqslant R.$$

This shows that the set $\{u \in E_{(k)} \times \mathbb{T}^n |\ \|u^-\| \geqslant R, \|u^+\| \leqslant \|u^-\|\}$ is positively invariant under the flow of $-\Phi'_{(k)}$. If $S^{(k)} \neq \emptyset$, the orbit of $u \in S^{(k)}$ cannot leave $B_{(k)}$ through $B^-_{(k)}$ in a positive direction, since its positive limit has to be a critical point of $\Phi_{(k)}$, being contained in $B_{(k)}$.

Similarly, we find that $\{u \in E_{(k)} \times \mathbb{T}^n |\ \|u^+\| \geqslant R, \|u^-\| \leqslant \|u^+\|\}$ is negatively invariant under the flow of $-\Phi'_{(k)}$, and therefore it follows that $S^{(k)} \subset B_{(k)}$. In particular, if $\|u^+\| = R$ and $\|u^-\| < R$, then the inequality (20) shows that the vector $\Phi'_k(u)$ points into the exterior of B_k. If $\|u^-\| = R$ and $\|u^+\| < R$, then the vector $-\Phi'_{(k)}(u)$ points into the exterior of $B_{(k)}$ by (21).

So far it would have been sufficient to choose the numbers $c = r = 0$ in the proofs of Lemmata 8 and 9. However, for a gradient-like vector field V_k for Φ_k we shall require that the flow of $-V_k$ admits an isolating block B_k with entrance set (18) and exit set (19). For this reason we shall use the slightly stronger estimates established above. This construction will be carried out at the end of the next paragraph.

9. *Proof of Theorem* 6

We now come to the existence proof of critical points of the action functional Φ. The proof will be carried out for the case that the gradient Φ' generates a flow.

Proposition 2. *Let $\Phi \in C^1(E \times \mathbb{T}^n, \mathbb{R})$ be the action functional defined by*

$$\Phi(u) = \tfrac{1}{2}(Tu, u) - \hat{\varphi}(u) + (v, u).$$

Assume that the gradient Φ' generates a unique flow on $E \times \mathbb{T}^n$. Then Φ has at least $n+1$ critical points. In particular, if Φ has only finitely many critical values $c_1 < \ldots < c_m$, and if

$$K_j := \{u \in E \times \mathbb{T}^n |\ \Phi(u) = c_j, \Phi'(u) = 0\}, \quad \text{for } j = 1, \ldots, m,$$

denotes the critical set at the level c_j, then

$$\sum_{j=1}^{m} \operatorname{cat}_{E\times\mathbb{T}^n}(K_j) \geqslant \operatorname{cuplength}(\mathbb{T}^n) + 1.$$

Proof. (i) If the action functional Φ has infinitely many critical values then there exist infinitely many critical points, and we have already finished.

(ii) Suppose there exist finitely many critical levels of Φ denoted by $c_1 < \ldots < c_m$. Since the mapping $\Phi'\colon E \times \mathbb{T}^n \to E \times E^0$ is proper by Lemmata 7 and 8, the critical set of Φ is compact. Consequently each of the components K_j is compact. By the continuity property of cat there exists $\delta > 0$ such that

$$(22)\qquad \operatorname{cat}_{E\times\mathbb{T}^n}(N_\delta(K_j)) = \operatorname{cat}_{E\times\mathbb{T}^n}(K_j),$$

where $N_\delta(K_j)$ denotes the closed δ-neighbourhood of K_j.

Fix $\bar{\varepsilon}>0$ such that $c_j+\bar{\varepsilon}<c_{j+1}-\bar{\varepsilon}$ for $j=1, \ldots, m-1$. According to this choice of $\bar{\varepsilon}$ we can assume that, in addition, δ is chosen such that, for $j=1, \ldots, m$,

$$c_j-\bar{\varepsilon}<\Phi(u)<c_j+\bar{\varepsilon} \quad \text{if } u \in N_\delta(K_j). \tag{23}$$

In particular, for this choice of δ the sets $N_\delta(K_j)$ are mutually disjoint.

As an immediate consequence of the Lemmata 6 and 8 we conclude that the following lemma holds.

LEMMA 10. *Let $K:=K_1 \cup \ldots \cup K_m$ denote the critical set of Φ. If $\delta>0$ is given, then there exist $k_1(\delta) \in \mathbb{N}$ and $\beta>0$ depending on δ, such that*

$$\|\Phi_k'(u_k)\| \geqslant \beta \quad \textit{for } k \geqslant k_1(\delta) \textit{ and } u_k \in W_k \backslash N_{\delta/4}(K).$$

Subsequently we shall always assume that δ is chosen such that the conditions (22) and (23) are satisfied.

If $k \geqslant k_1(\delta)$ then the critical points of Φ_k are contained in a $\frac{1}{4}\delta$-neighbourhood of the critical set K of Φ. In particular, for every critical point u of Φ_k there exists exactly one critical value c_j of Φ such that $|\Phi_k(u)-c_j|<\bar{\varepsilon}$. For $k \geqslant k_1(\delta)$ and $j=1, \ldots, m$, we define the following subsets of $E_k \times \mathbb{T}^n$:

$$S_j^k:=\left\{u \in E_k \times \mathbb{T}^n \,\Big|\, \lim_{|\tau| \mapsto \infty} u \cdot \tau \in N_\delta(K_j)\right\},$$

where $u \cdot \tau$ denotes the flow of $-\Phi_k'$ on $E_k \times \mathbb{T}^n$, that is,

$$\frac{d}{d\tau}(u \cdot \tau)=-\Phi_k'(u \cdot \tau).$$

Thus S_j^k consists precisely of the critical points of Φ_k contained in $N_\delta(K_j)$ together with the connecting orbits of these critical points.

LEMMA 11. *Suppose that $\delta>0$ is chosen such that* (22) *and* (23) *hold. There exists $k_0 \in \mathbb{N}$, depending on δ, such that for all $k \geqslant k_0$ we have*

$$S_j^k \subset N_\delta(K_j), \quad \textit{for } j=1, \ldots, m. \tag{24}$$

Proof. By the choice of $\bar{\varepsilon}$, δ, we have $\operatorname{cat}_{E \times \mathbb{T}^n}(N_\delta(K_j))=\operatorname{cat}_{E \times \mathbb{T}^n}(K_j)$, and $|\Phi(u)-c_j|<\bar{\varepsilon}$ for $u \in N_\delta(K_j)$.

Let $\beta>0$ and $k_1(\delta)$ be the δ-depending constants from Lemma 10. Now choose $\varepsilon>0$ such that $\varepsilon<\min\{\bar{\varepsilon}, \frac{1}{8}\beta\delta\}$. According to this choice of ε we can find $0<\delta_0 \leqslant \delta$ such that $|\Phi_k(u)-c_j|<\varepsilon$ if $k \geqslant k_1(\delta_0)$ and $u \in N_{\delta_0/4}(K_j)$. Corresponding to this δ_0 there exists by Lemma 10 an integer $k_1(\delta_0)$ such that for $k \geqslant k_1(\delta_0)$ the critical points of Φ_k have to be contained in $N_{\delta_0/4}(K)$. We define $k_0:=\max\{k_1(\delta), k_1(\delta_0)\}$.

Consider now a point $u \in E_k \times \mathbb{T}^n \backslash N_\delta(K_j)$ which satisfies $|\Phi_k(u)-c_j|<\varepsilon$. In a way similar to the proof of Theorem 2, we conclude that the orbit of this point u cannot enter $N_{\delta/2}(K_j) \supset N_{\delta_0/4}(K_j)$ in a positive direction without having passed the level $c_j-\varepsilon$, which shows that $u \notin S_j^k$. Hence the assertion of the lemma follows.

Recall the isolating block B_k defined in Lemma 9:

$$B_k = D_k^+ \times D_k^- \times \mathbb{T}^n \subset E_k^+ \times E_k^- \times \mathbb{T}^n$$

where $D_k^\pm = \{u^\pm \in E_k^\pm | \ \|u^\pm\| \leq R\}$ for some $R > 0$ independent of k.

We denote by S^k the maximal invariant set of the flow of $-\Phi_k'$ which is contained in the isolating block B_k:

$$S^k := \{u \in E_k \times \mathbb{T}^n | \ u \cdot \tau \in B_k \text{ for all } \tau \in \mathbb{R}\}.$$

In particular, the invariant set S^k is compact, and as a corollary of Lemma 11 we obtain another lemma.

LEMMA 12. *For $k \geq k_0$ the invariant sets S_j^k constitute a Morse decomposition of S^k. Moreover, $(S_1^k, \ldots, S_m^k)$ is an admissible ordering of the Morse decomposition.*

Now Proposition 1 comes in. For the present situation it reads as follows.

LEMMA 13. *For $k \geq k_0$ we have*

$$\operatorname{cat}_{E_k \times \mathbb{T}^n}(S^k) \leq \sum_{j=1}^{m} \operatorname{cat}_{E_k \times \mathbb{T}^n}(S_j^k). \tag{25}$$

Summarizing the steps of the proof of Proposition 2 which we have completed so far, we obtain the following.

COROLLARY 1. *For $k \geq k_0$ we have*

$$\operatorname{cat}_{E_k \times \mathbb{T}^n}(S^k) \leq \sum_{j=1}^{m} \operatorname{cat}_{E \times \mathbb{T}^n}(M_j).$$

Proof. Note that $S_j^k \subset E_k \times \mathbb{T}^n$ and $E_k \times \mathbb{T}^n$ is a retract of $E \times \mathbb{T}^n$ for every $k \in \mathbb{Z}^+$. Consequently

$$\operatorname{cat}_{E_k \times \mathbb{T}^n}(S_j^k) = \operatorname{cat}_{E \times \mathbb{T}^n}(S_j^k). \tag{26}$$

By the monotonicity of cat, (24) yields

$$\operatorname{cat}_{E \times \mathbb{T}^n}(S_j^k) \leq \operatorname{cat}_{E \times \mathbb{T}^n}(N_\delta(K_j)). \tag{27}$$

Using the relations (22), (25), (26) and (27), we have for $k \geq k_0$,

$$\sum_{j=1}^{m} \operatorname{cat}_{E \times \mathbb{T}^n}(M_j) \overset{(22)}{=} \sum_{j=1}^{m} \operatorname{cat}_{E \times \mathbb{T}^n}(N_\delta(K_j)) \overset{(27)}{\geq} \sum_{j=1}^{m} \operatorname{cat}_{E \times \mathbb{T}^n}(S_j^k)$$
$$\overset{(26)}{=} \sum_{j=1}^{m} \operatorname{cat}_{E_k \times \mathbb{T}^n}(S_j^k) \overset{(25)}{\geq} \operatorname{cat}_{E_k \times \mathbb{T}^n}(S^k).$$

It therefore remains to relate the category of S^k to the cuplength of $\mathbb{T}^n$:

PROPOSITION 3. *For $k \in \mathbb{N}$ let $S^k \subset E \times \mathbb{T}^n$ denote the maximal invariant set of the flow of $-\Phi_k'$ which is contained in the isolating block $B_k = D_k^+ \times D_k^- \times T^n$, where $D_k^\pm = \{u^\pm \in E_k^\pm | \ \|u^\pm\| \leq R\}$ for some $R > 0$ according to Lemma 8. Then*

$$\operatorname{cat}_{E_k \times \mathbb{T}^n}(S^k) \geq \operatorname{cuplength}(\mathbb{T}^n) + 1.$$

Proof. We assume, by contradiction, that $\operatorname{cat}_{E_k \times \mathbb{T}^n}(S^k) \leqslant \operatorname{cuplength}(\mathbb{T}^n) = n$. In the following we shall abbreviate $l := \operatorname{cat}_{E_k \times \mathbb{T}^n}(S^k)$.

By definition of the isolating block B_k, the compact invariant set $S^{(k)}$ is contained in the interior $\operatorname{int} B_k$. Also note that B_k is a retract of $E_k \times \mathbb{T}^n$, and hence it follows that

$$\operatorname{cat}_{B_k}(S^k) = \operatorname{cat}_{E_k \times \mathbb{T}^n}(S^k) = l.$$

Using the continuity property of cat, we can find an open neighbourhood N of S^k in B_k such that

$$\operatorname{cat}_{B_k}(N) = \operatorname{cat}_{B_k}(S^k) \quad \text{and} \quad \bar{N} \subset \operatorname{int} B_k.$$

(a) First we consider the case $l \geqslant 1$, which in particular implies that $S^k \neq \varnothing$. By definition of cat there exist l closed subsets $A'_j \subset B_k$ such that A'_j is contractible in B_k and $\bar{N} = A'_1 \cup \ldots \cup A'_l$. We note that $A'_j \cap S^+ \neq \varnothing$ and $A'_j \cap B_k^- = \varnothing$.

Now consider the maximal positively invariant subset contained in B_k:

$$S^+ := \{u \in B_k \mid u \cdot t \in B_k \text{ for all } t \geqslant 0\}.$$

We shall use the l covering sets A'_j and the flow to construct a covering of S^+ by l closed contractible sets having empty intersection with the exit set of B_k. Recall the entrance set B_k^+ and the exit set B_k^- of the isolating block B_k, given by

$$B_k^+ = \partial D_k^+ \times D_k^- \times \mathbb{T}^n \subset \partial B_k,$$
$$B_k^- = D_k^+ \times \partial D_k^- \times \mathbb{T}^n \subset \partial B_k.$$

Observe that $\partial B_k = B_k^+ \cup B_k^-$. For $u \in S^+$ we define the real number $\tau^+(u)$ by

$$\tau^+(u) := \inf\{\tau \in \mathbb{R} \mid u \cdot s \in \bar{N} \text{ for all } s \geqslant \tau\}.$$

We set

$$\tau^+ := \sup\{\tau^+(u) \mid u \in S^+\}.$$

The compactness of $S^+ \subset B_k$ yields $\tau^+ < \infty$, and we conclude that

$$S^+ \subset \bar{N} \cdot (-\tau^+) = \bigcup_{j=1}^{l} A'_j \cdot (-\tau^+).$$

Now $A'_j \cdot (-\tau^+) \subset E_k \times \mathbb{T}^n$ is a closed subset which is contractible in $E_k \times \mathbb{T}^n$. Note that $A'_j \cap S^k \neq \varnothing$, and $A'_j \cdot (-\tau^+) \cap B_k^- = \varnothing$. Now we define

$$A_j := A'_j \cdot (-\tau^+) \cap B_k \quad \text{for } j = 1, \ldots, l.$$

Then $A_j \subset B_k$ is closed and contractible in B_k, and

$$S^+ \subset A_1 \cup \ldots \cup A_l = \bar{N} \cdot (-\tau^+) \cap B_k.$$

Clearly, $A_j \cap S^k \neq \varnothing$, and $A_j \cap B_k^- = \varnothing$ for all $j = 1, \ldots, l$. All of B_k that is not covered by the A_j can be retracted to the exit set B_k^- as follows.

LEMMA 14. *There exists a closed set A satisfying $B_k^- \subset A \subset B_k$, such that B_k^- is a strong deformation retract of A, and such that $A \cup A_1 \cup \ldots \cup A_l = B_k$.*

Proof. Let $N^+ := N \cdot (-\tau^+)$ be the open neighbourhood of S^+ in $E_k \times \mathbb{T}^n$ determined above. Then for every $u \in B_k \backslash N^+$ there exists a unique $t^+(u) \geqslant 0$ such that $u \cdot t^+(u) \in B_k^-$. Since $B_k \backslash N^+$ is compact, $t^+ := \sup\{t^+(u) |\ u \in B_k \backslash N^+\} < \infty$. Consider

$$B_k^\infty := \{x \cdot t |\ x \in B_k^-, t \geqslant 0\}.$$

Observe that B_k^∞ is closed in $E_k \times \mathbb{T}^n$, and $B_k^\infty \cap B_k = B_k^-$ by the proof of Lemma 9. Now we define the closed subset of B_k:

$$A := (B_k^\infty \cdot (-t^+)) \cap B_k.$$

By construction, $A \cup A_1 \cup \ldots \cup A_l = B_k$.

It follows from Wazewski's principle (cf. Conley [**5**, p. 24]) that B_k^- is a strong deformation retract of A. The homotopy r: $A \times [0, 1] \to A$ which retracts A to B_k^- is defined as follows. For $u \in A$, let $t^+(u) \geqslant 0$ be the exit time of u as defined above. Then define $r(u, s) := u \cdot st^+(u)$.

In the following H^* and H_* will denote singular cohomology and homology with integer coefficients respectively.

LEMMA 15. (i) *The injections* i: $(B_k, \varnothing) \to (B_k, A_j)$ *for* $j = 1, \ldots, l$ *induce homomorphisms*

$$i^*\colon\ H^*(B_k, A_j) \to H^*(B_k)$$

which are surjective for $* \geqslant 1$.

(ii) *The injection* g: $(B_k, B_k^-) \to (B_k, A)$ *induces isomorphisms in relative cohomology*

$$g^*\colon\ H^*(B_k, A) \to H^*(B_k, B_k^-).$$

Proof. (i) Let ι: $A_j \to B_k$ denote the inclusion map. Since A_j is contractible in B_k, the inclusion ι is homotopic to a constant map

$$\iota \simeq \iota_0\colon\ A_j \to B_k, \quad \iota_0(u) = u_0 \text{ for all } u \in A_j,$$

and hence we have $\iota^* = \iota_0^*$: $H^*(B_k) \to H^*(A_j)$.

We denote by γ the map from A_j to $\{u_0\}$. Let p: $\{u_0\} \to B_k$ be the inclusion. We have a commutative diagram:

$$\begin{tikzcd} H^*(B_k) \arrow[r, "p^*"] \arrow[dr, "\iota^*"'] & H^*(\{u_0\}) \arrow[d, "\gamma^*"] \\ & H^*(A_j) \end{tikzcd}$$

Since $H^*(\{u_0\}) = 0$ for $* > 0$, we find that $\gamma^* = 0$, and by the commutativity of the diagram, $\iota^* = 0$. Consider the exact cohomology sequence

$$\ldots \longrightarrow H^{*-1}(A_j) \xrightarrow{\delta^*} H^*(B_k, A_j) \xrightarrow{i^*} H^*(B_k) \xrightarrow{\iota^*} H^*(A_j) \longrightarrow \ldots.$$

Now (i) follows from $\iota^* = 0$ and the exactness of the sequence.

(ii) Consider the commutative diagram

$$
\begin{tikzcd}
& {} \arrow[d] & {} \arrow[d] \\
& H^{*-1}(B_k) \arrow[r, "\mathrm{id}^*"] \arrow[d, "j_1^*"] & H^{*-1}(B_k) \arrow[d, "j_2^*"] \\
& H^{*-1}(A) \arrow[r, "i^*"] \arrow[d, "\delta^*"] & H^{*-1}(B_k^-) \arrow[d, "\delta^*"] \\
{} \arrow[r, "\delta^*"] & H^*(B_k, A) \arrow[r, "g^*"] \arrow[d, "\iota^*"] & H^*(B_k, B_k^-) \arrow[r, "f^*"] \arrow[d, "\iota^*"] & H^*(A, B_k^-) \arrow[r, "\delta^*"] & {} \\
& H^*(B_k) \arrow[r, "\mathrm{id}^*"] \arrow[d, "j_1^*"] & H^*(B_k) \arrow[d, "j_2^*"] \\
& H^*(A) \arrow[r, "i^*"] \arrow[d] & H^*(B_k^-) \arrow[d] \\
& {} & {}
\end{tikzcd}
$$

Observe that the long horizontal line is the exact cohomology sequence of the triple (B_k, A, B_k^-). The vertical lines are the exact cohomology sequences of the pairs (B_k, A) and (B_k, B_k^-) respectively. Note that B_k^- is a strong deformation retract of A, and hence the inclusion $i\colon B_k^- \to A$ induces an isomorphism $i^*\colon H^*(A) \to H^*(B_k^-)$. Since i^* and id^* are isomorphisms, it follows by the Five Lemma that g^* is an isomorphism, as claimed.

After these preparations we are ready to establish the argument which contradicts the assumption that $l \leqslant n$.

Observe that $\{0\} \times D_k^- \times \mathbb{T}^n$ is a strong deformation retract of B_k, and $\{0\} \times \{0\} \times \mathbb{T}^n$ is a strong deformation retract of B_k. Consequently, we have isomorphisms which commute with the cup-product

$$H^*(B_k) \cong H^*(D_k^- \times \mathbb{T}^n) \cong H^*(\mathbb{T}^n)$$

and

$$H^*(B_k, B_k^-) \cong H^*(D_k^- \times \mathbb{T}^n, \partial D_k^- \times \mathbb{T}^n).$$

Since $\mathrm{cuplength}(\mathbb{T}^n) = \mathrm{cuplength}(D_k^- \times \mathbb{T}^n)$, there exist cohomology classes

$$\omega_j \in H^{q_j}(D_k^- \times \mathbb{T}^n), \quad \text{for } j = 1, \ldots, n,\ q_j \geqslant 1,$$

such that $\omega_1 \cup \ldots \cup \omega_n \neq 0$. In particular, if $l < n$, we consider the first l factors of this cup-product only. Of course the cup-product $\omega_1 \cup \ldots \cup \omega_l \neq 0$. Let $q := q_1 + \ldots + q_l$. Consequently there exists a homology class $\alpha \in H_q(D_k^- \times \mathbb{T}^n)$ such that

$$\langle \omega_1 \cup \ldots \cup \omega_l, \alpha \rangle \neq 0, \tag{28}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing

$$H^*(D_k^- \times \mathbb{T}^n) \times H_*(D_k^- \times \mathbb{T}^n) \to \mathbb{Z}.$$

Recall that $\dim E_k^- = k$. Let $\xi \in H_{n+k}(D_k^- \times \mathbb{T}^n, \partial D_k^- \times \mathbb{T}^n)$ denote the fundamental class. By Lefschetz duality there is an isomorphism given by the cap-product

$$H^{n+k-q}(D_k^- \times \mathbb{T}^n, \partial D_k^- \times \mathbb{T}^n) \xrightarrow{\cap \xi} H_q(D_k^- \times \mathbb{T}^n).$$

Hence there exists $\omega_0 \in H^{n+k-q}(D_k^- \times \mathbb{T}^n, \partial D_k^- \times \mathbb{T}^n)$ such that $\alpha = \omega_0 \cap \xi$, and we have

$$\begin{aligned} \langle \omega_1 \cup \ldots \cup \omega_l, \alpha \rangle &= \langle \omega_1 \cup \ldots \cup \omega_l, \omega_0 \cap \xi \rangle \\ &= \langle \omega_0 \cup \omega_1 \cup \ldots \cup \omega_l, \xi \rangle. \end{aligned}$$

We claim that $\omega_0 \cup \omega_1 \cup \ldots \cup \omega_l = 0$. Consider the commutative diagram

$$\begin{array}{ccccc}
H^{n+k-q}(B_k, A) & \xrightarrow{g^*} & H^{n+k-q}(B_k, B_k^-) & \xrightarrow{\rho^{*^{-1}}} & H^{n+k-q}(D_k^- \times \mathbb{T}^n, \partial D_k^- \times \mathbb{T}^n) \\
\otimes & & \otimes & & \otimes \\
H^{q_1}(B_k, A_1) & \xrightarrow{i^*} & H^{q_1}(B_k) & \xrightarrow{r^{*^{-1}}} & H^{q_1}(D_k^- \times \mathbb{T}^n) \\
\otimes & & \otimes & & \otimes \\
\vdots & & \vdots & & \vdots \\
\otimes & & \otimes & & \otimes \\
H^{q_l}(B_k, A_l) & \xrightarrow{i^*} & H^{q_l}(B_k) & \xrightarrow{r^{*^{-1}}} & H^{q_l}(D_k^- \times \mathbb{T}^n) \\
\downarrow \cup & & \downarrow \cup & & \downarrow \cup \\
H^{n+k}(B_k, B_k) & \longrightarrow & H^{n+k}(B_k, B_k^-) & \xrightarrow{\rho^{*^{-1}}} & H^{n+k}(D_k^- \times \mathbb{T}^n, \partial D_k^- \times \mathbb{T}^n)
\end{array}$$

Here $i^*\colon H^*(B_k, A_j) \to H^*(B_k)$ denotes the surjective homomorphism from Lemma 15(i), and $g^*\colon H^*(B_k, A) \to H^*(B_k, B_k^-)$ is the isomorphism from Lemma 15(ii). Let $r\colon [0, 1] \times B_k \to B_k$ denote the deformation retract from B_k to $D_k^- \times \mathbb{T}^n$, and let r^* and ρ^* be the isomorphisms

$$r^*\colon H^*(D_k^- \times \mathbb{T}^n) \to H^*(B_k),$$

$$\rho^*\colon H^*(D_k^- \times \mathbb{T}^n, \partial D_k^- \times \mathbb{T}^n) \to H^*(B_k, B_k^-),$$

induced by $r(1, \cdot)\colon B_k \to D_k^- \times \mathbb{T}^n$. For $j = 1, \ldots, l$, we can find cohomology classes $\tilde{\omega}_j \in H^{q_j}(B_k, A_j)$ such that

$$\omega_j = (r^{*^{-1}} \circ \iota^*)(\tilde{\omega}_j) \in H^{q_j}(D_k^- \times \mathbb{T}^n).$$

Similarly we define

$$\tilde{\omega}_0 := (g^{*^{-1}} \circ \rho^*)(\omega_0) \in H^{n+k-q}(B_k, A).$$

Consequently we have

$$\tilde{\omega}_0 \cup \tilde{\omega}_1 \cup \ldots \cup \tilde{\omega}_l \in H^{n+k}(B_k, A \cup A_1 \cup \ldots \cup A_l) = H^{n+k}(B_k, B_k) = \{0\},$$

and therefore the cup-product vanishes. By the commutativity of the above diagram then also $\omega_0 \cup \omega_1 \cup \ldots \cup \omega_l = 0$, which contradicts (28).

(b) We still have to consider the case $l = 0$. Then $S^k = \emptyset$ and consequently $S^+ = \emptyset$, which by Wazewski's principle implies that B_k^- is a strong deformation retract of B_k. Consequently we can choose $A = B_k$ in the above proof, and therefore we have

$$H^*(B_k, B_k) \cong H^*(B_k, B_k^-) \cong H^*(D_k^- \times \mathbb{T}^n, \partial D_k \times \mathbb{T}^n),$$

which is obviously a contradiction. This completes the proof of Proposition 3.

(iii) Assuming finally that Φ has no critical values at all, we conclude by the A-properness of Φ with respect to the Galerkin approximation scheme that there has to be a $k_0 \in \mathbb{Z}^+$ such that for $k \geq k_0$ the functions Φ_k have no critical points. This, however, contradicts Proposition 3, and the proof of Proposition 2 is complete.

Finally we consider the case that the action functional Φ: $E \times \mathbb{T}^n \to \mathbb{R}$ is not Lipschitz-continuous. Then we have to ensure that for k sufficiently large there exists an appropriate gradient-like vector field V_k for Φ_k. Observe that in contrast to the situation considered in Theorem 2, the function Φ_k is not known to have at most finitely many critical values. However, the assumption on Φ to have only the critical values $c_1 < \ldots < c_m$, together with the approximation properties is sufficient.

Recall the choice of $\bar{\varepsilon}$ and δ in (22) and (23); that is, we have $|\Phi(u) - c_j| < \bar{\varepsilon}$ if $u \in N_\delta(K_j)$, and $\operatorname{cat}_{E\times\mathbb{T}^n}(K_j) = \operatorname{cat}_{E\times\mathbb{T}^n}(N_\delta(K_j))$ for $j = 1, \ldots, m$. Also recall constants β and $k_1(\delta)$ determined in Lemma 10. As in the proof of Lemma 11, fix $\varepsilon < \min\{\bar{\varepsilon}, \frac{1}{8}\beta\delta\}$, and choose δ_0 such that $|\Phi_k - c_j| < \varepsilon$ for $k \geq k_1(\delta_0)$ and $u \in N_{\delta_0/4}(K_j)$ for all j. It can be supposed that $k_1(\delta_0) \geq k_1(\delta)$. Finally we abbreviate $k_0 := k_1(\delta_0)$. In particular, if $k \geq k_0$ the critical points of Φ_k are contained in $N_{\delta_0/4}(K)$.

Now we define a $\frac{3}{8}\delta_0$-pseudo gradient vector field for Φ_k for a fixed $k \geq k_0$ according to the following conditions.

(i) For $u_0 \in E_k \times \mathbb{T}^n$ there exists $r^{(1)}_{u_0}$ such that

$$\|\tfrac{3}{2}\Phi'_k(u_0)\| \leq 2\,\|\Phi'_k(u)\| \quad \text{if } u \in B(u_0, r^{(1)}_{u_0}).$$

(ii) If $\Phi'_k(u_0) \neq 0$ then there exists $r^{(2)}_{u_0}$ such that

$$(\tfrac{3}{2}\Phi'_k(u_0), \Phi'_k(u)) > \|\Phi'_k(u)\|^2 \quad \text{if } u \in B(u_0, r^{(2)}_{u_0}).$$

We denote $u = (u^+ + u^-, u^0) \in (E^+_k \oplus E^-_k) \times \mathbb{T}^n$.

(iii) If $(\Phi'_k(u_0), u^+_0) \neq 0$ then there exists $r^+_{u_0} > 0$ such that

$$|(\tfrac{3}{2}\Phi'_k(u_0), u^+) - (\Phi'_k(u), u^+)| \leq \tfrac{3}{4}\,|(\Phi'_k(u), u^+)|$$

if $u \in B(u_0, r^+_{u_0})$.

(iv) If $(\Phi'_k(u_0), u^-_0) \neq 0$ then there exists $r^-_{u_0} > 0$ such that

$$|(\tfrac{3}{2}\Phi'_k(u_0), u^-) - (\Phi'_k(u), u^-)| \leq \tfrac{3}{4}\,|(\Phi'_k(u), u^-)|$$

if $u \in B(u_0, r^-_{u_0})$.

Here $B(u_0, r) \subset E_k \times \mathbb{T}^n$ denotes the open r-ball in $E_k \times \mathbb{T}^n$ around u_0. We define $\varrho_{u_0} := \min\{\frac{1}{8}\delta_0, r^{(1)}_{u_0}, r^{(2)}_{u_0}, r^+_{u_0}, r^-_{u_0}\}$ for all $u_0 \in E_k \times \mathbb{T}^n$. The $\frac{3}{8}\delta_0$-pseudo gradient vector field V_k can be defined similarly to that in the proof of Theorem 2. By the paracompactness of $E_k \times \mathbb{T}^n$, we can choose a locally finite subcover $\{B(u_i, \varrho_{u_i}) \mid i \in I\} \subset \{B(u, \varrho_u) \mid u \in E_k \times \mathbb{T}^n\}$. In view of the conditions (iii) and (iv), the flow of $-V_k$ admits the desired isolating block. We claim that the following conditions hold true. There exists $R' > 0$ such that

$$(-V_k(u), u^+) < 0 \tag{29}$$

if $\|u^+\| \geq R'$ and $\|u^+\| \geq \|u^-\|$, and

$$(-V_k(u), u^-) > 0 \tag{30}$$

if $\|u^-\| \geq R'$ and $\|u^-\| \geq \|u^+\|$.

We shall prove (29). Choose $r := \frac{1}{4}\delta_0$. Fix an arbitrary constant $c > 0$, and let $R > 0$ be such that (20) holds for this choice of c and r. Now consider $u = (u^+ + u^-, u^0) \in (E_k^+ \oplus E_k^-) \times \mathbb{T}^n$ satisfying $\|u^+\| \geq R + \frac{1}{8}\delta_0$ and $\|u^-\| \leq \|u^+\|$. Let $\{\psi_i \mid i \in I\}$ be a Lipschitz-continuous partition of the unity subordinated to the above subcover. Then

$$V_k(u) = \sum_{i \in I_u} \psi_i(u)(\tfrac{3}{2}\Phi_k'(u_i)),$$

where $I_u := \{i \in I \mid u \in B(u_i, \varrho_{u_i})\}$ is a finite subset. Consequently $\|u - u_i\| \leq \frac{1}{8}\delta_0$ by the definition of ϱ_{u_i}. Hence it follows that $\|u_i^+\| \geq R$ and $\|u_i^-\| \leq \|u_i^+\| + \frac{1}{4}\delta_0$, and consequently, by (20),

$$(\Phi_k'(u_i), u_i^+) > c > 0 \quad \text{for } i \in I_u.$$

From the above condition (iii) we now conclude that

$$|(\tfrac{3}{2}\Phi_k'(u_i), u^+) - (\Phi_k'(u), u^+)| \leq \tfrac{3}{4}\, |(\Phi_k'(u), u^+)|$$

for all $i \in I_u$. Summarizing we finally have

$$\begin{aligned}(-V_k(u), u^+) &\leq -(\Phi_k'(u), u^+) + \sum_{i \in I_u} \psi_i(u)\, |(\tfrac{3}{2}\Phi_k'(u_i) - \Phi_k'(u), u^+)| \\ &\leq -(\Phi_k'(u), u^+) + \sum_{i \in I_u} \psi_i(u)\tfrac{3}{4}\, |(\Phi_k'(u), u^+)| \\ &= -\tfrac{1}{4}(\Phi_k'(u), u^+) \leq -\tfrac{1}{4}c < 0.\end{aligned}$$

Hence (29) is proved with $R' = R + \frac{1}{8}\delta_0$. The inequality (30) is proved similarly, which finishes the proof of Theorem 6.

Appendix

Let M be a Hausdorff space. We collect some properties of cat_M:

(i) $\mathrm{cat}_M(A) = 1$ if and only if the closure $\bar{A}$ is contractible in M;

(ii) $\mathrm{cat}_M(A) = \mathrm{cat}_M(\bar{A})$;

(iii) if A is closed in M then $\mathrm{cat}_M(A) = k$ if and only if A is the union of k closed sets each contractible in M;

(iv) cat is monotone, that is, if $A \subset B \subset M$ then $\mathrm{cat}_M(A) \leq \mathrm{cat}_M(B)$;

(v) cat is subadditive, that is $\mathrm{cat}_M(A \cup B) \leq \mathrm{cat}_M(A) + \mathrm{cat}_M(B)$;

(vi) if A is closed and h_t: $A \to M$, for $0 \leq t \leq 1$, is a homotopy such that h_0 is the inclusion map of A into M, then $\mathrm{cat}_M(h_1(A)) \geq \mathrm{cat}_M(A)$;

(vii) if h_t: $M \to M$, for $0 \leq t \leq 1$, is a homotopy of M such that $h_0 = \mathrm{id}_M$, then $\mathrm{cat}_M(h_1(A)) \geq \mathrm{cat}_M(A)$ for every $A \subset M$;

(viii) if h: $M \to M$ is a homeomorphism of M, then $\mathrm{cat}_M(h(A)) = \mathrm{cat}_M(A)$ for every $A \subset M$;

(ix) let M be arcwise connected; then $\mathrm{cat}_M(A) = 1$ if $A \neq \varnothing$ is a finite set;

(x) let M be an absolute neighbourhood retract (ANR); if $A \neq \varnothing$ then $\mathrm{cat}_M(A) = \mathrm{cat}_M(N)$ for some neighbourhood N of A in M;

(xi) let M be a metric ANR; if $K \neq \varnothing$ is compact then there exists $\delta > 0$ such that $\mathrm{cat}_M(K) = \mathrm{cat}_M(N_\delta(K))$, where $N_\delta(K)$ denotes the δ-neighbourhood of K in M;

(xii) let $A \subset M \subset M'$; then $\operatorname{cat}_M(A) \geqslant \operatorname{cat}_{M'}(A)$; if M is a retract of M' then $\operatorname{cat}_M(A) = \operatorname{cat}_{M'}(A)$;

(xiii) let $\varnothing \neq A \subset M$ be closed, and suppose $\operatorname{cat}_M(A) < \infty$; if N is a neighbourhood of A in M such that $\operatorname{cat}_M(N) = \operatorname{cat}_M(A) = k$, and if $A_1 \cup \ldots \cup A_k \supset N$ is a covering of N by closed contractible subsets of M, then $A_j \cap A \neq \varnothing$.

Most of the proofs of these properties can be found in work of R. S. Palais **[28]** and K. Deimling **[9]**.

Acknowledgement

The results presented here are taken from the author's dissertation written under the direction of Eduard Zehnder. The author wishes to express his gratitude to his adviser for invaluable support and inspiration.

References

1. V. Benci and P. H. Rabinowitz, 'Critical point theorems for indefinite functionals', *Invent. Math.* 52 (1979) 241–273.
2. K. C. Chang, 'On the periodic nonlinearity and the multiplicity of solutions', *Nonlinear Anal.* 13 (1989) 527–538.
3. K. C. Chang, Y. Long, and E. Zehnder, 'Forced oscillations for the triple pendulum', *Analysis, et cetera* (eds P. H. Rabinowitz and E. Zehnder, Academic Press, San Diego, 1990).
4. W. Chen, 'Birkhoff periodic orbits for small perturbations of completely integrable Hamiltonian systems with nondegenerate Hessian', preprint, 1990.
5. C. C. Conley, *Isolated invariant sets and the Morse index,* CBMS Regional Conference Series in Mathematics 38 (American Mathematical Society, Providence, R. I., 1978).
6. C. C. Conley and E. Zehnder, 'The Birkhoff–Lewis fixed point theorem and a conjecture of V. I. Arnold', *Invent. Math.* 73 (1983) 33–49.
7. C. C. Conley and E. Zehnder, 'Morse-type index theory for flows and periodic solutions for Hamiltonian systems', *Comm. Pure Appl. Math.* 37 (1984) 207–253.
8. C. C. Conley and E. Zehnder, 'A global fixed point theorem for symplectic maps and subharmonic solutions of Hamiltonian systems on tori', *Nonlinear analysis and applications* (ed. F. E. Browder), Proceedings of Symposia in Pure Mathematics 45, Part 1 (American Mathematical Society, Providence, R. I., 1986), pp. 283–299.
9. K. Deimling, *Nonlinear functional analysis* (Springer, Berlin, 1985).
10. A. Floer, 'Proof of the Arnold conjecture for surfaces and generalizations to certain Kähler manifolds', *Duke Math. J.* 53 (1986) 1–32.
11. A. Floer, 'Cuplength estimates on Lagrangian intersections', *Comm. Pure Appl. Math.* 42 (1989) 335–356.
12. A. Floer and E. Zehnder, 'Fixed point results for symplectic maps related to the Arnold conjecture', *Dynamical systems and bifurcations,* Lecture Notes in Mathematics 1125 (Springer, Berlin, 1984), pp. 47–64.
13. P. L. Felmer, 'Periodic solutions of spatially periodic Hamiltonian systems', *J. Differential Equations* 98 (1992) 143–168.
14. P. L. Felmer, 'Multiple periodic solutions for Lagrangian systems in $\mathbb{T}^n$', *Nonlinear Anal.* 15 (1990) 815–831.
15. G. Fournier and M. Willem, 'Multiple solutions of the forced double pendulum equation', *Ann. Inst. H. Poincaré. Anal. Non Linéaire* 6, supplement (1989) 259–281.
16. G. Fournier and M. Willem, 'Relative category and the calculus of variations', preprint, 1989.
17. C. Golé, 'Periodic orbits for symplectomorphisms of $\mathbb{T}^n \times \mathbb{R}^n$', Ph.D. thesis, Boston University, 1989.
18. C. Golé, 'Monotone maps of $\mathbb{T}^n \times \mathbb{R}^n$ and their periodic orbits', *The geometry of Hamiltonian systems* (ed. T. Ratin), MSRI Publications 22 (Springer, New York, 1991), pp. 341–366.
19. C. Golé, 'Ghost tori for monotone maps', *Proceedings on twist maps, IMA,* to appear.
20. M. J. Greenberg and J. R. Harper, *Algebraic topology: a first course* (Benjamin/Cummings, Menlo Park, California, 1981).

21. M. W. HIRSCH, *Differential topology* (Springer, New York, 1976).
22. H. HOFER, 'On strongly indefinite functionals with applications', *Trans. Amer. Math. Soc.* 275 (1983) 185–213.
23. H. HOFER, 'Lagrangian embeddings and critical point theory', *Ann. Inst. H. Poincaré. Anal. Non Linéaire* 2 (1985) 407–462.
24. F. W. JOSELLIS, 'Global periodic orbits for Hamiltonian systems on $\mathbb{T}^n \times \mathbb{R}^n$', dissertation, ETH Zürich, 1991.
25. F. W. JOSELLIS, 'Morse theory for forced oscillations of Hamiltonian systems on $\mathbb{T}^n \times \mathbb{R}^n$', *J. Differential Equations*, to appear.
26. W. KLINGENBERG, *Riemannian geometry* (De Gruyter, Berlin, 1982).
27. J. Q. LIU, 'A generalized saddle point theorem', *J. Differential Equations* 82 (1989) 372–385.
28. R. S. PALAIS, 'Critical point theory and the minimax principle', *Global analysis*, Proceedings of Symposia in Pure Mathematics 15 (American Mathematical Society, Providence, R. I., 1970), pp. 185–212.
29. P. H. RABINOWITZ, 'On a class of functionals invariant under a $\mathbb{Z}^n$-action', *Trans. Amer. Math. Soc.* 310 (1988) 303–311.
30. P. H. RABINOWITZ, *Minimax methods in critical point theory and applications to differential equations*, CBMS Regional Conference Series in Mathematics 65 (American Mathematical Society, Providence, R. I., 1986).
31. J. T. SCHWARTZ, 'Generalizing the Lyusternik-Schnirelman theory of critical points', *Comm. Pure Appl. Math.* 17 (1964) 307–315.
32. J. T. SCHWARTZ, *Nonlinear functional analysis* (Gordon and Breach, New York, 1969).
33. J.-C. SIKORAV, 'Points fixes d'une application symplectique homologue à l'identité', *J. Differential Geom.* 22 (1985) 49–79.
34. A. SZULKIN, 'A relative category and applications to critical point theory for strongly indefinite functionals', *Nonlinear Anal.* 15 (1990) 725–739.
35. J. WEIDMANN, *Lineare Operatoren in Hilberträumen* (B. G. Teubner, Stuttgart, 1976).

ETH Zürich
Mathematikdepartement
CH-8092 Zürich
Switzerland