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Morse Theory for Forced Oscillations of Hamiltonian Systems on $\mathbb{T}^n \times \mathbb{R}^n$

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Considering a large class of periodically time dependent Hamiltonian systems on the cotangent bundle $T^*(\mathbb{T}^n)$ of the n-dimensional torus $\mathbb{T}^n$, we prove the existence of at least 2^n forced oscillations in every homotopy class of loops, provided these periodic solutions are non-degenerate. Moreover, given $\alpha \in \mathbb{Q}^n$ such that all the periodic solutions having rotation vector α are non-degenerate, there exist at least 2^n of them. We assume the Hamiltonian system to be asymptotically linear in the fibres, and the norm of the Hessian is required to have at most polynomial growth. © 1994 Academic Press, Inc.

1. Introduction and Results

We consider a non-autonomous Hamiltonian system

$$\dot{z} = J\nabla H(z, t), \qquad z \in \mathbb{R}^{2n}, \quad t \in \mathbb{R}, \tag{1}$$

where J is the skew-symmetric matrix

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \in \mathscr{L}(\mathbb{R}^n \times \mathbb{R}^n) \tag{2}$$

where I denotes the identity on $\mathbb{R}^n$. We shall assume that H depends periodically on the time t with period 1, and setting $z = (x, y) \in \mathbb{R}^n \times \mathbb{R}^n$ we shall also assume that H is periodic with respect to $x \in \mathbb{R}^n$:

$$H(x + j, y, t) = H(x, y, t) = H(x, y, t + 1) \qquad \text{for all} \quad j \in \mathbb{Z}^n. \tag{3}$$

Thus (1) is a periodically time dependent Hamiltonian system on the cotangent bundle $T^*\mathbb{T}^n \cong \mathbb{T}^n \times \mathbb{R}^n$ of the n-dimensinal flat torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$.

If the Hamiltonian H satisfies appropriate conditions asymptotically in the fibres, then the flow of the vector field $J\nabla H$ on $\mathbb{T}^n \times \mathbb{R}^n$ will contain

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periodic orbits. We shall assume the Hamiltonian vector field to be asymptotically linear, requiring that

$$\frac{1}{|y|}\,|\partial_y H(x, y, t) - A(t)\,y| \to 0 \qquad \text{and} \qquad \frac{1}{|y|}\,|\partial_x H(x, y, t)| \to 0 \tag{4}$$

as $|y| \to \infty$ uniformly in x and t, with a symmetric matrix $A(t)$ depending continuously and periodically on t

$$A(t) = A(t+1) \in \mathscr{L}(\mathbb{R}^n), \tag{5}$$

and satisfying the following non-resonance condition

$$\det\left(\int_0^1 A(t)\,dt\right) \neq 0. \tag{6}$$

Assuming $H \in C^1$ satisfies (4) and (6), we have shown in [15] the existence of at least $n+1 = \text{cuplength}(\mathbb{T}^n) + 1$ forced oscillations, i.e., 1-periodic solutions of (1), in every homotopy class of loops in $\mathbb{T}^n \times \mathbb{R}^n$. In the present paper we shall prove by the use of Morse theory that there exist at least 2^n forced oscillations in a homotopy class of loops in case that the 1-periodic orbits contained in this class are known to be non-degenerate. Clearly, in order to define the non-degeneracy of periodic orbits it is necessary to impose suitably stronger hypotheses on the Hamiltonian H.

Before giving a precise formulation of our main result we describe explicitly the periodic solutions we are looking for:

If $z(t) = (x(t), y(t)) = \mathbb{R}^n \times \mathbb{R}^n$ is a solution of (1) such that

$$\lim_{|t| \to \infty} \frac{x(t)}{t} = \alpha$$

for some $\alpha \in \mathbb{R}^n$, then the vector α is said to be the rotation vector of z. If in addition we assume this solution to be periodic with integer period p, requiring

$$x(t+p) = x(t) + j \qquad \text{and} \qquad y(t+p) = y(t)$$

for some $j \in \mathbb{Z}^n$, it follows that $\alpha p = j$ and

$$x(t) = \alpha t + \xi(t) \qquad \text{with} \qquad \xi(t+p) = \xi(t). \tag{7}$$

Definition 1. If $j = (j_1, ..., j_n) \in \mathbb{Z}^n$, and if $p \in \mathbb{N}$ such that $j_1, ..., j_n$, and p are relatively prime, then we call a solution $z(t) = (x(t), y(t))$ of (1) a j/p-solution if $x(t)$ is of the form (7) with $\alpha = j/p$.

Hence a j/p-solution considered in the universal cover $\mathbb{R}^n \times \mathbb{R}^n$ of the phase space $\mathbb{T}^n \times \mathbb{R}^n$ is not periodic unless $j=0$. However, its image under the canonical projection is a parameterized loop $\mathbb{R}/p\mathbb{Z} \to \mathbb{T}^n \times \mathbb{R}^n$. Recall that $\pi_1(\mathbb{T}^n \times \mathbb{R}^n) \cong \pi_1(\mathbb{T}^n) \cong \mathbb{Z}^n$, and the homotopy class of a j/p-loop is uniquely determined by the integer vector j, provided j and p are relatively prime.

Consider a fixed $\alpha = j/p \in \mathbb{Q}^n$. Given a j/p-solution $z(t) \in \mathbb{R}^n \times \mathbb{R}^n$, there exists a decomposition of z,

$$z(t) = \begin{pmatrix} \xi(t) + \alpha t \\ y(t) \end{pmatrix} =: \zeta(t) + e_\alpha t, \tag{8}$$

where $\zeta(t) = \zeta(t+p)$ is periodic, and where $e_\alpha = (\alpha, 0) \in \mathbb{R}^n \times \mathbb{R}^n$ contains the rotation vector $\alpha = j/p$.

Since $z(t)$ is a j/p-solution of the Hamiltonian equation, $\zeta(t)$ is a p-periodic solution of the following Hamiltonian equation which is periodic in t with period p:

$$\dot{\zeta}(t) = J\nabla H(\zeta(t) + e_\alpha t, t) - e_\alpha.$$

If the Hamiltonian $H(z, t)$ is twice continuously differentiable with respect to z then the Floquet multipliers of the periodic solution ζ are defined. Consequently we shall call a j/p-solution of the Hamiltonian equation (1) non-degenerate if its periodic part ζ according to the decomposition (8) is non-degenerate, i.e., ζ has no Floquet multipliers equal to 1.

Now we state our main result:

THEOREM 1. *Assume that $H \in C^2$ is periodic in x and t, and satisfies the asymptotic condition* (4), *such that H is non-resonant at infinity according to* (6).

We shall assume that the Hessian d^2H of the Hamiltonian satisfies the following growth condition with respect to y: there exist constants $a_1, a_2 \geqslant 0$ and $1 \leqslant r < \infty$ such that

$$|d^2H(x, y, t)| \leqslant a_1 + a_2\,|y|^r \tag{9}$$

uniformly in x and t.

If all j/p-solutions of the Hamiltonian equation (1) *are non-degenerate then*

$$\#\{j/p\text{-solutions}\} \geqslant SB(\mathbb{T}^n) = 2^n.$$

Here $SB(\mathbb{T}^n)$ stands for the sum of Betti numbers of $\mathbb{T}^n$.

Theorem 1 is related to a corresponding result due to Golé for monotone symplectomorphisms on $\mathbb{T}^n \times \mathbb{R}^n$, cf. [11, 12, 13]. His work includes the case of periodically time dependent Hamiltonians $H \in C^2$ which satisfy

$$H(x,y,t) = \tfrac{1}{2}\langle Ay, y\rangle + \langle c, y\rangle \qquad \text{if} \quad |y| \geqslant a$$

for some constant $a > 0$, where $A \in \mathscr{L}(\mathbb{R}^n)$ is a symmetric matrix with $\det A \neq 0$, and $c \in \mathbb{R}^n$. In addition, H is assumed to satisfy the interior condition

$$\det \frac{\partial^2 H}{\partial y^2}(x, y, t) \neq 0 \qquad \text{for all} \quad (x, y, t) \in \mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R},$$

which corresponds to the monotone twist condition in the case $n = 1$. Under the assumption that all the j/p-orbits are non-degenerate, Golé proves that there exist at least 2^n of them.

These assumptions are somewhat different from ours since we do not require any interior conditions. In particular, the existence of j/p-solutions can be established assuming only $H \in C^1$, cf. [15]. The hypothesis $H \in C^2$ is needed only in order to define the non-degeneracy. The non-resonance condition (6) is, however, crucial and cannot be dropped. It plays the role of the twist of the boundaries in the Poincaré–Birkhoff fixed point theorem, which states that every measure preserving homeomorphism of an annulus $\mathbb{S}^1 \times [a, b]$, rotating the boundaries $\mathbb{S}^1 \times \{a\}$ and $\mathbb{S}^1 \times \{b\}$ in opposite directions, has at least 2 fixed points in the interior. The above results can be seen as generalizations of this theorem to higher dimensions. Furthermore, a similar phenomenon arises in search of periodic orbits of asymptotically linear Hamiltonian systems in $\mathbb{R}^{2n}$, where existence results have been established under suitable non-resonance conditions, cf. [2, 6, 16, 19]. All these theorems may be understood as certain generalizations of the symplectic fixed point results on compact manifolds related to the Arnold conjecture, see [7, 9] for more details.

For the case of contractible loops in $\mathbb{T}^n \times \mathbb{R}^n$, the first result of this type is due to Conley and Zehnder [5, Theorem 3]. Although the non-degenerate case has not been considered explicitly by Conley and Zehnder, it is easily seen from their proofs that under the assumption of non-degeneracy on all the contractible 1-periodic solutions there exist at least 2^n of them. Including the case of periodic solutions having a rotation vector $\neq 0$, their result can be generalized as follows:

THEOREM 2. *Assume that $H \in C^2$ satisfies the condition* (4), *and let $A(t)$ satisfy* (5) *and* (6). *Moreover assume that there exists $R > 0$ such that*

$$H(x, y, t) = \tfrac{1}{2}\langle A(t)\, y, y\rangle + \langle b(t), y\rangle \qquad \text{if} \quad |y| \geqslant R$$

where b: $\mathbb{R} \to \mathbb{R}^n$ is a continuous periodic mapping with $b(t) = b(t+1)$. We denote the mean values by

$$[A] := \int_0^1 A(t)\, dt \in \mathscr{L}(\mathbb{R}^n), \qquad [b] := \int_0^1 b(t)\, dt \in \mathbb{R}^n.$$

Suppose $j \in \mathbb{Z}^n$ *and* $p \in \mathbb{N}$ *are relatively prime and satisfy*

$$|j/p - [b]| < \frac{R}{|[A]^{-1}|}. \tag{10}$$

If all the j/p-solutions are non-degenerate then there exist at least 2^n *of them. These orbits are contained in* $\mathbb{T}^n \times D_R$, *where* $D_R := \{y \in \mathbb{R}^n \mid |y| < R\}$ *is the disc of radius R.*

Theorem 2 is easily derived from Theorem 1, observing that the condition (10) implies that the j/p-solutions have to be contained in $\mathbb{T}^n \times D_R$, cf. [15].

Our assumptions on H include in particular the case of special Hamiltonians to the form

$$H(x, y, t) = \tfrac{1}{2}|y|^2 + V(x, t)$$

where V is periodic in $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$, and, moreover, $V \in C^2$, with respect to x. An example is the multiple pendulum, see Chang *et al.* [3]. Theorem 1 also holds true for not exact Hamiltonian systems of the form

$$\dot{z} = J[\nabla H(z, t) + f(t)] \tag{11}$$

with a continuous 1-periodic map $f(t) = (f_1(t), f_2(t)) \in \mathbb{R}^n \times \mathbb{R}^n$ which satisfies the mean value condition $\int_0^1 f_1(t)\,dt = 0$.

We briefly sketch the idea of the proof of Theorem 1. The claimed periodic orbits are characterized by a variational principle. The next section contains the setup for the variational formulation, which is the same as in [15]. For fixed $p \in \mathbb{N}$, the claimed j/p-solutions are characterized as the critical points of a functional Φ, being of the form

$$\Phi(u) = 1/2(Tu, u) - \hat{\varphi}(u) + (v, u),$$

defined on a suitable Hilbert space of loops $u : \mathbb{R}/p\mathbb{Z} \to \mathbb{R}^{2n}$. Here T is a bounded Fredholm operator of index 0, and dim $\ker(T) = n$. The nonlinearity $\hat{\varphi}$ and the linear part $(v, \cdot)$ depend on the particular integer vector j.

The action functional Φ is invariant under a free $\mathbb{Z}^n$-action on $\ker(T)$, such that mod $\mathbb{Z}^n$ we have $\Phi : \ker(T)^\perp \times \mathbb{T}^n \to \mathbb{R}$.

In Section 3 it is shown that the assumption of non-degeneracy on the j/p-solutions is equivalent to requiring that Φ is a Morse function.

In Section 4 we introduce a Galerkin approximation by finite-dimensional subspaces of $\ker(T)^\perp$, and the gradient of Φ is shown to be A-proper with respect to this projection scheme. For $k \in \mathbb{N}$ sufficiently large, the critical points of Φ and the critical points of the approximating functions Φ_k are in 1–1 correspondence, and, moreover, all the critical points of Φ_k are also non-degenerate. For the definition of A-proper mappings we refer to [8].

Finally, Section 5 contains the construction of an isolating block for the invariant set of bounded trajectories under the gradient flow of Φ_k, and the proof of Theorem 1 the follows by the Conley–Zehnder Morse theory.

Related results concerning the existence of contractible periodic solutions also have been obtained by Fournier and Willem [10]. As in [5], a Lyapunov–Schmidt reduction to finite dimension is used, which in particular requires that the norm of the Hessian of H be bounded.

2. The Variational Formulation

By $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ we denote the circle. Consider the Hilbert space $L^2(\mathbb{S}^1, \mathbb{R}^{2n})$ together with the inner product

$$(z, w)_2 := \int_0^1 \langle z(t), w(t) \rangle \, dt$$

where $\langle \cdot, \cdot \rangle$ is the Euclidean scalar product in $\mathbb{R}^{2n}$. The corresponding L^2-norm will be denoted by $\|z\|_2 = \sqrt{(z, z)_2}$.

If $\{e_i \mid i = 1, ..., 2n\}$ denotes the standard basis of $\mathbb{R}^{2n}$, an orthonormal basis of $L^2(\mathbb{S}^1, \mathbb{R}^{2n})$ is given by

$$\{u_{ki}(t) = \exp(2\pi k t J) \, e_i \mid k \in \mathbb{Z}, i = 1, ..., 2n\}$$

where J is the skew-symmetric matrix defined in (2). Every $z \in L^2(\mathbb{S}^1, \mathbb{R}^{2n})$ has the Fourier coefficients

$$z_k = \sum_{i=1}^{2n} (u_{ki}, z)_2 \, e_i \in \mathbb{R}^{2n}, \qquad k \in \mathbb{Z},$$

and is represented by the Fourier expansion

$$z(t) = \sum_{k \in \mathbb{Z}} \sum_{i=1}^{2n} (u_{ki}, z)_2 \, u_{ki}(t) = \sum_{k \in \mathbb{Z}} \exp(2\pi k t J) \, z_k$$

for almost every $t \in \mathbb{S}^1$. Using the Fourier expansion we introduce the following Sobolev space

$$W^{1/2, 2}(\mathbb{S}^1, \mathbb{R}^{2n}) = \left\{ z \in L^2(\mathbb{S}^1, \mathbb{R}^{2n}) \,\middle|\, \sum_{k \in \mathbb{Z}} |k| \, |z_k|^2 < \infty \right\}$$

which is a Hilbert space with the inner product

$$(z, w) = \langle z_0, w_0 \rangle + 2\pi \sum_{k \in \mathbb{Z}} |k| \, \langle z_k, w_k \rangle$$

and corresponding norm $\|z\| = \sqrt{(z, z)}$.

In the following we will abbreviate $W := W^{1/2,2}(\mathbb{S}^1, \mathbb{R}^{2n})$. There is an orthogonal decompostion $W = W^+ \oplus W^- \oplus W^0$ into closed subspaces according to the subscript $k > 0$, $k < 0$, $k = 0$. By P^+, P^-, P^0 we denote the orthogonal projectors on W^+, W^-, W^0.

Now we define a differential operator $L : \mathrm{dom}(L) \subset L^2(\mathbb{S}^1, \mathbb{R}^{2n}) \to L^2(\mathbb{S}^1, \mathbb{R}^{2n})$ on the domain of definition

$$\mathrm{dom}(L) = \left\{ z \in L^2(\mathbb{S}^1, \mathbb{R}^{2n}) \,\middle|\, \sum_{k \in \mathbb{Z}} |k|^2 \, |z_k|^2 < \infty \right\}$$

by

$$(Lz)(t) := -J \frac{d}{dt} z(t) = \sum_{k \in \mathbb{Z}} 2\pi k \exp(2\pi k t J) \, z_k$$

for almost every $t \in \mathbb{S}^1$. Note that $\mathrm{dom}(L)$ coincides with the Sobolev space $W^{1,2}(\mathbb{S}^1, \mathbb{R}^{2n})$ considered as a linear subspace of $L^2(\mathbb{S}^1, \mathbb{R}^{2n})$. Moreover, L is self-adjoint, and $\ker(L)$ consists precisely of the constant loops.

The spectrum of L is $\sigma(L) = \sigma_{pp}(L) = 2\pi\mathbb{Z}$. In particular, each u_{ki} is an eigenvector of L corresponding to the eigenvalue $2\pi k$. The Sobolev space W is the form-domain of the operator L, and by direct computation it is seen that

$$((P^+ - P^-)z, w) = (Lz, w)_2 \qquad \text{if} \quad z, w \in \mathrm{dom}(L).$$

Let $j \in \mathbb{Z}^n$ be a fixed integer vector. We define

$$e(t) := e_j(t) := (jt, 0) \in \mathbb{R}^n \times \mathbb{R}^n.$$

Consider the non linear functional $\varphi : W \to \mathbb{R}$ containing the Hamiltonian H:

$$\varphi(z) := \int_0^1 H(z(t) + e(t), t) \, dt. \tag{12}$$

From our assumptions (4) it follows that $\nabla H(x, y, t)$ is asymptotically linear with respect to y. This hypothesis implies at most linear growth for the gradient ∇H in each fibre; i.e., there exist constants $c_1, c_2 > 0$ such that $|\nabla H(x, y, t)| \leqslant c_1 + c_2 \, |y|$ uniformly in x and t. We have

LEMMA 1. (i) *The functional $\varphi : W \to \mathbb{R}$ defined in* (12) *is continuously differentiable, and the derivate of φ is represented by*

$$d\varphi(z)\xi = \int_0^1 \langle \nabla H(z + e, t), \xi \rangle \, dt \qquad \text{for} \quad z, \xi \in W. \tag{13}$$

(ii) *The condition* (9) *on the Hessian of H implies that $\varphi \in C^2(W, \mathbb{R})$, and the Hessian $d^2\varphi(u)$ of φ is represented by*

$$d^2\varphi(z)(\xi, \eta) = \int_0^1 \langle d^2H(z+e, t)\, \xi(t), \eta(t)\rangle \, dt \qquad \textit{for} \quad \xi, \eta \in W. \tag{14}$$

For a proof of $\varphi \in C^1$ we refer to Rabinowitz [18, Prop. B 37]. The proof of $\varphi \in C^2$ is similar, also cf. [18, Prop. B 34].

The corresponding gradient $\varphi' : W \to W$, defined by

$$(\varphi'(z), w) = d\varphi(z)\, w \qquad \text{for all} \quad w \in W,$$

is a compact map, see Rabinowitz [18, Prop. B 37]. The proof of the compactness of φ' is similar to the proof of the subsequent Lemma 8. For the compactness properties of $d^2\varphi$ see Lemmata 4 and 8 below.

Recall the matrix $A(t)$ from (5), (6) and define a symmetric matrix

$$Q(t) := \begin{pmatrix} 0 & 0 \\ 0 & A(t) \end{pmatrix} \in M(2n \times 2n, \mathbb{R}).$$

Define a symmetric bounded linear operator $\hat{Q} \in \mathscr{L}(L^2(\mathbb{S}^1, \mathbb{R}^n))$ by

$$(\hat{Q}z)(t) := Q(t)\, z(t).$$

Then there exists a unique symmetric operator $K \in \mathscr{L}(W)$, defined by

$$(Kz, w) := (\hat{Q}z, w)_2 = \int_0^1 \langle Q(t)\, z(t), w(t)\rangle \, dt.$$

From the compactness of the embedding $W^{1/2,2}(\mathbb{S}^1, \mathbb{R}^{2n}) \to L^2(\mathbb{S}^1, \mathbb{R}^{2n})$ it follows that $K \in \mathscr{L}(W)$ is compact.

Hence the operator $P^+ - P^- - K$ is a bounded symmetric linear operator on W. Since K is a relatively compact perturbation of $P^+ - P^-$, the essential spectrum remains unchanged. Therefore

$$\sigma_{\text{ess}}(P^+ - P^- - K) = \sigma_{\text{ess}}(P^+ - P^-) = \{-1, +1\}.$$

Moreover we have the identity

$$((P^+ - P^- - K)\, z, w) = ((L - \hat{Q})\, z, w)_2 \qquad \text{if} \qquad z, w \in \text{dom}(L).$$

Since $\hat{Q}$ is a bounded symmetric operator it follows that $L - \hat{Q}$ is self-adjoint on $\text{dom}(L - \hat{Q}) = \text{dom}(L)$.

In the following we shall denote

$$T := P^+ - P^- - K.$$

Since $P^+ - P^-$ is a Fredholm operator with index 0, it follows by the compactness of K that T is also Fredholm with $\text{index}(T) = 0$. In particular, $\text{ran}(T) = \ker(T)^\perp$, and in view of the non-resonance condition (6) we have

$$\ker(T) = \{z \in W \mid z(t) = (x(t), y(t)) \in \mathbb{R}^n \times \mathbb{R}^n \text{ with } x = \text{const.}, y = 0\}.$$

In order to include the case of a not exact Hamiltonian vector field of the form (11), we note that the vanishing mean value of f_1 implies that $f \in \text{ran}(L - \hat{Q})$, which is seen by elementary calculations. In particular, we then have

$$\int_0^1 \langle f(t), z(t) \rangle \, dt = (f, z)_2 = ((L - \hat{Q}) w, z)_2 = (Tw, z). \tag{15}$$

Finally we have to consider the contribution inherited from the rotation vector. Let $j \in \mathbb{Z}^n$ be fixed and $e(t) = (jt, 0) \in \mathbb{R}^n \times \mathbb{R}^n$. Then we define

$$v_j := -J\dot{e} = (0, j) \in W. \tag{16}$$

Obviously we have the identity

$$\int_0^1 \langle v_j, z \rangle \, dt = (v_j, z)_2 = \langle v_j, z_0 \rangle = (v_j, z) \qquad \text{for} \quad z \in W. \tag{17}$$

Combining the contributions from (15) and (17), we define a continuous linear function $(v, \cdot) : W \to \mathbb{R}$ by

$$(v, z) := (v_j - Tw, z) = \int_0^1 \langle -J\dot{e} - f(t), z(t) \rangle \, dt.$$

Now we define the action functional Φ on W by

$$\begin{aligned} \Phi(z) &= \tfrac{1}{2}((P^+ - P^-)z, z) - \varphi(z) + (v, z) \\ &= \tfrac{1}{2}(Tz, z) - \hat{\varphi}(z) + (v, z) \end{aligned} \tag{18}$$

where

$$\hat{\varphi}(z) := \varphi(z) - \tfrac{1}{2}(Kz, z) = \int_0^1 \{H(z + e, t) - \tfrac{1}{2}\langle \hat{Q}z, z \rangle\} \, dt.$$

Obviously $\hat{\varphi} \in C^2(W, \mathbb{R})$. Since φ' and K are compact it follows that $\hat{\varphi}' : W \to W$ is compact. The derivate of $\Phi \in C^2(W, \mathbb{R})$ is given by

$$d\Phi(z)\, w = (Tz, w) - d\hat{\varphi}(z)\, w + (v, w) \qquad \text{for} \quad w \in W,$$

and the corresponding gradient is defined by

$$(\Phi'(z), w) = (Tz - \hat{\varphi}'(z) + v, w) \qquad \text{for} \quad w \in W.$$

For the Hessian form of Φ we find

$$d^2\Phi(u)(\xi, \eta) = (T\xi, \eta) - d^2\hat{\varphi}(u)(\xi, \eta).$$

The function $\Phi : W \to \mathbb{R}$ extends the classical action functional defined on $W^{1,2}$-loops by

$$\Phi(z) = \int_0^1 \{\tfrac{1}{2}\langle -J\dot{z}, z\rangle - H(z+e, t) + \langle v_j - f(t), z\rangle\}\, dt$$

for $z \in W^{1,2}(\mathbb{S}^1, \mathbb{R}^{2n})$.

The setup on the Sobolev space $W^{1/2,2}(\mathbb{S}^1, \mathbb{R}^{2n})$ permits a variational characterization of the required periodic solutions. Using standard regularity arguments, it follows that the critical points of Φ on W are precisely the classical periodic solutions we are looking for:

LEMMA 2. *$z \in W$ is a critical point of Φ if and only if $z \in C^1(\mathbb{S}^1, \mathbb{R}^{2n})$ and z is a 1-periodic solution of*

$$-J\dot{z} = \nabla H(z+e, t) + f(t) - v_j.$$

The periodicity of the Hamiltonian H with respect to x implies the invariance of Φ and Φ' under a free action of the group $\mathbb{Z}^n$ on W which is defined as follows:

Identify $g = (g_1, ..., g_n) \in \mathbb{Z}^n$ with $(g_1, ..., g_n, 0, ..., 0) \in \mathbb{R}^{2n}$, which can be considered as a constant loop, i.e., an element of W. Now define the group action by

$$g \cdot z := z + g \qquad \text{for} \quad z \in W.$$

Then $\Phi(g \cdot z) = \Phi(z)$ and $\Phi'(g \cdot z) = \Phi'(z)$. Passing to the quotient we obtain

$$W/\mathbb{Z}^n \cong E \times \mathbb{T}^n$$

where $E = \ker(T)^{\perp} \cong W/\ker(T)$. We will consider E equipped with the scalar product induced from the inner product on W. In particular, because of the $\mathbb{Z}^n$-invariance we can consider Φ as an element of $C^2(E \times \mathbb{T}^n, \mathbb{R})$.

Subsequently we shall make crucial use of the fact that there exists a splitting

$$W = E^+ \oplus E^- \oplus E^0 =: E \oplus E^0$$

into closed orthogonal subspaces E^+, E^-, E^0 according to the positive, the negative, and the zero eigenspaces of the operator T.

The variational setup described so far is suited for forced oscillations, i.e., the j-solutions of Theorem 1. In order to establish the existence of j/p-solutions with $j \in \mathbb{Z}^n$ and $p \in \mathbb{N}$ relatively prime we define $e_p(t) := (p^{-1}jt, 0) \in \mathbb{R}^n \times \mathbb{R}^n$, and we have to consider the action functional

$$\Phi_p(z) := \frac{1}{p} \int_0^p \left\{ \frac{1}{2} \langle -J\dot{z}, z \rangle - H(z + e_p, t) + \langle -J\dot{e}_p - f(t), z \rangle \right\} dt$$

on the loop space $W^{1,2}(\mathbb{R}/p\mathbb{Z}, \mathbb{R}^{2n})$. The variational formulation on the Sobolev space $W^{1/2,2}(\mathbb{R}/p\mathbb{Z}, \mathbb{R}^{2n})$ then can be carried out analogous to the case $p = 1$.

3. Non-degenerate Loops and Non-degenerate Critical Points

Considering now the case $p = 1$, we state precisely the definition of non-degeneray for periodic solutions of the Hamiltonian equation. The basic facts on ordinary differential equations can be found, e.g., in [1].

Fix a rotation vector $j \in \mathbb{Z}^n$ and set $v_j = (0, j) \in \mathbb{R}^n \times \mathbb{R}^n$, according to (16). Consider a 1-periodic solution $\tilde{u}(t) \in \mathbb{T}^n \times \mathbb{R}^n$ of

$$\frac{d}{dt} \tilde{u} = J[\nabla H(\tilde{u}, t) + f(t)] \qquad \text{for} \quad \tilde{u}(t) \in \mathbb{T}^n \times \mathbb{R}^n, \tag{19}$$

having rotation vector j. Lifting $\tilde{u}$ to the universal cover $\mathbb{R}^n \times \mathbb{R}^n$, we can write $\tilde{u}(t) = u(t) + e(t)$ where u is a 1-periodic solution of

$$\frac{d}{dt} u = J \nabla H(u + e, t) + J\{f(t) - v_j\}, \qquad u(t) \in \mathbb{R}^{2n}. \tag{20}$$

We denote by $\alpha^t(z)$ the flow of the vector field in (20):

$$\begin{cases} \dfrac{d}{dt} \alpha^t(z) = J \nabla H(\alpha^t(z) + e(t), t) + J\{f(t) - v_j\} \\ \alpha^0(z) = z. \end{cases} \tag{21}$$

Since the Hamiltonian $H \in C^2$, the flow $\alpha^t(z)$ is differentiable with respect to z, and the derivative $d\alpha^t(z) \in \mathscr{L}(\mathbb{R}^{2n})$ satisfies the linearized equation

$$\begin{cases} \dfrac{d}{dt} d\alpha^t(z) = J\, d^2H(\alpha^t(z) + e(t), t)\, d\alpha^t(z) \\ d\alpha^0(z) = id_{\mathbb{R}^{2n}} \end{cases} \tag{22}$$

along the solution $\alpha^t(z)$. Here d denotes the derivative with respect to z. Consider the above 1-periodic solution $u(t)$ of (20). Setting $z := u(0)$ we have $u(t) = \alpha^t(z)$, and the Hessian $d^2H(\alpha^t(z) + e(t), t)$ is periodic with 1, which follows from the periodicity condition (3) on the Hamiltonian H.

Then the matrix $U(t) := d\alpha^t(z) \in \mathscr{L}(\mathbb{R}^{2n})$ is a fundamental system of the periodically t-dependent linear equation (22). The eigenvalues of $U(1)$ are called the Floquet multipliers of the 1-periodic solution $\alpha^t(z)$ of the Hamiltonian equation (20).

DEFINITION 2. A 1-periodic solution $u(t) = \alpha^t(z)$ of the Hamiltonian equation (20) is called non-degenerate if it has no Floquet multiplier equal to 1.

We recall the following well known fact from Floquet theory:

LEMMA 3. *The linearized equation*

$$\frac{d}{dt}\xi = J\, d^2H(\alpha^t(z) + e(t), t)\, \xi \tag{23}$$

has a nontrivial 1-periodic solution if and only if 1 is a Floquet multiplier of $\alpha^t(z)$.

In view of Lemma 2, the 1-periodic solutions of the Hamiltonian equation having rotation vector j are in one-to-one correspondence with the critical points of the action functional Φ defined in (18). We show that non-degenerate periodic solutions correspond to non-degenerate critical points of the action functional.

From Lemma 1 recall the Hessian form $d^2\varphi(u)$, which is a bounded quadratic form on W. To see this, we use the growth condition (9) to estimate

$$|d^2\varphi(u)(\xi,\eta)| \leqslant \int_0^1 |d^2H(u+e, t)|\, |\xi|\, |\eta|\, dt \tag{24}$$

$$\leqslant a_1 \int_0^1 |\xi|\, |\eta|\, dt + a_2 \int_0^1 |y|^r\, |\xi|\, |\eta|\, dt, \tag{25}$$

where $u(t) = (x(t), y(t))$. To estimate the second integral we apply the generalized Hölder inequality with $p_1 = (r+1)/r$ and $p_2 = p_3 = 2r+2$. Then $p_1^{-1} + p_2^{-1} + p_3^{-1} = 1$, and

$$\int_0^1 |y|^r\, |\xi|\, |\eta|\, dt \leqslant \left(\int_0^1 |y|^{r+1}\, dt\right)^{r/(r+1)} \|\xi\|_{2r+2}\, \|\eta\|_{2r+2}$$

$$= \|y\|_{r+1}^r\, \|\xi\|_{2r+2}\, \|\eta\|_{2r+2}$$

where $\|\cdot\|_q$ denotes the norm on $L^q(\mathbb{S}^1, \mathbb{R}^{2n})$. We have compact embeddings (cf. [17], or Appendix (b) of [14])

$$W = W^{1/2,2}(\mathbb{S}^1, \mathbb{R}^{2n}) \to L^q(\mathbb{S}^1, \mathbb{R}^{2n}), \qquad 1 \leqslant q < \infty. \tag{26}$$

In particular there exist constants c_q such that for every $\xi \in W$

$$\|\xi\|_q \leqslant c_q \|\xi\|, \qquad 1 \leqslant q < \infty. \tag{27}$$

Finally, using the Cauchy–Schwarz inequality to estimate the first integral in (25), we conclude that there exists a constant $c = c(u)$ depending only on u and the constants a_1, a_2, and r given by (9), such that

$$|d^2\varphi(u)(\xi, \eta)| \leqslant c(u) \|\xi\| \|\eta\|. \tag{28}$$

By Riesz' theorem there exists a uniquely determined symmetric bounded linear operator $\varphi''(u) \in \mathscr{L}(W)$, defined by

$$(\varphi''(u)\xi, \eta) := d^2\varphi(u)(\xi, \eta), \qquad \xi, \eta \in W. \tag{29}$$

Subsequently we shall make crucial use of the compactness properties of φ''. The first observation is

LEMMA 4. *The linear operator $\varphi''(u) \in \mathscr{L}(W)$ defined in* (29) *is compact for every $u \in W$.*

Proof. For $\xi \in W$ we have

$$\begin{aligned}
\|\varphi''(u)\xi\|^2 &= |d^2\varphi(u)(\xi, \varphi''(u)\xi)| \\
&\leqslant a_1 \|\xi\|_2 \|\varphi''(u)\xi\|_2 + a_2 \|y\|_{r+1}^r \|\xi\|_{2r+2} \|\varphi''(u)\xi\|_{2r+2} \\
&\leqslant \{a_1 + a_2 \|y\|_{r+1}^r\} \|\xi\|_{2r+2} \|\varphi''(u)\xi\|_{2r+2} \\
&=: C'(u) \|\xi\|_{2r+2} \|\varphi''(u)\xi\|_{2r+2}
\end{aligned}$$

where $u(t) = (x(t), y(t)) \in \mathbb{R}^n \times \mathbb{R}^n$. Note that we have used $\|\xi\|_2 \leqslant \|\xi\|_{2r+2}$. By use of (27) we have with $C(u) := c_{2r+2} C'(u)$

$$\|\varphi''(u)\xi\| \leqslant C(u) \|\xi\|_{2r+2} \qquad \text{for all } \xi \in W. \tag{30}$$

Since the embedding $W \to L^{2r+2}(\mathbb{S}^1, \mathbb{R}^{2n})$ is compact, the lemma follows. ∎

Recall from Section 2 the nonlinear functional $\hat{\varphi}(u) = \varphi(u) - \frac{1}{2}(Ku, u) \in C^2(W)$. For every $u \in W$ the linear operator $\hat{\varphi}''(u) := \varphi''(u) - K \in \mathscr{L}(W)$ is compact in view of Lemma 4. For $u \in W$ we define the bounded linear operator Φ'' by

$$\Phi''(u) := T - \hat{\varphi}''(u) \in \mathscr{L}(W),$$

which obviously satisfies the identity

$$d^2\Phi(u)(\xi, \eta) = (\Phi''(u)\xi, \eta).$$

DEFINITION 3. A critical point u of Φ is called non-degenerate if the Hessian form $d^2\Phi(u)$ is non-degenerate.

Equivalently, a critical point u of Φ is non-degenerate if $\ker \Phi''(u) = 0$. From

$$\Phi'(u+\xi) = \Phi''(u)\xi + o(|\xi|)$$

we conclude

LEMMA 5. *The non-degenerate critical points of Φ are isolated.*

In fact, we have the following lemma which will be needed for the Galerkin approximation of non-degenerate critical points in the next section:

LEMMA 6. *If $u \in W$ is a non-degenerate critical point of Φ then $\Phi''(u) \in \mathscr{L}(W)$ is an isomorphism of W.*

Proof. Recall that $T \in \mathscr{L}(W)$ is Fredholm with $\mathrm{index}(T) = 0$ and $\dim \ker(T) = n$. Since $\hat{\varphi}''(u)$ is compact, $\Phi''(u)$ is Fredholm with $\mathrm{index}(\Phi''(u)) = 0$. Consequently $\mathrm{ran}(\Phi''(u))$ is closed and $\dim \ker(\Phi''(u)) = \dim \mathrm{coker}(\Phi''(u))$.

If $d^2\Phi(u)$ is non-degenerate then $\ker(\Phi''(u)) = 0$. Hence $\Phi''(u)$ is onto and has a bounded inverse by the open mapping principle. ∎

After these preparations we can show the claimed equivalence between Definitions 2 and 3 of non-degeneracy:

LEMMA 7. *A 1-periodic solution of the equation* (20) *is non-degenerate if and only if u is a non-degenerate critical point of the action functional Φ.*

Proof. Let u be a non-degenerate periodic solution of the Hamiltonian equation, and let

$$d^2\Phi(u)(\xi, \eta) = 0 \qquad \text{for all} \quad \eta \in W. \tag{31}$$

In particular then (31) holds for $\eta \in \mathrm{dom}(L) = W^{1,2}(\mathbb{S}^1, \mathbb{R}^{2n})$, and we have

$$\begin{aligned} |(\xi, L\eta)_2| &= |(\xi, [P^+ - P^-]\,\eta)| \\ &\leqslant |(\xi, [P^+ - P^- - \varphi''(u)]\,\eta)| + |(\xi, \varphi''(u)\,\eta)| \\ &= |d^2\Phi(u)(\xi, \eta)| + \left| \int_0^1 \langle d^2H(u+e, t)\,\xi, \eta\rangle\, dt \right| \end{aligned}$$

$$\leqslant \int_0^1 |d^2H(u+e, t)|\, |\xi|\, |\eta|\, dt$$

$$\leqslant \int_0^1 \{a_1 + a_2 |y|^r\}\, |\xi|\, |\eta|\, dt$$

$$\leqslant a_1 \|\xi\|_2 \|\eta\|_2 + a_2 \int_0^1 |y|^r\, |\xi|\, |\eta|\, dt.$$

The remaining integral can be estimated using the generalized Hölder inequality:

$$\int_0^1 |y|^r\, |\xi|\, |\eta|\, dt \leqslant \left(\int_0^1 |y|^{4r}\, dt\right)^{1/4} \left(\int_0^1 |\xi|^4\, dt\right)^{1/4} \left(\int_0^1 |\eta|^2\, dt\right)^{1/2}$$
$$= \|y\|_{4r}^r\, \|\xi\|_4\, \|\eta\|_2.$$

Consequently

$$|(\xi, L\eta)_2| \leqslant \{a_1 \|\xi\|_2 + a_2 \|y\|_{4r}^r\, \|\xi\|_4\}\, \|\eta\|_2, \tag{32}$$

and therefore we conclude $\xi \in \operatorname{dom}(L^*) = \operatorname{dom}(L)$. Then we have for every $\eta \in W$:

$$0 = d^2\Phi(u)(\xi, \eta) = ([T - \hat{\varphi}''(u)]\,\xi, \eta)$$
$$= ([P^+ - P^- - \varphi''(u)]\,\xi, , \eta)$$
$$= (L\xi, \eta)_2 - \int_0^1 \langle d^2H(u+e, t)\xi, \eta\rangle\, dt$$
$$= \int_0^1 \langle -J\dot{\xi} - d^2H(u+e, t)\xi, \eta\rangle\, dt.$$

We conclude that ξ is a 1-periodic solution of the linearized Hamiltonian equation

$$\dot{\xi} = J\, d^2H(u+e, t)\xi \tag{33}$$

and, since u is a non-degenerate solution of (20), we have $\xi = 0$ by Lemma 3.

Conversely, if 1 is a Floquet multiplier of u, then, by Lemma 3, there exists a nontrivial solution $\xi \in C^1(\mathbb{S}^1, \mathbb{R}^{2n}) \subset W$ of the linearized Hamiltonian equation, and consequently

$$\int_0^1 \langle -J\dot{\xi} - d^2H(u+e, t)\xi, \eta\rangle\, dt = (\Phi''(u)\xi, \eta) = 0 \tag{34}$$

for all $\eta \in W$. Hence $\xi \neq 0$ and $\xi \in \ker \Phi''(u)$. Then, by Lemma 7.5, u has to be degenerate. This finishes the proof of the lemma. ∎

4. Approximation of Non-degenerate Critical Points

Before we introduce the Galerkin approximation, we consider the following lemma which is of subsequent importance.

Lemma 8. *The mapping* $\varphi'' : W \to \mathscr{L}(W)$ *is compact.*

Proof. Let u_m be a bounded sequence in W. Then u_m has a weakly convergent subsequence, and thus we may assume $u_{m_i} \rightharpoonup u \in W$ weakly. Let $\xi \in W$ with $\|\xi\| \leqslant 1$. We have

$$|(\varphi''(u_{m_i}) - \varphi''(u))\xi, \xi)| \leqslant \int_0^1 |d^2H(u_{m_i} + e, t) - d^2H(u + e, t)| \, |\xi|^2 \, dt.$$

We write $w(t) = (x(t), y(t)) \in \mathbb{R}^n \times \mathbb{R}^n$, $w \in W$. Choose $\alpha > 1$. Then the assumption on the norm of the Hessian gives

$$|d^2H(w(t) + e(t), t)| \leqslant a_1 + a_2 \, |y(t)|^{\alpha r/\alpha}. \tag{35}$$

As in [18, Prop. B1], we conclude that the map $W \mapsto d^2H(w + e, \cdot)$ is in $C(L^{\alpha r}(\mathbb{S}^1, \mathbb{R}^{2n}), L^{\alpha}(\mathbb{S}^1, \mathscr{L}(\mathbb{R}^{2n})))$. Consequently,

$$\begin{aligned}
&\int_0^1 |d^2H(u_{m_i} + e, t) - d^2H(u + e, t)| \, |\xi|^2 \, dt \\
&\quad \leqslant \left(\int_0^1 d^2H(u_{m_i} + e, t) - d^2H(u + e, t)|^{\alpha} \, dt \right)^{1/\alpha} \left(\int_0^1 |\xi|^{2\alpha/(\alpha - 1)} dt \right)^{(\alpha - 1)/\alpha} \\
&\quad = \|d^2H(u_{m_i} + e, \cdot) - d^2H(u + e, \cdot)\|_{L^{\alpha}(\mathbb{S}^1, \mathscr{L}(\mathbb{R}^{2n}))} \, \|\xi\|^2_{2\alpha/(\alpha - 1)}.
\end{aligned}$$

By the continuous embedding $W \to L^{2\alpha/(\alpha - 1)}(\mathbb{S}^1, \mathbb{R}^{2n})$ we conclude

$$\begin{aligned}
\|\varphi''(u_{m_i}) - \varphi''(u)\| &= \sup_{\|\xi\| \leqslant 1} |(\varphi''(u_{m_i}) - \varphi''(u)\xi, \xi)| \\
&\leqslant a_3 \, \|d^2H(u_{m_i} + e, \cdot) - d^2H(u + e, \cdot)\|_{L^{\alpha}(\mathbb{S}^1, \mathscr{L}(\mathbb{R}^{2n}))}
\end{aligned}$$

with some constant a_3 depending on α. Note we have used that $\varphi''(w)$ is a symmetric operator.

Since $u_m \rightharpoonup u$ weakly in W, we have $u_m \to u$ in $L^{\alpha r}(\mathbb{S}^1, \mathbb{R}^{2n})$ by the compactness of the embedding $W \to L^{\alpha r}(\mathbb{S}^1, \mathbb{R}^{2n})$. By the above continuity the right hand side tends to zero as $i \to \infty$, which proves the lemma. ∎

We introduce a Galerkin approximation scheme to obtain a finite-dimensional approximation. Recall the orthogonal splitting $W = E^+ \oplus E^- \oplus E^0$ according to the positive, the negative, and the zero eigenspaces of the operator $T=P^+-P^--K$. In E^+ and in E^- we choose orthonormal bases consisting of eigenvectors of T:

$$\{u_i \mid i\in\mathbb{Z}^+\}\subset E^+ \quad \text{and} \quad \{u_i \mid i\in\mathbb{Z}^-\}\subset E^-.$$

Define the finite dimensional subspaces E_k of $E=E^+\oplus E^-$ by

$$E_k:=\operatorname{span}\{u_i \mid 0<|i|\leqslant k\}, \qquad k=1,2,\ldots.$$

By P_k we denote the corresponding orthogonal projectors $P_k: E\oplus E^0\to E_k\oplus E^0$. The projection scheme $(E_k\oplus E^0, P_k)$ is projectionally complete; i.e., $P_k u\to u$ as $k\to\infty$ for every $u\in E\oplus E^0$. We define the functionals $\Phi_k: E_k\oplus E^0\to\mathbb{R}$ by restriction:

$$\Phi_k(u):=\Phi(u) \qquad \text{for} \quad u\in E_k\oplus E^0.$$

The gradient of Φ_k is given by

$$\Phi_k'(u)=P_k\Phi'(u)=Tu-P_k\hat{\varphi}'(u)+v \qquad \text{for} \quad u\in E_k\oplus E^0.$$

We shall also abbreviate $W_k:=E_k\oplus E^0$. The next proposition will give us the approximation properties we need:

PROPOSITION 1. (i) *Let $T=P^+-P^--K$, and let $F: W\to W$ be a compact map. Then $T-F$ is A-proper with respect to (W_k, P_k); i.e., every bounded sequence $u_{k_i}\in W_{k_i}$ which satisfies $Tu_{k_i}-P_{k_i}F(u_{k_i})\to u^*$ for some $u^*\in W$ as $i\to\infty$ contains a convergent subsequence $u_{k_i}\to u\in W$, and the limit u satisfies $Tu-F(u)=u^*$.*

(ii) *The map $T-F: W\to W$ is proper on bounded subsets of W.*

Proof. Let (u_{k_i}) be a bounded sequence in W such that $u_{k_i}\in W_{k_i}$, and

$$P_{k_i}(Tu_{k_i}-F(u_{k_i}))=Tu_{k_i}-P_{k_i}F(u_{k_i})\to u^*\in W$$

as $i\to\infty$. Since F is compact and (u_{k_i}) is bounded, the sequence $F(u_{k_i})$ contains a convergent subsequence. We can assume $F(u_{k_i})\to w\in W$. Consequently

$$\begin{aligned}\|P_{k_i}F(u_{k_i})-w\| &\leqslant \|P_{k_i}F(u_{k_i})-P_{k_i}w\|+\|P_{k_i}w-w\| \\ &\leqslant \|F(u_{k_i})-w\|+\|P_{k_i}w-w\|\to 0 \qquad \text{as} \quad i\to\infty.\end{aligned}$$

We therefore conclude $Tu_{k_i}\to u^*+w$ as $i\to\infty$. Recall the orthogonal decomposition $u_{k_i}=u_{k_i}^++u_{k_i}^-+u_{k_i}^0\in E^+\oplus E^-\oplus E^0=W$. Then $u_{k_i}^0$ is a bounded sequence in $E^0=\ker(T)$. Since $\dim E^0=n<\infty$, we may assume $u_{k_i}^0\to u^0\in E^0$. Thus consider

$$Tu_{k_i}=T(u_{k_i}^++u_{k_i}^-)\to u^*+w.$$

Since T is Fredholm, $\operatorname{ran}(T)$ is closed, and consequently, $u^* + w \in \operatorname{ran}(T)$. Moreover, $T|_{E^+ \oplus E^-}$ is continuously invertible, and therefore $u_{k_i}^+ + u_{k_i}^- \to T^{-1}(u^* + w)$. Thus we have found a convergent subsequence $u_{k_i} = u_{k_i}^+ + u_{k_i}^- + u_{k_i}^0 \to T^{-1}(u^* + w) + u^0 =: u$. Of course we have

$$Tu - F(u) = (u^* + w) - \lim_{i \to \infty} F(u_{k_i}) = u^*.$$

(ii) The proof of the properness of $T - F$ on bounded subsets is almost the same; simply delete the projections to the spaces W_{k_i} in the above proof. ▌

As a consequence we obtain the following

COROLLARY 1. (i) *For every* $u \in W$ *the operator* $\Phi''(u) \in \mathscr{L}(W)$ *is A-proper with respect to* (W_k, P_k).

(ii) $\Phi' : W \to W$ *is A-proper with respect to* (W_k, P_k).

Proof. (i) Set $F = \hat{\varphi}''(u) \in \mathscr{L}(W)$.

(ii) Set $F(u) = \hat{\varphi}'(u) - v$. ▌

We are now prepared to approximate the non-degenerate critical points of Φ:

LEMMA 9. *If* $u_0 \in W$ *is a non-degenerate critical point of* Φ, *then there exist* $\varepsilon > 0$ *and* $k_0 \in \mathbb{N}$ *such that for every* $k \geqslant k_0$ *the function* Φ_k *possesses exactly one critical point* $u_k \in B(u_0, \varepsilon) \cap W_k$. *Moreover, this critical point* u_k *of* Φ_k *is non-degenerate.*

Proof. (a) Recall from Lemma 5 that $\Phi''(u_0) \in \mathscr{L}(W)$ is an isomorphism, and we have for all $\xi \in W$

$$\|\Phi''(u_0)\xi\| \geqslant \frac{1}{\|\Phi''(u_0)^{-1}\|} \|\xi\|. \tag{36}$$

Since $\Phi'' : W \to \mathscr{L}(W)$ is continuous, there exists $\varepsilon_1 > 0$ such that

$$\frac{\|\Phi''(u_0)^{-1}\|}{\|\Phi''(u)^{-1}\|} \geqslant \frac{1}{2} \tag{37}$$

if $u \in B(u_0, \varepsilon_1)$. Consequently for these u we have an estimate

$$\|\Phi''(u)\xi\| \geqslant \frac{1}{\|\Phi''(u)^{-1}\|} \|\xi\| \geqslant \frac{1}{2\|\Phi''(u_0)^{-1}\|} \|\xi\|. \tag{38}$$

(b) We claim there exist $c>0$ and $k_1 \in \mathbb{Z}^+$ such that for $\xi \in W_k$, $k \geqslant k_1$ and $u \in B(u_0, \varepsilon_1)$ the estimate

$$\|P_k \Phi''(u)\xi\| \geqslant c \|\xi\| \tag{39}$$

holds true. Assuming, in contradiction, that such numbers c and k_1 do not exist, we can find sequences u_{k_i} in $B(u_0, \varepsilon_1)$ and $\xi_{k_i} \in W_{k_i}$, $\|\xi_{k_i}\| = 1$, such that

$$P_{k_i} \Phi''(u_{k_i}) \xi_{k_i} \to 0 \quad \text{as} \quad i \to \infty. \tag{40}$$

Recall $\Phi''(u_{k_i}) = T - \hat{\varphi}''(u_{k_i})$, and $\hat{\varphi}''(u_{k_i}) = \varphi''(u_{k_i}) - K$. By Lemma 8 the map φ'' is compact, and therefore $\hat{\varphi}'' : W \to \mathscr{L}(W)$ is compact. Since (u_{k_i}) is a bounded sequence we can choose a subsequence, again denoted by (u_{k_i}), such that $\hat{\varphi}''(u_{k_i})$ converges to $F \in \mathscr{L}(W)$. The set of compact linear operators on W is closed in $\mathscr{L}(W)$, and hence F is compact. Consequently

$$\Phi''(u_{k_i}) \to T - F =: S \in \mathscr{L}(W) \quad \text{as} \quad i \to \infty. \tag{41}$$

Note that

$$\begin{aligned} \|P_{k_i} S \xi\| &\leqslant \|P_{k_i} \Phi''(u_{k_i}) \xi_{k_i}\| + \|P_{k_i} S \xi_{k_i} - P_{k_i} \Phi''(u_{k_i}) \xi_{k_i}\| \\ &\leqslant \|P_{k_i} \Phi''(u_{k_i}) \xi_{k_i}\| + \|S - \Phi''(u_{k_i})\| \to 0 \quad \text{as} \quad i \to \infty. \end{aligned}$$

By Corollary 1 the operator $S = T - F$ is A-proper. Hence (ξ_{k_i}) contains a convergent subsequence. Thus we may assume $\xi_{k_i} \to \xi \in W$, and from $\|\xi_{k_i}\| = 1$ we conclude $\|\xi\| = 1$. By the A-properness, ξ satisfies $S\xi = 0$. Consequently

$$\|\Phi''(u_{k_i})\xi\| = \|\Phi''(u_{k_i})\xi - S\xi\| \leqslant \|\Phi''(u_{k_i}) - S\| \to 0 \quad \text{as} \quad i \to \infty.$$

On the other hand, (38) yields

$$\|\Phi''(u_{k_i})\xi\| \geqslant \frac{1}{2 \|\Phi''(u_0)^{-1}\|} \|\xi\| = \frac{1}{2 \|\Phi''(u_0)^{-1}\|} > 0.$$

Thus we arrive at a contradiction, and the claim is proved.

(c) In particular we have with c, k_1 as in (b)

$$\|P_k \Phi''(u_0)\xi\| \geqslant c \|\xi\| \quad \text{if} \quad k \geqslant k_1. \tag{42}$$

Hence $P_k \Phi''(u_0)|_{W_k} \in \mathscr{L}(W_k)$ is an isomorphism of W_k. Moreover, we have

$$\Phi'(u) = \Phi''(u_0)(u - u_0) + o(\|u - u_0\|). \tag{43}$$

Consequently we can choose $0<\varepsilon_2\leqslant\varepsilon_1$ such that $\Phi'(u)\neq 0$ whenever $u\in\overline{B(u_0,\varepsilon_2)}\setminus\{u_0\}$. By (ii) of Corollary 1 we can find $k_2\geqslant k_1$ such that $P_k\Phi'(u)\neq 0$ if $u\in\partial B(u_0,\varepsilon_2)$ and $k\geqslant k_2$.

The Brouwer mapping degree is therefore defined and by the homotopy given by

$$P_k\Phi''(u_0)(u-u_0)+to(\|u-u_0\|),\qquad 0\leqslant t\leqslant 1$$

we find

$$\deg(P_k\Phi', B(u_0,\varepsilon_2)\cap W_k, 0)=\deg(P_k\Phi''(u_0)(\cdot-u_0), B(u_0,\varepsilon_2)\cap W_k, 0).$$

Note that $P_k\Phi''(u_0)$ is an isomorphism for all $k\geqslant k_1$. Hence the equation

$$P_k\Phi''(u_0)u=P_k\Phi''(u_0)u_0 \tag{44}$$

has exactly one solution $\tilde{u}_k\in W_k$ for all $k\geqslant k_1$. Since $P_k\Phi''(u_0)u_0\to\Phi''(u_0)u_0$ as $k\to\infty$ it follows by the A-properness of $\Phi''(u_0)$ that $\tilde{u}_k\to u_0$ as $k\to\infty$. Consequently there exists $k_3\geqslant k_2$ such that $\tilde{u}_k\in B(u_0,\varepsilon_2)\cap W_k$ for $k\geqslant k_3$. Computing the Brouwer degree we obtain

$$\begin{aligned}&\deg(P_k\Phi''(u_0)(\cdot-u_0), B(u_0,\varepsilon_2)\cap W_k, 0)\\&\quad=\deg(P_k\Phi''(u_0), B(u_0,\varepsilon_2)\cap W_k), P_k\Phi''(u_0)\,u_0)\\&\quad=\operatorname{sign}(\det[P_k\varphi''(u_0)])=\pm 1\end{aligned}$$

and therefore

$$\deg(P_k\Phi', B(u_0,\varepsilon_2)\cap W_k, 0)=\pm 1. \tag{45}$$

Consequently there exists a critical point $u_k\in B(u_0,\varepsilon_2)\cap W_k$ of Φ_k, if $k\geqslant k_3$.

(d) We finally claim that there exists $k_4\geqslant k_3$ such that there is only one single critical point u_k of Φ_k in $B(u_0,\varepsilon_2)\cap W_k$.

By contradiction we assume there exist $u_k, u_k'\in B(u_0,\varepsilon_2)\cap W_k$, $u_k\neq u_k'$, such that $P_k\Phi'(u_k')=P_k\Phi'(u_k)=0$ for infinitely many $k\geqslant k_3$. The identity

$$\Phi'(u_k')=\Phi'(u_k)+\Phi''(u_k)(u_k'-u_k)+o(\|u_k'-u_k\|) \tag{46}$$

implies

$$0=P_k\Phi''(u_k)(u_k'-u_k)+o(\|u_k'-u_k\|) \tag{47}$$

Since Φ' is A-proper and since $u_k, u_k'\in B(u_0,\varepsilon_2)$ there exist subsequences u_{k_i}, u_{k_i}' which converge to u_0 as $i\to\infty$. Hence $\|u_{k_i}'-u_{k_i}\|\to 0$, and combining (39) and (47) leads to

$$c<\left\|P_{k_i}\Phi''(u_{k_i})\frac{u_{k_i}'-u_{k_i}}{\|u_{k_i}'-u_{k_i}\|}\right\|\leqslant\frac{o(\|u_{k_i}'-u_{k_i}\|)}{\|u_{k_i}'-u_{k_i}\|}\to 0 \tag{48}$$

as $i\to\infty$, in contradiction to $c>0$, and the claim is proved.

We define $k_0 := k_4$, $\varepsilon := \varepsilon_2$ in the statement of the lemma, and note that the equation (39) shows that the critical points u_k of Φ_k are non-degenerate. This finishes the proof of the lemma. ∎

5. Proof of Theorem 1

Passing to the quotient $W/\mathbb{Z}^n$, we consider $\Phi : E \times \mathbb{T}^n \to \mathbb{R}$. Our hypotheses on the Hamiltonian H guarantee the existence of at least $n+1$ critical points of Φ according to Theorem 6 in [15]. Assuming that all the critical points of Φ are non-degenerate, we recall from Lemma 5 that the critical set of Φ consists of isolated points. In Lemma 10 we give a priori estimates for the critical points of Φ and Φ_k, which show that the critical set is bounded independent of k. From Proposition 1 (ii) we find that Φ' is proper, and hence the number of critical points of Φ has to be finite. Let $u^1, ..., u^m$, denote the finitely many critical points of Φ. For each of these points u^i we can find by Lemma 9 some $\varepsilon^i > 0$ and $k_0^i \in \mathbb{N}$ such that for $\varepsilon := \min\{\varepsilon^i\}$ and $k_0 := \max\{k_0^i\}$ we have

1. $B(u^i, \varepsilon) \cap B(u^j, \varepsilon) = 0$ if $i \neq j$, $1 \leqslant i, j \leqslant m$.
2. There exists exactly one critical point u_k^i of Φ_k in each $B(u^i, \varepsilon) \cap (E_k \times \mathbb{T}^n)$ if $k \geqslant k_0$. Moreover, the u_k^i are non-degenerate, and $u_k^i \to u^i$ as $k \to \infty$, since u^i is the only critical point of Φ contained in $B(u^i, \varepsilon)$.

Assume we already have the a priori bounds; it follows by the A-properness of Φ' that k_0 can be chosen so large that for $k \geqslant k_0$ all the critical points of Φ_k have to be contained in the union of the ε-balls around the u^i. Hence for k sufficiently large the critical points of Φ are in 1–1 correspondence to the critical points of Φ_k, and consequently a lower bound for the number of critical points of Φ_k is also a lower bound for the number of critical points of Φ.

To carry out this strategy we first prove the required a priori estimate for the critical points of Φ on $E \times \mathbb{T}^n$ and Φ_k on $E_k \times \mathbb{T}^n$.

From now on let $\Phi_{(k)}$ and $E_{(k)}$ denote either Φ or Φ_k and E or E_k, respectively. Consider the negative gradient flow $u \cdot s$, $s \in \mathbb{R}$, of the vector field $-\Phi'_{(k)}$ on $E_{(k)} \times \mathbb{T}^n$, i.e.,

$$\frac{d}{ds}(u \cdot s) = -\Phi'_{(k)}(u \cdot s).$$

Lemma 10. *There exists $R > 0$ such that the restpoints of the flow of $-\Phi'_{(k)}$ are contained in*

$$B_{(k)} = D^+_{(k)} \times D^-_{(k)} \times \mathbb{T}^n$$

where $D^{\pm}_{(k)} = \{u^{\pm} \in E^{\pm}_{(k)} \mid \|u^{\pm}\| \leqslant R\}$ *denotes the closed disc of radius* R *in* $E^{\pm}_{(k)}$.

Proof. Recall that the restriction of the operator T to $E^+ \oplus E^-$ is a linear isomorphism. Hence there exists $\lambda > 0$ such that $(Tu^+, u^+) \geqslant \lambda \|u^+\|^2$ for all $u^+ \in E^+_{(k)}$, and such that $(Tu^-, u^-) \leqslant -\lambda \|u^-\|^2$ for $u^- \in E^-_{(k)}$.

We consider $u = u^+ + u^- + u^0 \in (E^+_{(k)} \oplus E^-_{(k)}) \times (E^0/\mathbb{Z}^n)$ such that $\|u^-\| \leqslant \|u^+\|$. For these u we have

$$\begin{aligned}(\Phi'_{(k)}(u), u^+) &\geqslant \lambda \|u^+\|^2 - \int_0^1 \langle \nabla H(u+e, t) - \hat{Q}(t)\, u, u^+ \rangle \, dt + (v, u^+) \\ &\geqslant \lambda \|u^+\|^2 - \{c_1(\varepsilon) + \varepsilon \|u^+ + u^-\|\} \|u^+\| - \|v\| \, \|u^+\| \\ &\geqslant (\lambda - 2\varepsilon) \|u^+\|^2 - c_2(\varepsilon, r) \|u^+\|,\end{aligned}$$

where we have used the asymptotic condition

$$|\partial_x H(x, y, t)| + |\partial_y H(x, y, t) - A(t) y| \leqslant c_1(\varepsilon) + \varepsilon \, |y|.$$

We choose $0 < \varepsilon < \lambda/2$. Consequently there exists $R > 0$ such that with this choice of ε:

$$\begin{aligned}&(-\Phi'_{(k)}(u), u^+) < 0 \quad \text{where} \quad u = u^+ + u^- + u^0 \\ &\text{satisfying } \|u^-\| \leqslant \|u^+\| \quad \text{and} \quad \|u^+\| \geqslant R.\end{aligned} \tag{49}$$

By a similar argument we can find $R > 0$ such that

$$\begin{aligned}&(-\Phi'_{(k)}(u), u^-) > 0 \quad \text{where} \quad u = u^+ + u^- + u^0 \\ &\text{satisfying } \|u^+\| \leqslant \|u^-\| \quad \text{and} \quad \|u^-\| \geqslant R.\end{aligned} \tag{50}$$

Thus the restpoints of the flow of $-\varphi'_{(k)}$ have to be contained in $D^+_{(k)} \times D^-_{(k)} \times E^0$. ∎

If $\|u^+\| = R$ and $\|u^-\| < R$, then the inequality (49) shows that the vector $\Phi'_{(k)}(u)$ points into the exterior of B_k, and conversely if $\|u^-\| = R$ and $\|u^+\| < R$, then the vector $-\Phi'_{(k)}(u)$ points into the exterior of B_k by (50). Thus it follows that the set B_k is an isolating block in the sense of Conley, cf. [4, Ch.I.3]. In particular,

$$B^+_k := \partial D^+_k \times D^-_k \times \mathbb{T}^n$$

is the entrance set of B_k for the flow of $-\Phi'_k$, and

$$B^-_k := D_k \times \partial D^-_k \times \mathbb{T}^n$$

is the corresponding exit set. Indeed, it follows from (49) and (50) that the compact invariant set S_k which is contained in the isolating block B_k consists precisely of the critical points of Φ_k together with their connecting orbits, see also Lemma 9 in [15].

We are ready now to apply the Conley–Zehnder Morse theory to the flow of $-\Phi_k'$ on $E_k \times \mathbb{T}^n$. Here we make use of the fact that $E_k \times \mathbb{T}^n$ is a finite-dimensional manifold.

DEFINITION 4. The Morse index $m(u)$ of a non-degenerate critical point $u \in E_k \times \mathbb{T}^n$ of Φ_k is defined to the number of the negative eigenvalues of the Hessian matrix $P_k \Phi''(u)$.

PROPOSITION 2 (Morse theory for Φ_k). *If $k \in \mathbb{N}$ is sufficiently large then all critical points of Φ_k on $E_k \times \mathbb{T}^n$ are non-degenerate and there exist only finitely many of them. Denote by $m(u)$ the Morse index of a critical point u, then*

$$\sum_{\{u^i \mid \Phi_k'(u^i)=0\}} t^{m(u^i)} = p(t, \mathbb{T}^n)\, t^N + (1+t)\, Q(t)$$

where $p(t, \mathbb{T}^n)$ is the Poincaré polynomial of the n-dimensional torus $\mathbb{T}^n$, N is a positive integer depending on k, and $Q(t)$ is a polynomial having non-negative integer coefficients.

Proof. We merely describe the main ideas and refer to [5, 6] for the details. Let S_k denote the maximal invariant set of the gradient flow for Φ_k which is contained in the isolating set $D_k^+ \times D_k^- \times \mathbb{T}^n$ having $D_k^+ \times \partial D_k^- \times \mathbb{T}^n$ as exit set. The set S_k is compact and consists of finitely many critical points together with all their connecting orbits. It has a Conley index which is the homotopy type of a pointed compact space, denoted by $h(S_k)$. From the above isolating block and the exit set one computes readily for the Poincaré polynomial of $h(S_k)$ that

$$p(t, h(S_k)) = \sum_q t^q \dim \check{H}^q(B_k, B_k^-) = p(t, \mathbb{T}^n)\, p(t, \dot{\mathbb{S}}^N) = p(t, \mathbb{T}^n)\, t^N$$

for some integer N. Here $\check{H}^*$ denotes the Čech cohomology with real coefficients, and $\dot{\mathbb{S}}^N$ is the N-dimensional sphere with one distinguished point. Denote the Conley index of a critical point u by $h(u)$. Then, since the critical points constitute a Morse decomposition of S_k, we have the following Morse equation, relating the global index of S_k with the local invariants of the critical points by

$$\sum_{i=1}^{m} p(t, h(u^i)) = p(t, h(S_k)) + (1+t)\, Q(t)$$

with a polynomial Q having non-negative integer coefficients. Observe now that if a critical point is non-degenerate, then its Conley index is a pointed sphere of dimension equal to the Morse index $m(u)$ of u, so that

$$p(t, h(u^i)) = p(t, \mathbb{S}^{m(u^i)}) = t^{m(u^i)}$$

and hence the proposition is proved. ▮

From the Morse equation it is now easy to derive a lower bound for the number of critical points. Observe that the Poincaré polynomial is given by

$$p(t, \mathbb{T}^n) = \sum_{j=0}^{n} \binom{n}{j} t^j.$$

Setting $t = 1$ we obtain

$$p(1, \mathbb{T}^n) = 2^n$$

which is equal to the sum of Betti numbers of $\mathbb{T}^n$. Consequently, setting $t = 1$ in the Morse equation, we obtain the

COROLLARY 2.

$$\#\{\textit{critical points of } \Phi_k\} \geqslant 2^n$$

This proves Theorem 1.

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