ZETA FUNCTIONS OF RESTRICTIONS OF THE SHIFT TRANSFORMATION[1]

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1. **Introduction.** Let S be a finite set with $N \geq 2$ elements, and consider $X = S^{\mathbf{Z}}$. We may regard X as the set of all two-sided sequences $(\ldots, a_{-1}, a_0, a_1, a_2, \ldots)$ of elements of S. Give S the discrete topology and X the product topology; this makes X into a compact space. Let α denote the shift transformation on X, i.e., the mapping defined by $(\alpha a)_n = a_{n+1}$. Evidently, α is a homeomorphism of X onto itself. An element a at X is a fixed point of α^m if and only if the sequence a is periodic with period m. The number of fixed points of α^m is therefore N^m, the number of finite sequences of length m.

Now let Y be a closed subset of X which is mapped *onto* itself by α. We will denote the number of fixed points of α^m in Y by $N_m(\alpha|Y)$. The *zeta function* of $\alpha|Y$ is by definition

$$\zeta_{\alpha|Y}(z) = \exp\left(\sum_m \frac{N_m(\alpha|Y)}{m} z^m\right).$$

Since $N_m(\alpha|Y) \leq N^m$, $\zeta_{\alpha|Y}$ is holomorphic in the disc $|z| < 1/N$. We will investigate under what circumstances the function $\zeta_{\alpha|Y}$ is rational.

Let $\sigma = (\sigma_0, \sigma_1, \ldots, \sigma_n)$ be a finite sequence in S. We define

$$Y_\sigma = \{a \in X : \text{There is no } i \text{ such that } a_i = \sigma_0, a_{i+1} = \sigma_1, \ldots, a_{i+n} = \sigma_n\}.$$

In other words, Y_σ is the set of all two-sided sequences which nowhere contain a segment of $n + 1$ elements of the form $(\sigma_0, \sigma_1, \ldots, \sigma_n)$. Any Y_σ is a closed subset of X mapped onto itself by α, and the same is true of any intersection, finite or infinite, at Y_σ's. (It is an amusing fact, which we will not have occasion to use, that every closed α-invariant subset of X is the intersection of the Y_σ's which contain it.) We say that a closed α-invariant subset of X is of *finite type*[2] if it is the intersection of finitely many Y_σ's. Our main result, to be proved in §2, is that the zeta function of the restriction of α to any closed invariant subset of finite type is rational.

Not every closed invariant subset of X has a rational zeta function however. In §3, we show that there are only countably many distinct rational zeta functions, but that there is an uncountable family of closed α-invariant subsets of X with pairwise distinct zeta functions. Finally, in §4, we give an example of a closed invariant subset of X whose zeta function may be explicitly shown to be non-rational.

[1] Supported in part by NSF contract GP-7176.

[2] See W. Parry, *Intrinsic Markov chains*, Trans. Amer. Math. Soc. **112** (1964), 55–66.

2. The zeta function of an invariant subset of finite type is rational. Let $\{\sigma_1, \ldots, \sigma_m\}$ be a finite collection at finite sequences in S. For the purposes of this section, we will say that a sequence, finite or infinite, is *acceptable* if it contains none of the σ_i's as a segment. To prove the rationality of the zeta function of α restricted to $Y = Y_{\sigma_1}, \ldots, Y_{\sigma_m}$, we need a technique for computing the number of periodic acceptable sequences with various periods. We proceed in the following way: To describe a periodic sequence a of period n, it is evidently enough to specify $(a_1, \ldots, a_n)$. However, to determine whether the corresponding infinite periodic sequence is acceptable, it is not enough to look for forbidden segments in $(a_1, \ldots, a_n)$; one must look at some more terms in the sequence. Let $n_0 + 1$ be the length of the longest sequence in $\{\sigma_1, \ldots, \sigma_m\}$. It is easy to see that, if a is an infinite sequence which is periodic with period n, then a is acceptable if and only if $(a_1, \ldots, a_{n+n_0})$ is. The sequence $(a_1, \ldots, a_{n+n_0})$ must be restricted appropriately if it is to come from a periodic sequence of period n; for this it is necessary and sufficient that $(a_1, \ldots, a_{n_0}) = (a_{n+1}, \ldots, a_{n+n_0})$, i.e., that the initial and terminal segments of length n_0 are the same. Thus, we have the following lemma:

LEMMA 1. *The periodic sequences in Y of period n are in a one-one correspondence with the acceptable sequences $(a_1, \ldots, a_{n_0+n})$ such that $a_1 = a_{n+1}, \ldots, a_{n_0} = a_{n+n_0}$.*

What we have to do, then, is to compute the number of such finite sequences. To do this, it is actually convenient to compute something more. Let τ, τ' be any two acceptable sequences of length n_0. We will let $N(n, \tau, \tau')$ denote the number of acceptable sequences of length $n_0 + n$ with τ as their initial sequence and τ' as their terminal sequence. It is easy to write a recurrence relation for $N(n, \tau, \tau')$: the idea is simply that an acceptable sequence of length $n_0 + n + 1$ is obtained by adjoining a new first element to an acceptable sequence of length $n + n_0$, being careful not to produce in this way one of the forbidden sequences $\{\sigma_1, \ldots, \sigma_m\}$ as an initial segment. This adjunction does not change the terminal segment, and whether or not a given element of S can be adjoined to a given acceptable sequence depends only on the initial segment of length n_0 of that sequence.

With these remarks in mind, we define a square matrix $T_{\tau\tau'}$ whose rows and columns are labelled by the acceptable sequences of length n_0:

$T_{\tau\tau'} = 1$ if the initial segment of length $n_0 - 1$ in τ' is the same as the terminal segment of length $n_0 - 1$ of τ, and if the sequence obtained by adjoining the first element of τ to τ' is acceptable.
$\phantom{T_{\tau\tau'}} = 0$ otherwise.

Examination of this definition shows that we could alternatively have defined $T_{\tau\tau'} = N(1, \tau, \tau')$.

EXAMPLE. $S = \{0, 1\}$, $m = 1$, $\sigma_1 = (0, 0)$, $n_0 = 1$, $T_{(0),(0)} = 0$, $T_{(0),(1)} = T_{(1),(0)} = T_{(1),(1)} = 1$.

LEMMA 2. *For $n \geq 1$,*

$$N(n, \tau, \tau') = (T^n)_{\tau\tau'},$$

where T^n means the matrix T raised to the nth power.

PROOF. We argue by induction on n. We have already remarked that the lemma is true for $n = 1$. Suppose it is true for n; we will prove it for $n + 1$. Every acceptable sequence of length $n_0 + n + 1$ is obtained by adjoining a first element to an acceptable sequence of length $n_0 + n$. To get an acceptable sequence with initial segment τ by this process it is necessary and sufficient to start with an acceptable sequence with an initial segment τ'' such that $T_{\tau,\tau''} = 1$ (and, of course, to adjoin the right first element). Hence,

$$\begin{aligned} N(n + 1, \tau, \tau') &= \sum_{\tau''} T_{\tau,\tau''} \cdot N(n, \tau'', \tau') \\ &= \sum_{\tau''} T_{\tau,\tau''} (T^n)_{\tau'',\tau'} \text{ by the induction hypothesis} \\ &= (T^{n+1})_{\tau,\tau'}, \end{aligned}$$

so the induction step is proved.

THEOREM 1. *Let $\{\alpha_1, \ldots, \alpha_m\}$ be a finite collection of finite sequences; let $Y = Y_{\sigma_1} \cap \ldots \cap Y_{\sigma_m}$. Then $\zeta_{\alpha|Y}$ is a rational function; in fact, if T is the matrix introduced above and if $\lambda_1, \ldots, \lambda_J$ are the nonzero eigenvalues of T, then*

$$\zeta_{\alpha|Y}(z) = \prod_{j=1}^{J} \frac{1}{(1 - \lambda_j z)}.$$

PROOF. By Lemma 1, the number of periodic sequences in Y of period n is just the number of acceptable sequences of length $n_0 + n$ with the same initial and terminal segments of length n_0. By the definition of $N(n, \tau, \tau')$ and Lemma 2, we have therefore

$$N_n(\alpha|Y) = \sum_{\tau} N(n, \tau, \tau) = \sum_{\tau} (T^n)_{\tau\tau} = \operatorname{Tr}(T^n) = \sum_{j=1}^{J} \lambda_j^n.$$

The theorem now follows by an elementary calculation from the definition of the zeta function.

In the example given above, where T is the 2×2 matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix},$$

the eigenvalues of T are $\frac{1}{2} \pm \frac{1}{2}\sqrt{5}$, so

$$\zeta_{\alpha|Y}(z) = \frac{1}{1 - z + z^2}.$$

REMARK. Our procedure for calculating $\zeta_{\alpha|Y}$ is frequently very inefficient. For example, if we consider Y_σ for a single σ of length $n_0 + 1$, T is a $N^{n_0} \times N^{n_0}$ matrix, but it can easily be seen that T^{n_0} has at most $n_0 + 1$ distinct columns and that therefore T has at most $n_0 + 1$ nonzero eigenvalues (counting multiplicity). It would be interesting to have a computation procedure which removes such redundancy.

3. **Many invariant subsets have nonrational zeta-functions.**

THEOREM 2. (a) *The set of all rational functions given in a neighborhood of zero by a convergent expansion of the form* $\exp(\sum_m (N_m/m)z^m)$, *where the* N_m *are integers, is countable.*

(b) *There exists a noncountable collection* $\{Y^\gamma\}$ *of closed, shift-invariant subsets of* X *such that* $\zeta_{\alpha|Y^\gamma} \neq \zeta_{\alpha|Y^{\gamma'}}$ *if* $\gamma \neq \gamma'$.

PROOF. (a) follows immediately from the fact that any rational function of the sort described has a representation as a finite product of factors of the form $(1 - \lambda_i z)^{\pm 1}$, where the λ_i are algebraic integers. (For the fact that the λ_i must be algebraic integers, see Appendix.)

To prove (b), consider sequences $\gamma = (\gamma_1, \gamma_2, \gamma_3, \ldots)$ of integers such that $1 < \gamma_1 < \gamma_2 < \ldots$. The set of all such sequences is noncountable. Let $s \in S$, and for any γ let Y^γ be the subset of X, consisting of all sequences having no block of precisely γ_1 successive s's, no block of precisely γ_2 successive s's, etc. Then Y^γ is a closed subset of X which is mapped onto itself by α.

We claim that, if $\gamma \neq \gamma'$, then $\zeta_{\alpha|Y^\gamma} \neq \zeta_{\alpha|Y^{\gamma'}}$. Thus, suppose $\gamma \neq \gamma'$ and choose the first n such that $\gamma_n \neq \gamma'_n$. We can suppose $\gamma_n > \gamma'_n$. We will show that the number of periodic sequences in Y^γ of period $\gamma'_n + 1$ is strictly larger than the number of such sequences in $Y^{\gamma'}$. Since $\gamma_1 = \gamma'_1, \ldots, \gamma_{n-1} = \gamma'_{n-1}$ every periodic sequence in $Y^{\gamma'}$ of period $\gamma'_n + 1$ is also in Y^γ. However, if $t \in S$ is different from s, then any periodic sequence whose minimal period contains γ'_n s's and one t is in Y^γ but not in $Y^{\gamma'}$. Thus, $\zeta_{\alpha|Y^\gamma} \neq \zeta_{\alpha|Y^{\gamma'}}$, and the proof of the theorem is complete.

4. **An example.** Theorem 2 shows that many closed shift-invariant subsets of X have irrational zeta functions, but it does not exhibit a single subset for which the zeta function can be shown to be irrational. We remedy this situation in this section by giving an example in which the zeta function may be computed fairly explicitly and shown to be irrational. The example is the following: Let $S = \{0, 1\}$ and let Y be the subset of X consisting of all periodic sequences (arbitrary period) with at most one "1" in a minimal period, together with all sequences with only one "1". It is clear that Y is shift-invariant; we will prove that it is closed. Thus, let a be in the closure of Y. If a contains at most one "1", then it is in Y. Otherwise, choose $m > n$ such that $a_n = 1, a_{n+1} = \ldots = a_{m-1} = 0, a_m = 1$, and consider the neighborhood $\mathscr{U}$ of a consisting of all sequences b with $b_n = 1, b_{n+1} = \ldots = b_{m-1} = 0, b_m = 1$. Then $\mathscr{U}$ contains only one point of Y; since a is in the closure of Y, it follows that a is actually in Y.

It is easy to see that the number of periodic sequences of period m in Y is the sum of the divisors of m, plus one. Let $\sigma(m)$ denote the sum of the divisors of m; then

$$\zeta_{\alpha|Y}(z) = \exp\left(\sum_m \left(\frac{1 + \sigma(m)}{m}\right) z^m\right)$$

$$= \frac{1}{1 - z} \exp\left(\sum_m \frac{\sigma(m)}{m} z^m\right).$$

It is known [1] that

$$\sum_{m=1}^{\infty} \frac{\sigma(m)}{m} z^m = -\log s(z),$$

where $s(z) = 1 - z - z^2 + z^5 + z^7 - \dots$ (the exponents which appear are those of the form $\frac{1}{2}(3k^2 \pm k)$). Thus we have $s(z)$ exhibited as a power series with arbitrarily long sequences of coefficients equal to zero; such a function cannot be rational unless it is a polynomial. Since

$$\zeta_{\alpha|Y}(z) = \frac{1}{(1 - z)s(z)},$$

$\zeta_{\alpha|Y}$ also cannot be rational.

Appendix. In the proof of Theorem 2 we used the fact that poles and zeroes of a rational zeta function can occur only at reciprocals of algebraic integers. This fact is well known, but we have been unable to find a satisfactory reference. For the convenience of the reader, we will supply a proof. We start by making a simple reduction. If $\exp(\sum_m (N_m/m)z^m)$ is rational, so is its logarithmic derivative $\sum_m N_m z^{m-1}$, and the logarithmic derivative has a pole wherever the original function has a pole or a zero. Hence, the statement about zeta functions follows from:

PROPOSITION 1. *Let f be a rational function regular at zero and let the power series expansion of f at zero have the form*

$$f(z) = \sum_m N_m z^m,$$

where the N_m are integers. Then the poles of f occur at reciprocals of algebraic integers.

This proposition follows at once from two lemmas:

LEMMA 3. *Let f be a rational function regular at zero; suppose that all derivatives of f at zero are rational numbers. Then f may be written in the form P/Q, where P and Q are polynomials with integral coefficients.*

Note that this lemma implies that the poles and zeroes of a rational zeta function must occur at algebraic numbers and that this fact would have been sufficient for the purposes of Theorem 2.

LEMMA 4. *Let f be a rational function which can be written as the quotient of two polynomials with integral coefficients. Suppose that f is regular at zero and that the power series expansion of f at zero has integral coefficients. Then the poles of f occur at reciprocals of algebraic integers.*

PROOF OF LEMMA 3. It evidently suffices to show that f can be written as P/Q, where P and Q have rational coefficients. Since f is a rational function regular at zero we can certainly write

$$f(z) = \frac{P(z)}{Q(z)} = \frac{P_0 + P_1 z + \dots + P_n z^n}{1 + Q_1 z + \dots + Q_m z^m}$$

where P and Q have no common factor. We also have $f(z) = \sum_n f_n z^n$, where the f_n are rational numbers, so if we can prove that $Q_1, \ldots, Q_m$ are rational it will follow that $P_0, \ldots, P_n$ are rational.

Since $Q \cdot f = P$, we have

$$\sum_{j=1}^{m} Q_j f_{k-j} = -f_k, \quad k = n+1, n+2, \ldots.$$

If we consider these equations for $k = n+1, \ldots, n+m$, we obtain a system of m linear equations with rational coefficients satisfied by $Q_1, \ldots, Q_m$. If these equations have a unique solution, then $Q_1, \ldots, Q_m$ are rational, and we are through. Hence, let $Q'_1, \ldots, Q'_m$ be such that

$$\sum_{j=1}^{m} Q'_j f_{k-j} = -f_k, \quad k = n+1, \ldots, n+m,$$

and let $Q'(z) = 1 + Q'_1 z + \ldots + Q'_m z^m$. Then

$$Q'P/Q = Q'f = P' + f',$$

where P' is a polynomial of degree at most n and f' has a zero of order at least $n+m+1$ at zero. Hence, $Qf' = Q'P - QP'$. The left-hand side of this equation has a zero of order at least $n+m+1$; the right-hand side is a polynomial of degree at most $n+m$, so both sides must be identically zero and we get $f' = 0$. From this and the fact that Q and P have no common factor it follows that $Q' = Q$ and the lemma is proved.

Proof of Lemma 4. This lemma is due to Fatou [2]; we reproduce his proof. We can write $f = P/Q$, where P and Q are polynomials with integral coefficients and no common factor. Since P, Q have no common factor we can find other polynomials A, B with integral coefficients such that $AP + BQ = N$, N an integer. Then the power series expansion at zero of $N/Q = Af + B$ has integral coefficients. Let us suppose, without changing our notation, that we have made all possible cancellations, i.e., that we have

$$\frac{N}{Q_0 + Q_1 z + \ldots + Q_m z^m} = C_0 + C_1 z + C_2 z^2 + \ldots,$$

where $N, Q_0, \ldots, Q_m, C_0, \ldots$ are all integers but where

$$1 = \text{G.C.D.}\,(N, Q_0, \ldots, Q_m) = \text{G.C.D.}\,(N, C_0, C_1, \ldots).$$

It will suffice to show that $Q_0 = 1$. Suppose that this is not the case, i.e., suppose that some prime number p divides Q_0. Then, since $N = C_0 Q_0$, p also divides N. Hence, if we cross-multiply and reduce mod p we get

$$0 \equiv (Q_0 + \ldots + Q_m z^m)(C_0 + C_1 z + \ldots) \mod (p)$$

so either all the Q_1's are divisible by p or else all the C_i's are. In either case we have a contradiction to the assumption that all possible cancellations have been made, so the proof is complete.

Acknowledgements. We are grateful to Professor S. Smale for suggesting the problem and for helpful discussions. We also benefited from some discussions with M. Schlessinger concerning the appendix.

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UNIVERSITY OF CALIFORNIA, BERKELEY

Proceedings of Symposia in

PURE MATHEMATICS

Volume 14

Global Analysis

Shiing-Shen Chern
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Editors

PROCEEDINGS OF THE SYMPOSIUM IN PURE MATHEMATICS
OF THE AMERICAN MATHEMATICAL SOCIETY

HELD AT THE UNIVERSITY OF CALIFORNIA
BERKELEY, CALIFORNIA
JULY 1–26, 1968

Prepared by the American Mathematical Society
under National Science Foundation Grant GP-8410

2010 *Mathematics Subject Classification.* Primary 57-XX.

Library of Congress Cataloging-in-Publication Data

Symposium in Pure Mathematics, University of California at Berkeley, 1968.
Global analysis. [Shiing-Shen Chern and Stephen Smale, editors]
p. cm. — (Proceedings of symposia in pure mathematics, v. 14–16
"Proceedings grew out of lectures given at the fifteenth Summer Mathematical Institute of the American Mathematical Society."
Includes bibliographical references.
ISBN 0-8218-1414-1 (v. 1) varies
1. Global analysis (Mathematics)—Congresses. I. Chern, Shiing-Shen, 1911–ed. II. Smale, Stephen, 1930–ed. III. American Mathematical Society. IV. Title. V. Series.

QA614 .S9 1968
513′.83

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