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The Classical Mechanics of One-Dimensional Systems of Infinitely Many Particles

I. An Existence Theorem

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Abstract. We prove a global existence and uniqueness theorem for solutions of the classical equations of motion for a one-dimensional system of infinitely many particles interacting by finite-range two-body forces which satisfy a Lipschitz condition.

§ 1. Introduction

In this paper, we prove an existence and uniqueness theorem for solutions of the equations of motion of a system of infinitely many classical point particles, constrained to move in one dimension, interacting by two-body forces of finite range. Thus, let (q_i, p_i) be a sequence of pairs of real numbers representing the positions and velocities of an infinite set of particles. We assume that each bounded interval in $\boldsymbol{R}$ contains only finitely many particles, and we want to solve the differential equations:

$$\frac{dq_i(t)}{dt} = p_i(t) \qquad \frac{dp_i(t)}{dt} = \sum_{j \neq i} F(q_i(t) - q_j(t)) \tag{1.1}$$

with the initial conditions:

$$q_i(0) = q_i \; ; \qquad p_i(0) = p_i \, .$$

(For simplicity, we are taking the particles to be identical and to have mass one). The interparticle force F will be assumed to be bounded and to have compact support. As long as each bounded interval in $\boldsymbol{R}$ contains only finitely many $q_j(t)$'s, the sum on the right of the second equation has only finitely many non-zero terms for each i and the equations therefore make sense. It is clear, however, that for some initial configurations we must expect the Eq. (1.1) to lead in finite time to a catastrophic situation with infinitely many particles in some bounded interval. To take a trivial example, if there are no interparticle forces and if $p_i = - q_i$ for each i, then all the particles are at the origin at time one. The crux of the problem of proving an existence theorem is to find a set of initial configurations for which such catastrophies can be shown not to happen.

A first result in this direction was obtained by J. GINIBRE (unpublished) who proved that the Eq. (1.1) have a solution valid for all values of t if the initial configuration admits an upper bound on the absolute values of the velocities of the various particles and on the numbers of particles in the various intervals of unit length; furthermore, he proved a local existence theorem for systems of particles moving in $\boldsymbol{R}^\nu$ (instead of $\boldsymbol{R}$), with the analogous restrictions on the initial momenta and densities. Our theorem, which holds only in one dimension, gives existence for initial configurations in which, roughly speaking, the velocities increase at most logarithmically with distance from the origin and the number of particles in an interval of unit length increases at most logarithmically with the distance from that interval to the origin. (The precise condition we impose on the initial densities is actually a bit more restrictive; see § 2.)

The main interest of the existence and uniqueness theorem which we will prove lies in its application to the time-evolution problem in the classical statistical mechanics of infinite systems. We will discuss this application in detail in a subsequent publication. Nevertheless, we give here a brief sketch of how the application is made, as motivation for our theorem and to explain why it is important to be able to solve the equations of motion for a set of initial configurations more general than those with bounded velocities and densities.

We have first to explain what is meant by a state of classical statistical mechanics. By a *locally finite configuration of labelled particles* we will mean either:

a) An n-tuple $x = (q_1, p_1; q_2, p_2; \ldots; q_n, p_n)$ of pairs of real numbers or

b) A sequence $x = (q_i, p_i)$ of pairs of real numbers such that each bounded set in $\boldsymbol{R}$ contains only finitely many q_i's, i.e., such that $\lim_{i\to\infty} |q_i| = \infty$.

We will denote the set of all such configurations by $\mathscr{X}$. A *locally finite configuration of unlabelled particles* will mean an equivalence class of locally finite configurations of labelled particles, where two configurations are equivalent if they differ only by a permutation of the indexing set; we will denote by $[x]$ the equivalence class of x and by $[\mathscr{X}]$ the set of equivalence classes. Space translations act on an obvious way on $\mathscr{X}$ and on $[\mathscr{X}]$. The space $[\mathscr{X}]$ may be equipped with a topology in a natural way [1]; then a *state of classical statistical mechanics* is a Borel probability measure on $[\mathscr{X}]$ which is invariant under the action of space translations.

Now suppose that the equations of motion (1.1) have a unique solution for every initial configuration x in some subset $\hat{\mathscr{X}}$ of $\mathscr{X}$, and that the solution curve

$$\{x(t) = (q_i(t), p_i(t)) : -\infty < t < \infty\}$$

in contained in $\hat{\mathscr{X}}$. By the invariance of Eq. (1.1) under permutations of the indexing set, it is clear that $\hat{\mathscr{X}}$ can be taken to be a union of equivalence classes; let $[\hat{\mathscr{X}}]$ denote the corresponding set of unlabelled configurations. For any t, we can define a mapping T^t of $[\hat{\mathscr{X}}]$ into itself by setting $T^t[x]$ equal to the equivalence class of the value at time t of a solution of the equations of motion whose value at time zero belongs to the equivalence class $[x]$. By the uniqueness of the solution, this definition does not depend on the choices made, and $\{T^t\}$ is a one-parameter group of mappings of $[\hat{\mathscr{X}}]$ onto itself.

If ϱ is a measure on $[\mathscr{X}]$ which is concentrated on $[\hat{\mathscr{X}}]$, i.e. which has

$$\varrho([\mathscr{X}]\backslash[\hat{\mathscr{X}}]) = 0\,,$$

and if each T^t is a measurable mapping of $[\hat{\mathscr{X}}]$ onto itself, then we can define for each t a measure ϱ^t by

$$\varrho^t([\mathscr{X}]\backslash[\hat{\mathscr{X}}]) = 0\,,$$

$$\varrho^t(A) = \varrho(T^{-t}A) \quad \text{if} \quad A \subset [\hat{\mathscr{X}}]\,.$$

Thus, we get a satisfactory time evolution for those states which are concentrated on $[\hat{\mathscr{X}}]$, and the usefulness of an existence theorem depends on whether or not interesting states are concentrated on the set of allowed initial configurations. In the subsequent publication referred to above, we will give a criterion for states to be concentrated on our set of initial configurations which implies in particular that

a) states obtained by taking the infinite volume limit of thermodynamic ensembles at low activity [2], and

b) states obtained by taking an infinite volume limit of thermodynamic ensembles with non-negative potentials, at any activity have this property. On the other hand, since any state obtained by taking an infinite volume limit of canonical or grand-canonical ensembles has a Maxwellian velocity distribution[1], it is easy to see that no such state can be concentrated on the set of configurations with bounded velocities (unless it is the trivial state which is concentrated on the configuration with no particles).

In fact, the logarithmic rate of increase of density fluctuations is almost the slowest increase which can be allowed if we are to have a sufficient set of initial configurations for applications to statistical

[1] A state is said to have a Maxwellian velocity distribution if the velocity of any given particle is independent of the position of that particle and of the positions and velocities of the other particles, and if the velocity of a single particle is distributed with probability density $\sqrt{\frac{\beta}{2\pi}}\,e^{-\beta p^2/2}$, where β is some positive real number.

mechanics; a typical configuration of non-interacting particles has density fluctuations which increase like $\log(d)/\log(\log(d))$, where d is the distance to the origin. To make this statement precise, we define a function on $[\mathscr{X}]$ as follows: For any configuration $[x]$, take the number of particles in the interval $[n, n+1)$, divide by $\log(n)/\log(\log(n))$, and take the lim. sup. as n goes to infinity. This gives either a non-negative real number or $+\infty$. An elementary calculation shows that, for the state obtained by taking the infinite volume limit of the grand canonical ensemble for non-interacting particles, with any (non-zero) temperature and any chemical potential, this function takes on the value one on the complement of a set of measure zero.

Having made these remarks by way of motivation, we will devote our attention for the rest of this article to the problem of solving the differential Eq. (1.1), without considering further the application to statistical mechanics. In § 2, we give a precise definition of the set of initial configurations for which we can solve these equations, state the main result, and sketch the ideas underlying the proof. In § 3, we reduce the equations of motion to a non-linear evolution equation:

$$\frac{d\zeta(t)}{dt} = A(\zeta(t)), \tag{1.2}$$

on a Banach space isomorphic to l^∞ (where the derivative is to be taken in the sense of the product tolopogy), and we estimate the norm of $A(\zeta)$. In § 4, we show that, although the non-linear operator A does not satisfy a norm Lipschitz condition, it does satisfy a Lipschitz condition with respect to a family of semi-norms defining the product topology. We then use this Lipschitz condition, together with norm estimates on the operator A, to prove the existence and uniqueness of solutions of (1.2), and to show that these solutions can be obtained by solving the integral equation

$$\zeta(t) = \zeta(0) + \int_0^t d\tau A(\zeta(\tau))$$

by iteration.

§ 2. The Set of Allowed Initial Configurations

Before defining the set of initial configurations for which we can solve the equations of motion, we need some notation. First, to cut off the logarithm function for small values of its argument, we make the definition:

$$\log_+(q) = \log(|q| \vee e) \tag{2.1}$$

where the symbol $\vee$ denotes "supremum". We shall make frequent use of the elementary inequalities:

$$\begin{aligned} \log_+(a+b) &\leqq \log_+(a) + \log_+(b)\ ; \\ \log_+(a\,.\,b) &\leqq \log_+(a) + \log_+(b)\ ; \\ \log_+(a) &\leqq |a| \quad \text{if} \quad |a| \geqq 1\ . \end{aligned} \tag{2.2}$$

Second, for any $x = (q_i, p_i)$ in $\mathscr{X}$, and any bounded set Λ in $\boldsymbol{R}$, we define $N_\Lambda(x)$ to be the number of i's such that $q_i \in \Lambda$, i.e., the number of particles in the region Λ for the configuration x.

The set of initial configurations $x = (q_i, p_i)$ which we want to consider will be those satisfying the following two conditions:

1) There is a constant K_1, such that, for each i,

$$|p_i| \leqq K_1 \log_+(q_i) . \tag{2.3}$$

2) There is a constant K_2 such that, if (α, β) is any bounded open interval whose length $\beta - \alpha$ is larger than $\log_+((\alpha + \beta)/2)$, then

$$N_{(\alpha, \beta)}(x) \leqq K_2(\beta - \alpha) . \tag{2.4}$$

Condition 2) may be reformulated by saying that there is an upper bound for the mean density of particles in any interval of length greater than one whose length is also greater than the logarithm of the distance from its center to the origin. It implies in particular that the number of particles in any interval of unit length is bounded by a constant times the logarithm of the distance from its center to the origin.

We will denote by $\hat{\mathscr{X}}$ the set of labelled configurations satisfying 1) and 2), and by $[\hat{\mathscr{X}}]$ the corresponding set of unlabelled configurations. The set $\hat{\mathscr{X}}$, although not defined in a manifestly translation-invariant way, is easily seen to be mapped into itself by translations.

For any $x \in \hat{\mathscr{X}}$, we will let $|x|$ denote the smallest number which will work for both K_1 and K_2 in (2.3) and (2.4) respectively, i.e.,

$$|x| = \left[\sup_i \left\{\frac{|p_i|}{\log_+(q_i)}\right\}\right] \vee \left[\sup \left\{\frac{N_{(\alpha,\beta)}(x)}{\beta - \alpha} : \beta - \alpha > \log_+\left(\frac{\beta + \alpha}{2}\right)\right\}\right] . \tag{2.5}$$

We can now formulate our main result:

Theorem 2.1. *Let F be a real-valued function with compact support, satisfying a Lipschitz condition:*

$$|F(q_1) - F(q_2)| \leqq K \cdot |q_1 - q_2| \tag{2.6}$$

and let $x = (q_i, p_i)$ belong to $\hat{\mathscr{X}}$. Then there is one and only one function $x(t) = (q_i(t), p_i(t))$, defined for $-\infty < t < \infty$, with values in $\mathscr{X}$, satisfying

1. $\dfrac{dq_i(t)}{dt} = p_i(t) \quad \dfrac{dp_i(t)}{dt} = \sum\limits_{j \neq i} F(q_i(t) - q_j(t))$ (1.1)

2. $q_i(0) = q_i\,; \quad p_i(0) = p_i$

3. $|x(t)|$ *is a locally bounded function of t, i.e., is bounded on any bounded interval.*

We will say that a solution of the Eq. (1.1) is *regular* if it satisfies condition 3. Note that the uniqueness statement of the theorem is not as strong as one might hope, since it does not rule out the possibility of non-regular solutions.

The formulation of the theorem supposes tacitly that we use the same labelling set for all t and in particular that the total number of particles does not change with time. The result is well-known if the initial configuration has only finitely many particles. We will therefore give the proof only for initial configurations with infinitely many particles; the argument can readily be adapted, at the expense of complicating the notation, to apply simultaneously to the two cases.

The proof of the theorem is obscured by technical problems and by straight forward but tedious estimates. It seems worthwhile, therefore, to give a brief and unencumbered description of the underlying idea. The central difficulty is that of showing that the differential equations cannot drive infinitely many particles into a finite region of space in finite time. One can convince oneself of this by showing that the differential Eq. (1.1) imply integral inequalities for the quantity $|x(t)|$ which prevent its going to infinity in finite time. These inequalities are gotten as follows: If we know $|x(\tau)|$ for $0 \leqq \tau \leqq t$, then we have in particular estimates on the density for that interval of time. Majorizing the force on the i^{th} particle by the maximum of $|F|$ times the number of particles within a distance R (the range of the force) of q_i, we can convert our density estimates to estimates on the forces and thus, by integrating the second differential equation, to estimates on the velocities at time t. Similarly, from knowing $|x(\tau)|$ for $0 \leqq \tau \leqq t$, we get bounds on the velocities and therefore on the distances travelled. Using these bounds we can find, for a given interval (α, β), a larger interval from which a particle has to start if it is to be in (α, β) at time t. Knowing $|x(0)|$ enables us to majorize the number of particles in this larger interval at time zero and therefore the number of particles in (α, β) at time t.

Combining all these estimates gives a bound for $|x(t)|$ in terms of $|x(\tau)|$ for $0 \leqq \tau \leqq t$, and this bound may be seen to imply that $|x(t)|$ cannot go to infinity in finite time. Unfortunately, the estimates are very tedious to write out since the density and velocity bounds vary with position as well with time. Furthermore, while these inequalities are convincing evidence that well-behaved solutions to the equations exist, there remains the problem of constructing a proof. We have found it convenient to bypass these inequalities and to organize the proof in a different way, by reducing the Eq. (1.1) to a non-linear evolution equation on a Banach space. The estimates described above then reappear in the form of norm estimates on the "infinitesimal generator".

§ 3. Reformulation of the Differential Equations

Given any configuration $x \in \hat{\mathscr{X}}$, we will introduce a space of "neighboring configurations" in which the evolution takes place. Let $\mathscr{Y}_x$ denote the Banach space of sequences of pairs of real numbers $\zeta = (\xi_i, \eta_i)$

such that

$$\|\zeta\|_x = \sup_i \frac{|\xi_i| \vee |\eta_i|}{\log_+(q_i)} < \infty . \tag{3.1}$$

Given ζ in $\mathscr{Y}_x$, we denote by $x + \zeta$ the sequence of pairs of numbers $(q_i + \xi_i, p_i + \eta_i)$. The following two lemmas motivate the introduction of the space $\mathscr{Y}_x$: The first shows that, for any ζ in $\mathscr{Y}_x$, $x + \zeta$ is in $\hat{\mathscr{X}}$, and the second shows that any regular solution of the equations of motion with x as initial value stays inside the set of configurations of this form.

Lemma 3.1. *Let $x \in \hat{\mathscr{X}}$ and $\zeta \in \mathscr{Y}_x$. Then $x + \zeta \in \hat{\mathscr{X}}$. Moreover, $|x + \zeta|$ is bounded on bounded sets in $\mathscr{Y}_x$.*

Lemma 3.2. *Let $x \in \hat{\mathscr{X}}$, and let $x(t) = (q_i(t), p_i(t))$ be defined for t in a bounded open interval I containing zero. Assume:*

1. $\frac{d q_i(t)}{dt} = p_i(t)$ *for all* $t \in I$.
2. $x(0) = x$.
3. $|x(t)|$ *is bounded on* I .

Then, for each $t \in I$, we can write

$$x(t) = x + \zeta(t)$$

with $\zeta(t)$ in $\mathscr{Y}_x$, and $\|\zeta(t)\|_x$ in bounded on I.

We postpone the proofs of these lemmas. By Lemma 3.2, to find regular solutions of the differential equations, we can concentrate our attention on ones of the form $x + \zeta(t)$, with $\zeta(t) \in \mathscr{Y}_x$. In terms of the new dependent variable $\zeta(t) = (\xi_i(t), \eta_i(t))$, the differential equations become:

$$\frac{d\xi_i(t)}{dt} = p_i + \eta_i(t) \,; \quad \frac{d\eta_i(t)}{dt} = \sum_{j \neq i} F(q_i + \xi_i(t) - q_j - \xi_j(t))$$

or, schematically,

$$\frac{d\zeta(t)}{dt} = A_x(\zeta(t)) \,, \tag{3.2}$$

where the derivative is to be taken a co-ordinate at a time and $A_x(\zeta)$ is the sequence of pairs of real numbers

$$A_x(\zeta) = \Big(p_i + \eta_i, \sum_{j \neq i} F(q_i + \xi_i - q_j - \xi_j)\Big) \,. \tag{3.3}$$

The following proposition shows that A_x defines a bounded non-linear operator on $\mathscr{Y}_x$:

Proposition 3.3. *Let F be a bounded function vanishing outside $(-R, R)$, and let $\zeta \in \mathscr{Y}_x$. Let $A_x(\zeta)$ be defined by* (3.3). *Then $A_x(\zeta) \in \mathscr{Y}_x$; moreover, there exist constants C, D (depending on $|x|$) such that*

$$\|A_x(\zeta)\|_x \leqq C + D \|\zeta\|_x \log_+(\|\zeta\|_x) \tag{3.4}$$

for all $\zeta \in \mathscr{Y}_x$.

Again we postpone the proof. Combining Lemma 3.1, Lemma 3.2, and Proposition 3.3, we see that Theorem 2.1 is equivalent to:

Theorem 2.1′. *Let F, x be as in Theorem 2.1, and let A_x be defined by (3.3). Then there is one and only one function $\zeta(t) = (\xi_i(t), \eta_i(t))$, defined for $-\infty < t < \infty$, with values in $\mathcal{Y}_x$, satisfying:*

1. $\frac{d\zeta(t)}{dt} = A_x(\zeta(t))$.
2. $\zeta(0) = 0$.
3. $\|\zeta(t)\|_x$ *is a locally bounded function of t.*

In 1., the derivative is to be understood in the sense of the product topology on $\mathcal{Y}_x$.

We will now give the proofs of Lemma 3.1, Lemma 3.2, and Proposition 3.3. Let us begin with Lemma 3.2. We have to show that

$$\frac{|q_i(t) - q_i|}{\log_+(q_i)} \quad \text{and} \quad \frac{|p_i(t) - p_i|}{\log_+(q_i)}$$

are bounded with respect to i and t. The differential equation and the boundedness of $|x(t)|$ imply that, for some K,

$$\left|\frac{d}{dt} q_i(t)\right| \leqq K \log_+(q_i(t)) .$$

Therefore,

$$|q_i(t) - q_i| \leqq K \int_I dt \log_+(q_i(t)) \leqq K' \log_+(|q_i| + \Delta Q_i) ,$$

where

$$\Delta Q_i = \sup_{t \in I} |q_i(t) - q_i| ,$$

and K' is K times the length of I.

Thus

$$\Delta Q_i \leqq K' \log_+(|q_i| + \Delta Q_i) \leqq K' [\log_+(q_i) + \log_+(\Delta Q_i)] ,$$

where, to get the second inequality, we have used (2.2). Rearranging this inequality, we get

$$\frac{\Delta Q_i}{\log_+(q_i)} \leqq K' \left[1 + \frac{\log_+(\Delta Q_i)}{\log_+(q_i)}\right] ,$$

which implies that $\frac{\Delta Q_i}{\log_+(q_i)}$ is bounded, i.e., that $\frac{|q_i(t) - q_i|}{\log_+(q_i)}$ is bounded with respect to i and t. The boundedness of $\frac{|p_i(t) - p_i|}{\log_+(q_i)}$ follows at once, since

$$\frac{|p_i(t) - p_i|}{\log_+(q_i)} \leqq \frac{|p_i(t)|}{\log_+(q_i(t))} \cdot \frac{\log_+(|q_i| + \Delta Q_i)}{\log_+(q_i)} + \frac{|p_i|}{\log_+(q_i)}$$

$$\leqq |x(t)| \cdot \sup_i \frac{\log_+(|q_i| + \Delta Q_i)}{\log_+(q_i)} + |x(0)| .$$

This completes the proof of Lemma 3.2.

The proofs of Lemma 3.1 and Proposition 3.3 both involve some tedious particle-tracing estimates which are isolated in the following lemma.

Lemma 3.4. *There exists a constant K such that, for all $x \in \mathscr{X}$, all $\beta > \alpha$, all $\lambda \geqq 1$, and all sequences (ξ_i) of real numbers such that*

$$\sup_i \{|\xi_i|/\log_+(q_i)\} \leqq \lambda\,,$$

the inequality

$$\#\{j : q_j + \xi_j \in [\alpha, \beta]\} \leqq |x|\,[\beta - \alpha + K\lambda(\log_+(\lambda) + \log_+(|\alpha| \vee |\beta|))] \tag{3.5}$$

holds. (The notation $\# X$, X a set, denotes the number of elements in the set).

Proof. It is enough to prove the lemma with the added restriction that α and β have the same sign, since we can prove the general result from this more restricted one by breaking up any interval into a piece to the right of the origin and a piece to the left. By symmetry, we can assume $\alpha \geqq 0$.

Let

$$W = \{q \in \boldsymbol{R} : q + \xi \in [\alpha, \beta] \text{ for some } \xi \text{ with } |\xi| \leqq \lambda \log_+(q)\}\,.$$

We will proceed by estimating the length of W and then applying the definition of $|x|$ to estimate the number of q_j's in W.

The first remark we need is the following: If

$$a \geqq 1 + 2\lambda\,, \tag{3.6}$$

then

$$q - \lambda \log_+(q) \leqq a \quad \text{implies} \quad q - a < 2\lambda \log_+(a)\,. \tag{3.7}$$

To prove this remark, we first observe that $q - a \leqq \lambda \log_+(q)$, so it suffices to prove that $q < a^2$. But $q - \lambda \log_+(q)$ is a strictly increasing function of q for $q \geqq \lambda$, and, by the hypotheses of (3.7), $a \geqq q - \lambda \log_+(q)$, so we have only to prove that

$$a^2 - \lambda \log_+(a^2) > a\,.$$

Dividing this inequality by a, using the fact that $\log_+(a^2) = 2\log_+(a)$, and transposing, we see that it suffices to show:

$$a > 1 + \frac{2\lambda \log_+(a)}{a}\,.$$

But $1 + \frac{2\lambda \log_+(a)}{a} < 1 + 2\lambda \leqq a$ [by (3.6)], so this inequality holds and our remark is proved.

Now let $q \leqq 0$ and suppose

$$q + \lambda \log_+(q) \geqq \alpha\,, \quad \text{i.e.,} \quad |q| - \lambda \log_+(|q|) \leqq -\alpha \leqq 0\,.$$

Applying the above remark, with $a = 1 + 2\lambda$ and $|q|$ replacing q, we see that

$$|q| < (1 + 2\lambda) + 2\lambda \log_+(1 + 2\lambda)\,. \tag{3.8}$$

To save writing, we denote the right-hand side of this inequality by h.

Similarly, if $q - \lambda \log_+(q) \leqq \beta$, then

a) If $\beta < 1 + 2\lambda, q < h$. (3.9)

b) If $\beta \geqq 1 + 2\lambda, q < \beta + 2\lambda \log_+(\beta)$. (3.10)

Combining (3.8), (3.9), and (3.10), we see that

$$W \subset (-h, h) \cup (0, \beta + 2\lambda \log_+(\beta)) .$$

We can reduce the second interval as follows: If $q \in (0, \beta) \cap W$, then

$$\alpha \leqq q + \lambda \log_+(q) < q + \lambda \log_+(\beta) ,$$

i. e.

$$q > \alpha - \lambda \log_+(\beta) .$$

Thus,

$$W \subset (-h, h) \cup (\alpha - \lambda \log_+(\beta), \beta + 2\lambda \log_+(\beta)) .$$

Applying the definition of $|x|$, we see that

$$\# \{j : q_j \in W\} \leqq |x| [\beta - \alpha + 2h + 3\lambda \log_+(\beta)] ;$$

inserting the value of h and making some elementary re-arrangements completes the proof of the lemma.

We can now give the proofs of Lemma 3.1 and Proposition 3.3. To prove Lemma 3.1, we have to find bounds on

$$\sup_i \left\{\frac{|p_i + \eta_i|}{\log_+(q_i)}\right\} \text{ and } \sup \left\{\frac{N_{(\alpha,\beta)}(x + \zeta)}{\beta - \alpha} : \beta - \alpha > \log_+ \frac{(\alpha + \beta)}{2}\right\}$$

valid for all ζ with $\|\zeta\|_x \leqq \lambda$, where we can assume $\lambda \geqq 1$.

The momentum bound is immediate since

$$\frac{|p_i + \eta_i|}{\log_+(q_i)} \leqq \frac{|p_i|}{\log_+(q_i)} + \frac{|\eta_i|}{\log_+(q_i)} \leqq |x| + \|\zeta\|_x . \tag{3.11}$$

To get the density bounds, we apply Lemma 3.4 to show that for any $\beta > \alpha$,

$$\frac{N_{(\alpha,\beta)}(x + \zeta)}{\beta - \alpha} \leqq |x| \left[1 + K\lambda \left(\frac{\log_+(\lambda)}{\beta - \alpha} + \frac{\log_+(|\alpha| \vee |\beta|)}{\beta - \alpha}\right)\right] .$$

Using the equation

$$|\alpha| \vee |\beta| = \left|\frac{\alpha + \beta}{2}\right| + \left|\frac{\alpha - \beta}{2}\right|$$

and the sub-additivity of $\log_+$, we see that, if $\beta - \alpha > \log_+\left(\frac{\beta + \alpha}{2}\right)$, we have

$$\frac{N_{(\alpha,\beta)}(x + \zeta)}{\beta - \alpha} \leqq |x| \left[1 + K\lambda \left(\log_+(\lambda) + \frac{\log_+\left(\frac{\beta - \alpha}{2}\right)}{\beta - \alpha} + 1\right)\right]$$

which gives the desired bound on the density and completes the proof of Lemma 3.1.

To prove Proposition 3.3, it suffices to find constants C and D such that

a) $\sup_i \frac{|p_i + \eta_i|}{\log_+(q_i)} \leqq C + D\lambda \log_+(\lambda)$,

b) $\sup_i \frac{\left|\sum_{j\neq i} F(q_i + \xi_i - q_j - \xi_j)\right|}{\log_+(q_i)} \leqq C + D\lambda \log_+(\lambda)$

whenever $\|\zeta\|_x \leqq \lambda$ and $\lambda \geqq 1$. (We have introduced λ just to avoid having to discuss separately what happens for $\|\zeta\|_x$ small).

We have already made the necessary momentum estimate (3.11). To get the estimate on the forces, we first write:

$$\left|\sum_{j\neq i} F(q_i + \xi_i - q_j - \xi_j)\right| \leqq M \# \{j : q_j + \xi_j \in [q_i + \xi_i - R, q_i + \xi_i + R]\}.$$

But

$$\log_+(|q_i + \xi_i| + R) \leqq \log_+(q_i) + \log_+(\lambda \log_+(q_i)) + \log_+(R) \leqq 2\log_+(q_i) + \log_+(\lambda) + \log_+(R).$$

Hence, here are constants C and D such that

$$\left|\sum_{j\neq i} F(q_i + \xi_i - q_j - \xi_j)\right| \leqq [C + D\lambda \log_+(\lambda)] \log_+(q_i),$$

so inequality b) is satisfied and the proposition is proved.

We will now make a brief digression to show in a heuristic way how the norm estimates of Proposition 3.3 imply that the differential equations cannot drive infinitely many particles into a finite region in finite time. Although the argument we will give is not a necessary part of the proof of the main theorem, it illuminates the role played by the choice of a logarithmic rate of growth of velocities and densities in the proof of a global existence theorem; we will also obtain an intermediate result needed in § 4.

From the differential equation $\frac{d\zeta(t)}{dt} = A_x(\zeta(t))$ and the initial condition $\zeta(0) = 0$, it is at least plausible that the inequality

$$\|\zeta(t)\|_x \leqq \int_0^t d\tau \, \|A_x(\zeta(\tau))\|_x$$

holds for $t \geqq 0$. Using the estimate

$$\|A_x(\zeta)\|_x \leqq C + D\, \|\zeta\|_x \log_+(\|\zeta\|_x)$$

we get:

$$\|\zeta(t)\|_x \leqq \int_0^t d\tau\, [C + D\|\zeta(\tau)\|_x \log_+(\|\zeta(\tau)\|_x)].$$

Hence, if $h(t)$ is the solution of the integral equation:

$$h(t) = \int_0^t d\tau\, [C + Dh(\tau) \log_+(h(\tau))], \tag{3.12}$$

it is again at least plausible that

$$\|\zeta(t)\|_x \leqq h(t)\,,$$

for all $t \geqq 0$ for which $h(t)$ is defined. Thus, to show that $\|\zeta(t)\|_x$ cannot go to infinity in a finite time, it suffices to prove that $h(t)$ is defined for all t, i.e., that the solution of (3.12) does not go to infinity in a finite time. But by elementary calculus it is easily seen that $h(t)$ is given implicitly by

$$t = \int_0^h \frac{ds}{C + Ds \log_+(s)}$$

and that, since $1/s \log s$ is not integrable at infinity, $h(t)$ does not go to infinity unless t does also. Thus, we have an *a priori* estimate on the norm of $\zeta(t)$ valid for all t, so we can expect to be able to prove a global existence theorem if we can prove a local one.

If, instead of allowing density and velocity fluctuations to increase like the logarithm of the distance from the origin, we allow a faster increase (e.g., like some power of the distance), we can carry through most of the constructions and estimates of this section. However, $\|A(\zeta)\|$ increases more rapidly with $\|\zeta\|$ in this case; the reciprocal of the bound is no longer non-integrable at infinity; and our technique for proving a global existence theorem fails. The choice of a logarithmic growth rate is thus to a large extent determined by two conflicting requirements: on the one hand, if we take a growth rate which is significantly faster, we are unable to prove a global existence theorem; on the other hand, if we take a growth rate which is significantly slower, we do not get enough allowed configurations for our intended applications to statistical mechanics.

§ 4. Proof of the Main Theorem

If the non-linear operator A_x satisfied a norm Lipschitz condition on each bounded set in $\mathscr{Y}_x$, standard theorems would enable us to conclude the existence and uniqueness of solutions of the equation:

$$\frac{d\zeta(t)}{dt} = A_x(\zeta(t))\,.$$

Unfortunately, the operator A_x almost never satisfies a norm Lipschitz condition (no matter how regular the potential is assumed to be), and it is not even norm-continuous in general. We will show, however, that A_x satisfies a Lipschitz condition of a very special kind in the product topology on $\mathscr{Y}_x$, and that this Lipschitz condition allows the standard existence proofs to be carried out almost exactly as in the Banach-space case.

To simplify the notation in this section, we will assume that we are dealing with a definite initial configuration $x = (q_i, p_i)$, and we will therefore write $\|\zeta\|$ instead of $\|\zeta\|_x$; $\mathscr{Y}$ instead of $\mathscr{Y}_x$, and A instead of A_x.

For any positive real number m, define a semi-norm ${}_m\|\ \|$ on $\mathscr{Y}$ by

$$\left.\begin{aligned} {}_m\|\zeta\| &= \sup\left\{\frac{|\xi_i| \vee |\eta_i|}{\log_+(q_i)} : |q_i| \leqq m\right\} \text{ if this set is non-empty} \\ &= 0 \text{ otherwise} . \end{aligned}\right\} \tag{4.1}$$

Evidently, the set of semi-norms $\{{}_m\|\ \|\}$ defines the product topology on $\mathscr{Y}$; furthermore,

$$\|\zeta\| = \sup {}_m\|\zeta\| .$$

The following lemma gives the Lipschitz condition satisfied by A:

Lemma 4.1. *Let F satisfy the hypotheses of Theorem 2.1, and let a real number d be given. Then there exists a constant B such that, for any $\alpha > 1$, there exists an m_0 such that, for all $m \geqq m_0$ and all ζ, ζ' with $\|\zeta\| \leqq d$, $\|\zeta'\| \leqq d$, we have:*

$$ {}_m\|A(\zeta) - A(\zeta')\| \leqq B \log_+(m)\, {}_{\alpha m}\|\zeta - \zeta'\| . \tag{4.2}$$

Some interpretation of this lemma may by helpful. What is asserted is that, on any norm-bounded set in $\mathscr{Y}$ (the ball of radius d), the m-norm of $A(\zeta) - A(\zeta')$ may be majorized by a constant multiple of the larger αm-norm of $\zeta - \zeta'$, provided that m is large enough. The "constant" can be taken to increase no faster than logarithmically with m, and to be independent of α.

Proof. By the definition of A and ${}_m\|\zeta\|$,

$$ {}_m\|A(\zeta) - A(\zeta')\| = \sup\left\{\frac{|\eta_i - \eta_i'| \vee |\Delta F_i|}{\log_+(q_i)} : |q_i| \leqq m\right\},$$

where

$$\Delta F_i = \sum_{j \neq i} [F(q_i + \xi_i - q_j - \xi_j) - F(q_i + \xi_i' - q_j - \xi_j')] . \tag{4.3}$$

Since

$$\frac{|\eta_i - \eta_i'|}{\log_+(q_i)} \leqq {}_m\|\zeta - \zeta'\| \quad (\text{for } |q_i| \leqq m) ,$$

we have only to estimate ΔF_i.

We first choose m_0 so that, if $m \geqq m_0$, and if $|q_j| \geqq \alpha m$, $|q_i| \leqq m$, $\|\zeta\| \leqq d$, then

$$|q_i + \xi_i - q_j - \xi_j| > R .$$

(Here, R is some number such that $F(q) = 0$ for $|q| \geqq R$.) This can be done by choosing m_0 so that

$$2d \cdot \log_+(\alpha m_0) + R < (\alpha - 1)\, m_0 .$$

Next, using Lemma 3.1, we see that there is an E such that

$$\#\{j : |q_i + \xi_i - q_j - \xi_j| \leqq R\} \leqq E \log_+(q_i) \tag{4.5}$$

whenever $\|\zeta\| \leqq d$.

In the sum defining ΔF_i, we can evidently omit all j's such that $F(q_i + \xi_i - q_j - \xi_j)$ and $F(q_i + \xi'_i - q_j - \xi'_j)$ are both zero. By (4.5), the number of terms left is no greater than $2E \log_+(q_i)$. We estimate each remaining term using the Lipschitz condition satisfied by F; thus:

$$|F(q_i + \xi_i - q_j - \xi_j) - F(q_i + \xi'_i - q_j - \xi'_j)| \leqq K[|\xi_i - \xi'_i| + |\xi_j - \xi'_j|] .$$

But now, if $m \geqq m_0$ and $|q_i| \leqq m$, (4.4) implies that $|q_j| \leqq \alpha m$ (or else the term in question would have been zero). Hence

$$|\xi_j - \xi'_j| \leqq \log_+(\alpha m)\, {}_{\alpha m}\|\zeta - \zeta'\| ,$$

and

$$|\Delta F_i| \leqq 2E \log_+(q_i) \cdot K \cdot [\log_+(m) + \log_+(\alpha m)] \cdot {}_{\alpha m}\|\zeta - \zeta'\| .$$

This inequality immediately implies the lemma.

We can now proceed to construct the solution of the equations of motion. It is easy to see that the differential Eq. (3.2) and the boundary condition $\zeta(0) = 0$ are equivalent to the integral equation

$$\zeta(t) = \int_0^t A(\zeta(\tau))\, d\tau , \tag{4.6}$$

where the integral is to be evaluated co-ordinate by co-ordinate. We will solve this equation by sucessive approximations: Let

$$\zeta_0(t) = 0 ; \qquad \zeta_{n+1}(t) = \int_0^t A(\zeta_n(\tau))\, d\tau \quad \text{for} \quad n \geqq 0 . \tag{4.7}$$

Proposition 4.2. *Let F satisfy the hypotheses of Theorem* 2.1, *and let $\zeta_n(t)$ be defined by* (4.7). *Then:*

1. *For each t, $\zeta_n(t)$ converges in the product topology on $\mathscr{Y}$ to a limit $\zeta(t)$.*

2. *For any m, ${}_m\|\zeta_n(t) - \zeta(t)\|$ converges to zero as n goes to infinity; the convergence is uniform in t on any bounded set.*

3. *The function $\zeta(t)$ is a solution of* (4.6).

4. *$\|\zeta(t)\|$ is a locally bounded function of t; moreover $\|\zeta_n(t)\|$ is bounded in t on any bounded interval, uniformly in n.*

Proof. We will consider only $t \geqq 0$; the proof for $t < 0$ is obtained from the argument we give here by changing some signs. Let C, D be as in Proposition 3.3, i.e., such that

$$\|A(\zeta)\| \leqq C + D\|\zeta\| \log_+(\|\zeta\|) .$$

Let h be the solution of the integral equation

$$h(t) = \int_0^t d\tau [C + Dh(\tau) \log_+(h(\tau))] ;$$

we saw in § 3 that $h(t)$ is defined for all positive t.

We now claim that

$$\|\zeta_n(t)\| \leqq h(t)$$

for all $t \geqq 0$. The assertion is clearly true for $n = 0$. On the other hand, it is easy to see that

$$\|\zeta_{n+1}(t)\| \leqq \int_0^t d\tau \, \|A(\zeta_n(\tau))\|$$

$$\leqq \int_0^t d\tau [C + D \, \|\zeta_n(\tau)\| \log_+(\|\zeta_n(\tau)\|)] \, .$$

Hence, if $\|\zeta_n(\tau)\| \leqq h(\tau)$, we have

$$\|\zeta_{n+1}(t)\| \leqq \int_0^t d\tau [C + Dh(\tau) \log_+(h(\tau))] = h(t) \, ,$$

and (4.8) follows by induction.

Now for any $T > 0$, apply Lemma 4.1 to get B such that

$${}_m\|A(\zeta) - A(\zeta')\| \leqq B \log_+(m) \, {}_{\alpha m}\|\zeta - \zeta'\| \, ,$$

whenever $\|\zeta\|$ and $\|\zeta'\|$ are not larger than $h(T)$ and m is large enough. Choose $\alpha > 1$ so that $B \log(\alpha) T < 1$. If m is large enough and if $0 \leqq t \leqq T$, we have:

$${}_m\|\zeta_{n+1}(t) - \zeta_n(t)\| \leqq \int_0^t d\tau \, {}_m\|A(\zeta_n(\tau)) - A(\zeta_{n-1}(\tau))\|$$

$$\leqq B \log_+(m) \int_0^t d\tau \, {}_{\alpha m}\|\zeta_n(\tau) - \zeta_{n-1}(\tau)\| \, .$$

Repeating this argument n times, we get:

$${}_m\|\zeta_{n+1}(t) - \zeta_n(t)\| \leqq B^n \log_+(m) \ldots \log_+(\alpha^{n-1} m)$$

$$\cdot \int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \ldots \int_0^{\tau_{n-1}} d\tau_n \, {}_{\alpha^n m}\|\zeta_1(\tau)\| \, .$$

Since

$${}_{\alpha^n m}\|\zeta_1(t)\| \leqq \|\zeta_1(t)\| \leqq h(T) \, ,$$

we get finally

$${}_m\|\zeta_{n+1}(t) - \zeta_n(t)\| \leqq \frac{B^n \cdot \log_+(m) \ldots \log_+(\alpha^{n-1} m) \cdot h(T) \cdot T^n}{n!} \, .$$

The ratio of succeeding terms on the right is

$$\frac{B \log_+(\alpha^n T) \cdot T}{n+1} \, ;$$

as n goes to infinity, this ratio approaches $B \log(\alpha) \, T$ which, by the choice of α, is less than one. Hence

$$\sum_{n=0}^{\infty} {}_m\|\zeta_{n+1}(t) - \zeta_n(t)\| \, ,$$

converges uniformly in t on $[0, T]$. This proves statements 1., 2., and 4. of Proposition 4.2. Statement 3. follows from statements 2. and 4., and the Lipschitz condition.

Remark 4.3. All the above estimates have been made for a definite choice of the initial configuration x. It is easy to see that the convergence of ${}_m\|\zeta_n(t) - \zeta(t)\|$ to zero and the bound on $\|\zeta_n(t)\|$ can be taken to be uniform in x on $\{x : |x| \leqq \delta\}$ for any real number δ.

Proposition 4.4. *Let $\zeta(t)$ and $\zeta'(t)$ be solutions of the integral Eq.* (4.6); *suppose $\|\zeta(t)\| \leqq M$ and $\|\zeta'(t)\| \leqq M$ for $|t| \leqq T$. Then $\zeta(t) = \zeta'(t)$ for $|t| \leqq T$.*

Proof. Again we consider only $t \geqq 0$. Choose B so that the Lipschitz condition (4.2) holds for $\|\zeta\| \leqq M$; $\|\zeta'\| \leqq M$. Choose $\alpha > 1$ so that $B\,T \log(\alpha) < 1$; then for m large enough and $0 \leqq t \leqq T$

$$\begin{aligned} {}_m\|\zeta(t) - \zeta'(t)\| &\leqq \int_0^t d\tau\, {}_m\|A(\zeta(\tau)) - A(\zeta'(\tau))\| \\ &\leqq B \log_+(m) \int_0^t d\tau\, {}_{\alpha m}\|\zeta(\tau) - \zeta'(\tau)\| . \end{aligned}$$

Iterating n times and using the fact that $\|\zeta(\tau) - \zeta'(\tau)\| \leqq 2\,M$ for $0 \leqq \tau \leqq T$, we get:

$${}_m\|\zeta(t) - \zeta'(t)\| \leqq \frac{B^n T^n \log_+(m) \ldots \log_+(\alpha^{n-1} m) \cdot 2M}{n!} .$$

As before, the ratio of succeeding terms on the right approaches a limit which is less than one, so

$${}_m\|\zeta(t) - \zeta'(t)\| = 0 .$$

This is true for any large m, so $\zeta(t) = \zeta'(t)$.

Combining Propositions 4.2 and 4.4 gives Theorem 2.1′ which, by the discussion in § 3, is equivalent to Theorem 2.1.

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