Commun. math. Phys. 11, 257—292 (1969)

The Classical Mechanics of One-Dimensional Systems of Infinitely Many Particles

II. Kinetic Theory

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Received August 1, 1968

Abstract. We apply the existence theorem for solutions of the equations of motion for infinite systems to study the time evolution of measures on the set of locally finite configurations of particles. The set of allowed initial configurations and the time evolution mappings are shown to be measurable. It is shown that infinite volume limit states of thermodynamic ensembles at low activity or for positive potentials are concentrated on the set of allowed initial configurations and are invariant under the time evolution. The total entropy per unit volume is shown to be constant in time for a large class of states, if the potential satisfies a stability condition.

§ 1. Introduction

In [1], we proved an existence and uniqueness theorem for solutions of the equations of motion for systems of infinitely many particles. In this article, we will apply this theorem to the study of the time-evolution of states of classical statistical mechanics. Let us recall briefly the notation and results of [1]. We denote by $\mathscr{X}$ the set of locally finite configurations of labelled particles and by $[\mathscr{X}]$ the corresponding set of configurations of unlabelled particles. A state of classical statistical mechanics is a probability measure on $[\mathscr{X}]$ invariant under space translations. Let $\hat{\mathscr{X}}$ denote the set of labelled configurations satisfying conditions 1) and 2) of [1]. Theorem 2.1 of [1] asserts the existence of a solution of the equations

$$\frac{dq_i(t)}{dt} = p_i(t)$$

$$\frac{dp_i(t)}{dt} = \sum_{j \neq i} F(q_i(t) - q_j(t))$$

and the initial conditions

$$q_i(0) = q_i\,, \quad p_i(0) = p_i\,,$$

provided that F has compact support and satisfies a Lipschitz condition and that the initial configuration (q_i, p_i) is in $\hat{\mathscr{X}}$; it also asserts the uni-

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queness of the solution in the class of trajectories satisfying a certain regularity condition. This theorem enables us to define a one-parameter group T^t of evolution operators mapping $[\hat{\mathscr{X}}]$ (the set of unlabelled configurations corresponding to the set $\hat{\mathscr{X}}$ of labelled configurations) onto itself. If the mappings T^t are measurable, they define a time evolution of measures on $[\hat{\mathscr{X}}]$ and in particular of states of classical statistical mechanics which are concentrated on $[\hat{\mathscr{X}}]$ (i.e., for which $[\mathscr{X}]\setminus[\hat{\mathscr{X}}]$ has measure zero). This time evolution will be the object of our investigations.

In § 2, we develop some notation and tools which will be needed in the course of this article, and we restate in a convenient form the results we will need from [1]. Section 3 is devoted to some measurability questions which are technically important if not very interesting; we prove that $[\hat{\mathscr{X}}]$ is a Borel subset of $[\mathscr{X}]$ and that the time-evolution mapping $(t, x) \longmapsto T^t x$ is a Borel mapping from $\boldsymbol{R} \times [\hat{\mathscr{X}}]$ to $[\hat{\mathscr{X}}]$. In § 4 we show that for a state ϱ of classical statistical mechanics to be concentrated on $[\hat{\mathscr{X}}]$ it is sufficient that ϱ:

i) has a Maxwellian velocity distribution,

ii) has correlation functions $\bar{\varrho}_n(q_1, \ldots, q_n)$ of all orders (see [2]) admitting a majorization of the form

$$\bar{\varrho}_n(q_1, \ldots, q_n) \leqq \lambda^n$$

with λ independent of $n, q_1, \ldots, q_n$.

These two conditions are satisfied if ϱ is an infinite volume limit state obtained from the grand canonical ensemble at small activity and, for a non-negative potential, at arbitrarily large activity.

In § 5, we prove an approximation theorem which will be our main technical device for the rest of this article and which asserts that the time evolution of the part of an infinite system contained in a bounded region can be arbitrarily well approximated by the evolution of the corresponding part of a system with a large but finite number of particles. We apply this approximation theorem in § 6 to show that states obtained by taking infinite volume limits of grand canonical ensembles with a given twice-continuously-differentiable stable[1] potential of compact support are invariant under the time-evolution defined by that potential, provided again either that the activity is small or the potential non-negative.

In § 7, we show that the entropy per unit volume is conserved by the time evolution. Here, for the first time in our investigations, thermodynamic stability properties of the potential defining the interparticle

[1] A function Φ on $\boldsymbol{R}$ is a *stable* potential if there exists B such that, for all n and all $q_1, \ldots, q_n$,

$$\sum_{1 \leqq i < j \leqq n} \Phi(q_i - q_j) \geqq -nB .$$

force play an essential role. We assume that the interparticle force is the derivative of a potential Φ with compact support which may be written in the form

$$\Phi = \Phi_1 + \Phi_2$$

with Φ_1 stable and of compact support and Φ_2 non-negative, continuous, and strictly positive at the origin. We consider a state ϱ which is concentrated on $[\hat{\mathscr{X}}]$ and which has:

i) finite mean kinetic energy in $(0, 1)$,

ii) finite mean square number of particles in $(0, 1)$,

and we let ϱ^t denote the time-evolved measure $\varrho \circ T^{-t}$.
Under all these hypotheses we show that, for any t, the entropy per unit volume of ϱ^t is equal to that of ϱ.

§ 2. Preliminaries

We will need, unfortunately, a rather complicated set of tools; these are developed in this section. Most of the results given here are of limited originality. In particular, Sections 2.1 and 2.2 draw heavily on [2], and Section 2.3 on [3].

2.1. The Space of Locally Finite Configurations

Recall that the set $\mathscr{X}$ of locally finite labelled configurations is defined as the set of all mappings (q_i, p_i) from an index set [which is either $(1, 2, 3, \ldots, n)$ or $(1, 2, 3, \ldots)$] to $\boldsymbol{R} \times \boldsymbol{R}$, subject to the restriction that $\lim_{i\to\infty} |q_i| = \infty$ if the index set is not finite, and that $[\mathscr{X}]$ denotes the set of all equivalence classes of such mappings, two mappings being equivalent if they differ only by a permutation of the index set. We will define a topology on $[\mathscr{X}]$ by specifying a class of functions on $[\mathscr{X}]$ and giving $[\mathscr{X}]$ the weakest topology making each function in this class continuous.

Let $\mathscr{K}_1$ denote the set of all continuous real-valued functions f on $\boldsymbol{R} \times \boldsymbol{R}$ whose supports have bounded projections onto the first factor. In other words, a continuous real-valued function f is in $\mathscr{K}_1$ if and only if there is a bounded set $\Lambda \subset \boldsymbol{R}$ such that $f(q, p) = 0$ whenever $q \notin \Lambda$. For f in $\mathscr{K}_1$, we define a function Sf on $[\mathscr{X}]$

$$Sf(x) = \sum_i f(q_i, p_i)$$

if (q_i, p_i) is a representative of x. The sum has only finitely many non-zero terms because of the support properties of f and the local finiteness of x; moreover, $Sf(x)$ evidently does not depend on the choice of the representative (q_i, p_i) of x. We give $[\mathscr{X}]$ the weakest topology making Sf continuous for every f in $\mathscr{K}_1$.

We can give another description of this topology. For any x in $[\mathscr{X}]$, let (q_i, p_i) be a representative of x, and define a measure μ_x on $\boldsymbol{R} \times \boldsymbol{R}$ by

$$\mu_x = \sum_i \delta_{q_i} \otimes \delta_{p_i} .$$

In other words, μ_x is the measure which assigns, to every subset of $\boldsymbol{R} \times \boldsymbol{R}$, the number of particles whose position and momentum lie in that set. It is easy to see that μ_x determines x, so $[\mathscr{X}]$ may be thought of as a set of measures on $\boldsymbol{R} \times \boldsymbol{R}$. From the formula

$$Sf(x) = \int f \, d\mu_x$$

it follows that the topology on $[\mathscr{X}]$ is just the weak topology on measures defined by the space $\mathscr{K}_1$ of functions. Using measure theory, one proves the following:

Lemma 2.1. *The set of positive linear functionals on $\mathscr{K}_1$ of the form $f \longmapsto Sf(x)$ is closed in the weak topology in the algebraic dual of $\mathscr{K}_1$.*

By TYCHONOV's theorem, this implies the following compactness criterion:

Proposition 2.2. *A closed subset X of $[\mathscr{X}]$ is compact if and only if each Sf is bounded on X.*

We want to transform this criterion into one which is more directly applicable. Before doing this, we will define some notation. For any bounded set $\Lambda \subset \boldsymbol{R}$, we define three functions on $[\mathscr{X}]$:

$$N_\Lambda(x) = \# \{i : q_i \in \Lambda\} ,$$

$$K.E._\Lambda(x) = \frac{1}{2} \sum \{p_i^2 : q_i \in \Lambda\} ,$$

$$\overline{P}_\Lambda(x) = 0 \vee \sup \{|p_i| : q_i \in \Lambda\} .$$

Here, as usual, (q_i, p_i) is a representative of x. N_Λ, $K.E._\Lambda$, and $\overline{P}_\Lambda$ are respectively the number of particles in Λ, the total kinetic energy of the particles in Λ, and the maximum velocity of any particle in Λ. The following proposition is easily obtained from Proposition 2.2.

Proposition 2.3. *A closed subset X of $[\mathscr{X}]$ is compact if and only if, for every bounded open set Λ, N_Λ and $\overline{P}_\Lambda$ are bounded on X.*

We also have:

Proposition 2.4. *Let Λ be a bounded open subset of $\boldsymbol{R}$. Then N_Λ, $K.E._\Lambda$, and $\overline{P}_\Lambda$ are lower semi-continuous functions on $[\mathscr{X}]$.*

Proof. Let ψ_n be an increasing sequence of non-negative continuous functions on $\boldsymbol{R} \times \boldsymbol{R}$ converging pointwise to the characteristic function of $\Lambda \times \boldsymbol{R}$. Then $N_\Lambda = \sup_n S\psi_n$; since each $S\psi_n$ is continuous by definition, N_Λ is lower semi-continuous. A similar argument shows that $K.E._\Lambda$ and that

$$\overline{P}_\Lambda^n = [\sum \{|p_i|^n : q_i \in \Lambda\}]^{1/n}$$

are lower semi-continuous. Every point of $[\mathscr{X}]$ has a neighborhood on which N_Λ is bounded; on such a neighborhood, $\bar{P}^n_\Lambda$ converges uniformly to $\bar{P}_\Lambda$ as n goes to infinity. Hence, $\bar{P}_\Lambda$ is lower semi-continuous.

The following proposition is proved by showing that the topology of $[\mathscr{X}]$ may be defined by a suitably chosen countable subset of the functions of the form Sf, f in $\mathscr{K}_1$.

Proposition 2.5. *The space* $[\mathscr{X}]$ *is a Polish space, i.e. it is separable and its topology is compatible with a metric with respect to which it is complete.*

2.2. Borel Measures on $[\mathscr{X}]$

For any bounded non-empty set $\Lambda \subset \boldsymbol{R}$, we let $[\mathscr{X}](\Lambda)$ denote the set of configurations of finitely many unlabelled particles in Λ:

$$[\mathscr{X}](\Lambda) = \coprod_{n=0}^{\infty} (\Lambda \times \boldsymbol{R})^n_{\text{symm}}$$

where $\coprod$ denotes disjoint union and $(\Lambda \times \boldsymbol{R})^n_{\text{symm}}$ the symmetric product of n copies of $\Lambda \times \boldsymbol{R}$, i.e., the set of all equivalence classes of n-tuples of points in $\Lambda \times \boldsymbol{R}$, two n-tuples being equivalent if they differ only by a permutation of their labels. Since $[\mathscr{X}](\Lambda)$ is a disjoint union of quotients of products of copies of $\Lambda \times \boldsymbol{R}$, it has a natural topology. This topology in itself is not very useful, and we will use it only to define the Borel subsets of $[\mathscr{X}](\Lambda)$.

Given any bounded non-empty subset Λ of $\boldsymbol{R}$, there is a natural mapping from $[\mathscr{X}]$ to $[\mathscr{X}](\Lambda)$ which simply forgets about all particles outside of Λ. We will refer to this mapping as the *restriction* from $\boldsymbol{R}$ to Λ; it is a Borel mapping[2] if Λ is a Borel set[3].

Let $[a, b)$ be a non-trivial bounded interval in $\boldsymbol{R}$; then we can decompose $\boldsymbol{R}$ into a countable union of disjoint translates of $[a, b)$. It is easy to see that this decomposition gives a bijective mapping from $[\mathscr{X}]$ to $\coprod_{n=-\infty}^{\infty} [\mathscr{X}]([a_n, b_n))$ $(a_n = a + n(b-a),\ b_n = b + n(b-a))$ and that this mapping is in fact a Borel isomorphism if $\coprod_{n=-\infty}^{\infty} [\mathscr{X}]([a_n, b_n))$ is given the product topology. Thus, we have a fairly simple description of the Borel structure on $[\mathscr{X}]$.

[2] A mapping from one topological space to another is *Borel* if the inverse image of every Borel set is Borel.

[3] The restriction mapping is not continuous: A continuous trajectory in $[\mathscr{X}]$ can have a varying number of particles in Λ (since particles can move in and out of Λ), whereas the number of particles is constant on continuous trajectories in $[\mathscr{X}](\Lambda)$. This is why the topology we have defined on $[\mathscr{X}](\Lambda)$ is not very useful. If Λ is open, a better topology can be defined on $[\mathscr{X}](\Lambda)$ by imitating the definition of the topology on $[\mathscr{X}]$; this topology makes the restriction mapping continuous and has the same Borel sets as the topology we are using.

The representation of $[\mathscr{X}]$ as a product space gives a useful technique for constructing measures on $[\mathscr{X}]$. Let μ be a probability measure on $[\mathscr{X}]([a, b))$. For each n, translation by $n(b-a)$ gives a Borel isomorphism of $[\mathscr{X}]([a_n, b_n))$ with $[\mathscr{X}]([a, b))$, and we get therefore a probability measure on each $[\mathscr{X}]([a_n, b_n))$. Taking the product of all these measures gives a probability measure on $[\mathscr{X}]$. We will refer to this procedure for passing from a measure on $[\mathscr{X}]([a, b))$ to a measure on $[\mathscr{X}]$ as the *product measure construction*. It gives a measure which is periodic under translations, with period $b-a$.

We can apply this construction in particular to thermodynamic ensembles. Thus, let a stable two-body potential Φ, an inverse temperature β, and a chemical potential μ be given. For any interval $[a, b)$, we define a measure on $([a, b) \times \boldsymbol{R})^n$ as:

$$\frac{1}{n!} \exp\left\{\beta\left[\mu - \frac{1}{2}\sum_i p_i^2 - \sum_{i<j} \Phi(q_i - q_j)\right]\right\} dq_1, \ldots, dq_n\, dp_1, \ldots, dp_n$$

(where $dq_i\, dp_i$ is Lebesgue measure on $[a, b) \times \boldsymbol{R}$). Because of the stability of the potential Φ, this collection of measures defines a finite measure on $[\mathscr{X}]([a, b))$, and by normalizing we get a probability measure, which we will call the grand canonical ensemble on $[a, b)$. Applying the product measure construction to this measure gives a probability measure on $[\mathscr{X}]$ which, physically, corresponds to the grand canonical ensemble for the infinite system with insulating walls at the points a_n.

Let $\mathscr{M}^1[\mathscr{X}]$ denote the set of Borel probability measures on $[\mathscr{X}]$. We will introduce two topologies on $\mathscr{M}^1[\mathscr{X}]$, each of which is a weak topology defined by regarding $\mathscr{M}^1[\mathscr{X}]$ as a subset of the dual of a space of bounded measurable functions on $[\mathscr{X}]$. Thus, let $\mathfrak{A}$ be the C^* algebra of functions on $[\mathscr{X}]$ generated by the set of all functions of the form $\varphi(Sf_1, \ldots, Sf_k)$, where $f_1, \ldots, f_k$ belong to $\mathscr{K}_1$ and φ is a bounded continuous function on $\boldsymbol{R}^k$. The C^* algebra $\mathfrak{A}$ defines a topology on $\mathscr{M}^1[\mathscr{X}]$ which we will refer to as the $\mathfrak{A}$ topology. When we speak of convergence in $\mathscr{M}^1[\mathscr{X}]$ without specifying a topology, we will always mean convergence with respect to the $\mathfrak{A}$ topology. It is sometimes useful to consider the topology defined by the C^* algebra $\mathfrak{A}_\infty$ generated by all functions obtained by composing a bounded Borel function on $[\mathscr{X}](\Lambda)$ (Λ some bounded Borel set) with the restriction mapping from $[\mathscr{X}]$ to $[\mathscr{X}](\Lambda)$. We will refer to this topology as the $\mathfrak{A}_\infty$ topology; it is evidently strictly stronger than the $\mathfrak{A}$ topology.

We may regard $[\mathscr{X}]$ as a subset of the spectrum of $\mathfrak{A}$, and it may be seen that the $\mathfrak{A}$ topology on $[\mathscr{X}]$ coincides with the initial topology. Since $\mathfrak{A}$ is defined as an algebra of functions on $[\mathscr{X}]$, it is clear that $[\mathscr{X}]$ is dense in the spectrum of $\mathfrak{A}$. The following proposition further clarifies the way $[\mathscr{X}]$ lies in the spectrum of $\mathfrak{A}$.

Proposition 2.6. *There exists a family* $(\varphi_{m,n})$ *of elements of* $\mathfrak{A}$ $0 \leqq \varphi_{m,n} \leqq \varphi_{m+1,n} \leqq 1$, *such that, if the* $\varphi_{m,n}$ *are regarded as functions on the spectrum of* $\mathfrak{A}$, *then the characteristic function of* $[\mathscr{X}]$ *is*

$$\inf_n \sup_m \varphi_{m,n} .$$

In particular $[\mathscr{X}]$ *is a Baire set*[4] *in the spectrum of* $\mathfrak{A}$.

This proposition is due to Ruelle ([2], Proposition 4.2 and Corollary 4.4). The proof in this reference is inseparable from other and more complicated considerations; for the convenience of the reader we will give here a direct proof. The idea of the proof is simple: A point of the spectrum of $\mathfrak{A}$ which does not belong to $[\mathscr{X}]$ heuristically represents a situation in which some bounded interval, which we can take to be of the form $(-n, n)$, contains either infinitely many particles or a particle with infinite velocity; we will therefore construct $\varphi_{m,n}$ so that $\lim_{m \to \infty} \varphi_{m,n} = 1$ on $[\mathscr{X}]$ but such that $\varphi_{m,n} = 0$ for all m if there are infinitely many particles, or a particle with infinite velocity, in $(-n, n)$.

Let χ be a continuous non-increasing function on $\boldsymbol{R}$ such that $\chi(t) = 0$ for $t \geqq 1$ and $\chi(t) = 1$ for $t \leqq 0$. Let ψ_n be a continuous non-negative function on $\boldsymbol{R}$ which has compact support and which is equal to one on $(-n, n)$. We will prove that we may take:

$$\varphi_{m,n}(x) = \chi(Sf_n(x) - m) ,$$

with $f_n(q, p) = \psi_n(q)(1 + p^2)$. Since f_n is in $\mathscr{K}_1$ and χ is bounded and continuous, it follows from the definition that $\varphi_{m,n}$ belongs to $\mathfrak{A}$. Since $0 \leqq \chi \leqq 1$, and since χ is non-decreasing, we have

$$0 \leqq \varphi_{m,n} \leqq \varphi_{m+1,n} \leqq 1 .$$

Clearly, $\lim_{m \to \infty} \varphi_{m,n}(x) = 1$ for all x in $[\mathscr{X}]$.

It remains to be shown that the $\varphi_{m,n}$'s have the desired property of separating $[\mathscr{X}]$ from the rest of the spectrum of $\mathfrak{A}$. Thus, let x be a point of the spectrum of $\mathfrak{A}$ which does not belong to $[\mathscr{X}]$, and let x_α be a net in $[\mathscr{X}]$ converging to x. We claim that, for some n, $\limsup_\alpha Sf_n(x_\alpha) = \infty$.

If this were not the case, it would follow from Proposition 2.3 that the net (x_α) has a cluster point in $[\mathscr{X}]$, and this would contradict the assumption that $\lim_\alpha x_\alpha \notin [\mathscr{X}]$. Thus, for that value of n, we must have

$$\varphi_{m,n}(x) = \lim_\alpha \varphi_{m,n}(x_\alpha) = 0$$

[4] On any topological space, we define the set of Baire functions to be the smallest set of functions containing the continuous functions and closed under pointwise limits, and the Baire sets to be those sets whose characteristic functions are Baire functions. Every Baire set is also a Borel set; the converse is true if the topological space in question is metrizable.

for all m. Hence, $\inf_n \sup_m \varphi_{m,n} = 0$ on $\complement [\mathscr{X}]$, whereas we have seen that $\inf_n \sup_m \varphi_{m,n} = 1$ on $[\mathscr{X}]$.

Let $E(\mathfrak{A})$ denote the set of states of $\mathfrak{A}$; $E(\mathfrak{A})$ may be identified with the set of regular Borel probability measures on the spectrum of $\mathfrak{A}$. Any Borel probability measure on $[\mathscr{X}]$ may be regarded as a Borel probability measure on the spectrum of $\mathfrak{A}$ which assigns measure zero to the complement of $[\mathscr{X}]$. Moreover, such a measure is automatically regular. To see this we remark first that, since any finite Borel measure on a compact space is regular on the Baire sets [4], it suffices to show that any Borel set in $[\mathscr{X}]$ is a Baire set in the spectrum of $\mathfrak{A}$. This last assertion follows from the fact that $[\mathscr{X}]$ is a Baire set in the spectrum of $\mathfrak{A}$ and the fact, easily verified, that there is a countable set in $\mathfrak{A}$ which generates the topology of $[\mathscr{X}]$. We may therefore, when convenient, regard $\mathscr{M}^1[\mathscr{X}]$ as a subset of $E(\mathfrak{A})$. It is easy to construct integrals of functions with values in $E(\mathfrak{A})$; we will need some technical results which assure us that, if we consider such a function whose values actually lie in $\mathscr{M}^1[\mathscr{X}]$, then the integral is also a measure on $[\mathscr{X}]$.

Proposition 2.7. *Let (X, ν) be a probability measure space and $x \longmapsto \varrho_x$ a mapping from X to $E(\mathfrak{A})$ such that $x \longmapsto \varrho_x(\psi)$ is measurable for every ψ in $\mathfrak{A}$. Define a state ϱ of $\mathfrak{A}$ by*

$$\varrho(\psi) = \int d\nu(x)\, \varrho_x(\psi)\,. \tag{2.1}$$

Then, for all bounded Baire functions f on the spectrum of $\mathfrak{A}$, $x \longmapsto \int f\, d\varrho_x$ is measurable and

$$\int f\, d\varrho = \int d\nu(x) \int f\, d\varrho_x\,.$$

Proof. Let $\mathscr{F}$ denote the class of all bounded Baire functions on the spectrum of $\mathfrak{A}$ for which the proposition holds. Since the continuous functions on the spectrum of $\mathfrak{A}$ are just the elements of $\mathfrak{A}$, $\mathscr{F}$ contains the continuous functions by the definition of ϱ. On the other hand, if (f_n) is a uniformly bounded sequence of functions in $\mathscr{F}$ converging pointwise to f, then a double application of the dominated convergence theorem shows that f is in $\mathscr{F}$. Hence, $\mathscr{F}$ contains all bounded Baire functions.

Corollary 2.8. *Let the notation be as in Proposition 2.7. Then ϱ belongs to $\mathscr{M}^1[\mathscr{X}]$ if and only if ϱ_x belongs to $\mathscr{M}^1[\mathscr{X}]$ for almost all x.*

Proof. By Proposition 2.6, the characteristic function $\chi_{[\mathscr{X}]}$ of $[\mathscr{X}]$ is a Baire function. The assertion therefore follows from (2.1) and the remark that ϱ belongs to $\mathscr{M}^1[\mathscr{X}]$ if and only if

$$\int d\varrho\,[1 - \chi_{[\mathscr{X}]}] = 0\,.$$

Corollary 2.8a. *Let the notation be as in Proposition 2.7, but assume that ϱ_x is in $\mathscr{M}^1[\mathscr{X}]$ for almost all x. Let f be a bounded or non-negative*

Borel function on $[\mathscr{X}]$. *Then*

$$\int f\, d\varrho = \int d\nu(x) \int f\, d\varrho_x\,. \tag{2.2}$$

Proof. The assertion for non-negative functions follows from the assertion for bounded functions by the monotone convergence theorem. The assertion for bounded functions follows from the corresponding assertion for characteristic functions of sets, by a standard approximation argument. The assertion for characteristics functions of sets follows from Proposition 2.7 and the fact that every Borel set in $[\mathscr{X}]$ is a Baire set in the spectrum of $\mathfrak{A}$.

If ϱ belongs to $\mathscr{M}^1[\mathscr{X}]$, we define a translated measure $\tau_s\varrho$ by

$$(\tau_s\varrho)(\varphi) = \varrho(\varphi \circ \tau_s)$$

for φ in $\mathfrak{A}$. It is easy to see that, for any φ in $\mathfrak{A}$ and any x in $[\mathscr{X}]$, $s \longmapsto \varphi(\tau_s x)$ is continuous. Hence, by the dominated convergence theorem, $s \longmapsto \tau_s\varrho$ is continuous in the $\mathfrak{A}$ topology. We may therefore construct $1/a \int\limits_0^a ds\, \tau_s\varrho$, which belongs to $\mathscr{M}^1[\mathscr{X}]$. If we temporarily denote this measure by $\bar{\varrho}$, then by Corollary 2.9 we have

$$\int f\, d\bar{\varrho} = \frac{1}{a}\int\limits_0^a ds \left[\int d\varrho(f \circ \tau_s)\right] \tag{2.3}$$

for any semi-bounded Borel function f on $[\mathscr{X}]$. If ϱ is periodic with period a, then $\bar{\varrho}$ is translation invariant. We will refer to this measure as the *average of ϱ over translations*. We can apply this construction in particular when ϱ is obtained by the product measure construction from a measure ϱ_1 on $[\mathscr{X}]([a, b))$; we will refer to the operation of passing from ϱ_1 to this invariant measure as the *averaged product measure construction*.

2.3. Entropy

We will need the notion of the total entropy per unit volume of a measure on $[\mathscr{X}]$ which is periodic under translations. In our discussion we will follow, roughly, the work of Robinson and Ruelle [3]. The infinite volume of momentum space, however, introduces some complications not present in the theory of the configurational entropy.

We will start by defining, abstractly, the entropy of a probability measure with respect to an arbitrary σ-finite measure. Let X be a set and $\mathscr{S}$ a σ-algebra of subsets of X. We will consider the σ-algebra $\mathscr{S}$ to be fixed and suppress it from our notation, i.e., we will speak of measures on X rather than of measures defined on $\mathscr{S}$ and of measurable functions on X rather than of functions measurable with respect to $\mathscr{S}$. Let σ be a σ-finite measure on X. If ϱ is a probability measure on X, we want to define the entropy of ϱ with respect to σ as $-\infty$ if ϱ is not absolutely continuous with respect to σ and as

$$-\int \left(\frac{d\varrho}{d\sigma}\right) \log \left(\frac{d\varrho}{d\sigma}\right) d\sigma$$

if ϱ is absolutely continuous with respect to σ. Unfortunately, this integral need not make any sense; the positive and negative parts of the integrand may both have infinite integrals. If σ is a finite measure, this difficulty does not arise because $x \log x$ is bounded below. If σ is not a finite measure, we must restrict the class of probability measures ϱ that we consider. This we do by choosing a non-negative measurable function ϕ which is rapidly increasing in the sense that $\int e^{-\phi}\, d\sigma < \infty$, and considering only ϱ's such that $\int \phi \, d\varrho < \infty$.

Lemma 2.9. *Let X, σ, ϕ be as above; let f be a non-negative measurable function on X such that $\int f\, d\sigma = 1$ and $\int f \phi \, d\sigma < \infty$. Then the positive part of $-f \log f$ has finite σ-integral, and*

$$-\int f \log f \, d\sigma \leqq \int f . \phi \, d\sigma + \log(\textstyle\int e^{-\phi}\, d\sigma) . \tag{2.4}$$

Proof. The first statement of the lemma follows at once from the identity

$$-f \log f = f . \phi - e^{-\phi} \, (f/e^{-\phi}) \log(f/e^{-\phi}) \tag{2.5}$$

and the integrability of $f . \phi$ and $e^{-\phi}$. The inequality (2.4) is proved by integrating this identity and using the concavity of $-x \log x$.

We now make the following definition: Let ϱ be a probability measure on X such that $\int \phi\, d\varrho < \infty$. Then we define $s(\varrho, \sigma)$, the *entropy of ϱ relative to σ*, by

$$s(\varrho, \sigma) = -\int \left(\frac{d\varrho}{d\sigma}\right) \log \left(\frac{d\varrho}{d\sigma}\right) d\sigma$$

if ϱ is absolutely continuous with respect to σ,

$$= -\infty \text{ otherwise.}$$

(We could have given a more general definition by defining the entropy for any probability measure ϱ for which there exists a ϕ with the desired properties. For our purposes, it will be convenient to work with a fixed ϕ.)

We will denote by $\mathscr{M}^1(X)$ the set of probability measures on X. There is an obvious pairing between $\mathscr{M}^1(X)$ and the space of all bounded measurable functions on X; we will refer to the weak topology induced on the set of measures by this pairing as the $\mathscr{L}^\infty$ topology. Any statements implying a topology on $\mathscr{M}^1(X)$ are to be understood in the $\mathscr{L}^\infty$ topology. If we identify probability measures which are absolutely continuous with respect to σ with elements of $L^1(\sigma)$, then the $\mathscr{L}^\infty$ topology corresponds to the weak topology on $L^1(\sigma)$.

Proposition 2.10. *Let σ be a finite measure, and let N be a real number. Then $s(\varrho, \sigma)$ is an upper semi-continuous function of ϱ and*

$$\{\varrho : s(\varrho, \sigma) \geqq -N\}$$

is compact.

Proof. Using the concavity of $-x \log x$, one checks easily that

$$s(\varrho, \sigma) = \inf\left\{-\sum_i \varrho(A_i) \log \frac{\varrho(A_i)}{\sigma(A_i)}\right\}$$

where the infimum is to be taken over all partitions of X into a finite number of disjoint measurable sets $\{A_1, \ldots, A_n\}$ each of which has strictly positive σ-measure. Since $\varrho \longmapsto \varrho(A_i)$ is continuous, $s(\varrho, \sigma)$ is the infimum of a collection of continuous functions and is therefore upper semi-continuous. In particular, $\{\varrho : s(\varrho, \sigma) \geqq -N\}$ is closed in $\mathscr{M}^1(X)$.

Let $\mathscr{P}_1(\sigma)$ be the set of non-negative elements of $L^1(\sigma)$ with integral one; to complete the proof of the proposition it will suffice to prove that

$$\mathscr{K} = \{f \in \mathscr{P}_1(\sigma) : \int f \log f \, d\sigma \leqq N\}$$

is relatively compact for the weak topology on $L^1(\sigma)$.

To prove this, it is enough to show that

$$\lim_{\sigma(E) \to 0} \int_E f \, d\sigma = 0$$

uniformly for f in $\mathscr{K}$ [5].

Let λ be a real number greater than one. Then

$$\int_E f \, d\sigma = \int_{E \cap \{f \leqq \lambda\}} f \, d\sigma + \int_{E \cap \{f > \lambda\}} f \, d\sigma \leqq \lambda \, \sigma(E) + \int_{\{f > \lambda\}} f \, d\sigma \, .$$

We therefore want to show that

$$\lim_{\lambda \to \infty} \int_{\{f > \lambda\}} f \, d\sigma = 0$$

uniformly for f in $\mathscr{K}$. But

$$N \geqq \int f \log f \, d\sigma \geqq \log(\lambda) \int_{\{f > \lambda\}} f \, d\sigma - \frac{1}{e} \int_X d\sigma \, ,$$

or

$$N + \frac{\sigma(X)}{e} \geqq \log(\lambda) \int_{\{f > \lambda\}} f \, d\sigma \, ,$$

which completes the proof of the proposition.

Proposition 2.11. *Let (X, σ) be a measure space, and let ϕ be a non-negative measurable function on X such that $\int e^{-\alpha\phi} \, d\sigma < \infty$ for all $\alpha > 0$. Let M, N be real numbers. Then $\varrho \longmapsto s(\varrho, \sigma)$ is an upper semi-continuous function on $\{\varrho \in \mathscr{M}^1(X) : \int \phi \, d\varrho \leqq M\}$, and $\{\varrho \in \mathscr{M}^1(X) : \int \phi \, d\varrho \leqq M$ and $s(\varrho, \sigma) \geqq -N\}$ is compact.*

Proof. It follows from the hypotheses on ϕ that there exists a non-negative measurable function $\hat{\phi}$ on X such that $\int e^{-\hat{\phi}} \, d\sigma < \infty$ and such that

$$\lim_{\phi(X)\to\infty} \frac{\hat{\phi}(X)}{\phi(X)} = 0 . \tag{2.6}$$

Let σ' denote the finite measure $e^{-\hat{\phi}}\,\sigma$. Then we have

$$s(\varrho, \sigma) = s(\varrho, \sigma') + \int \hat{\phi}\, d\varrho$$

by (2.5). Proposition 2.10 asserts that $s(\varrho, \sigma')$ is an upper semi-continuous function on $\mathscr{M}^1(X)$. We will prove the upper semi-continuity of $s(\varrho, \sigma)$ by proving that $\varrho \longmapsto \int \hat{\phi}\, d\varrho$ is continuous on

$$\{\varrho \in \mathscr{M}^1(X) : \int \phi\, d\varrho \leqq M\} .$$

Let $\hat{\phi}_n = \hat{\phi} \wedge n$; then $\hat{\phi}_n$ is a bounded measurable function on X, so $\varrho \longmapsto \int \hat{\phi}_n\, d\varrho$ is continuous. On the other hand,

$$\left|\int \hat{\phi}\, d\varrho - \int \hat{\phi}_n\, d\varrho\right| \leqq \int\limits_{\{\phi \geqq n\}} \hat{\phi}\, d\varrho \leqq \sup_{\hat{\phi}(X) \geqq n} \left\{\frac{\hat{\phi}(X)}{\phi(X)}\right\} \cdot \int \phi\, d\varrho ,$$

and, by (2.6), the right-hand side goes to zero as n goes to infinity uniformly for ϱ in $\{\varrho : \int \phi\, d\varrho \leqq M\}$. Hence, $\varrho \longmapsto \int \hat{\phi}\, d\varrho$ is a uniform limit of continuous functions and is therefore continuous.

It remains to prove the compactness of

$$\{\varrho \in \mathscr{M}^1(X) : \int \phi\, d\varrho \leqq M, s(\varrho, \sigma) \geqq -N\} .$$

Since $\int \phi\, d\varrho = \sup_n \int (\phi \wedge n)\, d\sigma$, $\varrho \longmapsto \int \phi\, d\varrho$ is lower semi-continuous, so $\{\varrho \in \mathscr{M}^1(X) : \int \phi\, d\varrho \leqq M, s(\varrho, \sigma) \geqq -N\}$ is closed. Let $\sigma'' = e^{-\phi}\,\sigma$; then

$$s(\varrho, \sigma) = s(\varrho, \sigma'') + \int \phi\, d\sigma .$$

It therefore suffices to prove that $\{\varrho \in \mathscr{M}^1(X) : s(\varrho, \sigma'') \geqq -(M+N)\}$ is compact; this follows from Proposition 2.10 since σ'' is a finite measure.

Proposition 2.12. *Let $\varrho_1, \varrho_2 \in \mathscr{M}^1(X)$, and suppose $\int \phi\, d\varrho_1 < \infty$ and $\int \phi\, d\varrho_2 < \infty$. Let $0 \leqq \alpha \leqq 1$. Then*

$$\begin{aligned} \alpha\, s(\varrho_1, \sigma) + (1-\alpha)\, s(\varrho_2, \sigma) &\leqq s(\alpha\, \varrho_1 + (1-\alpha)\, \varrho_2, \sigma) \\ &\leqq \alpha\, s(\varrho_1, \sigma) + (1-\alpha)\, s(\varrho_2, \sigma) + \log 2 . \end{aligned} \tag{2.7}$$

Proof. This follows, just as in [3], from the concavity of $-x\log x$ and the monotonicity of $\log x$.

Proposition 2.13. *Let T be a one-one measurable mapping of X onto itself with a measurable inverse. Suppose that σ is invariant under T, i.e., that $\sigma(T^{-1}E) = \sigma(E)$ for every measurable subset E of X. Let ϱ be a probability measure on X such that*

$$\int \phi\, d\varrho < \infty , \qquad \int \phi\, d(\varrho \circ T^{-1}) < \infty .$$

Then

$$s(\varrho, \sigma) = s(\varrho \circ T^{-1}, \sigma) .$$

Proof. This proposition follows at once from the definitions.

We now apply this general construction to statistical mechanics. For any bounded Borel set Λ, let σ_Λ be the Borel measure on $[\mathscr{X}]\,(\Lambda)$ whose restriction to each $(\Lambda \times \boldsymbol{R})^n_{\text{symm}}$ is given by the measure

$$\frac{1}{n!}\,dq_1, \ldots, dq_n\, dp_1, \ldots, dp_n$$

on $(\Lambda \times \boldsymbol{R})^n$. If ϱ is a probability measure on $[\mathscr{X}]$, the image of ϱ under the restriction mapping is a probability measure ϱ_Λ on $[\mathscr{X}]\,(\Lambda)$. The role of the function ϕ of the preceeding propositions will be played by the kinetic energy in Λ; note that, for any $\alpha > 0$, we have

$$\int e^{-\alpha K.E._\Lambda}\, d\sigma_\Lambda = e^{\sqrt{\frac{2\pi}{\alpha}}\, V(\Lambda)} \tag{2.8}$$

where $V(\Lambda)$ is the Lebesgue measure of Λ. If $\int K.E._\Lambda\, d\varrho < \infty$, we will define $s_\Lambda(\varrho)$, the *entropy of* ϱ in Λ, to be $s(\varrho_\Lambda, \sigma_\Lambda)$.

Proposition 2.14. *Let ϱ be a probability measure on $[\mathscr{X}]$ such that $\int K.E._\Lambda\, d\varrho < \infty$ for all bounded Borel sets Λ. Then $s_\Lambda(\varrho)$ is a subadditive set function, i.e.,*

$$s_{\Lambda_1 \cup \Lambda_2}(\varrho) \leqq s_{\Lambda_1}(\varrho) + s_{\Lambda_2}(\varrho) \tag{2.9}$$

for all pairs Λ_1, Λ_2 of disjoint bounded Borel sets, and we have

$$s_\Lambda(\varrho) \leqq 3\left(\frac{\pi}{2}\right)^{1/3} V(\Lambda)\left[\frac{1}{V(\Lambda)} \int K.E._\Lambda\, d\varrho\right]^{1/3} \tag{2.10}$$

for all bounded Borel sets Λ.

Proof. The subadditivity is proved in exactly the same way as in Proposition 1 of [3]; the inequality (2.10) is obtained by applying (2.4) of Lemma 2.9 and (2.8) to get

$$s_\Lambda(\varrho) \leqq \alpha \int K.E._\Lambda\, d\varrho + \sqrt{\frac{2\pi}{\alpha}}\, V(\Lambda)$$

for any $\alpha > 0$, then minimizing with respect to α.

Let a be a positive real number, and let $\mathscr{M}^1_a[\mathscr{X}]$ denote the set of probability measures on $[\mathscr{X}]$ which are periodic under space translations with period a. If $\varrho \in \mathscr{M}^1_a[\mathscr{X}]$, and if $\int d\varrho\, K.E._{[0,a)} < \infty$, then $\int d\varrho\, K.E._\Lambda < \infty$ for every bounded Borel set Λ. Thus, we can define $s_\Lambda(\varrho)$ for all such Λ.

Proposition 2.15. *Let $\varrho \in \mathscr{M}^1_a[\mathscr{X}]$, and suppose that $\int K.E._{[0,a)}\, d\varrho < \infty$. Then:*

1. $\lim\limits_{\beta - \alpha \to \infty} \frac{1}{\beta - \alpha}\, s_{[\alpha, \beta)}(\varrho)$ *exists. We denote this limit by $\bar{s}(\varrho)$; it is the entropy of ϱ per unit volume.*
2. $\bar{s}(\varrho) = \inf\limits_n \frac{1}{na}\, s_{[0, na)}(\varrho)$.
3. *$\bar{s}(\varrho)$ is an affine function of ϱ.*

4. $\bar{s}(\varrho) \leqq 3\left(\frac{\pi}{2}\right)^{1/3}\left[\frac{1}{a}\int K.E._{[0,a)}\,d\varrho\right]^{1/3}$.

5. *For any pair of real numbers* M, N,

$$\mathscr{K} = \{\varrho \in \mathscr{M}_a^1[\mathscr{X}] : \int K.E._{[0,a)}\,d\varrho \leqq M, \bar{s}(\varrho) \geqq -N\}$$

is compact for the $\mathfrak{A}_\infty$ *topology, and the* $\mathfrak{A}$ *topology agrees with the* $\mathfrak{A}_\infty$ *topology on* $\mathscr{K}$.

6. *For any real number* M, $\bar{s}(\varrho)$ *is an upper-semi continuous function for the* $\mathfrak{A}$ *topology on* $\{\varrho \in \mathscr{M}_a^1[\mathscr{X}] : \int K.E._{[0,a)}\,d\varrho \leqq M\}$.

Proof. By Proposition 2.14, $s_{[0,na)}(\varrho)$ is a sub-additive function of n, so $\lim_{n\to\infty}\frac{1}{na}s_{[0,na)}(\varrho)$ exists and is equal to $\inf_n \frac{1}{na}s_{[0,na)}(\varrho)$. Now let $\beta > \alpha$ be given and let n, n' be chosen so that $na \leqq \alpha < (n+1)a$, $n'a \leqq \beta < (n'+1)a$. Then by subadditivity

$$s_{[\alpha,\beta)}(\varrho) \leqq s_{[(n+1)a,n'a)}(\varrho) + s_{[\alpha,(n+1)a)}(\varrho) + s_{[n'a,\beta)}(\varrho)\,.$$

By (2.10) the right-hand side is not greater than

$$s_{[(n+1)a,n'a)}(\varrho) + 6\left(\frac{\pi}{2}\right)^{1/3} a^{2/3}\left[\int K.E._{[0,a)}\,d\varrho\right]^{1/3},$$

so

$$\limsup_{\beta-\alpha\to\infty}\frac{1}{\beta-\alpha}s_{[\alpha,\beta)}(\varrho) \leqq \lim_{n\to\infty}\frac{1}{na}s_{[0,na)}(\varrho)\,.$$

Similarly,

$$\liminf_{\beta-\alpha\to\infty}\frac{1}{\beta-\alpha}s_{[\alpha,\beta)}(\varrho) \geqq \lim_{n\to\infty}\frac{1}{na}s_{[0,na)}(\varrho)\,,$$

so statements 1. and 2. are proved.

The fact that $s(\varrho)$ is an affine function of ϱ follows from inequality (2.7) of Proposition 2.12. The bound 4. follows from 2. and inequality (2.10) of Proposition 2.14.

To prove statement 5., let ϱ_α be a universal net[5] in $\mathscr{K}$. Then since we have:

$$s(\varrho_{\alpha,[-na,na)}, \sigma_{[-na,na)}) \geqq -2naN$$
$$\int K.E._{[-na,na)}\,d\varrho_\alpha \leqq 2nM$$

it follows from Proposition 2.11 that $\varrho_{\alpha,[-na,na)}$ converges in the $\mathscr{L}^\infty$ topology for measures on $[\mathscr{X}]([-na,na))$. Let $\hat{\varrho}_{[-na,na)}$ denote the limiting measure. Then the collection of measures $(\hat{\varrho}_{[-na,na)})$ is consistent and therefore defines a unique measure $\hat{\varrho}$ on $[\mathscr{X}]$; evidently ϱ_α converges to $\hat{\varrho}$ in the $\mathfrak{A}_\infty$ topology. It remains to be checked that $\hat{\varrho}$ belongs to $\mathscr{K}$. It is clear that $\hat{\varrho}$ is in $\mathscr{M}_a^1[\mathscr{X}]$. Since for any $\varrho \in \mathscr{M}^1[\mathscr{X}]$ $\int K.E._{[0,a)}\,d\varrho = \sup \int [K.E._{[0,a)} \wedge n]\,d\varrho$, $\varrho \mapsto \int K.E._{[0,a)}\,d\varrho$ is a lower semi-contin-

[5] See [6], for the definition of a universal net. A universal net is roughly one which is a refined as possible. Every net has a universal subnet; the image of a universal net under any mapping is universal; a Hausdorff topological space is compact if and only if every universal net in it converges.

uous function on $\mathscr{M}^1[\mathscr{X}]$ with the $\mathfrak{A}_\infty$ topology; hence, $\int K.E._{[0,a)} \cdot d\hat{\varrho} \leqq M$.

By Proposition 2.11, each $s_{[-na,na)}(\varrho)$ is an $\mathfrak{A}_\infty$ upper semi-continuous function on $\{\varrho \in \mathscr{M}_a^1[\mathscr{X}] : \int K.E._{[0,a)}\, d\varrho \leqq M\}$ and hence, by 2., $\bar{s}(\varrho)$ is also $\mathfrak{A}_\infty$ upper semi-continuous on this set. Hence, in particular, $\bar{s}(\hat{\varrho}) \geqq -N$, so $\hat{\varrho}$ belongs to $\mathscr{K}$ and $\mathscr{K}$ is $\mathfrak{A}_\infty$-compact. From the compactness of $\mathscr{K}$ in the $\mathfrak{A}_\infty$ topology, and the fact that the $\mathfrak{A}$ topology is a Hausdorff topology which is no finer than the $\mathfrak{A}_\infty$ topology, it follows that the $\mathfrak{A}$ topology coincides with the $\mathfrak{A}_\infty$ topology on $\mathscr{K}$.

We still have to prove 5., and we know already that $\bar{s}(\varrho)$ is $\mathfrak{A}_\infty$ upper semi-continuous on $\mathscr{K}' = \{\varrho \in \mathscr{M}_a^1[\mathscr{X}] : \int K.E._{[0,a)}\, d\varrho \leqq M\}$. To prove $\mathfrak{A}$ upper semi-continuity, it is enough to prove that, if ϱ_α is a net in $\mathscr{K}'$ which converges to ϱ, then $\bar{s}(\varrho) \geqq \limsup_\alpha \bar{s}(\varrho_\alpha)$. There is evidently nothing to be proved if $\limsup_\alpha \bar{s}(\varrho_\alpha) = -\infty$. If this is not the case, then we can pass to a subnet for which $\bar{s}(\varrho_\alpha)$ is bounded below, without changing the lim sup. In other words, we can assume that all the ϱ_α are contained in

$$\{\varrho \in \mathscr{M}_a^1[\mathscr{X}] : \int K.E._{[0,a)}\, d\varrho \leqq M, \bar{s}(\varrho) \geqq -N\},$$

for some choice of N. But, on this set, the $\mathfrak{A}$ topology coincides with the $\mathfrak{A}_\infty$ topology, so ϱ_α converges to ϱ in the $\mathfrak{A}_\infty$ topology, and

$$\bar{s}(\varrho) \geqq \limsup_\alpha s(\varrho_\alpha)$$

follows from the $\mathfrak{A}_\infty$ upper semi-continuity of $\bar{s}$.

Proposition 2.16. *Let X be a compact topological space, $x \longmapsto \varrho_x$ a continuous mapping from X to $\mathscr{M}_a^1[\mathscr{X}]$, and ν a Radon probability measure on X. Suppose $\int K.E._{[0,a)}\, d\varrho_x$ is bounded with respect to x. Let $\bar{\varrho} = \int d\nu(x)\, \varrho_x$. Then*

$$\int K.E._{[0,a)}\, d\varrho = \int d\nu(x) \int K.E._{[0,a)}\, d\varrho_x$$

and

$$\bar{s}(\varrho) = \int d\nu(x)\, \bar{s}(\varrho_x).$$

Proof. The first formula follows immediately from Corollary 2.9. Replacing X, ν by their images under $x \longmapsto \varrho_x$ we can suppose that $X \subset \mathscr{M}_a^1[\mathscr{X}]$ and that $\bar{\varrho}$ is the barycenter of ν. But $\bar{s}(\varrho)$ is affine and upper semi-continuous on any set on which $\int K.E._{[0,a)}\, d\varrho$ is bounded; hence, by the theorem of the barycenter[6]

$$\bar{s}(\bar{\varrho}) = \int d\nu(x)\, \bar{s}(\varrho_x).$$

[6] The theorem of the barycenter asserts that, if $\mathscr{K}$ is a compact convex set in a locally convex topological vector space, ν a probability measure on $\mathscr{K}$ with barycenter $r(\nu)$, and f an affine upper semi-continuous function on $\mathscr{K}$, then $f(r(\nu)) = \int f(x)\, d\nu(x)$. See [7].

(To justify the application of the theorem of the barycenter, we have to know that the closed convex hull of the image of X in $\mathscr{M}^1[\mathscr{X}]$ is compact, or equivalently, that the closed convex hull of the image of X in $E(\mathfrak{A})$ is contained in $\mathscr{M}^1[\mathscr{X}]$. This follows easily from Corollary 2.8.)

Corollary 2.17. *Let* $\varrho \in \mathscr{M}^1_a[\mathscr{X}]$, *and assume* $\int K.E._{[0,a)}\, d\varrho < \infty$. *Let* $\bar{\varrho} = \frac{1}{a}\int_0^a ds(\tau_s \varrho)$. *Then*

$$\int K.E._{[0,a)}\, d\bar{\varrho} = \int K.E._{[0,a)}\, d\varrho < \infty, \quad \textit{and} \quad \bar{s}(\bar{\varrho}) = \bar{s}(\varrho)\,.$$

Proof. By Proposition 2.16, we have only to prove

$$\int K.E._{[0,a)} \circ \tau_s\, d\varrho = \int K.E._{[0,a)}\, d\varrho \quad \text{for} \quad 0 \leqq s \leqq a\,.$$

Now $K.E._{[0,a)} \circ \tau_s = K.E._{[-s,s-a)} = K.E._{[-s,0)} + K.E._{[0,a-s)}$. By the periodicity of ϱ,

$$\int K.E._{[-s,0)}\, d\varrho = \int K.E._{[a-s,a)}\, d\varrho\,.$$

Reassembling gives:

$$\int K.E._{[0,a)} \circ \tau_s\, d\varrho = \int [K.E._{[0,a-s)} + K.E._{[a-s,a)}]\, d\varrho = \int K.E._{[0,a)}\, d\varrho$$

2.4. The Existence Theorem

In this section we summarize and reformulate the main results of [1] in a form which will be convenient for our purposes in this article.

For any $x = (q_i, p_i)$ in $\mathscr{X}$, we define

$$|x| = \sup_i \left(\frac{|p_i|}{\log_+(q_i)}\right) \quad \vee \sup \quad \left\{\frac{N_{(\alpha,\beta)}(x)}{\beta-\alpha} : \beta - \alpha > \log_+\left(\frac{\beta+\alpha}{2}\right)\right\},$$

where $\log_+(q) = \log(|q| \vee e)$. The quantity $|x|$ is either a non-negative real number or $+\infty$. We will regard $|\;|$ either as a function on $\mathscr{X}$ or on $[\mathscr{X}]$ as convenient.

The set $\hat{\mathscr{X}}$ is $\{x \in \mathscr{X} : |x| < \infty\}$. For any non-negative real number δ, we define $\hat{\mathscr{X}}_\delta = \{x \in \mathscr{X} : |x| \leq \delta\}$. We denote by $[\hat{\mathscr{X}}]$ and by $[\hat{\mathscr{X}}_\delta]$ the corresponding sets of equivalence classes.

We want to solve the equations of motion

$$\frac{dq_i(t)}{dt} = p_i(t); \quad \frac{dq_i(t)}{dt} = \sum_{j \neq i} F(q_i(t) - q_j(t))$$

with initial data in $\hat{\mathscr{X}}$. Throughout this article, we will assume that F has compact support and satisfies a Lipschitz condition. We will always use R to denote the range of F, i.e., the smallest number such that $F(q) = 0$ whenever $|q| \geqq R$.

To solve the equations of motion, we introduce for each initial configuration $x = (q_i, p_i)$ the Banach space $\mathscr{Y}_x$ of sequences $\zeta = (\xi_i, \eta_i)$ of pairs of real numbers such that

$$\|\zeta\|_x = \sup_i \frac{|\xi_i| \vee |\eta_i|}{\log_+(q_i)} < \infty\,.$$

For ζ in $\mathscr{Y}_x$, we let $x + \zeta$ denote the configuration $(q_i + \xi_i, p_i + \eta_i)$. The equations of motion with initial configuration x can be reformulated as an evolution equation in $\mathscr{Y}_x$:

$$\frac{d\zeta_x(t)}{dt} = A_x(\zeta_x(t));$$

the solution of the original equations is then obtained as

$$x(t) = x + \zeta_x(t) .$$

We may obtain the solution of the evolution equation by an iterative procedure. Define

$$\zeta_{0,x}(t) \equiv 0$$
$$\zeta_{n,x}(t) \equiv \int_0^t d\tau \, A_x(\zeta_{n-1,x}(\tau)) \quad \text{for} \quad n = 1, 2, 3, \ldots,$$

i. e.,

$$\begin{aligned} \xi_{i,n,x}(t) &= \int_0^t d\tau \, [p_i + \eta_{i,n-1,x}(\tau)] \\ \eta_{i,n,x}(t) &= \int_0^t d\tau \left[\sum_{j \neq i} F(q_i + \xi_{i,n-1,x}(\tau) - q_j - \xi_{j,n-1,x}(\tau)) \right]. \end{aligned} \tag{2.12}$$

Let $x_n(t) = x + \zeta_{n,x}(t)$. For each positive real number m, define a seminorm ${}_m\| \ \|_x$ on $\mathscr{Y}_x$ by

$${}_m\|\zeta\|_x = 0 \vee \sup \left\{ \frac{|\zeta_i| \vee |\eta_i|}{\log_+(q_i)} : |q_i| \leqq m \right\} .$$

The following proposition is a more explicit version of Remark 4.3 of [1]:

Proposition 2.18. *There exist functions $h(\delta, T)$ and $\varepsilon(n, m, \delta, T)$ such that*

i) $\|\zeta_{n,x}(t)\|_x \leqq h(\delta, T)$ *for all* n *whenever* $|x| \leqq \delta$ *and* $|t| \leqq T$.

ii) $\lim_{n\to\infty} \varepsilon(n, m, \delta, T) = 0$ *for all* m, δ, T

and

$${}_m\|\zeta_{n,x}(t) - \zeta_x(t)\|_x \leqq \varepsilon(n, m, \delta, T)$$

whenever $|x| \leqq \delta$ *and* $|t| \leqq T$.

We define T^t to be the mapping of $[\mathscr{X}]$ into itself which takes the equivalence class of x to the equivalence class of $x(t)$. The mappings T^t form a one-parameter group of transformations on $[\hat{\mathscr{X}}]$. We let T^t_n denote the mapping of $[\mathscr{X}]$ into itself which takes the equivalence class of x to the equivalence class of $x_n(t)$. From Proposition 2.18, one easily obtains the following:

Proposition 2.19. *For any pair of positive numbers δ, T, and any ψ in $\mathfrak{A}$,*

$$\lim_{n\to\infty} \psi(T^t_n x) = \psi(T^t x)$$

uniformly for x in $[\hat{\mathscr{X}}_\delta]$ and $|t| \leqq T$.

2.5. Space-Periodized Systems

We will have occasion to consider systems of a finite number of particles moving in a finite interval $[-a, b)$ "with periodic boundary conditions". From a fundamental point of view, this is a matter of studying a second order differential equation on torus. For our purposes, it is convenient to formulate the equations in a slightly different way.

For any function f defined on $\boldsymbol{R}$, and any positive real number α, we let $\tilde{f}_\alpha$ denote the function on $\boldsymbol{R}$ which is periodic with period α and which agrees with f on $[-\alpha/2, \alpha/2)$. To find the motion of a system of n particles moving on $[-a, b)$ with interparticle force F and with periodic boundary conditions, it is enough to solve the system of ordinary differential equations:

$$\begin{aligned} \frac{dq_i(t)}{dt} &= p_i(t); \\ \frac{dp_i(t)}{dt} &= \sum_{j \neq i} \tilde{F}_{a+b}(q_i(t) - q_j(t)) . \end{aligned} \tag{2.13}$$

(Note that, if $a + b \geqq 2R$, then $\tilde{F}_{a+b}$ satisfies a Lipschitz condition since F does.) The $p_i(t)$'s are the correct velocities, but the $q_i(t)$'s are not necessarily the correct positions; these latter are obtained from the $q_i(t)$'s by subtracting appropriate integral multiples of $a + b$ to give values in $[-a, b)$.

The solution of this system of equations gives a one-parameter group of mappings of $[\mathscr{X}]([-a, b))$ onto itself; we denote these mappings by $\tilde{T}^t_{[-a,b)}$. If we identify

$$[\mathscr{X}] = \prod_{n=-\infty}^{\infty} [\mathscr{X}]\big([-a + n(a+b), b + n(a+b))\big)$$

and if we consider the separate evolution of each factor, we get a one-parameter group of mappings of $[\mathscr{X}]$ onto itself which we will also denote by $\tilde{T}^t_{[-a,b)}$.

We also need to adapt some ideas from statistical mechanics to the framework of periodized systems. A two-body potential Φ will be said to be *P-stable* if there exist constants B and D such that

$$\sum_{1 \leqq i < j \leqq n} \tilde{\Phi}_d(q_i - q_j) \geqq -Bn \tag{2.14}$$

for all $n, q_1, \ldots, q_n$, whenever $d \geqq D$. Passing to the limit $d \to \infty$ in (2.14) with the q_i held fixed gives

$$\sum_{1 \leqq i < j \leqq n} \Phi(q_i - q_j) \geqq -Bn ,$$

i.e., any P-stable potential is stable.

Although the notion of P-stability is a useful tool, it is not a very pleasing hypothesis from an aesthetic point of view. Fortunately, for potentials of compact support, it reduces to the ordinary notion of stability.

Proposition 2.20.[7] *Any stable potential of compact support is P-stable.*

Proof. Suppose $\Phi(q) = 0$ for $|q| \geqq R$, and suppose that

$$\sum_{1 \leqq i < j \leqq n} \Phi(q_i - q_j) \geqq -Bn$$

for all $n, q_1, \ldots, q_n$. We will show that

$$\sum_{1 \leqq i < j \leqq n} \tilde{\Phi}_d(q_i - q_j) \geqq -Bn$$

for all $n, q_1, \ldots, q_n$, if $d \geqq 2R$. We can assume that $q_1, \ldots, q_n \in [0, d)$. For any positive integer N, define $q_{Kn+i} = Kd + q_i$ for $K = 0, 1, \ldots, N-1$ and $i = 1, 2, \ldots, n$.
Then

$$\sum_{1 \leqq i < j \leqq n} \tilde{\Phi}_d(q_i - q_j) = \lim_{N \to \infty} \frac{1}{N} \sum_{1 \leqq i < j \leqq Nn} \Phi(q_i - q_j)$$

$$\geqq \frac{1}{N}(-B\,Nn) = -Bn\,.$$

If Φ is P-stable and if $a + b$ is large enough, we can construct the grand canonical ensemble on $[-a, b)$ for the potential $\tilde{\Phi}_{a+b}$. We will refer to this measure on $[\mathscr{X}]([-a, b))$ as *the periodized grand canonical ensemble* on $[-a, b)$. If $F(q) = -\frac{d}{dq}\Phi(q)$, where Φ is an even P-stable potential, and if $a + b$ is larger than $2R$, then the measure on $[\mathscr{X}]$ obtained by the product measure construction from the periodized grand canonical ensemble on $[-a, b)$ is invariant under the one-parameter group $\tilde{T}^t_{[-a,b)}$.

§ 3. Measurability of $[\hat{\mathscr{X}}]$ and T^t

Proposition 3.1. *Each* $[\hat{\mathscr{X}}_\delta]$ *is a compact subset of* $[\mathscr{X}]$.

Proof. By definition, $[\hat{\mathscr{X}}_\delta]$ is the set of all x in $[\mathscr{X}]$ such that

i) $\overline{P}_{(-q,q)}(x) \leqq \delta \log_+(q)$ for all positive real numbers q.

ii) $N_{(\alpha,\beta)}(x) \leqq \delta(\beta - \alpha)$ for all α, β with $\beta - \alpha > \log_+\left(\frac{\alpha+\beta}{2}\right)$.

By Proposition 2.5, for any q, α, β,

$$\{x : \overline{P}_{(-q,q)}(x) \leqq \delta \log_+(q)\}$$

and

$$\{x : N_{(\alpha,\beta)}(x) \leqq \delta(\beta - \alpha)\}$$

[7] This proposition is due to D. Ruelle (unpublished).

are closed in $[\mathscr{X}]$. Hence, $[\hat{\mathscr{X}}_\delta]$ is the intersection of a collection of closed sets and is therefore closed. On the other hand, for any bounded open set Λ, $\overline{P}_\Lambda$ and N_Λ are bounded on $[\hat{\mathscr{X}}_\delta]$. Hence, by Proposition 2.3, $[\hat{\mathscr{X}}_\delta]$ is compact.

Corollary 3.2. *$[\hat{\mathscr{X}}]$ is a Borel subset of $[\mathscr{X}]$.*

Next we investigate the measurability of the time-evolution mappings T^t.

Proposition 3.3. *For each δ, the mapping $(t, x) \longmapsto T^t x$ is continuous from $\boldsymbol{R} \times [\hat{\mathscr{X}}_\delta]$ to $[\mathscr{X}]$.*

Proof. It suffices to prove that $(t, x) \longmapsto \psi(T^t x)$ is continuous for every ψ in $\mathfrak{A}$. We know from Proposition 2.19 that

$$\lim_{n\to\infty} \psi(T_n^t x) = \psi(T^t x)$$

and that the convergence is uniform as t runs over any bounded interval and x runs over $[\hat{\mathscr{X}}_\delta]$. Hence, it will be enough to prove that $(t, x) \longmapsto \psi(T_n^t x)$ is continuous. Furthermore, we can suppose that ψ depends only on the co-ordinates of the particles in some bounded interval $[-\lambda, \lambda]$. By Proposition 2.18 there is, for any $T > 0$, a constant H such that

$$|\xi_{i,n,x}(t)| \leqq H \log_+(q_i)$$

whenever $|t| \leqq T$ and $x = (q_i, p_i) \in \hat{\mathscr{X}}_\delta$.

Now choose λ_0 so that $|q| - H \log_+(q) \leqq \lambda$ implies $|q| < \lambda_0$; if $x \in \hat{\mathscr{X}}_\delta$, if $|t| \leqq T$, and if $|q_i + \xi_{i,n,x}(t)| \leqq \lambda$ for some n, then $|q_i| < \lambda_0$. Choose successively $\lambda_1, \lambda_2, \ldots$ so that $|q| \leqq \lambda_j$ and $|q'| \geqq \lambda_{j+1}$ implies

$$|q| + H \log_+(q) + R < |q'| - H \log_+(q') \qquad j = 0, 1, 2, \ldots$$

Then if $x \in \hat{\mathscr{X}}_\delta$ and if $|q_i| \leqq \lambda_k$, $|q_j| \geqq \lambda_{k+1}$, we have

$$|q_{i,n,x}(t) - q_{j,n,x}(t)| > R$$

for all n and all $|t| \leqq T$. [We have introduced the notation $q_{i,n,x}(t) = q_i + \xi_{i,n,x}(t)$.]

From the formula (2.12) for $\xi_{i,n,x}(t)$, $\eta_{i,n,x}(t)$, we see that, if $|t| \leqq T$ and if $|q_i| \leqq \lambda_k$, then $q_{i,n,x}(t)$ and $p_{i,n,x}(t)$ depend only on t, on the values of $p_{i,n-1,x}(\tau)$, τ between 0 and t, and on the values of $q_{j,n-1,x}(\tau)$, τ between 0 and t, for those j's with $|q_j| \leqq \lambda_{k+1}$. By induction, and using the fact that $q_{i,0,x}(t) = q_i$, $p_{i,0,x}(t) = p_i$, we see that, if $|q_i| \leqq \lambda_0$, and if $|t| \leqq T$, then $q_{i,n,x}(t)$ and $p_{i,n,x}(t)$ depend only on t and on those q_j's and p_j's with $|q_j| \leqq \lambda_n$. Furthermore, from the continuity of F we see that the $(q_{i,n,x}(t), p_{i,n,x}(t))$, and hence $\psi(x_n(t))$, are continuous functions of these variables. (Here, we regard ψ as a function on $\mathscr{X}$ rather than on $[\mathscr{X}]$).

It is now an elementary exercise in the topology of $[\mathscr{X}]$ to conclude from these statements that $\psi(T_n^t x)$ varies continuously with t and x for $|t| \leqq T$ and $x \in [\hat{\mathscr{X}}_\delta]$.

Corollary 3.4. *The mapping* $(t, x) \longmapsto T^t x$ *is a Borel mapping from* $\boldsymbol{R} \times [\hat{\mathscr{X}}]$ *to* $[\hat{\mathscr{X}}]$.

§ 4. Measures Concentrated on $[\hat{\mathscr{X}}]$

If ϱ is a probability measure on $[\mathscr{X}]$, we let ϱ_Λ denote the probability measure on $[\mathscr{X}]\,(\Lambda)$ which is the image of ϱ under the restriction mapping. Specifying a probability measure on $[\mathscr{X}]\,(\Lambda)$ is equivalent to specifying, for each n, a finite permutation-invariant measure on $(\Lambda \times \boldsymbol{R})^n$ such that the sum of the total masses of these measures is one. We say that ϱ has a Maxwellian velocity distribution with inverse temperature β if, for each Λ, the component of ϱ_Λ on $(\Lambda \times \boldsymbol{R})^n$ has the form

$$d\hat{\varrho}_\Lambda^n(q_1, \ldots, q_n) \exp\left\{-\frac{\beta}{2} \sum_i p_i^2\right\} dp_1, \ldots, dp_n$$

where $\hat{\varrho}_\Lambda^n$ is a permutation-invariant measure on Λ^n.

Proposition 4.1. *Let* $\varrho \in \mathscr{M}^1[\mathscr{X}]$. *For* ϱ *to be concentrated on* $[\hat{\mathscr{X}}]$ *it is sufficient (but not necessary) that the following two conditions both hold:*

A. ϱ *has a Maxwellian velocity distribution (with some inverse temperature* β*).*

B. *There exists a real number* λ *such that, for any interval* $[a, b)$ *of length at least one, and any* $n = 0, 1, 2, \ldots,$

$$\int d\varrho\, N_{[a,b)}(N_{[a,b)} - 1) \ldots (N_{[a,b)} - n) \leqq [\lambda(b-a)]^{n+1}. \quad (4.1)$$

Moreover, we can put the estimates in a more quantitative form: There exists a function $\varepsilon(\delta, \beta, \lambda)$ *such that* $\lim_{\delta \to \infty} \varepsilon(\delta, \beta, \lambda) = 0$ *for all* (β, λ) *and such that*

$$\varrho(\complement[\hat{\mathscr{X}}_\delta]) \leqq \varepsilon(\delta, \beta, \lambda)$$

whenever ϱ *satisfies A. and B.*

Proof. Let P be a real number, and let

$$Y_{1,P} = \{x \in [\mathscr{X}]\colon \text{ for some integer } m \text{ there is a particle with } m \leqq q_i < m+1 \text{ and } |p_i| \geqq P \log_+(m)\};$$

$$Y_{2,P} = \{x \in [\mathscr{X}]\colon \text{ for some integers } m, j \text{ with } 2j \geqq \log_+(m), \text{ the interval } [m-j, m+j) \text{ contains more than } 2\,Pj \text{ particles}\}.$$

We will estimate $\varrho(Y_{1,P})$ and $\varrho(Y_{2,P})$ and show that they both go to zero as P goes to infinity; this will prove the proposition.

Let $\varrho_{m,n}$ be the ϱ measure of the set of configurations with precisely n particles in $[m, m+1)$, and let

$$\phi(\xi) = \sqrt{\frac{2}{\pi}} \int_{\xi}^{\infty} e^{-p^2/2}\, dp\,.$$

An elementary calculation shows that the ϱ measure of the set of configurations having at least one particle in $[m, m+1)$ with velocity at least $P \log_+(m)$ is

$$\begin{aligned}\sum_{n=0}^{\infty} \varrho_{m,n}&[1-(1-\phi(\sqrt{\beta}\, P \log_+(m)))^n] \\ &\leqq \phi(\sqrt{\beta}\, P \log_+(m)) \sum_{n=0}^{\infty} \varrho_{m,n} \cdot n \\ &= \phi(\sqrt{\beta}\, P \log_+(m)) \int d\varrho\, N_{[m,m+1)} \\ &\leqq \phi(\sqrt{\beta}\, P \log_+(m)) \cdot \lambda\,.\end{aligned}$$

Hence,

$$\begin{aligned}\varrho(Y_{1,P}) &\leqq \sum_{m=-\infty}^{\infty} \varrho\{x: \text{There is a particle in } [m, m+1) \text{ with velocity at least } P \log_+(m)\} \\ &\leqq \lambda \sum_{m} \phi(\sqrt{\beta}\, P \log_+(m))\,.\end{aligned}$$

Using the fact that $\phi(\xi)$ decreases more rapidly at infinity than $e^{-\xi^2/2}$ it is easy to verify that the right-hand side of this inequality goes to zero as P goes to infinity.

To estimate the measure of $Y_{2,P}$, we let $\varrho_{m,j,n}$ denote the ϱ measure of the set of configurations having exactly n particles in the interval $[m-j, m+j)$, and we let

$$\begin{aligned}\sigma_{m,j,k} &= \int d\varrho\, N_{[m-j,m+j)}\,, (N_{[m-j,m+j)} - 1) \ldots (N_{[m-j,m+j)} - k + 1) \\ &= \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} \varrho_{m,j,n}\,.\end{aligned}$$

By condition B.,

$$\sigma_{m,j,k} \leqq (2\lambda j)^k\,.$$

It is straightforward to verify, using this inequality, that[8]

$$\varrho_{m,j,n} = \frac{1}{n!} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \sigma_{m,j,l+n} \leqq \frac{1}{n!} e^{2\lambda j} (2\lambda j)^n\,.$$

[8] The possibility of using such an identity to estimate particle number probabilities was suggested to me by D. RUELLE.

Thus, the measure of the set of configurations with more than $2Pj$ particles in $[m-j, m+j)$ is not greater than

$$e^{2\lambda j} \sum_{n>2Pj} \frac{1}{n!}(2\lambda j)^n .$$

If $P \geqq 2\lambda$, the ratio of succeeding terms in this sum is not greater than 1/2, so the sum is not greater than twice its first term. Using STIRLING's formula, we see the sum is majorized by $C \cdot \left(\frac{e^{P+\lambda} \cdot \lambda^P}{P^P}\right)$. Letting $f(P,\lambda) = \left(\frac{e^{P+\lambda}\lambda^P}{P^P}\right)$, we have

$$\varrho(Y_{2,P}) \leqq \sum_{m=-\infty}^{\infty} \sum_{2j \geqq \log_+(m)} C[f(P,\lambda)]^{2j}$$

Since $\lim_{P\to\infty} f(P,\lambda) = 0$, the right-hand side goes to zero as P goes to infinity, so the proof of the proposition is complete.

Condition B. holds, in particular, if ϱ has correlation functions of all orders $\bar{\varrho}_n(q_1, \ldots, q_n)$ and if there exists a constant λ such that

$$\bar{\varrho}_n(q_1, \ldots, q_n) \leqq \lambda^n \tag{4.2}$$

for all n and all $q_1, \ldots, q_n$. In fact, we have:

$$\int d\varrho\, N_{(a,b]}(N_{(a,b]}-1)\ldots(N_{(a,b]}-n+1) = \int_{[a,b)^n} dq_1, \ldots, dq_n\, \bar{\varrho}_n(q_1, \ldots, q_n) .$$

There are two cases in which inequalities of the form (4.2) are known to hold:

i) ϱ is a state obtained by taking the infinite-volume limit of the grand-canonical ensemble at low activity[9] [8].

ii) ϱ is a state obtained by taking any infinite volume cluster point of the grand-canonical ensemble for a system with a non-negative potential, at any value of the temperature and chemical potential.

In both these cases, ϱ also has a Maxwellian velocity distribution and is therefore concentrated on $[\hat{\mathscr{X}}]$.

Case ii) requires some explanation. Suppose we have fixed a temperature and a chemical potential, and suppose Φ is a non-negative two-body potential. For any positive number m, let $\bar{\varrho}_m$ be the measure on $[\mathscr{X}]$ obtained from the grand canonical ensemble on $[-m, m)$ by the averaged product measure construction, and let $\bar{\bar{\varrho}}_m$ be the corresponding measure obtained from the periodized grand canonical ensemble. The measures $\bar{\varrho}_m$ and $\bar{\bar{\varrho}}_m$ are translation invariant, and it may be seen that they both have correlation functions satisfying (4.2) with λ equal to the activity $\mathfrak{z}$. Furthermore, $\int K.E._{[0,1)}\, d\bar{\varrho}_m$ and $-s(\bar{\varrho}_m)$, and the corre-

[9] The *activity* $\mathfrak{z}$ corresponding to the inverse temperature β and chemical potential μ is $e^{\beta\mu}\sqrt{2\pi/\beta}$.

sponding quantities for $\bar{\bar{\varrho}}_m$, are bounded if m stays away from zero. Hence, by statement 5. of Proposition 2.15, $\{\bar{\varrho}_m : m \geqq 1\}$ and $\{\bar{\bar{\varrho}}_m : m \geqq 1\}$ have compact closures in $\mathscr{M}^1[\mathscr{X}]$. Any cluster point of the net $(\bar{\varrho}_m)$, or of the net $(\bar{\bar{\varrho}}_m)$, has a Maxwellian velocity distribution and satisfies (4.2) with $\lambda = \mathfrak{z}$, and is therefore concentrated on $[\hat{\mathscr{X}}]$. It is to such cluster points that ii) refers. We will see in § 6 that, if Φ has compact support and has a first derivative satisfying a Lipschitz condition, so that we can solve the equations of motion with $F(q) = -\frac{d}{dq}\Phi(q)$, then any cluster point of the net $(\bar{\bar{\varrho}}_m)$ is invariant under the time evolution given by this F.

§ 5. Approximation by Space-Periodized Systems

In this section we show that, if we consider the time-evolution of an infinite configuration x, but look only at those particles in some bounded interval, we find a motion which is well approximated by the evolution of a corresponding system which is space-periodized with respect to some much larger interval. To be more explicit, we will show that, for any ψ in $\mathfrak{A}$, any x in $[\hat{\mathscr{X}}]$ and any t, $\psi(T^t x) = \lim \psi(\tilde{T}^t_{[-a,b)} x)$ as a, b go to infinity in an appropriate way. Since the space-periodized evolution is constructed by putting together infinitely many independent finite systems, this result enables us to approximate infinite system by finite ones. It therefore makes possible the use of the classical mechanics of finite systems, notably of LIOUVILLE's theorem and energy conservation, to obtain information about the infinite system.

This approximation theorem will be proved as follows: we start from the equations for a finite periodized system in the form given in § 2.5, Eq. (2.13). These equations are formally identical with the equations for a non-periodized system, and we will study them in the same way. Thus, we convert these equations to a non-linear evolution equation, which we solve by successive approximations. By keeping track of the way the estimates depend on the interval $[-a, b)$ of periodization, we find that the convergence of the solution by successive approximations is uniform in a, b, provided that they are large enough and that one is not too much larger than the other. Thus, all we have to do is to show that the nth approximation for the periodized system is close to the nth approximation for the original infinite system. Now the evolutions of the two systems differ only by "boundary effects" having to do with the behavior of particles near the ends of the periodicity interval. Using the finite range of the forces, and bounds on the distances particles can travel, we can control the propagation of these boundary effects and show that, for any n and any finite interval (α, β), the n-th approximations to the motion of the particles inside (α, β) for the periodized system and the non-periodized system are identical, provided that the ends of the periodicity interval are far enough from the origin.

The details of the proof will consist primarily of the rewriting of the estimates of [1] in the slightly different context of finite periodized systems. We will frequently have to impose some restrictions on the periodicity intervals we consider, and it is convenient to have an abbreviation for this set of restrictions. We will say that an interval $[-a, b)$ is *allowable* if $a \geqq e^e$, $b \geqq e^e$, $a+b \geqq 2R$, and $1/2 \leqq \log(a)/\log(b) \leqq 2$. (No special importance should be assigned to the number e^e; it is simply a conveniently large number. A similar remark holds for the bounds on $\log(a)/\log(b)$.)

Let $[-a, b)$ be a finite interval, and let $x = (p_1, p_1; \ldots; q_n, p_n)$ be a configuration of particles in $[-a, b)$. We may regard x as a configuration in $\boldsymbol{R}$ which happens to be finite and to be contained in $[-a, b)$; with this convention we will use the definitions given in § 2.5 for $|x|$, $\mathscr{Y}_x$, $\| \ \|_x$, ${}_m\| \ \|_x$, etc. If $\tilde{x}(t)$ denotes the solution of the Eqs. (2.13) with $\tilde{x}(0) = x$, we write $\tilde{x}(t) = x + \tilde{\zeta}_x(t)$, with $\tilde{\zeta}_x(t)$ in $\mathscr{Y}_x$. The differential equations become

$$\frac{d\tilde{\zeta}(t)}{dt} = \tilde{A}_{x, a+b}(\tilde{\zeta}_x(t)) \tag{5.1}$$

where

$$\tilde{A}_{x, a+b}(\zeta) = \left(p_i + \eta_i, \sum_{j \neq i} \tilde{F}_{a+b}(q_i + \xi_i - q_j - \xi_j)\right). \tag{5.2}$$

The following lemma generalizes Lemma 3.4 of [1]:

Lemma 5.1. *There exists a constant K such that, for all allowable intervals $[-a, b)$, for all finite configurations $x = (q_1, p_1; \ldots; q_n, p_n)$ in $[-a, b)$, for all closed intervals $[\alpha, \beta] \subset [-a, b]$, for all $\lambda \geqq 1$, and for all n-tuples of numbers (ξ_i) with $\sup_i \frac{|\xi_i|}{\log_+(q_i)} \leqq \lambda$, we have the inequality:*

$$\begin{aligned} &\# \left\{ j : q_j + \xi_j \in \bigcup_{k=-\infty}^{\infty} \{[\alpha, \beta] + k(a+b)\} \right\} \\ &\leqq |x| \left\{ \beta - \alpha + K\lambda[\log_+(\lambda) + \log_+(|\alpha| \vee |\beta|)] \right\}. \end{aligned}$$

Proof. By Lemma 3.4 of [1], we have an estimate of the desired form on $\#\{j : q_j + \xi_j \in [\alpha, \beta]\}$. Thus, we want to consider j's such that $q_j + \xi_j \in [\alpha, \beta] + k(a+b)$ for some $k \neq 0$. Since $[\alpha, \beta] \subset [-a, b]$, we must at least have $q_j + \xi_j \notin (-a, b)$. But $|q_j| \leqq a \vee b$, so q_j must be within a distance $\lambda \log_+(a \vee b)$ of the boundary of $(-a, b)$, and the number of such q_j's is not greater than $2\,|x|\,\lambda \log_+(a \vee b)$. On the other hand, the interval $[\alpha, \beta] + k(a+b)$ must also come within a distance $\lambda \log_+(a \vee b)$ of the boundary of $(-a, b)$, and this implies that

$$|\alpha| \vee |\beta| \geqq a \wedge b - \lambda \log_+(a \vee b).$$

Using the inequalities

$$\log_+(a \vee b) \leqq 2 \log_+(a \wedge b)$$

and

$$\frac{1}{2}\log_+(a \vee b) - \log_+(\log_+(a \vee b)) \geqq \frac{1}{10}\log_+(a \vee b)$$

(which follows from $a \vee b \geqq e^e$), and also using the sub-additivity of $\log_+$, we get

$$\begin{aligned}\log_+(|\alpha| \vee |\beta|) &\geqq \log_+(a \wedge b) - \log_+(\lambda \log_+(a \vee b)) \\ &\geqq \frac{1}{2}\log_+(a \vee b) - \log_+(\lambda) - \log_+(\log_+(a \vee b)) \\ &\geqq \frac{1}{10}\log_+(a \vee b) - \log_+(\lambda)\,.\end{aligned}$$

Hence,

$$\begin{aligned}&\# \Big\{j : q_j + \xi_j \in \bigcup_{k \neq 0} ([\alpha, \beta] + k(a+b))\Big\} \\ &\leqq 20\,|x|\,\lambda\,[\log_+(|\alpha| \vee |\beta|) + \log_+(\lambda)]\end{aligned}$$

so the proof of the lemma is complete.

Proposition 5.2. *There exist constants C, D such that for all allowable $[-a, b)$, all configurations x in $[-a, b)$, and all ζ in $\mathscr{Y}_x$, we have*

$$\|\tilde{A}_{x,a+b}(\zeta)\| \leqq (1 + |x|)\,[C + D\,\|\zeta\|_x \log_+(\|\zeta\|_x)]\,.$$

The proof of this proposition, using Lemma 5.1, is nearly identical with the proof of Proposition 3.3 of [1]; we omit the details.

Lemma 5.3. *Let a real number d be given. Then there exists a constant B such that, for all $\alpha > 0$ there exists an m_0 such that, for all $m \geqq m_0$, all allowable $[-a, b)$, all configurations x in $[-a, b)$, and all ζ, ζ' in $\mathscr{Y}_x$ with $\|\zeta\| \leqq d$, $\|\zeta'\| \leqq d$, we have:*

$${}_m\|\tilde{A}_{x,a+b}(\zeta) - \tilde{A}_{x,a+b}(\zeta')\|_x \leqq B\,|x|\,\log_+(m)\,{}_{\alpha m}\|\zeta - \zeta'\|_x\,.$$

The proof is essentially the same as that of Lemma 4.1 of [1].

We now define:

$$\tilde{\xi}_{0,x}(t) = 0\,,$$

$$\tilde{\xi}_{n,x}(t) = \int_0^t d\tau\,\tilde{A}_{x,a+b}(\tilde{\xi}_{n-1,x}(\tau)) \quad \text{for} \quad n = 1, 2, 3, \ldots$$

Proposition 5.4. *There exist functions $\tilde{h}(\delta, T)$ and $\tilde{\varepsilon}(n, m, \delta, T)$, with*

$$\lim_{n\to\infty} \tilde{\varepsilon}(n, m, \delta, T) = 0$$

for all m, δ, T, such that:

i) $\|\tilde{\xi}_{n,x}(t)\|_x \leqq \tilde{h}(\delta, T)$

ii) ${}_m\|\tilde{\xi}_{n,x}(t) - \tilde{\xi}_x(t)\|_x \leqq \tilde{\varepsilon}(n, m, \delta, T)$

for all n, m, whenever $[-a, b)$ is an allowable interval, x is a configuration in $[-a, b)$ with $|x| \leqq \delta$, and $|t| \leqq T$.

The proof is essentially the same as that of Proposition 4.2 of [1].

We now state the principal result of this section:

Proposition 5.5. *Let $\gamma > 0$, δ, and T be given, and let ψ belong to $\mathfrak{A}$. Then there exists a real number A such that, whenever $a \geqq A$, $b \geqq A$, $1/2 \leqq \log(a)/\log(b) \leqq 2$, we have*

$$|\psi(T^t x) - \psi(\tilde{T}^t_{[-a,b)} x)| \leqq \gamma$$

if $x \in [\hat{\mathscr{X}}_\delta]$ and $|t| \leqq T$.

Proof. It suffices to prove the proposition for ψ of the form $\phi(Sf_1, \ldots, Sf_k)$, with ϕ a bounded continuous function on $\boldsymbol{R}^n$ and $f_1, \ldots, f_k$ in $\mathscr{K}_1$. Choose m so that $f_i(q, p) = 0$ for all i if $|q| \geqq m$.

For any labelled configuration x belonging to $\hat{\mathscr{X}}$ and any finite interval $[-a, b)$, let $\tilde{x}$ be the part of x in $[-a, b)$. [The index set for the finite labelled configuration $\tilde{x}$ may not be of the form $(1, 2, 3, \ldots, n)$, but this is inessential.] Note that $|\tilde{x}| \leqq |x|$. Using Propositions 2.18 and 5.4, we see that there is a constant H such that

$$\|\zeta_{n,x}(t)\|_x \leqq H, \qquad \|\tilde{\zeta}_{n,\tilde{x}}(t)\|_{\tilde{x}} \leqq H \tag{5.3}$$

whenever $[-a, b)$ is allowable, $|x| \leqq \delta$, and $|t| \leqq T$.

Choose m_0 so that

$$|q| - 2H \log_+(q) \leqq m \quad \text{implies} \quad |q| \leqq m_0 \,. \tag{5.4}$$

Now these inequalities imply that, if $[-a, b)$ is allowable, if $|x| \leqq \delta$, if $|t| \leqq T$, and if $a \geqq m$, $b \geqq m$, the sums defining $Sf_j(T^t x)$ and $Sf_j(\tilde{T}^t_{[-a,b)} x)$ can be restricted to those i's with $|q_i| \leqq m_0$. The number of terms in such a sum is not greater than $2\,\delta m_0$, and (5.3) enables us to put an upper bound on the velocities of the particles entering the sum. Hence, the f_j, and also ϕ, are uniformly continuous on the relevant ranges of variables. To prove the proposition, then, it will suffice to prove the following assertion: For all m_0, δ, T and $\varepsilon > 0$, there exists a real number $A > m_0$ such that, if $1/2 \leqq \log(a)/\log(b) \leqq 2$, if $a \geqq A$, $b \geqq A$, if $|x| \leqq \delta$, and if $|t| \leqq T$, then

$$\sup\left\{\frac{|\tilde{\xi}_{i,\tilde{x}}(t) - \xi_{i,x}(t)| \vee |\tilde{\eta}_{i,\tilde{x}}(t) - \eta_{i,x}(t)|}{\log_+(q_i)} : |q_i| \leqq m_0\right\} \leqq \varepsilon \,. \tag{5.5}$$

Again using Propositions 2.18 and 5.4, we see that there exists n such that

$$_{m_0}\|\zeta_{n,x}(t) - \zeta_x(t)\|_x \leqq \varepsilon/2$$

$$_{m_0}\|\tilde{\zeta}_{n,x}(t) - \tilde{\zeta}_x(t)\|_x \leqq \varepsilon/2$$

whenever $|t| \leqq T$, $|x| \leqq \delta$, and $[-a, b)$ is allowable. Comparing these inequalities with (5.5) shows that it suffices to find, for any given n, a constant $A \geqq m_0$ such that, whenever $[-a, b)$, t, x are as above, with

$a \geqq A$, $b \geqq A$, we have

$$\tilde{\xi}_{i,n,\tilde{x}}(t) = \xi_{i,n,x}(t), \qquad \tilde{\eta}_{i,n,\tilde{x}}(t) = \eta_{i,n,x}(t) \tag{5.6}$$

for all i with $|q_i| \leqq m_0$.

We choose successively $m_1, m_2, \ldots, m_n$ so that, if $|q| \leqq m_i$, $|q'| \geqq m_{i+1}$,

$$|q| + H \log(q) + R < |q'| - 2H \log_+(q') . \tag{5.7}$$

We assert that we can take $A = m_n$. We will prove (5.6) by showing by induction that, for $[-a, b)$, t, x as above, for $a \geqq m_n$ and $b \geqq m_n$, and for $0 \leqq k \leqq n$, if $|q_i| \leqq m_{n-k}$, then

$$\tilde{\xi}_{i,k,\tilde{x}}(t) = \xi_{i,k,x}(t) , \qquad \tilde{\eta}_{i,k,\tilde{x}}(t) = \eta_{i,k,x}(t) . \tag{5.8}$$

This is clearly true for $k = 0$ since everything is then identically zero. Suppose it is true for k; we will prove it for $k + 1$. Now

$$\tilde{\xi}_{i,k+1,x}(t) = \int_0^t d\tau \, [p_i + \tilde{\eta}_{i,k,x}(\tau)] , \tag{5.9}$$

$$\tilde{\eta}_{i,k+1,\tilde{x}}(t) = \int_0^t d\tau \left[\sum_{j \neq i} \tilde{F}_{a+b}(q_i + \tilde{\xi}_{i,k,\tilde{x}}(\tau) - q_j - \tilde{\xi}_{j,k,\tilde{x}}(\tau)) \right], \tag{5.10}$$

and corresponding Eq. (2.12) hold for $\xi_{i,k+1,x}(t)$ and $\eta_{i,k+1,x}(t)$. If $|q_i| \leqq m_{n-k-1}$, then $|q_i| \leqq m_{n-k}$, so $\tilde{\eta}_{i,k,\tilde{x}}(\tau) = \eta_{i,k,x}(\tau)$ by the induction hypothesis. Hence, by (5.9) and the first part of (2.12), $\tilde{\xi}_{i,k+1,\tilde{x}}(t) = \xi_{i,k+1,x}(t)$.

From the inequalities (5.3) and (5.7) it follows that, if $|q_i| \leqq m_{n-k-1}$, the sums over j in (5.10) and in the second part of (2.12) may be restricted to those j's with $|q_j| \leqq m_{n-k}$, and $\tilde{F}_{a+b}$ may be replaced by F. But from the induction hypothesis

$$\tilde{\xi}_{j,k,\tilde{x}}(\tau) = \xi_{j,k,x}(\tau)$$

for all j with $|q_j| \leqq m_{n-k}$. This proves the induction step (5.8) and therefore the proposition.

§ 6. Equilibrium States

In this section, we prove two propositions which imply that many infinite-volume limits of thermodynamic ensembles are invariant under the time-evolution defined by the corresponding potentials.

Proposition 6.1. *Let* (a_α) *be a net of positive numbers, with* $\lim_\alpha a_\alpha = \infty$. *For each* α, *let* ϱ_α *be a probability measure on* $[\hat{\mathscr{X}}]$ *which is invariant under* $\tilde{T}^t_{[-a_\alpha, a_\alpha)}$. *Suppose that*

$$\lim_{\delta \to \infty} \varrho_\alpha(\complement[\hat{\mathscr{X}}_\delta]) = 0$$

uniformly in α, *and that*

$$\lim_\alpha \int d\varrho_\alpha \psi = \varrho(\psi)$$

for all ψ in $\mathfrak{A}$. *Then the state* ϱ *of* $\mathfrak{A}$ *is a probability measure concentrated on* $[\mathscr{X}]$ *and is invariant under* T^t.

Proposition 6.2. *Let* (a_α) *be a net of positive numbers, with* $\lim_\alpha a_\alpha = \infty$. *For each* α, *let* ϱ_α *be a probability measure on* $[\hat{\mathscr{X}}]$ *which is invariant under* $\tilde{T}^t_{[-a_\alpha, a_\alpha)}$. *Let* $\bar{\varrho}_\alpha = \frac{1}{2a_\alpha} \int_{-a_\alpha}^{a_\alpha} ds\,(\tau_s\,\varrho_\alpha)$ *and suppose that*

$$\lim_{\delta \to \infty} \bar{\varrho}_\alpha(\complement\,[\hat{\mathscr{X}}_\delta]) = 0$$

uniformly in α *and that*

$$\lim_\alpha \int d\bar{\varrho}_\alpha \psi = \varrho(\psi)$$

for every ψ *in* $\mathfrak{A}$. *Then the state* ϱ *of* $\mathfrak{A}$ *is a probability measure concentrated on* $[\hat{\mathscr{X}}]$ *and is invariant under* T^t.

We will give the details only for Proposition 6.2; the proof of Proposition 6.1 is similar but less complicated. Let us first dispose of showing that ϱ is concentrated on $[\hat{\mathscr{X}}]$. Choose δ so that $\bar{\varrho}_\alpha(\complement\,[\hat{\mathscr{X}}_\delta]) \geqq 1 - \varepsilon$ for all α. Since ϱ is a measure on the spectrum of $\mathfrak{A}$, and since, by Proposition 3.1, $[\hat{\mathscr{X}}_\delta]$ is a compact subset of the spectrum of $\mathfrak{A}$, we have:

$$\varrho([\hat{\mathscr{X}}_\delta]) = \inf\{\varrho(\psi) : \psi \in \mathfrak{A},\ \psi \geqq 0,\ \psi \geqq 1 \text{ on } [\hat{\mathscr{X}}_\delta]\}\,.$$

But for any such ψ, $\bar{\varrho}_\alpha(\psi) \geqq 1 - \varepsilon$ for all α, so $\varrho(\psi) \geqq 1 - \varepsilon$; hence, $\varrho([\hat{\mathscr{X}}_\delta]) \geqq 1 - \varepsilon$. We can make this argument for any $\varepsilon > 0$, so $\varrho([\hat{\mathscr{X}}]) = 1$.

To prove the rest of the proposition, it suffices to show that

$$\int d\varrho\ \psi \circ T^t = \int d\varrho\ \psi \tag{6.1}$$

for all ψ in $\mathfrak{A}$ with $\|\psi\| \leqq 1$. The first thing we want to show is that

$$\int d\varrho\ \psi \circ T^t = \lim_\alpha \int d\bar{\varrho}_\alpha\ \psi \circ T^t\,. \tag{6.2}$$

This is not immediate since $\psi \circ T^t$ is not in $\mathfrak{A}$. However, we can argue as follows: Let δ be chosen large enough so that $\bar{\varrho}_\alpha(\complement\,[\mathscr{X}_\delta]) \leqq \varepsilon$ for all α. By Proposition 3.3, $\psi \circ T^t$ is continuous on $[\hat{\mathscr{X}}_\delta]$. Since $[\hat{\mathscr{X}}_\delta]$ is compact in the spectrum of $\mathfrak{A}$, the Tietze extension theorem asserts that there exists $\hat{\psi}$ in $\mathfrak{A}$ with $\|\hat{\psi}\| \leqq 1$, such that $\hat{\psi} = \psi \circ T^t$ on $[\hat{\mathscr{X}}_\delta]$. Then

$$\left|\int d\bar{\varrho}_\alpha(\psi \circ T^t - \hat{\psi})\right| \leqq 2\,\varepsilon$$

for all α, and similarly

$$\left|\int d\varrho(\psi \circ T^t - \hat{\psi})\right| \leqq 2\,\varepsilon\,.$$

Hence, whenever α is large enough so that $|\varrho(\hat{\psi}) - \bar{\varrho}_\alpha(\hat{\psi})| \leqq \varepsilon$, we have

$$\begin{aligned}\left|\int d\varrho(\psi \circ T^t) - \int d\varrho_\alpha(\psi \circ T^t)\right| &\leqq \left|\int d\varrho(\psi \circ T^t - \hat{\psi})\right| \\ &+ |\varrho(\hat{\psi}) - \bar{\varrho}_\alpha(\hat{\psi})| + \left|\int d\bar{\varrho}_\alpha(\psi \circ T^t - \hat{\psi})\right| \leqq 5\,\varepsilon\,,\end{aligned}$$

so (6.2) is proved.

Next observe that Eq. (2.3), applied with f equal to the characteristic function of $[\hat{\mathscr{X}}]$, implies that $\tau_s(\varrho_\alpha)$ is concentrated on $[\hat{\mathscr{X}}]$ for almost all (and therefore for all) s between $-a_\alpha$ and a_α. Also notice that:

$$\begin{aligned}
\int d\bar{\varrho}_\alpha\,\psi &= \frac{1}{2a_\alpha}\int_{-a_\alpha}^{a_\alpha} ds \int d\varrho_\alpha \psi\circ\tau_s \\
&= \frac{1}{2a_\alpha}\int_{-a_\alpha}^{a_\alpha} ds \int d\varrho_\alpha \psi\circ\tau_s\circ \tilde{T}^t_{[-a_\alpha, a_\alpha)} \\
&= \frac{1}{2a_\alpha}\int_{-a_\alpha}^{a_\alpha} ds \int d\varrho\ \psi\circ \tilde{T}^t_{[-a_\alpha+s, a_\alpha+s)}\circ\tau_s \\
&= \int d\bar{\varrho}_\alpha(\psi\circ T^t) + \frac{1}{2a_\alpha}\int_{-a_\alpha}^{a_\alpha} ds\int d\varrho_\alpha \\
&\quad\cdot\left[\psi\circ\tilde{T}^t_{[-a_\alpha+s, a_\alpha+s)} - \psi\circ T^t\right]\circ\tau_s .
\end{aligned}$$

[The first equality is just the definition of $\bar{\varrho}_\alpha$; the second follows from the invariance of ϱ_α under $\tilde{T}^t_{[-a_\alpha, a_\alpha)}$; the third follows from:

$$\tau_s\circ\tilde{T}^t_{[-a_\alpha, a_\alpha)} = \tilde{T}^t_{[-a_\alpha+s, a_\alpha+s)}\circ\tau_s ;$$

and the fourth uses Eq. (2.3) with $f=\psi\circ T^t$.]

Let $b_\alpha = \sup\{s<a_\alpha : 2\log(a_\alpha - s) \geqq \log(a_\alpha+s)\}$; then $\lim_\alpha \frac{b_\alpha}{a_\alpha} = 1$, and proving (6.1) reduces to proving:

$$\lim_\alpha \frac{1}{2a_\alpha}\int_{-b_\alpha}^{b_\alpha} ds \int d\varrho_\alpha\left[\psi\circ\tilde{T}^t_{[-a_\alpha+s, a_\alpha+s)} - \psi\circ T^t\right]\circ\tau_s = 0 . \quad (6.3)$$

Also, $\lim_\alpha(a_\alpha - b_\alpha)=\infty$, and, if $|s|\leqq b_\alpha$, $1/2\leqq \log(a_\alpha-s)/\log(a_\alpha+s)\leqq 2$.

We are therefore in a position to apply Proposition 5.5. For any δ, define χ_δ on $[\mathscr{X}]$ by

$$\begin{aligned}
\chi_\delta &= \frac{1}{\delta} \text{ on } [\hat{\mathscr{X}}_\delta] \\
&= 2 \text{ on } \complement[\hat{\mathscr{X}}_\delta] .
\end{aligned}$$

Then for sufficiently large α

$$\sup_{|s|\leqq b_\alpha}\left|\psi\circ T^t_{[-a_\alpha+s, a_\alpha+s)} - \psi\circ T^t\right| \leqq \chi_\delta .$$

Hence,

$$\begin{aligned}
&\limsup_\alpha \frac{1}{2a_\alpha}\left|\int_{-b_\alpha}^{b_\alpha} ds \int d\varrho_\alpha\left[\psi\circ\tilde{T}^t_{[-a_\alpha+s, a_\alpha+s)} - \psi\circ T^t\right]\circ\tau_s\right| \\
&\leqq \limsup_\alpha \int d\bar{\varrho}_\alpha\,\chi_\delta
\end{aligned}$$

for all δ. Since $\lim_{\delta\to\infty} \int d\bar{\varrho}_\alpha \chi_\delta = 0$ uniformly in α, (6.3) is proved and the proof of the proposition is complete.

We now describe two applications of these propositions to proving that thermodynamic limit states are invariant under the time evolution. Assume that $F(q) = -\frac{d}{dq}\Phi(q)$, where Φ is even, of compact support, and stable. We choose an inverse temperature and a chemical potential, and we construct, for every real number $m \geqq R$ which is large enough so that the P-stability inequality holds, the periodized grand canonical ensemble on $[-m, m]$. Recall that, in our terminology, this is a probability measure on $[\mathscr{X}]([-m, m))$. Let $\tilde{\varrho}_m$ be the measure on $[\mathscr{X}]$ obtained from this measure by the product measure construction. By LIOUVILLE's theorem and energy conservation, $\tilde{\varrho}_m$ is invariant under $\tilde{T}^t_{[-m,m)}$.

A straightforward adaptation of the arguments in [8] shows that, if the activity is sufficiently small, the measures $\tilde{\varrho}_m$:

i) have correlation functions satisfying a bound of the form (4.2), where λ may be taken to be independent of m;

ii) converge as m goes to infinity to a translation-invariant measure ϱ on $[\mathscr{X}]$.

Moreover, ϱ is the same state as is obtained in [8] as the infinite volume limit of non-periodized grand canonical ensembles. The discussion in this reference is in fact formulated in terms of correlation functions and not in terms of measures, but, because of the bound (4.2), statements about correlation functions may easily be translated into statements about measures[10].

Taking into account Proposition 4.1, we see that

$$\lim_{\delta\to\infty} \tilde{\varrho}_m(\complement[\hat{\mathscr{X}}_\delta]) = 0$$

uniformly in m. Hence, Proposition 6.1 applies and shows that ϱ is invariant under the time evolution.

Now assume that Φ is non-negative, rather than merely stable, and consider $\bar{\tilde{\varrho}}_m$, the average of $\tilde{\varrho}_m$ over translations, for any fixed values of the inverse temperature and chemical potential (i.e., not necessarily for

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[10] If a measure $\hat{\varrho}$ on the space of locally finite position configurations has correlation functions $\bar{\varrho}$ satisfying (4.2), then the measure may be recovered from the correlation functions as follows: Let Λ be a bounded Borel subset in $\boldsymbol{R}$, and let E be a symmetric Borel subset of Λ^n. The $\hat{\varrho}$-measure of the set of all configurations having precisely n particles in Λ and those n particles distributed so that their co-ordinates form a point of E is

$$\frac{1}{n!}\sum_{j=0}^{\infty}\frac{(-1)^j}{j!}\int_E dq_1,\ldots,dq_n \int_{\Lambda^j} dq_1',\ldots,dq_j'\, \bar{\varrho}_{n+j}(q_1,\ldots,q_n,q_1',\ldots,q_j')\,.$$

This formula, which does not seem to appear in the literature, was pointed out to me by D. RUELLE.

small activity). By the last paragraph of § 4, $\{\bar{\bar{\varrho}}_m\}$ has compact closure in $\mathscr{M}^1[\mathscr{X}]$ and

$$\lim_{\delta \to \infty} \bar{\bar{\varrho}}_m(\complement[\hat{\mathscr{X}}_\delta]) = 0$$

uniformly in m. Hence, Proposition 6.2 implies that any cluster point of the net $(\bar{\bar{\varrho}}_m)$ is invariant under the time evolution.

§ 7. Conservation of Entropy

In this section, we will assume that the interparticle force F is of the form $-\frac{d}{dq}\Phi(q)$, where Φ is even, of compact support, and stable, and we will consider probability measures ϱ on $[\mathscr{X}]$ which are translation invariant and concentrated on $[\hat{\mathscr{X}}]$, and which have, in addition,

$$\int d\varrho\, N^2_{[0,1)} < \infty\,, \qquad \int d\varrho\, K.E._{[0,1)} < \infty\,.$$

For such a measure ϱ, we denote by ϱ^t the measure $\varrho \circ T^t$. The main result of this section is the following:

Proposition 7.1. *Let ϱ, F be as above. Then, for any t,*

$$\int d\varrho^t\, K.E._{[0,1)} < \infty$$

so $\bar{s}(\varrho^t)$ is defined, and we have:

$$\bar{s}(\varrho^t) \geqq \bar{s}(\varrho)\,.$$

This would immediately imply that $s(\varrho^t) = s(\varrho)$ if we knew that

$$\int d\varrho^t\, N^2_{[0,1)} < \infty\,;$$

we will also prove this, but under more restrictive assumptions on the potential Φ.

In outline, the proof of Proposition 7.1 goes as follows:

1. We consider, instead of ϱ^t, the measure $\varrho^t_n = \varrho \circ \tilde{T}^t_{[-n,n)}$, which should be a good approximation to ϱ^t for large n, by Proposition 5.5. Although it need not be translation invariant, ϱ^t_n is periodic with period 2 n.

2. Using conservation of energy for the periodized system. together with the P-stability of the potential, we obtain a bound

$$\frac{1}{2n}\int d\varrho^t_n\, K.E._{[-n,n)} \leqq M$$

valid for large n. In particular $\bar{s}(\varrho^t_n)$ is defined for such n.

3. Using LIOUVILLE's theorem, we show that

$$\bar{s}(\varrho^t_n) = \bar{s}(\varrho)\,.$$

4. Denoting by $\bar{\varrho}_n^t$ the average of ϱ_n^t over translations

$$\bar{\varrho}_n^t = \frac{1}{2n} \int_{-n}^{n} ds\, \tau_s \varrho_n^t ,$$

we have by Corollary 2.17 $\bar{s}(\bar{\varrho}_n^t) = \bar{s}(\varrho)$

$$\int d\bar{\varrho}_n^t\, K.E._{[0,1)} = \frac{1}{2n} \int K.E._{[-n,n)}\, d\varrho_n^t \leqq M .$$

5. Finally, using Proposition 5.5, we prove

$$\lim_{n\to\infty} \bar{\varrho}_n^t = \varrho^t$$

which implies by a semi-continuity argument:

$$\int K.E._{[0,1)}\, d\varrho^t \leqq \liminf_n \int K.E._{[0,1)}\, d\bar{\varrho}_n^t \leqq M$$

$$\bar{s}(\varrho^t) \geqq \limsup_n \bar{s}(\bar{\varrho}_n^t) = \bar{s}(\varrho) .$$

We now proceed to fill in the details. Step 1. merely defines the notation. For step 2., we define the energy in $[-n, n)$ for the periodized interaction as:

$$\tilde{E}_{[-n,n)} = K.E._{[-n,n)} + \widetilde{P.E.}_{[-n,n)}$$

$$\widetilde{P.E.}_{[-n,n)}(x) = \frac{1}{2} \sum_{\substack{i \neq j \\ q_i, q_j \in [-n,n)}} \tilde{\Phi}_{2n}(q_i - q_j)$$

where (q_i, p_i) is a representative of x. By conservation of energy for the periodized system,

$$\tilde{E}_{[-n,n)} \circ \tilde{T}^t_{[-n,n)} = \tilde{E}_{[-n,n)} .$$

We will prove that $\tilde{E}_{[-n,n)}$ is ϱ-integrable and estimate its integral. By the translation invariance of ϱ,

$$\int d\varrho\, K.E._{[-n,n)} = 2n \int d\varrho\, K.E._{[0,1)} ;$$

on the other hand, if K is an upper bound for $|\Phi|$, and if J is an integer not smaller than R, then

$$|\widetilde{P.E.}_{[-n,n)}| \leqq \frac{K}{2} \sum_{i=-n}^{n-1} \sum_{j=-J}^{J} N_{[i,i+1)}\, N_{[i \dotplus j,\, i \dotplus j+1)} ,$$

where $i \dotplus j$ means that integer in $[-n,n)$ which is equal to $i+j$ modulo $2n$. Applying the Schwarz inequality and translation invariance, we get

$$\int d\varrho\, \widetilde{P.E.}_{[-n,n)} \leqq K(2J+1)\, n \int d\varrho\, N^2_{[0,1)} .$$

Hence,

$$\int d\varrho_n^t\, \tilde{E}_{[-n,n)} = \int d\varrho\, \tilde{E}_{[-n,n)} \circ \tilde{T}^t_{[-n,n)} = \int d\varrho\, \tilde{E}_{[-n,n)} \leqq 2n\, M' ,$$

where M' does not depend on n.

Now suppose that n is large enough so that

$$\sum_{1 \leqq i < j \leqq m} \tilde{\Phi}_{2n}(q_i - q_j) \geqq -Bm$$

for all $m, q_1, \ldots, q_m$. Then

$$\widetilde{P.E.}_{[-n,n)} + B\, N_{[-n,n)} \geqq 0\,.$$

But we also have

$$\int d\varrho_n^t\, N_{[-n,n)} = \int d\varrho\, N_{[-n,n)} \circ \tilde{T}^t_{[-n,n)} = \int d\varrho\, N_{[-n,n)}$$
$$= 2n \int d\varrho\, N_{[0,1)}$$

and therefore

$$\int d\varrho_n^t\, K.E._{[-n,n)} \leqq \int d\varrho_n^t\, [\tilde{E}_{[-n,n)} + B\, N_{[-n,n)}] \leqq 2n\, M$$

where again M does not depend on n. This finishes the proof of 2.

To prove step 3., it suffices, by statement 2. of Proposition 2.15, to prove that

$$s_{[-jn,jn)}(\varrho_n^t) = s_{[-jn,jn)}(\varrho)$$

for all odd integers j. LIOUVILLE's theorem asserts that the measure $\sigma_{[-jn,jn)}$ is invariant under $\tilde{T}^t_{[-n,n)}$; our statement now follows from Proposition 2.13.

Step 4. is a straightforward application of Corollary 2.17. For step 5., we have first to prove

$$\lim_{n\to\infty} \frac{1}{2n} \int_{-n}^{n} ds \int d\varrho\, \psi \circ \tau_s \circ T^t_{[-n,n)} = \int d\varrho\, \psi \circ T^t \qquad (7.1)$$

for all ψ in $\mathfrak{A}$. Using the equation

$$\tau_s \circ \tilde{T}^t_{[-n,n)} = \tilde{T}^t_{[-n+s,n+s)} \circ \tau_s$$

and the translation invariance of ϱ, we have

$$\frac{1}{2n} \int_{-n}^{n} ds \int d\varrho\, \psi \circ \tau_s \circ \tilde{T}^t_{[-n,n)}$$
$$= \frac{1}{2} \int_{-1}^{1} da \int d\varrho\, \psi \circ \tilde{T}^t_{[-n(1-a),n(1+a))}\,.$$

By Proposition 5.5, if $-1 < a < 1$,

$$\lim_{n\to\infty} \psi \circ \tilde{T}^t_{[-n(1-a),n(1+a))} = \psi \circ T^t$$

on $[\hat{\mathscr{X}}]$; hence, applying the dominated convergence theorem twice gives (7.1).

To finish the proof of step 5., and hence of the proposition, we have only to show that

$$\int K.E._{[0,1)}\, d\varrho^t \leqq M\,;$$

then the statement about the mean entropy will follow from statement 6 of Proposition 2.15. Because ϱ^t is translation invariant, $K.E._{(0,1)} = K.E._{[0,1)}$ almost everywhere, so it will suffice to prove that

$$\varrho \longmapsto \int K.E._{(0,1)}\, d\varrho$$

is a lower semi-continuous function on $\mathscr{M}^1[\mathscr{X}]$. Let (χ_j) be an increasing sequence of non-negative continuous functions on $\boldsymbol{R}$ converging pointwise to the characteristic function of (0,1), and let $f_j(q, p) = \chi_j(q) \frac{p^2}{2}$. Then $\psi_j = (Sf_j) \wedge j$ is an increasing sequence in $\mathfrak{A}$ such that

$$\int K.E._{(0,1)}\, d\varrho = \sup_j \varrho(\psi_j)$$

for all ϱ in $\mathscr{M}^1[\mathscr{X}]$. Since $\varrho \longmapsto \varrho(\psi_j)$ is continuous, our assertion is proved.

Next, we take up the question of the integrability of $N^2_{(0,1)}$ with respect to ϱ^t.

Proposition 7.2. *Let ϱ be as in the first paragraph of this section; assume that $F(q) = -\frac{d}{dq}\Phi(q)$, where Φ has compact support and is of the form $\Phi_1 + \Phi_2$, with Φ_1 and Φ_2 both even, Φ_1 P-stable, and Φ_2 non-negative and bounded away from zero on a neighborhood of the origin. Then*

$$\int d\varrho^t\, N^2_{[0,1)} < \infty\,.$$

Proof. We will keep the notation of the proof of the preceding proposition. Since ϱ^t is translation invariant,

$$\int d\varrho^t\, N^2_{[0,1)} = \int d\varrho^t\, N^2_{(0,1)}\,.$$

Arguing as in the proof of step 5. of the preceding proposition, we see that $\varrho \longmapsto \int d\varrho\, N^2_{(0,1)}$ is lower semi-continuous on $\mathscr{M}^1[\mathscr{X}]$ and therefore that it suffices to obtain an upper bound on

$$\frac{1}{2n} \sum_{j=-n}^{n-1} \int d\bar{\varrho}^t_n\, N^2_{(j,j+1)} \text{ which is uniform in } n \text{ for large } n.$$

By Eq. (2.3),

$$\begin{aligned}\sum_{j=-n}^{n-1} \int d\bar{\varrho}^t_n\; N^2_{(j,j+1)} &= \frac{1}{2n} \int_0^{2n} ds \int d\varrho^t_n \sum_{j=-n}^{n-1} N^2_{(j,j+1)} \circ \tau_s \\ &= \frac{1}{2n} \int_0^{2n} ds \int d\varrho^t_n \sum_{j=-n}^{n-1} N^2_{(j-s,j-s+1)}\end{aligned}$$

For any value of s between 0 and $2n$, some of the intervals $(j-s, j-s+1)$ will be contained in $(-n, n)$ and some in $(-3n, -n)$. One, at most, will contain $-n$. If $j - s < -n < j - s + 1$, we can estimate

$$\begin{aligned}\int d\varrho^t_n\, N^2_{(j-s,j-s+1)} &= \int d\varrho^t_n\, (N_{(j-s,-n)} + N_{[-n,j-s+1)})^2 \\ &\leq 2 \int d\varrho^t_n\, [N^2_{(-n-1,-n)} + N^2_{[-n,-n+1)}]\,.\end{aligned}$$

From these two remarks, and the periodicity of ϱ_n^t, we see that to prove the proposition it will be sufficient to show that there is a constant K such that, whenever n is large enough,

$$\frac{1}{2n} \int d\varrho_n^t \sum_j N_{I_j}^2 \leqq K$$

for all pairwise-disjoint collections $\{I_1, I_2, \ldots\}$ of intervals of unit length contained in $[-n, n)$. Using the conservation of energy and particle number for the space-periodized evolution, we can get such an estimate if we can find constants C, C' such that

$$\sum_j N_{I_j}^2 \leqq C\, \tilde{E}_{[-n,n)} + C'\, N_{[-n,n)}$$

whenever n is large enough and $\{I_1, I_2, \ldots\}$ is as above.

From the P-stability of Φ_1, we have

$$2\, \tilde{E}_{[-n,n)}(x) \geqq -B\, N_{[-n,n)}(x) + \sum_{\substack{i \neq j \\ q_i, q_j \in [-n,n)}} \tilde{\Phi}_{2,2n}(q_i - q_j)$$

(where (q_i, p_i) is a representative of x).

Since Φ_2 is non-negative, we can omit any terms we like from the sum on the right; we keep only those pairs with q_i, q_j both belonging to the same one of the I_k's. The proposition now follows from the fact [9] that there exist B', B'', with $B'' > 0$, such that, for all m and all $q_1, \ldots, q_m$ in $[0,1]$,

$$\sum_{i \neq j} \Phi_2(q_i - q_j) \geqq -B'm + B''m^2 .$$

Acknowledgements. I am grateful to D. RUELLE for several helpful suggestions and for reading the manuscript, and to G. GALLAVOTTI and S. MIRACLE for useful discussions. I also thank Monsieur L. MOTCHANE for his hospitality at the Institut des Hautes Etudes Scientifiques.

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