A NOTE ON A PAPER OF GINSBURG

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Introduction. Let P be a partially ordered system and let S and T be non-empty subsets of P. If, for every $p \varepsilon S$, there exists a $q \varepsilon T$ such that $q \geq p$, T is said to be confinal in S. For every $p \varepsilon P$, we denote the set of successors of p in P by $A_P(p)$. If two partially ordered systems P and Q are order isomorphic with confinal subsets of some partially ordered system, they are said to be confinally similar. A partially ordered system P without maximal elements is said to have sufficiently many non-cofinal subsets if, for any two distinct elements p and q of P, either $A_P(p)$ is not cofinal in $A_P(q)$ or $A_P(q)$ is not cofinal in $A_P(p)$. The properties of sets having sufficiently many non-cofinal subsets have been investigated by Ginsburg [1], who poses the following question: "If P has sufficiently many non-cofinal subsets and Q is cofinally similar to P, does Q contain a cofinal subset S which has sufficiently many non-cofinal subsets?" It will be shown by example that the answer to this question is negative.

A subset S of a partially ordered system P is said to be a residual subset if, for every $p \varepsilon S$, $A_P(p)$ is contained in S. A subset S of P is said to be maximal residual if S is a residual subset which is not a proper cofinal subset of any residual subset of P. The set of maximal residual subsets of P, ordered by the dual of set inclusion, is denoted by $F(P)$. Ginsburg proves the following theorem (Theorem 5 of [1]): *If P has sufficiently many non-cofinal subsets, P is cofinally similar to $F(P)$.* It is shown that the proof given for this theorem is invalid, and a counterexample to the theorem is given.

1. An example. An example is to be given of two cofinally similar partially ordered systems, one of which has sufficiently many non-cofinal subsets and the other of which contains no cofinal subset having sufficiently many non-cofinal subsets.

Let ω_1 be the first non-denumerable ordinal, and let $W(\omega_1)$ be the set of ordinals less than ω_1. Associate with each $x \varepsilon W(\omega_1)$ an infinite subset A_x of the set of integers in such a way that distinct ordinals are assigned distinct sets of integers. Now, for any finite set of integers A, one of the following two cases occurs:

i. For each $x \varepsilon W(\omega_1)$ there exists an $s \varepsilon W(\omega_1)$, $s \geq x$, such that $A \subset A_s$.

ii. For some $x \varepsilon W(\omega_1)$, A is not contained in A_s for any $s \geq x$, while, for all $y < x$, there exists a $z \varepsilon W(\omega_1)$, $z \geq y$, such that $A \subset A_z$.

We now consider the set of all x's associated with sets of integers in the second category. This is a denumerable set of denumerable ordinals; hence, there exists a denumerable ordinal ω' which is greater than any of the ordinals

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in this set. Let W' denote the set of denumerable ordinals greater than ω'. From the definition of W' it follows that if A is a finite set of integers contained in some A_x with $x \in W'$, and if $y \in W'$, then there exists a $z \in W'$, $z \geq y$, such that $A \subset A_z$.

Define a partial ordering on the set of finite sequences of integers by $(a_1, \cdots, a_n) \geq (b_1, \cdots, b_m)$ if $n \geq m$ and $a_1 = b_1, \cdots, a_m = b_m$. For ease of writing we will write $(a_1, \cdots, a_n) \subset A$ if $a_i \in A$, $i = 1, \cdots, n$. We are not restricting ourselves to considering only sequences all of whose elements are distinct.

Consider the set P of elements of the form $(x, F, +)$ and $(x, F, -)$, where $x \in W'$ and F is a finite sequence of integers, $F \subset A_x$. We may define a partial ordering on P by

1. $(x, F, -) \geq (y, G, -)$ if $x \geq y$, $F \geq G$.
2. $(x, F, +) \geq (y, G, -)$ if $x \geq y$, $F \geq G$.
3. $p \geq (x, F, +)$ if either $p = (x, F, +)$ or $p \geq$ some $(x, G, -) \in P$, where $G > F$, $G \neq F$.

It is easy to show that this is indeed a partial ordering, i.e., that it is transitive, reflexive, and anti-symmetric. It is also evident that P has no maximal element.

We separate P into two subsets P_+ and P_- consisting of the elements of P with $+$ and $-$ signs respectively. Both are clearly cofinal subsets of P; hence, they are cofinally similar.

We shall first show that P_- contains no cofinal subset having sufficiently many non-cofinal subsets. Let S be a cofinal subset in P_-. Suppose S is denumerable. Then there exists a denumerable ordinal z such that $(x, F, -) \in S$ implies $x < z$. Then no successor of any element of P_- of the form $(z, G, -)$ belongs to S. This is a contradiction, since S is cofinal in P_-. Hence, S must be non-denumerable. Since the family of finite sequences of integers is denumerable, at least two of the elements of S must have the same sequence of integers and differ only in their ordinals. Let one such pair be $(x, F, -)$ and $(y, F, -)$ and suppose for definiteness that $x > y$. We will show that the sets of successors in S of these two elements are cofinal in each other.

Since $x > y$, $A_S((x, F, -))$ is contained in $A_S((y, F, -))$; hence, the latter set is cofinal in the former. Let $(z, G, -) \in A_S((y, F, -))$. Since $G \subset A_z$, it follows from the properties of W' that there exists a $v \in W'$, $v > \sup \{x, z\}$, such that $G \subset A_v$. Hence, $(v, G, -) \in P_-$. Since S is cofinal in P_-, there exists a successor p of $(v, G, -)$ belonging to S. It is easy to show that $(v, G, -) \geq (x, F, -)$ and $(v, G, -) \geq (z, G, -)$, so the same relations hold with $(v, G, -)$ replaced by p. Thus, $A_S((x, F, -))$ is cofinal in $A_S((y, F, -))$, so S does not have sufficiently many non-cofinal subsets.

Next, we consider P_+ and show that it has sufficiently many non-cofinal subsets. Let $(x, F, +)$ and $(y, G, +)$ be two distinct elements of P_+; we shall show that either $A_{P_+}((x, F, +))$ is not cofinal in $A_{P_+}((y, G, +))$ or $A_{P_+}((y, G, +))$

is not cofinal in $A_{P_+}((x, F, +))$. Assume first that neither $F \geq G$ nor $G \geq F$. It follows that F and G have no common successor in the set of finite sequences of integers, and hence that $A_{P_+}((x, F, +)) \cap A_{P_+}((y, G, +))$ is empty. Therefore, we need only consider the case in which F and G are comparable. Suppose that $F = G$. Then $x \neq y$, so $A_x \neq A_y$ and $D = (A_x - A_y) \cup (A_y - A_x)$ is non-empty. Let $b \in D$, and assume for definiteness that $b \in (A_y - A_x)$. If $G = (a_1, \cdots, a_n)$, let $G' = (a_1, \cdots, a_n, b)$. Then $(y, G', +) \in A_{P_+}((y, G, +))$ but $A_{P_+}((y, G', +)) \cap A_{P_+}((x, G, +))$ is empty, so $A_{P_+}((x, G, +))$ is not cofinal in $A_{P_+}((y, G, +))$. Now suppose $F \neq G$, and assume for definiteness that $F > G$. Let $F = (a_1, \cdots, a_n)$ and $G = (a_1, \cdots, a_m)$, $n > m$. Let $b \in A_y$, $b \neq a_{m+1}$. Such a b exists since A_y is infinite. Let $G' = (a_1, \cdots, a_m, b)$. Then $(y, G', +) \geq (y, G, +)$ but $A_{P_+}((y, G', +)) \cap A_{P_+}((x, F, +))$ is empty, so $A_{P_+}((x, F, +))$ is not cofinal in $A_{P_+}((y, G, +))$. This completes the proof that P_+ has sufficiently many non-cofinal subsets, so P_+ and P_- provide the desired example.

2. Cofinal similarity of P and $F(P)$. Ginsburg asserts that, if P has sufficiently many non-cofinal subsets, $F(P)$ is cofinally similar to P (Theorem 5 of [1]). The proof of this result is based on the assertion that the mapping f, which takes an element p of P into that maximal residual subset $f(p)$ which contains $A_P(p)$ as a cofinal subset, is an order isomorphism of P onto a cofinal subset of $F(P)$. This assertion is not correct. It may happen that $q \in f(p)$ (and hence $f(q) \subset f(p)$), even if $q \ngeq p$. This is in fact the case with certain pairs of elements of the set P_+ defined above. Indeed, it is not hard to show that $F(P_+)$ contains a denumerable cofinal subset and that consequently it cannot be cofinally similar to P_+. (See Appendix.)

However, the corollary to Theorem 5 of [1] is correct. Let P be a partially ordered system such that $F(P)$ has sufficiently many non-cofinal subsets; what is to be shown is that $F(P)$ is cofinally similar to $F(F(P))$. This is proved by observing that the proof of Theorem 5 is valid for $F(P)$ if it is shown that $F(P)$ has the property that $T \geq S$ if $A_{F(P)}(S)$ is cofinal in $A_{F(P)}(T)$. This suffices to guarantee that the mapping f constructed in the proof of Theorem 5 in [1] is an order isomorphism. Thus, let $S, T \in F(P)$ be such that $A_{F(P)}(S)$ is cofinal in $A_{F(P)}(T)$, and suppose that $T \ngeq S$. Then, by the definition of $F(P)$, there exists a $p \in P$ such that $p \in T - S$. Since $p \notin S$, and S is a maximal residual subset of P, S is not cofinal in $A_P(p)$. Let $q \geq p$ be such that $A_P(q) \cap S = \phi$, and let $f(q)$ denote the unique maximal residual subset of P in which $A_P(q)$ is cofinal. We shall show that $A_{F(P)}(f(q)) \cap A_{F(P)}(S) = \phi$, which contradicts the fact that $A_{F(P)}(S)$ is cofinal in $A_{F(P)}(T)$ as $f(q) \geq T$. If $A_{F(P)}(f(q)) \cap A_{F(P)}(S) \neq \phi$, there is a maximal residual subset of P contained in both $f(q)$ and S, so it suffices to show that $f(q) \cap S = \phi$. Hence, let $s \in f(q) \cap S$. Since $A_P(q)$ is cofinal in $f(q)$, there exists a $t \in A_P(q)$ such that $t \geq s$. But this implies that $t \in A_P(q) \cap S$. Since $A_P(q) \cap S = \phi$, this proves that $T \geq S$, and hence the corollary to Theorem 5 of [1].

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Appendix. Proof that P_+ is not cofinally similar to $F(P_+)$.

We begin by defining, for each finite sequence of integers F contained in some $A_x (x \in W')$, the set $B(F) = \{(x, G, +) \in P_+ \mid G \geq F\}$. It is easy to see that each $B(F)$ is a maximal residual subset of P_+. Next, we shall show that the set of such B's is cofinal in $F(P_+)$. To do this, let $S \in F(P_+)$, and let $(y, F, +) \in S$. Let $F' \subset A_y$, $F' > F$, $F' \neq F$. We shall show that $B(F')$ is contained in S. Because S is a maximal residual subset of P_+, it suffices to show that S is cofinal in $B(F')$. Let $(z, G, +) \in B(F')$ and let $G' > G$, $G' \neq G$, $G' \subset A_z$. From the properties of W' it follows that for some $v > \sup \{y, z\}$, $(v, G', +) \in P_+$. Now $(v, G', +) \geq (y, F, +)$ and consequently $(v, G', +) \in S$, since S is residual. Since $(v, G', +)$ is also a successor of $(z, G, +)$, we have shown that S is cofinal in $B(F')$. Hence, $B(F')$ is contained in S, and the set of B's is a cofinal subset of $F(P_+)$. Moreover, the set of B's is denumerable. It is easy to show, however, that any partially ordered system cofinally similar to a partially ordered system having a denumerable cofinal subset itself has a denumerable cofinal subset. Since P_+ contains no denumerable cofinal subset, $F(P_+)$ is not cofinally similar to P_+.

Reference

1. Seymour Ginsburg, *A class of everywhere branching sets*, this Journal, vol. 20(1953), pp. 521–526.

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