Physica 7D (1983) 124–125
North-Holland Publishing Company

PERIOD DOUBLING IN ONE AND SEVERAL DIMENSIONS

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Feigenbaum cascade—infinite sequences of successive period doublings–form a route from periodic to aperiodic behavior of dynamical systems. These sequences of bifurcations exhibit some striking universal features. The simplest of these features to formulate concerns the rate of accumulation of the bifurcations: If μ_n denotes the parameter value at which the nth doubling occurs, then, asymptotically,

$$\mu_n = \mu_\infty - c(4.6692\ldots)^{-n} + \text{“higher order terms”}.$$

The rate 4.6692... appears to be universal, i.e., it shows up in many apparently unrelated systems such as
–one-dimensional non-invertible mappings, such as the one-parameter family $x \to 1 - \mu x^2$ on $[-1, 1]$, where $0 < \mu < 2$;
–dissipative (volume-decreasing) invertible mappings such as the Hénon system (see below);
–dissipative differential equations, such as the Lorenz system and the five-component truncation of the two-dimensional Navier–Stokes equations studied by Franceschini et al.

The main point we want to make here is that, despite their apparent diversity, these are really all instances of *precisely the same* mathematical phenomenon, and can be understood relatively easily once one has understood period doubling for one-dimensional mappings. (There is, on the other hand, another period doubling process, occurring for area-preserving mappings of the plane, which, although analogous to dissipative period doubling, seems to be an independent mathematical phenomenon. See Eckmann, Koch, and Wittwer [1] and the references cited therein.)

To undergo dissipative period doubling, a family of mappings–or, more generally a restriction of some iterate of the mappings–must have a characteristic behavior illustrated by the Hénon family

$$(x, y) \to (1 - \mu x^2 + cy, -cx),$$

with c small. (We are restricting ourselves here, for definiteness, to the orientation-preserving case, and have called the parameters μ and $-c^2$ instead of the more traditional a and b. We think of the bifurcations as occurring as we vary μ with c held fixed.) These mappings can be visualized as acting by:
1) contracting vertically:

$$(x, y) \to (x, cy);$$

2) bending and stretching by an x-dependent vertical shift:

$$\to (x, 1 - \mu x^2 + cy);$$

3) rotating a quarter-turn clockwise:

$$\to (1 - \mu x^2 + cy, -x);$$

4) contracting again vertically:

$$\to (1 - \mu x^2 + cy, -cx).$$

The general feature we want to emphasize is that the mapping contracts its multi-dimensional domain to an almost one-dimensional one, then folds that approximately one-dimensional set back into the original domain. Furthermore, the region of strong

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†Work supported in part by NSF grant MCS81-07086AO1

folding–a vertical strip about the x-axis in the above example–is mapped away from itself and into a region of gentle folding. Finally, it is characteristic of mappings undergoing period doubling that the strong-folding region, although mapped away from itself by a first application of the mapping, is sent back into itself by a second.

The key to understanding repeated period doubling is the introduction of a *renormalization* or *doubling* operator $\mathscr{T}$ which carries a mapping F to one obtained by

–composing F with itself;
–restricting to an appropriate subdomain;
–making a change of coordinates to magnify the subdomain up to the original domain.

Roughly speaking, applying $\mathscr{T}$ divides the periods of all cycles by two but preserves their stability properties.

The idea now is to apply the renormalization group program to $\mathscr{T}$. To account for the observed universality, what one needs to show is that $\mathscr{T}$ has a fixed point and that, in the neighborhood of the fixed point, $\mathscr{T}$ is expanding in one direction and contracting in all others (i.e., that the linearization of $\mathscr{T}$ at the fixed point has a single simple eigenvalue with modulus greater than one and that the remainder of its spectrum is strictly inside the unit circle.) These facts have been established for one-dimensional mappings [2]; the proof rests on complicated numerical estimates verified (rigorously) by computer. Up to now, no one has succeeded in giving a conceptual proof.

By contrast, the theory of multi-dimensional period doubling can be reduced to the one-dimensional theory by a relatively simple conceptual argument. The argument goes roughly as follows: The space of one-dimensional mappings may be imbedded in the space of multi-dimensional mappings by associating with the one-dimensional mapping f the multi-dimensional mapping

$$F_0: (x, y) \to (f(x), 0).$$

(Here, y may have any number of components.) Such as F_0 is of course not invertible, but an arbitrarily small perturbation on F_0 can give an invertible mapping; the Hénon mapping with c small is an example. We can think of the space of F_0's as a surface M_0 in the space of all F's. What is now done is to construct a multi-dimensional doubling operator which

1) maps M_0 into itself;
2) agrees with the ordinary one-dimensional doubling operator on M_0;
3) is contractive in the directions transverse to M_0, i.e., when applied to an F near but not on M_0, gives a new mapping which is still closer to M_0.

In order to get 3) to hold, it is necessary to choose the change of variables in the construction of the doubling operator with some care.

A multi-dimensional doubling operator satisfying 1)–3) has as a fixed point the mapping

$$(x, y) \to (g(x), 0),$$

where g is the fixed point for the one-dimensional operator. Contractivity in directions transverse to M_0 guarantees that allowing the operator to act on mappings which are not strictly one-dimensional does not introduce any new expanding directions.

An analysis similar to the one described above was first given by Collet, Eckmann, and Koch [3]. In precisely this form, it is unpublished work of the author.

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