

A NUMERICAL STUDY OF THE LIKELIHOOD OF PHASE LOCKING

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We report the results of numerical computation of the size of the set of Ω 's for which the homeomorphism of the circle

$$\tilde{f}_{K,\Omega}(x) = x + \Omega + K/2\pi \sin(2\pi x) \pmod{1}$$

is phase locked, for values of K ranging from 0.1 to 1. The results suggest that for moderate values of K phase locking is unlikely except for small periods, but that for $K = 1$ almost all Ω 's give phase locking, with large periods accounting for a surprisingly large portion of the total.

1. Introduction

We are going to discuss *orientation-preserving homeomorphisms* of the circle, i.e., continuous one-to-one mappings of the circle to itself which are, intuitively, increasing. More precisely, we mean the following: Let f denote a continuous strictly increasing mapping of the real line to itself such that

$$f(x+1) = f(x) + 1.$$

Such a mapping defines by passing to quotients a continuous one-to-one mapping \tilde{f} of the circle—identified with the real numbers modulo the integers—to itself. We can *define* an orientation-preserving homeomorphism of the circle to be a mapping obtained in this way. The mapping f is not quite uniquely determined by the induced mapping \tilde{f} ; two f 's induce the same \tilde{f} if and only if they differ by an integer constant. We can use this arbitrariness to arrange, say, that $0 \leq f(0) < 1$; f will then be unique.

From the point of view of dynamics, orientation-preserving homeomorphisms of the circle can

be grouped broadly into two classes, those which have at least one periodic cycle and those which do not (or, equivalently, those with respectively rational and irrational rotation numbers). If an orientation-preserving homeomorphism has a cycle of some period, then every orbit for that homeomorphism is asymptotic to a cycle of that period (so in particular every cycle must have the same period). On the other hand, a (sufficiently smooth) *diffeomorphism* (differentiable mapping of the above kind, with f' strictly positive everywhere), which has no periodic cycles, can be converted, by a continuous change of coordinates, into an irrational rotation. (The precise smoothness requirement is that f' be of bounded variation. This is a theorem of Denjoy, a proof may be found in Cornfeld et al. [2], §3.4.)

A mapping with at least one cycle is frequently said to be *phase locked*. The terminology arises from the application of theory of homeomorphisms of the circle to the analysis of the behavior of a system of two weakly coupled oscillators. In connection with this and other applications, it is of interest to have some idea of how “likely” it is that an “arbitrary” homeomorphism will be phase locked. This is a question in which qualitative theory gives contradictory indications. From the

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topological standpoint, phase locking is generic, i.e., absence of periodic cycles is exceptional (Arnold [1]). From a measure-theoretic standpoint, on the other hand, absence of periodic cycles is *not* exceptional. In the paper referred to just above, for example, Arnold proves a theorem to the effect that, in a one-parameter family of *analytic* mappings which is sufficiently close (in an analytic-function topology) to a (non-constant) family of pure rotations, “most” (in the sense of Lebesgue measure) parameter values correspond to mappings which are not phase-locked. (Arnold’s results have been greatly extended by Herman [5, 6]. For useful reviews of this subject, see Rosenberg [8] and Deligne [3].)

The purpose of this note is to report some numerical results on a concrete example which

may give some feeling for what the quantitative situation is.

2. Statement of results

The example we study is the two-parameter family of mappings

$$f_{K,\Omega}(x) = x + \Omega + \frac{K}{2\pi} \sin(2\pi x).$$

We may restrict Ω to lie in $[0, 1)$. For $K = 0$, $\tilde{f}_{K,\Omega}$ is simply a rotation by $2\pi\Omega$ radians. We may think of K as a “non-linearity parameter” governing the deviation of $\tilde{f}_{K,\Omega}$ from a pure rotation. In order that $f_{K,\Omega}$ be non-decreasing we have to require $|K| \leq 1$; $K = 1$ corresponds to the maxi-

Table I

n	$e_n(0.1)$	$e_n(0.3)$	$e_n(0.5)$	$e_n(0.7)$	$e_n(0.8)$	$e_n(0.9)$	$e_n(1.0)$
1	0.031831	0.095493	0.159155	0.222817	0.254648	0.286479	0.318310
2	0.000398	0.007109	0.019498	0.037534	0.048500	0.060670	0.073966
3	0.000070	0.001871	0.008473	0.022486	0.032874	0.045725	0.061119
4	4.23e-06	0.000333	0.002443	0.008728	0.014274	0.021842	0.031705
5	5.51e-07	0.000129	0.001564	0.007733	0.014354	0.024503	0.039089
6	3.01e-08	0.000020	0.000352	0.002080	0.004063	0.007211	0.011891
7	5.64e-09	0.000011	0.000342	0.003002	0.006968	0.014454	0.027330
8	5.04e-10	2.70e-06	0.000115	0.001193	0.002949	0.006502	0.013059
9	7.89e-11	1.18e-06	0.000077	0.001042	0.002858	0.006884	0.014893
10	1.08e-11	4.09e-07	0.000034	0.000474	0.001327	0.003310	0.007515
11	2.14e-12	2.21e-07	0.000029	0.000612	0.002089	0.006200	0.016207
12	3.88e-13	9.32e-08	0.000013	0.000218	0.000665	0.001872	0.004953
13	8.91e-14	5.44e-08	0.000011	0.000311	0.001233	0.004309	0.013219
14	2.10e-14	2.92e-08	6.44e-06	0.000142	0.000508	0.001712	0.005321
15	5.81e-15	1.79e-08	5.09e-06	0.000137	0.000532	0.001856	0.005810
16	1.94e-15	1.11e-08	3.54e-06	0.000088	0.000354	0.001415	0.005323
17	1.33e-15	7.36e-09	2.94e-06	0.000101	0.000495	0.002295	0.009513
18		4.95e-09	2.12e-06	0.000049	0.000172	0.000691	0.003097
19		3.48e-09	1.81e-06	0.000063	0.000330	0.001729	0.008293
20			1.40e-06	0.000037	0.000148	0.000669	0.003070
21			1.19e-06	0.000038	0.000175	0.000839	0.004077
22			9.73e-07	0.000026	0.000107	0.000556	0.003178
23			8.42e-07	0.000030	0.000162	0.001033	0.006532
24			7.03e-07	0.000018	0.000066	0.000320	0.002099
25			6.16e-07	0.000021	0.000110	0.000697	0.004707
26			5.26e-07	0.000015	0.000060	0.000352	0.002628
27			4.65e-07	0.000016	0.000074	0.000461	0.003624
28			4.04e-07	0.000012	0.000046	0.000273	0.002122
29			3.61e-07	0.000013	0.000067	0.000518	0.004869
30			3.18e-07	8.83e-06	0.000028	0.000121	0.001320

imum permissible non-linearity. This family has been extensively studied. The article of Arnold cited above gives a picture of the region in K, Ω -space in which phase-locking occurs, and Arnold's theorem says that the intersection of this region with a line of constant K has Lebesgue measure going to zero with K . The (very modest) contribution of this note is to offer some detailed numerical data confirming Arnold's picture.

Given K and a positive integer n , we let $E_n(K)$ denote the set of Ω 's in $[0,1)$ for which $\tilde{f}_{K,\Omega}$ has a cycle of period n . We will see shortly that E_n is a finite union of intervals, and we will denote by $e_n(K)$ the total length of these intervals, i.e., the fraction of $[0,1)$ occupied by Ω 's such that $\tilde{f}_{K,\Omega}$ is phase-locked with period n . Table I shows the results of a numerical computation of $e_n(K)$ for several values of K and n up to 30.

3. Interpretation

One inference which can be drawn from these results is that, for moderately non-linear mappings, phase locking – especially with large periods – does not seem to be very likely. For example: When $K = 0.5$, i.e., with nonlinearity of half-maximum strength, the sum of the e_n 's for n up to 30 is only about 0.192, and the steady decrease of e_n with n makes it seem unlikely that adding on the contributions for n larger than 30 will increase the sum much. (This intuitive statement can be supported, somewhat, by quantitative estimates; see below.) Furthermore, most of the sum comes from small values of n ; e.g., the sum for n from 4 to 30 is only about 0.005.

The situation at $K = 1$ (maximal non-linearity) is quite different. The sum of the e_n for n up to 30 is about 0.709 and the decrease of e_n with increasing n is both slow and irregular, making direct extrapolation difficult. A little more regularity can, however, be discerned as follows: For $n > 1$, E_n is the union of $\phi(n)$ intervals, where $\phi(n)$ (the Euler ϕ function) is the number of integers between 1 and $n - 1$ prime relative to n . The ratio $e_n(1)/\phi(n)$

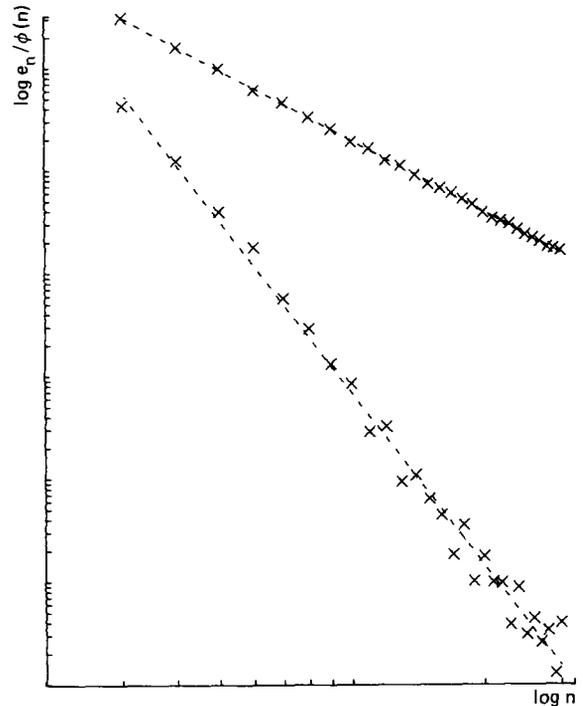


Fig. 1. Log-log plot of $e_n/\phi(n)$ vs. n for n from 1 to 30. The upper line represents $K = 1$, the lower line $K = 0.5$.

turns out to vary regularly with n for the values of n we have examined. The upper part of Fig. 1 shows a log-log plot of $e_n(1)/\phi(n)$ versus n for n from 3 to 30. The points lie remarkably close to a straight line, corresponding to a power-law dependence of $e_n(1)$ on n . (Why this should be true is a mystery.) A least-squares fit of a straight line to these points gives the empirical formula

$$e_n(1) \approx 0.377 \cdot \phi(n) \cdot n^{-2.288},$$

(shown as a dashed line in fig. 1) which has an error of no more than 6% for n from 3 to 30. Using this empirical formula to extrapolate $e_n(1)$ beyond $n = 30$, we can estimate the contributions of larger periods to the probability of phase locking. We have

$$\sum_{n=31}^N 0.377 \cdot \phi(n) \cdot n^{-2.288} = 0.2885 \dots,$$

(The sum converges too slowly to be evaluated directly. To evaluate it, first show analytically that

$$\sum_{n=1}^{\infty} \phi(n) \cdot n^{-\gamma} = \zeta(\gamma - 1) / \zeta(\gamma),$$

where ζ is the Riemann zeta function; then subtract off the finite sum from $n = 1$ to 30.)

Since the e_n for n from 1 to 30 add up to about 0.709, it appears very plausible that

$$\sum_{n=1}^{\infty} e_n(1) = 1,$$

i.e., that the mappings $\tilde{f}_{1,\Omega}$ are phase-locked for almost all Ω (in the sense of Lebesgue measure). It also appears, however, that large periods are quite common, e.g., that Ω 's which give phase locking with period greater than 100 comprise about 20% of the unit interval and those with period greater than 1000 about 8%.

Remarks:

1) D. Ruelle has shown me results of quite a different numerical investigation also tending to the conclusion that phase locking occurs for almost all Ω at $K = 1$. For a discussion of other related work, see section 5.

2) The empirical fit of a power of n to $e_n/\phi(n)$ works less well for $K = 0.5$ than for $K = 1$. The lower part of Fig. 1 is a log-log plot of $e_n(0.5)/\phi(n)$ against n , and the empirical formula obtained by fitting a straight line to the points shown is

$$e_n(0.5) \approx 2.47 \cdot \phi(n) \cdot n^{-5.55},$$

which is off by as much as a factor of 2.5. Furthermore, disturbingly, the fit is least good at $n = 30$. Nevertheless, for what it is worth, the empirical formula gives an estimate of only 2.3×10^{-6} for the sum of the $e_n(0.5)$ from n from 31 to infinity.

3) It is natural to ask whether the excellent fit of $e_n(1)$ by a power of n is a special property of the family of mappings considered here or is more general, and whether the exponent $2.288 \dots$ is "universal." To check on this, preliminary computations were done on two other families. In both

cases, the fit by a power law is not at all good. If, nevertheless, one insists on making a power law fit, the general trend is not inconsistent with an exponent of about 2.3.

4. Method of computation

The computation of the e_n rests on the following simple fact: In order that $\tilde{f}_{K,\Omega} \in [0, 1)$, have a cycle of period $n > 1$, it is necessary and sufficient that there exist an x_0 in $[0, 1)$ and an integer j between 1 and $n - 1$, prime relative to n , such that

$$f_{K,\Omega}^n(x_0) = x_0 + j.$$

(In this case, the rotation number of $\tilde{f}_{K,\Omega}$ is j/n .) We thus get, immediately, the following criterion: Let

$$\phi^+(n, K, \Omega) = \max_{0 \leq x < 1} \{ f_{K,\Omega}^n(x) - x \},$$

$$\phi^-(n, K, \Omega) = \min_{0 \leq x < 1} \{ f_{K,\Omega}^n(x) - x \}.$$

Then $\tilde{f}_{K,\Omega}$ has a cycle of period n if and only if there is an integer j between 1 and $n - 1$, prime relative to n , such that

$$\phi^-(n, K, \Omega) \leq j \leq \phi^+(n, K, \Omega).$$

A simple induction argument shows that ϕ^\pm are strictly increasing in Ω and indeed that the rate of increase is at least one, i.e.,

If $\Omega_1 > \Omega_2$ then

$$\phi^\pm(n, K, \Omega_1) \geq \phi^\pm(n, K, \Omega_2) + (\Omega_1 - \Omega_2).$$

Thus, for each j , there are uniquely determined $\Omega^\pm(n, j, K)$ such that

$$\phi^\pm(n, K, \Omega^\pm) = j,$$

and

$$E_n = \bigcup_j E_{n,j}, \quad E_{n,j} = [\Omega^+(n, j, K), \Omega^-(n, j, K)],$$

where the union is taken over j 's between 1 and $n - 1$ prime relative to n .

The preceding analysis does not quite apply to fixed points ($n = 1$), but it is easy to adapt the arguments given to show that $f_{K,\Omega}$ has a fixed point if and only if $0 \leq \Omega \leq K/2\pi$ or $1 - K/2\pi \leq \Omega < 1$ and hence that

$$e_1(K) = K/\pi.$$

The e_n 's for $n > 1$ were computed by solving the equations

$$\phi^\pm(n, K, \Omega^\pm) = j$$

for the Ω^\pm . These are non-linear equations whose left-hand side has to be computed numerically; they were solved using the FORTRAN subroutine ZEROIN (see Forsythe et al. [4], chap. 7), based on R. Brent's rootfinding algorithm. All calculations were done in double precision PDP-11 floating point arithmetic, i.e., with a precision of 56 bits or about 16 decimal digits. Because the ϕ^\pm increase with Ω with at least unit rate, the Ω^\pm can be computed as accurately as the ϕ^\pm can. Computing the ϕ^\pm is a global extremization problem. The approach used was to compute $f_{K,\Omega}^n(x) - x$ at each of 51 uniformly-spaced points running from 0 to 1; to find all places on this grid where the function is larger (smaller) than at both neighboring grid points; to locate with high accuracy a local maximum (minimum) somewhere in the two grid intervals around such a local grid extremum; and to take the largest (smallest) function value found in this way as $\phi^+(\phi^-)$. The local-extremum calculation was done using the one-dimensional minimum-finding routine FMIN (again, see Forsythe et al. [4], chap. 8) which reliably finds *some* local minimum in a specified interval with accuracy limited only by round-off error. It is possible, however, because of the finite grid spacing, that the global search will miss the true extremum. As a check on whether this is happening, some extrema were recomputed using a grid spacing of 1/200 instead of 1/50; in no case did this more refined search give a significantly different result. Nevertheless, the possibility that the true extremum is sometimes missed, and hence that the computed e_n 's are seriously in error, cannot be completely ruled out. If there are no such gross errors, the e_n

should be in error by no more than something like 10^{-14} .

5. Addendum

After completing this manuscript I learned of the work of Jensen, Bak and Bohr [7] who also study the fraction of the Ω interval corresponding to phase locking at $K = 1$. Jensen et al. analyze their data in a different way from what we have done; they examine the dependence on r of the fraction of the parameter interval not covered by phase-locking intervals of length at least r (rather than the dependence of n of the fraction of the parameter interval covered by phase-locking intervals of period n .) They find a power law dependence on r and present strong evidence that this power law is universal (in contrast to the power law dependence on n which we find in our particular example but which seems to fit other examples less well.) It is thus fair to say that their work provides stronger evidence than ours that phase-locking occurs except for a set of parameters of measure zero; it furthermore indicates that this is the general situation for critical circle mappings.

On the other hand, precisely because Jensen et al. don't analyze their numerical results in terms of the periods of the cycles considered, their work does not exhibit the fact that it is necessary to consider rather large periods in order to account for as much as, say, 90% of the parameter interval. It also seems to me to be worthy of note that the situation is quantitatively radically different even for K 's as large as 0.5.

References

- [1] V.I. Arnold, Small denominators I, Amer. Math. Soc. Translations, ser. 2, vol. 46 (1965) 213-284.
- [2] I.F. Cornfeld, S.V. Fomin and Ya.G. Sinai, Ergodic Theory (Springer, New York, 1982).
- [3] P. Deligne, Les difféomorphismes du cercle [d'après M.R.

- Herman], *Lecture Notes in Mathematics* 567 (1977) 99–121.
- [4] G.E. Forsythe, M.A. Malcolm and C.B. Moler, *Computer Methods for Mathematical Computations* (Prentice-Hall, Englewood Cliffs, 1977).
- [5] M.R. Herman, *Mesure de Lebesgue et nombre de rotation*, *Lecture Notes in Mathematics* 597 (1977) 271–293.
- [6] M.R. Herman, *Sur la conjugaison différentiable des difféomorphismes du cercle à des rotations*, *Publ. Math. IHES* 49 (1979) 5–233.
- [7] M. Høgh Jensen, P. Bak and T. Bohr, *Complete devil's staircase, fractal dimension, and universality of mode-locking structure in the circle map*, *Phys. Rev. Lett.* 50 (1983) 1637–1639.
- [8] H. Rosenberg, *Les difféomorphismes du cercle [d'après M.R. Herman]*, *Lecture Notes in Mathematics* 567 (1977) 81–98.