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Approach to Equilibrium of Free Quantum Systems

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Abstract. It is proved for fermi systems that each translationally invariant state ω with square integrable correlation functions approaches a limit under the free time evolution. The limit state is the gauge invariant quasi-free state with the same two-point function as ω and it is characterized by a maximum entropy principle. Various properties of the limit are discussed, and the extension of the results to bose systems is also given.

1. Introduction

The study of the structural properties of infinitely extended systems has played a useful role in the understanding of the nature and properties of equilibrium states. Similarly one would expect that analysis of the time development of such systems would aid the understanding of non-equilibrium phenomena. Very little work has however been done in this direction because it is notoriously difficult even to define the time-development of systems with an infinite number of degrees of freedom. In fact the only interacting systems for which one has a satisfactory definition are quantum spin systems [1] and a class of one-dimensional classical systems [2]. If, however, one turns to non-interacting systems the definition of time-development is relatively simple, and it is possible to analyse the properties of states of the system as they change with time. For example it has recently been shown that the equilibrium states of non-interacting classical systems have strong ergodic properties with respect to time and in fact provide examples of K-systems [3, 4]. Alternatively, for free systems of fermi particles, it has been argued that many states, possibly differing globally from equilibrium states, approach limiting equilibrium states as time progresses [5]. In this paper we will characterize a class of states of quantum systems which converge to a limit state as they evolve freely and examine properties of the convergence and the limiting "equilibrium states".

In the first part of this paper we consider exclusively fermions and in Sect. 2 we define the set of states whose time-development towards

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an equilibrium state is analyzed in Sect. 3. Properties of the approach to equilibrium are discussed in Sect. 4 and the method of extension of our results to bosons is presented in Sect. 5.

2. Square Integrable States

A system of fermi particles moving in the configuration space R^ν can be described in a well-known way by the C^*-algebra $\mathscr{A}$ associated with the canonical anti-commutation relations. We will adopt the standard notation and terminology used, for example, in Chapter VII of [6]. In particular, we denote by

$$f \in L^2(R^\nu) \to a(f) \in \mathscr{A}, \qquad g \in L^2(R^\nu) \to a^*(g) \in \mathscr{A}$$

the generating elements of $\mathscr{A}$ which satisfy the anti-commutation relations.

$$\{a(f), a^*(g)\} = (f, g) \qquad \text{etc.}$$

The group R^ν of space translations is represented as a group of strongly continuous automorphisms α of $\mathscr{A}$ whose action is defined by

$$\alpha_x(a(f)) = a(U_x f) \qquad x \in R^\nu$$

where

$$(U_x f)(y) = f(y - x) \qquad \text{etc.}$$

The one-parameter group of free time translations is also represented as a group of strongly continuous automorphisms τ of $\mathscr{A}$ and the action of this group is defined by

$$\tau_t(a(f)) = a(V_t f) \qquad t \in R$$

where

$$(V_t f)(x) = \frac{1}{(2\pi)^{\nu/2}} \int dp\, \hat{f}(p)\, e^{ip^2 t - ipx}$$

[$\hat{f}$ is the Fourier transform of f].

As $\mathscr{A}$ is generated by the $a(f)$, $a^*(g)$ each state ω over $\mathscr{A}$ is determined by the set of values

$$\{\omega(a^*(f_1) \dots a^*(f_n)\, a(g_1) \dots a(g_m))\,;\, f_1, \dots, f_n, g_1, \dots, g_m \in L^2(R^\nu)\}\,.$$

We introduce

$$W_{nm}(f_1, \dots, f_n;\, g_1, \dots, g_m) = \omega(a^*(f_1) \dots a^*(f_n)\, a(g_1) \dots a(g_m))\,.$$

The state ω is defined to be even if

$$W_{nm}(f_1, \dots, f_n;\, g_1, \dots, g_m) = 0$$

whenever $n + m$ is odd.

An even state ω over $\mathscr{A}$ is also completely determined by the truncated functions ω^T_{nm} which are defined recursively by the following formulae

$$\omega_{nm}(f_1, \dots, f_n; g_1, \dots, g_m) = \sum_{\pi} (-1)^{\sigma(\pi)} \omega^T_{r_1 s_1}(f_{i_1}, \dots, f_{i_{r_1}}; g_{j_1}, \dots, g_{j_{s_1}}) \dots$$

$$\dots \omega^T_{r_p s_p}(f_{k_1}, \dots, f_{k_{r_p}}; g_{l_1}, \dots, g_{l_{s_p}})$$

where the sum is taken over all partitions π of $\{f_1, \dots, f_n; g_1, \dots, g_m\}$ into disjoint subsets $\{f_{i_1}, \dots, f_{i_{r_1}}; g_{j_1}, \dots, g_{j_{s_1}}\}, \dots, \{f_{k_1}, \dots, f_{k_{r_1}}, g_{l_1}, \dots, g_{l_{s_p}}\}$. The f's and g's appear within each subset of a partition π in the same order that they appear in the original set and $\sigma(\pi)$ is the permutation of the set required to rearrange the f's and g's into the order in which they appear in the partitioned subsets. The requirement of evenness of ω ensures that this latter convention is unambiguous.

The state ω over $\mathscr{A}$ is said to be translationally invariant if

$$\omega(\alpha_x A) = \omega(A), \qquad A \in \mathscr{A}, \ x \in R^\nu.$$

Each translationally invariant state is automatically even [7]. Such a state ω is completely determined by the set of functions

$$\omega^T_{\{f_n\}\{g_m\}}(x_2 - x_1, \dots, x_{n+m} - x_{n+m-1})$$
$$= \omega^T_{nm}(U_{x_1} f_1, \dots, U_{x_n} f_n; U_{x_{n+1}} g_1, \dots, U_{x_{n+m}} g_m).$$

Definition 1. *The translationally invariant state ω over $\mathscr{A}$ is defined to be square integrable if*

$$\int d\xi_1 \dots d\xi_{n+m-1} \left|\omega_{\{f_n\}\{g_m\}}(\xi_1, \dots, \xi_{n+m-1})\right|^2 < +\infty \quad \textit{for} \quad n+m > 2$$

and for all f_i, g_j such that $\tilde{f}_i, \tilde{g}_j \in \mathscr{D}$.

Remark 1. Using the continuity and linearity properties of $f \to a^*(f)$ etc., one can associate with each translationally invariant state ω a set of tempered distributions W^T_{nm} such that

$$\omega^T_{nm}(f_1, \dots, f_n; g_1, \dots, g_m)$$
$$= \int dx_1 \dots dx_{n+m} W^T_{nm}(x_2 - x_1, \dots, x_{n+m} - x_{n+m-1}) f_1(x_1) \dots$$
$$\dots f_n(x_n)\, g_1(x_{n+1}), \dots, g_{n+m}(x_{n+m}).$$

If the W^T_{nm} are in fact square integrable then ω is a square integrable state in the sense of Definition 1. This definition actually allows the W^T_{nm} to have local singularities and corresponds to a condition of square integrability at infinity. It might seem natural to demand the square integrability condition to be satisfied for all $f_i, g_j \in \mathscr{D}$ but this is in fact a stronger condition than the specified requirement $\hat{f}_i, \tilde{g}_j \in \mathscr{D}$.

Remark 2. If ω is square integrable then it is automatically strongly mixing of all orders, i.e. for all $A_1, \ldots, A_n \in \mathscr{A}$

$$\lim_{\substack{\min|x_i - x_j| \to \infty \\ i \neq j}} \omega(\alpha_{x_1} A_1 \ldots \alpha_{x_n} A_n) = \omega(A_1) \ldots \omega(A_n),$$

and in particular R^ν ergodic. The existence of square integrable states is assured by the work of Ginibre [8] who shows that the equilibrium states of a large class of interacting systems have this property at low density. we suspect that the set of such states is weak*-dense among the set of all translationally invariant states over $\mathscr{A}$.

Before proceeding we recall that an even state ω over $\mathscr{A}$ is called quasi-free if

$$\omega_{nm}^T = 0 \quad \text{for} \quad n + m > 2$$

and is called a gauge invariant quasi-free state if

$$\omega_{nm}^T = 0 \quad \text{for} \quad (n, m) \neq (1, 1).$$

Further, if ω is an arbitrary state over $\mathscr{A}$ we can define the associated gauge invariant quasi-free state $\hat{\omega}$ by

$$\hat{\omega}(a^*(f)\, a(g)) = \omega(a^*(f)\, a(g)), \qquad f, g \in L^2(R^\nu)$$

and $\hat{\omega}_{nm}^T = 0$ for $(n, m) \neq (1, 1)$. It is well known that this definition actually does determine a state.

3. Approach to Equilibrium

We next wish to analyse the time development of square integrable states. Before this, however, we examine properties of two point functions of a general translationally invariant state.

Theorem 1. *If ω is a translationally invariant state over $\mathscr{A}$ then*

$$\omega(\tau_t(a^*(f)\, a(g))) = \omega(a^*(f)\, a(g)),$$

$$\lim_{t \to \infty} \omega(\tau_t(a(f)\, a(g))) = 0,$$

and

$$\lim_{t \to \infty} \omega(\tau_t(a^*(f)\, a^*(g))) = 0$$

for all $f, g \in L^2(R^\nu)$.

Proof. First, note that as $\|a(f)\| = \|a^*(f)\| = |f|_2$ one has

$$|\omega(a^*(f)\, a(g))| \leq |f|_2\, |g|_2 .$$

Thus there exists an operator A on $L^2(R^\nu)$ with $\|A\| \leq 1$ such that

$$\omega(a^*(f)\, a(g)) = (f, Ag).$$

Now as ω is translationally invariant A commutes with the group of translation operators $f \to U_x f$ on $L^2(R^\nu)$ and hence is a multiplication operator in momentum space. Thus there is a function $\tilde{\varrho}$, with $|\tilde{\varrho}| \leqq 1$, such that

$$\begin{aligned}\omega(a^*(f)\, a(g)) &= \int dp\, \tilde{\varrho}(p)\, \bar{\tilde{f}}(p)\, \tilde{g}(p) \\ &= \omega\big(\tau_t(a^*(f)\, a(g))\big)\end{aligned}$$

and the first part of the theorem is proved.

Similarly there is an operator B with $\|B\| \leqq 1$, such that

$$\omega(a(f)\, a(g)) = (\bar{f}, Bg)\,.$$

Again B commutes with translations and hence there is a function $\tilde{\sigma}$ with $|\tilde{\sigma}| \leqq 1$, such that

$$\omega(a(f)\, a(g)) = \int dp\, \tilde{\sigma}(p)\, \tilde{f}(-p)\, \tilde{g}(p)\,.$$

But then we have

$$\omega\big(\tau_t(a(f)\, a(g))\big) = \int dp\, \tilde{\sigma}(p)\, \tilde{f}(-p)\, \tilde{g}(p)\, e^{2ip^2t}$$

which goes to zero as $t \to \infty$ by the Riemann-Lebesgue Lemma.

Corollary 1. *If ω is a translationally invariant quasi-free state and $\hat{\omega}$ the associated gauge invariant quasi-free state then*

$$\lim_{t\to\infty} \omega(\tau_t A) = \hat{\omega}(A) \qquad A \in \mathscr{A}.$$

This follows simply by noting that the value of ω on each monomial $a^*(f_1) \ldots a^*(f_n)\, a(g_1) \ldots a(g_m)$ is a sum of products of the two point functions $\omega(a^*(f_i)\, a^*(f_j))$, $\omega(a^*(f_k)\, a(g_l))$, $\omega(a(g_p)\, a(g_q))$.

Remark 3. As positivity implies that

$$\omega\big((\lambda a^*(f) + a(g))^*\, (\lambda a^*(f) + a(g))\big) \geqq 0$$

one can deduce actually that

$$\tilde{\varrho}(p) \geqq 0 \quad \text{and} \quad \tilde{\varrho}(p)\,(1 - \tilde{\varrho}(-p)) \geqq |\tilde{\sigma}(p)|^2\,.$$

Further, one knows that if

$$\tilde{\varrho}(p)\,(1 - \tilde{\varrho}(-p)) = |\tilde{\sigma}(p)|^2$$

then ω is automatically a pure quasi-free state, but if this equality is not valid then the quasi-free state determined by $\tilde{\varrho}$ and $\tilde{\sigma}$ is a mixed state and in fact primary.

Theorem 2. *If ω is a square integrable state over $\mathscr{A}$ and $\hat{\omega}$ the associated gauge invariant quasi-free state then*

$$\lim_{t\to\infty} \omega(\tau_t A) = \hat{\omega}(A)\,, \qquad A \in \mathscr{A}.$$

Proof. The proof proceeds in four steps:

1. It suffices to prove that the truncated functions ω^T_{nm} satisfy the convergence property

$$\lim_{t\to\infty}\omega^T_{nm}(V_t f_1,\ldots,V_t f_n; V_t g_1,\ldots,V_t g_m)=0$$

for all $(n,m)\neq(1,1)$ and all f_i, g_i with Fourier transforms in $\mathscr{D}$. This follows because convergence of the state is equivalent to the convergence of the functions

$$\omega_{nm}(V_t f_1,\ldots,V_t f_n; V_t g_1,\ldots,V_t g_m)$$

for all $f,g\in L^2(R^\nu)$ which is in turn equivalent to the convergence of the truncated functions. To obtain this convergence it is sufficient to prove convergence for the f_i, g_i in a dense set of $L^2(R^\nu)$, for example $\tilde f_i, \tilde g_i\in\mathscr{D}$. Finally $\hat\omega$ is obtained from ω by setting the truncated functions with $(n,m)\neq(1,1)$ equal to zero and hence the necessity that ω^T_{nm} must converge to zero.

2. The appropriate convergence of the two point functions has been dealt with in Theorem 1.

3. Now with $\tilde f_1,\ldots,\tilde f_n, \tilde g_1,\ldots,\tilde g_m\in\mathscr{D}$ we can choose $\tilde h\in\mathscr{D}$ such that

$$\tilde h\tilde f_i=\tilde f_i\,,\qquad i=1,\ldots,n\,,$$
$$\tilde h\tilde g_j=\tilde g_j\,,\qquad j=1,\ldots,m\,.$$

But then we note that

$$\begin{aligned}V_t f_i&=V_t(f_i * h)\\&=f_i * V_t h\end{aligned}$$

where the star denotes the convolution product. Thus

$$\begin{aligned}\omega^T_{nm}(V_t f_1,\ldots,V_t g_m)&=\int dx_1\ldots dx_{n+m}V_t h(x_1)\ldots V_t h(x_{n+m})\\&\quad\cdot\omega^T_{nm}(U_{x_1}f_1\ldots U_{x_{n+m}}g_m)\\&=\int dx_1\ldots dx_{n+m}V_t h(x_1)\ldots V_t h(x_{n+m})\\&\quad\cdot\omega^T_{\{f_n\}\{g_m\}}(x_2-x_1,\ldots,x_{n+m}-x_{n+m-1})\,.\end{aligned}$$

Alternatively this last equation can be written

$$\omega^T_{nm}(V_t f_1\ldots V_t g_m)=\int d\xi_1\ldots d\xi_{n+m-1}H_t(\xi_1\ldots\xi_{n+m-1})\,\omega^T_{\{f_n\}\{g_m\}}(\xi_1\ldots\xi_{n+m-1})$$

where

$$\tilde H_t(p_1,\ldots,p_{n+m-1})=\exp\{itE(p)\}\,\tilde H_0(p_1,\ldots,p_{n+m-1})$$

and

$$E(p)=p_1^2+\sum_{i=2}^{n}(p_i-p_{i-1})^2-\sum_{i=n+1}^{n+m-1}(p_i-p_{i-2})^2-p_{n+m-1}^2\,.$$

Now $\omega^T_{\{f_n\}\{g_m\}}$ is square integrable by assumption and H_t by construction.

Thus the proof of the theorem is complete if we can show that H_t tends L_2-weakly to zero as $t \to \infty$. This is however accomplished by the following version of the Riemann-Lebesgue Lemma which constitutes the fourth and final step of the proof.

Lemma 1. *Let Φ be a Lebesgue integrable function on R^k and F a non-zero bilinear form then it follows that*

$$\lim_{t\to\infty} \int dr_1 \ldots dr_k \, \Phi(r_1 \ldots r_k) \exp\{itF(r_1 \ldots r_k)\} = 0 .$$

Proof. It suffices to show that for any $r = (r_1 \ldots r_k) \neq (0, 0, \ldots, 0)$ there is a neighbourhood N_r such that the lemma holds for any continuous Φ with support in N_r. Since $\nabla_r F \neq 0$ we may choose a non-singular system of local coordinates $u = (u_1, \ldots, u_r)$ at r with $u_1 = F$. If the support of Φ is contained in the coordinate neighbourhood we then have

$$\int dr \, \Phi(r) \, e^{itF(r)} = \int du \, \Phi(r(u)) \, |J(u)| \, e^{itu_1}$$

where J is the Jacobian of the transformation $(r_1, \ldots, r_j) \to (u_1, \ldots, u_j)$. But this latter expression converges to zero as $t \to \infty$ by the Riemann-Lebesgue Lemma.

Remark 4. If we had considered a more general time evolution

$$(V_t f)(x) = \frac{1}{(2\pi)^{\nu/2}} \int dp \, f(p) \, e^{i\omega(p)t - ipx}$$

then the foregoing results would be valid as long as the Jacobian of the transformation $(p_1, \ldots, p_{n+m-1}) \to (u_1, \ldots, u_{n+m-1})$ where

$$u_1 = \omega(p_1) + \sum_{i=2}^{n} \omega(p_i - p_{i-1}) - \sum_{i=n+1}^{n+m-1} \omega(p_i - p_{i-1}) - \omega(p_{n+m-1})$$

$$u_j = p_j, \qquad j = 2, 3, \ldots, n+m-1$$

is non-singular for all n, m.

A slightly stronger result than that given in Theorem 2 can be established with a slight elaboration of the above proof.

Theorem 3. *Let ω be a square integrable state over $\mathscr{A}$ and $\hat{\omega}$ the associated gauge invariant quasi-free state. Denote by π_ω the representation of $\mathscr{A}$, on the Hilbert space $\mathscr{H}_\omega$, associated with ω by the Gelfand-Segal construction. It follows that*

$$\lim_{t\to\infty} \omega(A(\tau_t B) C) = \omega(AC) \, \hat{\omega}(B), \qquad A, B, C \in \mathscr{A},$$

i.e.

$$\text{weak} \lim_{t\to\infty} \pi_\omega(\tau_t B) = \hat{\omega}(B) \, 1_\omega, \qquad B \in \mathscr{A}$$

where 1_ω is the identity operator on $\mathscr{H}_\omega$.

It further follows that the strong limit of $\pi_\omega(\tau_t B)$ does not exist for all $B \in \mathscr{A}$.

Proof. The proof of the first statement proceeds again in several steps.

1. It suffices to prove the statement with A, B, C monomials in $a^*(f_i)$, $a(g_i)$, where $\tilde{f}_i, \tilde{g}_i \in \mathscr{D}$, ordered with the a^* at the left and the a at the right. This follows because the linear hull of such monomials is uniformly dense in $\mathscr{A}$.

2. Next use the anti-commutation relations to order the monomial $A\tau_t(B)\,C$ with the a^* at the left and the a at the right. This process gives an ordered monomial of the same order as $A\tau_t(B)\,C$ plus lower order terms each of which is proportional to an anti-commutator of an a from A (or an a^* from C) with an a^* from B (an a from B), i.e. proportional to a factor of the form $(V_t f, g)$. As $t \to \infty$ this latter factor tends to zero and consequently the lower order terms do not contribute to the limit. Thus we need only study the highest order monomial, the ordered form of $A\tau_t(B)\,C$.

3. If we now write the value of this latter monomial in the state ω in terms of truncated functions we obtain, using the first statement of Theorem 1, a sum of terms exactly equal to $\omega(AC)\,\hat{\omega}(B)$ plus a number of t-dependent terms. It remains to prove that these latter terms tend to zero.

4. Introduce h as in step 3 of the proof of Theorem 2. Each of the relevant terms has a factor expressible in the form

$$\int d\xi_1 \ldots d\xi_{n+m-1}\, W^T_{\{f_n\}\{g_n\}}(\xi_1 \ldots \xi_{n+m-1})\, H_t(\xi_1 \ldots \xi_{n+m-1})$$

where

$$\tilde{H}_t(p_1 \ldots p_{n+m-1}) = \exp\{itE'(p)\}\, \tilde{H}_0(p_1 \ldots p_{n+m-1})$$

and E' is a non-zero bilinear form in a subset of the variables $p_1 \ldots p_{n+m-1}$. The proof of the desired result is then obtained from Lemma 1.

The last statement of the theorem is easily proved by contradiction. Assume $\pi_\omega(\tau_t\, a(f))$ converges strongly as $t \to \infty$; then it must tend strongly to zero because we have established that it converges weakly to zero. Thus $\pi_\omega(\tau_t(a^*(f)\,a(f)))$ converges weakly to zero. Similarly $\pi_\omega(\tau_t(a(f)\,a^*(f)))$ converges weakly to zero. But

$$\pi_\omega(\tau_t(a^*(f)\,a(f))) + \pi_\omega(\tau_t(a(f)\,a^*(f))) = |f|_2^2\, 1_\omega$$

by the anti-commutation relations which is a contradiction.

Remark 5. In the above theorems we have established the existence of pointwise limits of states or expectation values. One can also give general conditions under which ergodic averages exist. Firstly, if ω is

τ-invariant then the limits

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T dt\, \omega(A\tau_t(B)\,C) \qquad A, B, C \in \mathscr{A}$$

exist because $\mathscr{A}$ is R-abelian [7, 9]. The limit can be identified and in particular if ω is extremal τ-invariant then it gives $\omega(AC)\,\omega(B)$. This circumstance is of interest in examing the problem of return to equilibrium. Secondly, if the truncated functions are not square integrable in the sense of Definition 2 but are simply the Fourier transforms of measures then the limits

$$\lim_{t\to\infty} \frac{1}{T} \int_0^T dt\, \omega(\tau_t A), \qquad A \in \mathscr{A}$$

exist but are not necessarily equal to $\omega(A)$.

4. Properties of Approach to Equilibrium

In the previous section we have shown that a class of states approach a limit state, or equilibrium state, in the limit of infinite time as the system evolves freely. A priori one might be tempted to argue that the limit state should be identifiable with the Gibbs equilibrium state compatible with the given energy and particle densities; this is clearly not the case. The reason for this discrepancy is however immediate. The free evolution is pathological in the sense that it allows many too many constants of the motion. In particular Theorem 1 establishes that the gauge invariant two point function remains constant in time. Thus the information that could possibly be infered about the more realistic situation of an interacting system is limited. Nevertheless one can use the information gathered for the freely evolving system to check a number of general principles and provide examples and counter examples to conjectured behaviour. We now examine a few such points but it is first necessary to recall a few facts and definitions.

A state ω over $\mathscr{A}$ is called *locally normal* if its restriction to each $\mathscr{A}_\Lambda$ [the subalgebra generated by $\{a(f), a^*(g); f, g \in L^2(\Lambda)\}$] is normal with respect to the Fock representation of $\mathscr{A}_\Lambda$, i.e. a locally normal state is determined by a family of density matrices $\{\varrho_\Lambda\}$ on the Fock spaces $\{\mathscr{H}(\Lambda)\}$ in the following manner

$$\omega(A) = \mathrm{Tr}_{\mathscr{H}(\Lambda)}(\varrho_\Lambda A) \qquad A \in \mathscr{A}_\Lambda\,.$$

If ω is a translationally invariant locally normal state over $\mathscr{A}$ its *mean entropy* $S(\omega)$ is defined by

$$S(\omega) = \lim_{\Lambda\to\infty} - V(\Lambda)^{-1}\, \mathrm{Tr}_{\mathscr{H}(\Lambda)}(\varrho_\Lambda \log \varrho_\Lambda)$$

where $V(\Lambda)$ is the volume of Λ and the limit over the net of increasing parallelepipeds is known to exist [10].

A state ω is said to have *finite density* if for all $\Lambda \subset R^\nu$

$$N_\Lambda(\omega) = V(\Lambda)^{-1} \sum_{i \geqq 1} \omega(a^*(f_i)\, a(f_i)) < +\infty$$

where $\{f_i\}_{i \geqq 1}$ is an orthonormal basis of $L^2(\Lambda)$. A finite density state is locally normal, and, further,

$$N_\Lambda(\omega) = \mathrm{Tr}_{\mathscr{H}(\Lambda)}\left(\varrho_\Lambda \frac{N_\Lambda}{V(\Lambda)}\right)$$

where N_Λ is the unbounded operator which measures particle number on $\mathscr{H}(\Lambda)$. Clearly, the restriction of finite density is solely a condition on the gauge invariant two point function; if ω is translationally invariant it is readily shown [11] that $N_\Lambda(\omega)$ is independent of $\Lambda (= N(\omega))$ and

$$N(\omega) = \int dp\, \tilde{\varrho}(p)$$

in the notation of Theorem 1.

Next let us define the state ω to be *N-entire (N-analytic)* whenever ω is locally normal and

$$\mathrm{Tr}_{\mathscr{H}(\Lambda)}(\varrho_\Lambda\, e^{\alpha N_\Lambda}) < +\infty$$

for all $\Lambda \subset R^\nu$ and all $\alpha \in R$ (for some $\alpha > 0$). This definition is motivated by the following facts [12]. If ω is invariant under space translations and N-entire then the following quantity exists and is finite

$$e_N(\omega) = \sum_{r \geqq 0} \frac{|\alpha|^r}{r!} \lim_{\Lambda \to \infty} \mathrm{Tr}_{\mathscr{H}(\Lambda)}\left(\varrho_\Lambda \left(\frac{N_\Lambda}{V(\Lambda)}\right)^r\right), \qquad \alpha \in R$$

where the limit is again over increasing parallelepipeds; further, if ω is α-ergodic, i.e. extremal among the translationally invariant states, then $e_N(\omega)$ is identifiable as follows:

$$e_N(\omega) = \sum_{r \geqq 0} \frac{|\alpha|^r}{r!} \lim_{\Lambda \to \infty} \left[\mathrm{Tr}_{\mathscr{H}(\Lambda)}\left(\varrho_\Lambda \frac{N_\Lambda}{V(\Lambda)}\right)\right]^r.$$

Thus an α-ergodic state which is N-entire has small density fluctuations of all orders.

Finally recall that there are two notions of faithfulness of a state over a C^*-algebra. A state ω over $\mathscr{A}$ is *weakly faithful* if

$$\omega(A^* A) = 0$$

implies $A = 0$. The state ω is *strongly faithful* if the cyclic vector Ω_ω associated with it by the Gelfand-Segal construction is separating for the von Neumann algebra π''_ω generated by the corresponding representation π_ω (equivalently Ω_ω is cyclic for the commutant π'_ω of the representation π_ω).

We now examine how these properties are affected by the approach to equilibrium.

A. Maximum Entropy Principle

The Gibbs equilibrium state of a free fermi gas can be characterized as the translationally invariant state which maximises the mean entropy at fixed energy and particle density. Although the states we are considering do not necessarily approach the Gibbs state in the equilibrium limit we will show that the state which they do approach has the maximum mean entropy compatible with the constants of the motion.

Proposition 1a. *Let ω be a translationally invariant state of finite mean density which has the property that*

$$\lim_{t\to\infty} \omega(\tau_t A) = \hat{\omega}(A), \qquad A \in \mathscr{A}$$

where $\hat{\omega}$ is the gauge invariant quasi-free state associated with ω. Further let K_ω denote the set of all translationally invariant states with the same two point function $(f, g) \to \omega(a^*(f)\, a(g))$ as ω and $\hat{\omega}$. It follows that

$$S(\hat{\omega}) = \sup_{\omega' \in K_\omega} S(\omega').$$

Proof. It suffices to prove, for an arbitrary $\omega' \in K_\omega$, that $S(\omega') \geqq S(\hat{\omega})$. Now as ω has finite mean density

$$\begin{aligned} N(\omega) = N(\omega') &= V(\Lambda)^{-1} \sum_{i \geqq 1} \omega(a^*(f_i)\, a(f_i)) \\ &= V(\Lambda)^{-1} \sum_{i \geqq 1} (f_i, A f_i) \\ &= V(\Lambda)^{-1}\, \mathrm{Tr}_{L^2(\Lambda)}(A) < +\infty \end{aligned}$$

where A is the operator which determines the two-point function of ω (cf. proof of Theorem 1) restricted to $L^2(\Lambda)$. Thus A is of trace-class. Let $\{\lambda_i\}_{i \geqq 1}$ and $\{f_i\}_{i \geqq 1}$ be the eigenvalues and a complete orthonormal set of eigenfunctions of A respectively. Further let $\mathscr{H}_i$ be the Fock space (2-dimensional) associated with the algebra generated by $a(f_i)$ and $a^*(f_i)$. $\mathscr{H}(\Lambda)$ has a tensor decomposition of the form $\mathscr{H}_i \otimes R_i$ and in fact

$$\mathscr{H}(\Lambda) = \prod_{i \geqq 1}^{\otimes} \mathscr{H}_i .$$

Next define σ_i as the matrix on $\mathscr{H}_i$ given by

$$\begin{aligned}\sigma_i &= \mathrm{Tr}_{R_i}(\varrho'_\Lambda)\\ &= \lambda_i\, a^*(f_i)\, a(f_i) + (1-\lambda_i)\, a(f_i)\, a^*(f_i)\end{aligned}$$

where ϱ'_Λ is the density matrix associated with ω'. It is easily checked that the density matrix associated with $\hat{\omega}$ is given by

$$\hat{\varrho}_\Lambda = \prod_{i \geqq 1}^{\otimes} \sigma_i .$$

Now using Lemma 1 of [7] we have

$$\begin{aligned}-\mathrm{Tr}_{\mathscr{H}(\Lambda)}(\varrho'_\Lambda \log \varrho'_\Lambda) &\leqq -\mathrm{Tr}_{\mathscr{H}(\Lambda)}(\varrho'_\Lambda \log \hat{\varrho}_\Lambda)\\ &= -\sum_{i\geqq 1} \mathrm{Tr}_{\mathscr{H}(\Lambda)}(\varrho'_\Lambda \log \sigma_i)\\ &= -\sum_{i\geqq 1} \mathrm{Tr}_{\mathscr{H}(\Lambda)}(\hat{\varrho}_\Lambda \log \sigma_i)\\ &= -\mathrm{Tr}_{\mathscr{H}(\Lambda)}(\hat{\varrho}_\Lambda \log \hat{\varrho}_\Lambda) .\end{aligned}$$

Dividing by $V(\Lambda)$ and taking the limit $\Lambda \to \infty$ gives the desired result.

Remark 6. If ω is translationally invariant and ω_t is defined by $\omega_t(A) = \omega(\tau_t A)$, $A \in \mathscr{A}$ then in general $S(\omega_t) = S(\omega)$, i.e. the entropy is a constant of the motion. This however does not rule out the increase of the entropy in the limit $t \to \infty$ as S is usually only a semi-continuous function. An example in which the mean entropy increases strictly is given by a non-gauge invariant quasi-free state which in the limit approachs the associated gauge invariant quasi-free state [13]. We suspect that this strict increase is a general property, i.e. the supremum in Proposition 1a is attained by only one state namely $\hat{\omega}$.

B. Density Fluctuations and Local Normality

Proposition 1b. *Let ω be a locally normal translationally invariant state which has the property that*

$$\lim_{t\to\infty} \omega(\tau_t A) = \hat{\omega}(A), \qquad A \in \mathscr{A}$$

where $\hat{\omega}$ is the gauge invariant quasi-free state associated with ω.

It follows that $\hat{\omega}$ is locally normal if, and only if, ω has finite mean density and in the case $\hat{\omega}$ is N-entire.

Proof. That a quasi-free state is locally normal if and only if it has finite mean density is demonstrated for example in [14]. Assuming ω and hence $\hat{\omega}$ has finite mean density then we have with the notation of Sect. 4a.

$$\begin{aligned}\mathrm{Tr}_{\mathscr{H}(\Lambda)}(\hat{\varrho}_\Lambda e^{\alpha N\Lambda}) &= \prod_i \mathrm{Tr}_{\mathscr{H}_i}(\sigma_i e^{\alpha a^*(f_i)a(f_i)}) \\ &= \prod_{i\geqq 1}(1+(e^\alpha-1)\lambda_i) \\ &\leqq \exp\left\{(e^\alpha-1)\sum_{i\geqq 1}\lambda_i\right\} \\ &= \exp\{(e^\alpha-1)N(\omega)V(\Lambda)\}\end{aligned}$$

and hence $\hat{\omega}$ is N-entire.

C. Faithfulness

Proposition 1c. *If ω is a weakly faithful translationally invariant state which has the property that*

$$\lim_{t\to\infty}\omega(\tau_t A)=\hat{\omega}(A), \qquad A\in\mathscr{A}$$

where $\hat{\omega}$ is the gauge invariant quasi-free state associated with ω then it follows that $\hat{\omega}$ is strongly faithful.

Proof. The weak faithfulness of ω implies

$$\omega(a^*(f)\,a(f))>0, \qquad \omega(a(f)\,a^*(f))>0, \qquad f\in L^2(R^\nu)$$

and hence

$$1>\tilde{\varrho}>0$$

almost everywhere. Now one can define a one-parameter group of automorphisms of $\mathscr{A}$ by the following action on the generating elements

$$s\to\sigma_s(a(f))=a(X_s f)$$

where

$$(X_s f)(x)=\int dp\left(\frac{\tilde{\varrho}(p)}{1-\tilde{\varrho}(p)}\right)^{is}\tilde{f}(p)\,e^{-ipx}.$$

It is easily checked that the state $\hat{\omega}$ satisfies the K.M.S. boundary condition [16] with respect to this group of automorphisms, for example

$$\begin{aligned}\hat{\omega}(a^*(f)\,\sigma_i(a(g))) &= \int dp\,\tilde{f}(p)\,\tilde{g}(p)\,\tilde{\varrho}(p)\left(\frac{\tilde{\varrho}(p)}{1-\tilde{\varrho}(p)}\right)^{-1} \\ &= \hat{\omega}(a(g)\,a^*(f)).\end{aligned}$$

It then follows from [16] that $\hat{\omega}$ is strongly faithful.

D. Purity

Mathematically it is natural to ask if starting with a pure state ω one always obtains a limiting state $\hat{\omega}$ which is also pure. We will now demonstrate that this is not generally the case by citing an example.

Consider the quasi-free state whose two point functions are given by

$$\tilde{\varrho}(p) = e^{-p^2}/1 + e^{-p^2}$$

and

$$\tilde{\sigma}(p) = e^{-p^2/2}/1 + e^{-p2}.$$

As $|\tilde{\sigma}(p)|^2 = \tilde{\varrho}(p)(1 - \tilde{\varrho}(-p))$ this state is a pure translationally invariant state. Nevertheless the state attained in the equilibrium limit, the gauge invariant quasi-free state determined by $\tilde{\varrho}(p)$, is a type III factor state (and incidentally a Gibbs equilibrium state).

5. Bosons

Examination of the time development of free bose systems is technically more complicated because it is impossible to give an algebraic description in which the time evolution is realised as a group of strongly continuous automorphisms. By suitable choice of the underlying C^*-algebra one can retain the automorphism property but the continuity is lost. Nevertheless we will show that the time evolution can be defined as a continuous mapping of a subset of states and use this formalism to extend the foregoing results.

We will work with a C^*-algebra $\mathscr{A}$ defined in the following manner. On each Fock space $\mathscr{H}(\Lambda)$, $\Lambda \subset R^\nu$, we define $\mathscr{A}_\Lambda$ to be the C^*-algebra generated by the set of Weyl operators

$$\{U(f), V(g); f, g \in \mathscr{D} \cap L^2(\Lambda)\}.$$

Using the canonical identification of $\mathscr{A}_\Lambda$ as a subalgebra of $\mathscr{A}_{\Lambda'}$, whenever $\Lambda \subset \Lambda'$ we can construct the algebra $\mathscr{A}$ as the uniform closure of the union (over $\Lambda \subset R^\nu$) of the $\mathscr{A}_\Lambda$.

Note that the group R^ν is represented as a group of automorphisms α of $\mathscr{A}$ whose action is defined by

$$\alpha_x(U(f)) = U(U_x f) \quad \text{etc.}$$

where again

$$(U_x f)(y) = f(y - x)$$

but this group is not strongly continuous because of the easily established relation

$$\|U(f) - U(g)\| = \|U(f-g) - 1\| = 2 \quad \text{if} \quad f \neq g.$$

This latter relation also makes it impossible to translate the free evolution of the test functions

$$f \in \mathscr{D} \to V_t f \in \mathscr{S}, \qquad t \in R$$

into a group of strongly continuous automorphisms. This difficulty can however be dealt with as follows. First for convenience introduce (on Fock space)

$$W(f,g) = U(f)\,V(g)\,e^{-i(f,g)/2}$$
$$= e^{i[\Phi(f)+\pi(g)]}$$

where $\Phi(f)$ and $\pi(g)$ denote the infinitesimal generators of $U(f)$ and $V(g)$ respectively.

Definition 2. *If ω is a state over $\mathscr{A}$ such that for each pair $f, g \in \mathscr{D}$ the function*

$$(f,g) \to \omega(W(f,g))$$

is continuous on $\mathscr{S} \times \mathscr{S}$ we define the time evolved state ω_t by

$$\omega_t(W(f,g)) = \omega(W(V_t f, V_t g)) \qquad t \in R\,.$$

We will now prove that if ω is a translationally invariant state with finite mean density then it has the continuity property demanded by the definition and in fact ω_t also has finite mean density.

Lemma 2. *If ω is a state of finite density and $f, g \in \mathscr{D} \cap L^2(\Lambda)$ then*

$$\|(W_\omega(f,g)-1)\,\Omega_\omega\|^2 \leqq (2N_\Lambda(\omega)\,V(\Lambda)+1)\,[|f|_2+|g|_2]^2$$

where Ω_ω is the cyclic vector associated with ω and W_ω the representative of W.

Proof. As ω is locally normal we can write

$$\begin{aligned}\|(W_\omega(f,g)-1)\,\Omega_\omega\|^2 &= \omega((W(f,g)-1)^*\,(W(f,g)-1))\\ &= \mathrm{Tr}_{\mathscr{H}(\Lambda)}\,(\varrho_\Lambda(e^{i(\Phi(f)+\pi(g))}-1)^*\,(e^{i(\Phi(f)+\pi(g))}-1))\\ &\leqq \mathrm{Tr}_{\mathscr{H}(\Lambda)}\,(\varrho_\Lambda(\Phi(f)+\Phi(g))^2)\end{aligned}$$

because for a self-adjoint operator A on a Hilbert space

$$\|(e^{iA}-1)\,\Psi\| \leqq \|A\Psi\|\,, \qquad \Psi \in D(A)\,.$$

Hence as on the Fock space $\mathscr{H}(\Lambda)$ one has

$$\Phi(f)^2 \leqq (2N_\Lambda+1)\,|f|_2^2 \quad \text{etc.}$$

we have

$$\|(W_\omega(f,g)-1)\,\Omega_\omega\|^2 \leqq \mathrm{Tr}_{\mathscr{H}(\Lambda)}\,(\varrho_\Lambda(2N_\Lambda+1))\,[|f|_2^2+|g|_2^2+2|f|_2\,|g|_2]$$

where we have used the Schwarz inequality. This proves the Lemma.

Next let us introduce a norm $|||\,.\,|||$ on $\mathscr{S}$ as follows. Consider R^ν divided into cubic cells of unit size $\Lambda_1, \Lambda_2, \ldots$. Then the norm is defined by

$$|||f||| = \sum_{i\geqq 1}\left[\int_{\Lambda_i} dx\,|f(x)|^2\right]^{\frac{1}{2}}, \qquad f \in \mathscr{S}\,.$$

It is easily to verify that this norm is continuous. We now have

Lemma 3. *If ω is a translationally invariant state of finite mean density then for $f, g \in \mathscr{D}$,*

$$\|(\Phi_\omega(f) + \pi_\omega(g))\,\Omega_\omega\| \leqq \sqrt{2N_{\Lambda_1}(\omega)+1}\,[|||f||| + |||g|||]$$

and hence

$$\|(W_\omega(f, g) - 1)\,\Omega_\omega\| \leqq \sqrt{2N_{\Lambda_1}(\omega)+1}\,[|||f||| + |||g|||]\,.$$

Proof. We have immediately that

$$\begin{aligned}\|(\Phi_\omega(f) + \pi_\omega(g))\,\Omega_\omega\| &= \Big\|\sum_i (\Phi_\omega(f_i) + \pi_\omega(g_i))\,\Omega_\omega\Big\| \\ &\leqq \sum_i \big\|(\Phi_\omega(f_i) + \pi_\omega(g_i))\,\Omega_\omega\big\|\end{aligned}$$

where $f_i = f\chi_i$, $g_i = g\chi_i$ and χ_i is the characteristic function of Λ_i. The inequalities now follow as in Lemma 2 with the use of translational invariance to identify the various $N_{\Lambda_i}(\omega)$.

Lemma 4. *If ω is a translationally invariant state of finite mean density then the associated representation $W_\omega(f, g)$, which is defined for $f, g \in \mathscr{D}$, is strongly continuous with respect to the topology induced by $|||\,.\,|||$. Hence the representation W_ω of the commutation relations extends by continuity to a representation defined and continuous on $\mathscr{S}$. Further for $f, g \in \mathscr{S}$ the vector Ω_ω is in the domain of $\Phi_\omega(f) + \pi_\omega(g)$ and*

$$\|(\Phi_\omega(f) + \pi_\omega(g))\,\Omega_\omega\| \leqq \sqrt{2N_{\Lambda_1}(\omega)+1}\,[|||f||| + |||g|||]\,.$$

Proof. By the commutation relations and the cyclicity of Ω_ω the continuity of W_ω follows from its continuity on Ω_ω. It also follows from the commutation relations that W_ω on Ω_ω is continuous if it is continuous at the origin. This however follows from Lemma 3. The second inequality of Lemma 3 remains valid for $f, g \in \mathscr{S}$ by continuity; hence

$$\left\|\left(\frac{W_\omega(tf, tg) - 1}{t}\right)\Omega_\omega\right\| \leqq \sqrt{2N_{\Lambda_i}(\omega)+1}\,[|||f||| + |||g|||]\,.$$

But it follows by spectral theory (for an explicit proof see [15]) that if A is a self-adjoint operator on a Hilbert space then $\Psi \in D(A)$ if, and only if

$$\left\|\left(\frac{e^{iAt} - 1}{t}\right)\Psi\right\| \leqq C_\Psi$$

where C_Ψ is a constant independent of t. Further in such a case

$$\|A\Psi\| = \lim_{t\to 0}\left\|\left(\frac{e^{iAt} - 1}{t}\right)\Psi\right\| \leqq C_\Psi\,.$$

The remaining statements of the Lemma are then established by applying this result with $A = \Phi_\omega(f) + \pi_\omega(g)$.

Thus we have at this point established that each translationally invariant state with finite mean density satisfies the continuity conditions required to define its time evolution. But it is also possible to deduce the following result.

Theorem 4. *If ω is a translationally invariant state with finite mean density and ω_t its time translate, then $t \to \omega_t$ is a weak* continuous one parameter family of translationally invariant states with constant (finite) density.*

Proof. It is clear that ω_t is translationally invariant and the continuity follows from Lemma 3 but it remains to prove that the mean density is constant. Since however Ω_ω is in the domain of $\Phi_\omega(f) + \pi_\omega(g)$ for all $f, g \in \mathscr{S}$ we can introduce the bilinear form

$$f, g \in \mathscr{S} \to ((\Phi_\omega(f) + i\pi_\omega(f))\, \Omega_\omega, (\Phi_\omega(g) + i\pi_\omega(g))\, \Omega_\omega).$$

By Lemma 3 this form is separately continuous in f and g and hence is determined by a tempered distribution in two variables. But using translation invariance we see in fact that

$$((\Phi_\omega(f) + i\pi_\omega(f))\, \Omega_\omega, (\Phi_\omega(g) + i\pi_\omega(g))\, \Omega_\omega) = \int dp\, \tilde{\varrho}(p) \tilde{f}(p)\, g(\tilde{p})$$

where $\tilde{\varrho}$ is a tempered distribution. (As the form is positive one has $\tilde{\varrho} \geqq 0$.) From this formula it now follows that

$$\begin{aligned} \|(\Phi_\omega(V_t f) + i\pi_\omega(V_t f))\, \Omega_\omega\| &= \|(\Phi_\omega(f) + i\pi_\omega(f))\, \Omega_\omega\| \\ &= \|(\Phi_{\omega_t}(f) + i\pi_{\omega_t}(f))\, \Omega_{\omega_t}\|. \end{aligned}$$

Consequently Ω_t is a state of finite density and

$$N_\Lambda(\omega_t) = N_\Lambda(\omega)$$

which completes the proof of the theorem.

If we wish to derive theorems similar to Theorems 3 and 2 for fermions it is necessary to consider a more restricted class of states, namely those possessing Wightman functions of all orders. Thus we now consider the translationally invariant C_∞ states, i.e. the states ω for which Ω_ω is in the domain of all polynomials of $\Phi_\omega(f)$ and $\pi_\omega(g)$ and

$$(\Phi_\omega(f_1) + \pi_\omega(g_1)) \ldots (\Phi_\omega(f_n) + \pi_\omega(g_n))\, \Omega_\omega$$

is continuous in f_i, g_i, $i = 1, 2, \ldots, n$.

Remark 7. If ω is translationally invariant and a C_∞ state for the local number operators, i.e. if ω is locally normal and

$$\mathrm{Tr}_{\mathscr{H}(\Lambda_1)}(\varrho_\Lambda \mathrm{N}_\Lambda^m) < +\infty$$

for all $\Lambda \subset R^\nu$ and $m = 1, 2, \ldots$ then it follows that ω is C_∞ because one can show that

$$\|\Phi_\omega(f_1) \ldots \Phi_\omega(f_n)\,\Omega_\omega\|^2 \leqq \mathrm{Tr}_{\mathscr{H}(\Lambda)}\left(\varrho_{\Lambda_1}(2N_{\Lambda_1} + 2n - 1)^n\right) \prod_{i=1}^{n} |||f_i|||$$

and a similar inequality if some of the $\Phi_\omega(f_i)$ are replaced by $\pi_\omega(f_i)$.

Finally for C_∞ states of finite mean density an analysis of the approach to equilibrium can be carried out in a manner parallel to that of the previous sections but the following changes should be noted.

1. The one point functions $\omega(\Phi(f))$, $\omega(\pi(g))$ are not automatically zero but due to translation invariance do not change with time.

2. The truncated part of the non-gauge invariant two point functions

$$\omega\big((\Phi(f) - i\pi(f))\,(\Phi(g) - i\pi(g))\big)$$

must be assumed square integrable, in the sense of Definition 1, together with the higher functions.

3. The convergence of a square integrable state to the associated gauge invariant quasi-free state as $t \to \infty$ is no longer weak* convergence but convergence of the Wightman functions.

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