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Statistical Mechanics of Quantum Spin Systems. III

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Abstract. In the algebraic formulation the thermodynamic pressure, or free energy, of a spin system is a convex continuous function P defined on a Banach space $\mathfrak{B}$ of translationally invariant interactions. We prove that each tangent functional to the graph of P defines a set of translationally invariant thermodynamic expectation values. More precisely each tangent functional defines a translationally invariant state over a suitably chosen algebra $\mathfrak{A}$ of observables, i. e., an equilibrium state. Properties of the set of equilibrium states are analysed and it is shown that they form a dense set in the set of all invariant states over $\mathfrak{A}$. With suitable restrictions on the interactions, each equilibrium state is invariant under time-translations and satisfies the Kubo-Martin-Schwinger boundary condition. Finally we demonstrate that the mean entropy is invariant under time-translations.

1. Introduction

The purpose of this paper is to continue the general analysis of quantum spin systems which was presented in [1, 2] and [3]. In [2] we gave an algebraic formulation of the mathematical framework of quantum spin systems and showed that the thermodynamic pressure, or free energy, P could be considered as a convex continuous function defined on a Banach space of translationally invariant interactions. Further it was shown that the pressure also served as a generating functional of equilibrium states in the sense that the functional derivatives, i.e., the tangent functionals to the graph of P, determined translationally invariant states over a suitably chosen C^* algebra $\mathfrak{A}$ of observables. The states introduced in this manner play the same role as the more conventionally used correlation functions or thermodynamic expectation values. The results of [2] were, however, incomplete in the sense that we could only rigorously establish that P generated equilibrium states under certain restrictive conditions. In particular it was shown that if the interaction Φ were such that the tangent functional to the graph of P at Φ was unique then this tangent functional determined an equilibrium state. It was further shown that the equilibrium states obtained under such conditions described pure thermodynamic phases. This latter result was derived by establishing and using a variational principle for the pressure which involves the mean entropy introduced in [1]. In the following we complete the results of [2] by proving that each tangent functional to the graph of P determines an equilibrium state, thus covering the situation when mixtures of phases can occur. Further we establish a variational principle for the mean entropy which involves the pressure and also show that every translationally invariant state over $\mathfrak{A}$ can be approximated by physical equilibrium states. Next we extend the results of [3] by proving that if the interactions are such that time translations correspond to a one-parameter group of automorphisms of $\mathfrak{A}$ then the corresponding equilibrium states are invariant under such translations and satisfy the Kubo-Martin-Schwinger boundary condition. Finally, we demonstrate that the mean entropy is invariant under time-translations.

It should perhaps be pointed out that whilst we work in an essentially quantum mechanical setting the results we derive also have relevance for classical spin systems and lattice gases. In fact the analysis of [1, 2] was based on earlier works [4, 5, 6] in a classical framework; many of our present results can be directly transcribed to this framework.

2. Convexity Theorems

The aim of this Section is to derive two mathematical theorems concerning the tangent planes to the graph of a convex function; the physical application of these results will be dealt with in the following Section.

Lemma 1. *Let X and Y be complete metric spaces and let Y be separable. If $Z \subset X \times Y$ is a residual set, i.e., the complement of a set of first category, then there is a residual set $X_1 \subset X$ such that for all $x \in X_1$ the set $Z \cap (\{x\} \times Y)$ is a residual set in $\{x\} \times Y$.*

Proof. We may assume that Z is open and dense and then it is sufficient to find $X_1 \subset X$ such that $Z \cap (\{x\} \times Y)$ is dense in $\{x\} \times Y$ for all $x \in X_1$. Let $a_1, a_2 \ldots$ be a denumerable dense set in Y and define W_i by

$$W_i = \Pi_1 \left\{ z \in Z;\, d(\Pi_2(z); a_i) < \frac{1}{i} \right\}$$

where $\Pi_1(z)$, $\Pi_2(z)$ denote the co-ordinates of z and $d(.;.)$ the metric in Y. Clearly W_i is open and dense. If $x_0 \in \bigcap_i W_i$ it follows that for each i there is a $y_i \in Y$ such that $(x_0, y_i) \in Z$ and $d(y_i; a_i) < \frac{1}{i}$. Then $\{y_i\}$ is dense in Y.

Corollary. *Let $\mathfrak{X}$ be a Banach space and Y a subset of the closed unit ball in $\mathfrak{X}$ which is a residual set. Let $\omega \in \mathfrak{X}$ be a unit vector. It follows that for $\varepsilon > 0$ there is a unit vector ω' with $\|\omega - \omega'\| < \varepsilon$ such that*

$$\{\lambda;\, \lambda\omega' \in Y,\, -1 \leqq \lambda \leqq 1\}$$

is a residual set in $[-1, 1]$.

In the following we will need the notion of the tangent functional to the graph of a convex function; a tangent functional is essentially a tangent plane normalised suitably. If f is a convex continuous function defined on a Banach space $\mathfrak{X}$ an element $y_x \in \mathfrak{X}'$ is said to be a tangent functional to the graph of f at x if

$$f(x+\omega) \geqq f(x) + y_x(\omega)\,, \qquad \omega \in \mathfrak{X}\,.$$

If f is differentiable at x the only tangent functional at x is the derivative Df_x.

Theorem 1. *Let f be a convex function defined and continuous on a neighbourhood of zero in a separable Banach space $\mathfrak{X}$. Let $y \in \mathfrak{X}'$ be a tangent functional at zero to the graph of f. It follows that y is contained in the weak * closed convex hull of the set of tangent functionals $\mathfrak{z}$ defined by $\mathfrak{z} = \{z \in \mathfrak{X}'$; there exist $x_\alpha \to 0$ (in norm) such that f is differentiable at each x_α and weak * $\lim_\alpha Df x_\alpha = z\}$.*

Proof. From convexity we may directly deduce that for a sufficiently small neighbourhood $\mathscr{V}$ of zero there is an $M > 0$ such that $|f(x) - f(y)| \leqq M\,\|x-y\|$ for $x, y \,\varepsilon\, \mathscr{V}$. In particular it follows that $\|y\| \leqq M$ and $\|z\| \leqq M$ for all $z \,\varepsilon\, \mathfrak{z}$. Now assume the theorem is false; then there exists a weak * continuous linear functional on $\mathfrak{X}'$, i.e., an element of $\mathfrak{X}$, which strongly separates y from $\mathfrak{z}$. In particular there exists a unit vector $\omega \,\varepsilon\, \mathfrak{X}$ and a real number m such that $y(\omega) > m$ and $z(\omega) \leqq m$ for all $z \,\varepsilon\, \mathfrak{z}$. Since $\mathfrak{z}$ is bounded we can replace ω by any ω' sufficiently close to it and still obtain separation. But as f is convex it is differentiable on a residual set and hence, using the preceding corollary, we see that we may assume that f is differentiable at $\lambda\omega$ for all λ in a residual subset of $[-1, 1]$. By weak * compactness we can choose a net $\lambda_\alpha \to 0$ and $\lambda_\alpha \geqq 0$ such that f is differentiable at each $\lambda_\alpha\omega$ and $Df_{\lambda_\alpha\omega}$ converges in the weak * topology on $\mathfrak{X}'$. Since $\lim_\alpha Df_{\lambda_\alpha\omega} \in \mathfrak{z}$ we have $\lim_\alpha Df_{\lambda_\alpha\omega}(\omega) \leqq m$, i.e.,

$$\lim_\alpha \left[\frac{d}{d\lambda} f(\lambda\omega)\right]_{\lambda=\lambda_\alpha} \leqq m\,. \tag{1}$$

However, since $\lambda_\alpha \geqq 0$ the slope of any tangent to $f(\lambda\omega)$ at zero must be majorised by the left-hand side of (1). But, since y is a tangent functional to the graph of f at zero, there is a tangent line to the function $\lambda \to f(\lambda\omega)$ at $\lambda = 0$ with slope $y(\omega)$. Hence $y(\omega) \leqq m$. But this contradicts our assumption $y(\omega) > m$, and thus the theorem is proved.

Lemma 2[1]**.** *Let f be a non-negative C^∞ function defined on R^n; then the derivative Df of f satisfies the inequality*

$$\min_{\|x\| < a} (1 + \|x\|)\,\|Df\|(x) \leqq \frac{f(0)}{\log(1+a)}\,,\ a \in R^+$$

[1] The proofs of this and the following lemma are based upon suggestions by D. Ruelle.

and hence

$$\min_{x \in R^n} (1 + \|x\|) \|Df\| (x) = 0$$

where the $\|.\|$ *refers to the usual Euclidean norm on* R^n *which is also identified with its dual.*

Proof. We may assume $\|Df\| > 0$ for $\|x\| \leqq a$ because the contrary assumption leads trivially to the desired result. Now, let $x(t)$ be an arc in R^n with $x(0) = 0$ and such that

$$\frac{d}{dt} x(t) = - \frac{Df}{\|Df\|} (x(t)) .$$

We note that for $t > 0$ we have $\|x(t)\| \leqq t$ and

$$0 \leqq f(x(a)) = f(0) + \int_0^a dt \frac{d}{dt} f(x(t))$$

$$= f(0) - \int_0^a dt \|Df\| (x(t))$$

$$\leqq f(0) - \min_{\|x\| \leqq a} (1 + \|x\|) \|Df\| (x) \int_0^a \frac{dt}{1+t} .$$

A simple rearrangement yields the desired result.

Lemma 3. *Let* f *be a convex continuous non-negative function defined on* R^n *and let* $a > 0$ *be given. There is an* $x \in R^n$, *with* $\|x\| \leqq a$ *and a tangent functional* h_x *to the graph of* f *at* x *such that* $(1 + \|x\|) \|h_x\| < 2f(0)/\log(1+a)$.

Proof. Let ϱ_n be a sequence of positive C^∞ functions of compact support with the following properties

1. $\int dx \varrho_n(x) = 1$
2. $\varrho_n * f \to f$ uniformly on compact sets
3. $(\varrho_n * f)(0) \leqq 2f(0)$.

Now $\varrho_n * f$ is non-negative, C^∞, and convex; therefore, there exists an x_n with $\|x_n\| \leqq a$ such that

$$(1 + \|x_n\|) \|D_{x_n}(\varrho_n * f)\| \leqq \frac{2f(0)}{\log(1+a)}$$

by lemma 2. Next, possibly passing to a subsequence, we can assume $x_n \to x$ and $h_n = D_{x_n}(\varrho_n * f) \to h_x$. We then have

$$(1 + \|x\|) \|h_x\| \leqq \frac{2f(0)}{\log(1+a)} .$$

But, by convexity, we also have

$$(\varrho_n * f)(x_n + \delta) \geqq (\varrho_n * f)(x_n) + h_n(\delta) , \quad \delta \in R^n$$

and therefore

$$f(x+\delta) \geqq f(x) + h_x(\delta)$$

i.e., h_x is a tangent functional to the graph of f at x. This completes the proof of the lemma.

Theorem 2. *Let f be a convex continuous function defined on a separable Banach space $\mathfrak{X}$ and let $h \in \mathfrak{X}'$ have the properties that $h(x) \leqq f(x)$ for all $x \in \mathfrak{X}$. It follows that h is contained in the weak * closure of the set of tangent functionals to the graph of f.*

Proof. We can suppose, without loss of generality, that $h = 0$. Now let $\omega_1, \omega_2, \ldots, \omega_n \in \mathfrak{X}$ and $\varepsilon > 0$ be given. We have to find an $x \varepsilon \mathfrak{X}$ and a tangent functional y_x to the graph of f at x such that $|y_x(\omega_i)| < \varepsilon$ for $i = 1, 2, \ldots, n$. Now by the Hahn-Banach extension theorem, it suffices to find an x in the linear subspace $\tilde{\mathfrak{X}}$ of $\mathfrak{X}$ spanned by $\omega_1, \ldots, \omega_n$ and a tangent functional $\tilde{y}_x \,\varepsilon\, \tilde{\mathfrak{X}}'$ such that $|\tilde{y}_x(\omega_i)| < \varepsilon$ for $i = 1, 2, \ldots, n$, i.e., we can, effectively, assume that $\mathfrak{X}$ is finite dimensional. The proof of the theorem is thus immediately given by lemma 3.

Note that x and the tangent functional y_x can be chosen such that we not only have $|y_x(\omega_i)| < \varepsilon$ for $i = 1, 2, \ldots, n$ but also $|y_x(x)| < \varepsilon$. This remark, which will be of importance in the next Section, follows from the estimate given in lemma 3.

3. Equilibrium States

In this Section we apply the foregoing results to the characterization of the equilibrium states of a quantum spin system and to the derivation of certain properties of these states. The characterization we obtain completes earlier results obtained in [2] and [3]. We begin by recalling the mathematical framework associated with a quantum spin system.

A quantum spin system is described in terms of a simple separable C^* algebra $\mathfrak{A}$ of quasi-local observables and a collection $\{\mathfrak{A}(\Lambda)\}$ of C^* sub-algebras of $\mathfrak{A}$, where Λ takes values on the finite subsets of Z^ν. Elements of the $\mathfrak{A}(\Lambda)$ are called strictly local observables. The algebras $\mathfrak{A}$ and $\mathfrak{A}(\Lambda)$, $\Lambda \subset Z^\nu$, satisfy the following properties

1. $\mathfrak{A}(\Lambda_1) \subset \mathfrak{A}(\Lambda_2)$ if $\Lambda_1 \subset \Lambda_2$
2. $\mathfrak{A}$ is the norm closure of $\bigcup_{\Lambda \in Z^\nu} \mathfrak{A}(\Lambda)$
3. $[\mathfrak{A}(\Lambda_1), \mathfrak{A}(\Lambda_2)] = 0$ if $\Lambda_1 \cap \Lambda_2 = \emptyset$
4. the group Z^ν of space translations is a subgroup of the automorphism group of $\mathfrak{A}$ and the action of these automorphisms is such that

$$A \,\varepsilon\, \mathfrak{A}(\Lambda) \to \tau_x A \,\varepsilon\, \mathfrak{A}(\Lambda + x)\,, \qquad x \in Z^\nu$$

and

$$\|[A, \tau_x B]\| \xrightarrow[|x| = \infty]{} 0\,, \qquad A, B \,\varepsilon\, \mathfrak{A} \quad \text{and} \quad x \in Z^\nu$$

5. for each $\Lambda \subset Z^\nu$, $\mathfrak{A}(\Lambda)$ is isomorphic to the matrix algebra of bounded operators $\mathfrak{B}(\mathfrak{H}_\Lambda)$ on a finite dimensional Hilbert space $\mathfrak{H}_\Lambda$.

The states, i.e., the normalized positive linear functionals over $\mathfrak{A}$, form a weakly compact convex subset E of $\mathfrak{A}'$ and the translationally invariant states, i.e., the states such that

$$\varrho(\tau_x A) = \varrho(A), \quad A \in \mathfrak{A}, \quad x \in Z^\nu$$

form a weakly compact convex subset $E \cap L^\perp_{Z^\nu}$ of E. The extremal elements $\mathscr{E}(E \cap L^\perp_{Z^\nu})$ of this latter subset enjoy many remarkable properties of an ergodic nature (see for example [7] and [8]) which allow the physical interpretation that they describe single thermodynamic phases. If we consider a state ϱ restricted to any subalgebra $\mathfrak{A}(\Lambda)$ then, by property 5. above, the state defines a positive operator ϱ_Λ on $\mathfrak{H}_\Lambda$ such that

$$\mathrm{Tr}_{\mathfrak{H}_\Lambda}(\varrho_\Lambda) = 1 \quad \text{and} \quad \mathrm{Tr}_{\mathfrak{H}_\Lambda}(\varrho_\Lambda A) = \varrho(A)$$

for $A \in \mathfrak{A}(\Lambda)$ [here and in the sequel, we tacitly identify $\mathfrak{A}(\Lambda)$ and $\mathfrak{B}(\mathfrak{H}_\Lambda)$]. The density matrices ϱ_Λ are related by certain compatibility conditions, but for our present purposes it suffices to note that we can define a local entropy $S_\varrho(\Lambda)$ of a state via

$$S_\varrho(\Lambda) = -\,\mathrm{Tr}_{\mathfrak{H}_\Lambda}(\varrho_\Lambda \log \varrho_\Lambda)$$

and, if ϱ is an invariant state, i.e., $\varrho \in E \cap L^\perp_{Z^\nu}$, a mean entropy via

$$S(\varrho) = \lim_{\Lambda \to \infty} \frac{S_\varrho(\Lambda)}{N(\Lambda)} = \inf_\Lambda \frac{S_\varrho(\Lambda)}{N(\Lambda)}$$

where $N(\Lambda)$ is the number of points in the set $\Lambda \subset Z^\nu$ and, for simplicity, here, and in the following, we take the limits over parallelepipeds whose sides each tend to infinity. The mean entropy defined in this manner is a non-negative affine upper semi-continuous function on $E \cap L^\perp_{Z^\nu}$ (for details, and proofs of these statements, see [1]).

Physically we consider the points $x \in Z^\nu$ as sites of particles or "spins", which interact together. In our rather abstract setting we introduce an interaction Φ as a function from the finite sets $X \subset Z^\nu$ to $\mathfrak{A}$ with values $\Phi(X) \in \mathfrak{A}(X)$. We assume

1. $\Phi(X)$ is Hermitian

2. $\Phi(X + a) = \tau_a \Phi(X)$ for $a \in Z^\nu$

and 3. $\|\Phi\| = \sum_{X \ni 0} \frac{\|\Phi(X)\|}{N(X)} < +\infty$.

With respect to the norm introduced in the last conditions the interactions Φ form a separable Banach space $\mathfrak{B}$. The finite range interactions, i.e., those interactions such that for $X \ni 0$ $\Phi(X) = 0$ unless $X \subset \Lambda$ for some finite Λ, form a dense subset $\mathfrak{B}_0 \subset \mathfrak{B}$. It is convenient to introduce an auxiliary Banach space $\mathfrak{B}_1$, which we leave arbitrary up to the assumption that $\mathfrak{B}_0 \subset \mathfrak{B}_1 \subset \mathfrak{B}$ and $\mathfrak{B}_0$ is dense in $\mathfrak{B}_1$. The interaction energy of a spin system confined to the finite set Λ is defined for $\Phi \in \mathfrak{B}_1$ by

$$U_\Phi(\Lambda) = \sum_{X \subset \Lambda} \Phi(X) .$$

We also introduce the "interaction energy" at the origin by

$$A_\Phi = \sum_{X \ni 0} \frac{\Phi(X)}{N(X)} .$$

The following theorem gives information concerning the equilibrium states of spin systems with interactions $\Phi \in \mathfrak{B}_1$; in part the theorem summarises results already derived in [2].

Theorem 3. 1. *If $\Phi \in \mathfrak{B}_1$ then the thermodynamic pressure*

$$P(\Phi) = \lim_{\Lambda \to \infty} \frac{1}{N(\Lambda)} \log \operatorname{Tr}_{\mathfrak{H}_\Lambda}(e^{-U_\Phi(\Lambda)})$$

exists. The function $\Phi \to P(\Phi)$ is convex continuous on the Banach space $\mathfrak{B}_1$ and

$$|P(\Phi) - P(\Psi)| \leqq \|\Phi - \Psi\| , \quad \Phi, \Psi \in \mathfrak{B}_1 .$$

2. *If $\alpha_\Phi \in \mathfrak{B}_1'$ is a tangent functional to the graph of P at Φ, i.e.,*

$$P(\Phi + \Psi) \geqq P(\Phi) - \alpha_\Phi(\Psi) \quad \textit{for all} \quad \Psi \in \mathfrak{B}_1$$

then α_Φ determines a state $\varrho_\Phi \in E \cap L_{Z^\nu}^{\perp}$ through the relation

$$\alpha_\Phi(\Psi) = \varrho_\Phi(A_\Psi) .$$

The states ϱ_Φ defined in this way will be called equilibrium states.

3. *If $T \subset \mathfrak{B}_1$ is the set of Φ such that the graph of P has a unique tangent functional at Φ then T is a residual set in $\mathfrak{B}_1$ and for $\Phi \in T$ the equilibrium state ϱ_Φ determined by the tangent functional α_Φ is ergodic i.e., $\varrho_\Phi \in \mathscr{E}(E \cap L_{Z^\nu}^{\perp})$. Further we have for $\Phi \in T$ the relation*

$$\alpha_\Phi(\Psi) = \varrho_\Phi(A_\Psi) = \lim_{\Lambda \to \infty} \frac{1}{Tr_{\mathfrak{H}_\Lambda}(e^{-U_\Phi(\Lambda)})} \operatorname{Tr}_{\mathfrak{H}_\Lambda}\left((e^{-U_\Phi(\Lambda)} \frac{U_\Psi(\Lambda)}{N(\Lambda)}\right) . \quad (2)$$

4. *The pressure P, the mean entropy S, and the set of equilibrium states are related as follows*

$$P(\Phi) = S(\varrho_\Phi) - \varrho_\Phi(A_\Phi) = \sup_{\varrho \in E \cap L_{Z^\nu}^{+}} \{S(\varrho) - \varrho(A_\Phi)\} , \quad \Phi \in \mathfrak{B}_1 , \quad (3)$$

where ϱ_Φ is any equilibrium state associated with Φ. The supremum in the last expression is reached by a unique state ϱ_Φ if, and only if, $\Phi \in T$.

5. *The pressure P, the mean entropy S, and the space $\mathfrak{B}_1$ of interactions are related as follows*

$$S(\varrho) = \inf_{\Phi \in \mathfrak{B}_1} \{P(\Phi) + \varrho(A_\Phi)\} \quad \textit{for} \quad \varrho \in E \cap L_{Z^\nu}^{\perp} .$$

6. *The equilibrium states are weak * dense in the set $E \cap L^{\perp}_{Z^\nu}$ of all translationally invariant states over $\mathfrak{A}$.*

Proof. Statements 1. and 3. together with parts of statement 4. are proved in [2]. In particular it is shown in this reference that the maximum principle (3) holds and that, for $\Phi \varepsilon T$, the tangent functional α_Φ determines an ergodic equilibrium state ϱ_Φ, the relation (2) is valid, and ϱ_Φ gives the unique supremum in (3). However it now follows directly from theorem 1 that a general tangent functional α_Φ determined an equilibrium state ϱ_Φ; in the present context theorem 1 states that a tangent functional α_Φ with $\Phi \notin T$ can be approximated weakly by convex combinations of tangent functionals α_Ψ with $\Psi \varepsilon T$. The facts that in general ϱ_Φ gives the maximum in (3) and that this maximum is unique only if $\Phi \varepsilon T$ follow from considerations reproduced in [2] and [3]. It remains to prove statements 5. and 6.; we begin with the latter.

Let $\varrho \varepsilon E \cap L^{\perp}_{Z^\nu}$ be any invariant state; then from (3) we see that

$$P(\Phi) \geqq S(\varrho) - \varrho(A_\Phi) \geqq -\varrho(A_\Phi)$$

where we have used the non-negativity of S to obtain the second inequality. Thus the function $\Phi \to \alpha(\Phi) = \varrho(A_\Phi)$ is linear and its graph lies below the graph of P. Hence by theorem 2 α lies in the weak * closure of the set of tangent functionals to P and thus by statement 2. of the above theorem we obtain the desired result.

To prove statement 5. we note that by (3)

$$P(\Phi) + \varrho(A_\Phi) - S(\varrho) \geqq 0 \tag{4}$$

for $\Phi \in \mathfrak{B}_1$ and $\varrho \varepsilon E \cap L^{\perp}_{Z^\nu}$. However, given $\varepsilon > 0$ we can choose $\Phi \in \mathfrak{B}_1$ and ϱ_Φ such that

$$S(\varrho) + \frac{\varepsilon}{2} > S(\varrho_\Phi) = P(\Phi) + \varrho_\Phi(A_\Phi) \tag{5}$$

and

$$|\varrho_\Phi(A_\Phi) - \varrho(A_\Phi)| < \frac{\varepsilon}{2}\,. \tag{6}$$

Here we have used the upper semi-continuity of S and the remark at the end of the proof of theorem 2. Combining (4), (5) and (6) we find with this choice of Φ

$$\varepsilon > P(\Phi) + \varrho(A_\Phi) - S(\varrho) \geqq 0\,.$$

This establishes the desired property and completes the proof of the theorem.

In the foregoing we have left a certain arbitrariness in the definition of the Banach space $\mathfrak{B}_1$. In the following, however, we will consider one specific Banach space which we define as the set of interactions $\Phi \in \mathfrak{B}$ which have the property that

$$\|\Phi\|_1 = \sum_{X \ni 0} \|\Phi(X)\| \exp\{N(X)\} < +\infty\,. \tag{7}$$

For this space of interactions it is possible to discuss the time development of the spin system. In particular, for each $\Phi \varepsilon \mathfrak{B}_1$ there exists a one-parameter group of automorphisms of the algebra $\mathfrak{A}$ of quasilocal observables corresponding to time translations. We denote the action of this group by $A \varepsilon \mathfrak{A} \to \tau_t^\Phi A \varepsilon \mathfrak{A}$ for $t \varepsilon R$; the action is defined by

$$\tau_t^\Phi A = \lim_{\Lambda \to \infty} e^{itU_\Phi(\Lambda)} A e^{-itU_\Phi(\Lambda)} \quad t \varepsilon R, \quad A \varepsilon \mathfrak{A}, \quad \Phi \varepsilon \mathfrak{B}_1 .$$

(The existence of this limit was established in [3] for a dense subset of $\mathfrak{B}_1$; Ruelle [9] has shown that the arguments of [3] can be improved to establish the existence for all $\Phi \varepsilon \mathfrak{B}_1$.)

Theorem 4. *If $\Phi \varepsilon \mathfrak{B}_1$, the space of interactions whose norm is given by (7), then any equilibrium state ϱ_Φ, defined by a tangent functional to the graph of the pressure P at Φ, has the following properties;*

1. *ϱ_Φ is invariant under time-translations, i.e.*

$$\varrho_\Phi(\tau_t^\Phi A) = \varrho_\Phi(A) \quad \textit{for all} \quad A \varepsilon \mathfrak{A},\ t \varepsilon R .$$

2. *ϱ_Φ satisfies the Kubo-Martin-Schwinger boundary condition. Explicitly, for $A, B \varepsilon \mathfrak{A}$, the function $t \to \varrho_\Phi(A(\tau_t^\Phi B))$ extends to a bounded continuous function on the strip $0 \leqq \operatorname{Im}\{t\} \leqq 1$ which is analytic on the interior of the strip, and we have*

$$\varrho_\Phi(A(\tau_{t+i}^\Phi B)) = \varrho_\Phi((\tau_t^\Phi B) A) .$$

Proof. Let $T \subset \mathfrak{B}_1$ be the set of interactions at which the graph of P has a unique tangent plane. For Φ in T the properties stated in the theorem have already been proved in [3]; we will obtain the general statement from this result by an approximation argument using theorem 1. It is easy to see that weak limits of convex combinations of states satisfying 1. and 2. again satisfy 1. and 2.; hence, by theorem 1, it will suffice to prove the theorem in the special case in which

$$\varrho_\Phi = \lim_\alpha \varrho_{\Phi_\alpha}$$

where Φ_α is a net in T converging in norm to Φ and ϱ_{Φ_α} is the state determined by the unique tangent plane to the graph of P at Φ_α. Moreover, we can assume that A and B are strictly local; the assertions for general elements of $\mathfrak{A}$ are then obtained by a straightforward limiting argument.

It follows easily from the estimates in [9] that

$$\lim_\alpha \|\tau_t^\Phi A - \tau_t^{\Phi_\alpha} A\| = 0 .$$

uniformly for t in any bounded interval. Hence, using the invariance of ϱ_{Φ_α} under $\tau_t^{\Phi_\alpha}$, we get

$$|\varrho_\Phi(\tau_t^\Phi A) - \varrho_\Phi(A)| \leqq |\varrho_\Phi(\tau_t^\Phi A) - \varrho_{\Phi_\alpha}(\tau_t^\Phi A)| + \|\tau_t^\Phi A - \tau_t^{\Phi_\alpha} A\| + |\varrho_{\Phi_\alpha}(A) - \varrho_\Phi(A)|$$

and the right-hand side goes to zero as $\alpha \to \infty$. This proves 1.

To prove 2., we first remark that

$$\frac{d}{dt}\tau_t^{\Phi_\alpha} B = \tau_t^{\Phi_\alpha}(B'_\alpha)$$

where

$$B'_\alpha = i \lim_{\Lambda\to\infty} [U_{\Phi_\alpha}(\Lambda), B]$$

and that $\|B'_\alpha\|$ is bounded with respect to α for any fixed B. Hence,

$$\varrho_{\Phi_\alpha}(A(\tau_t^{\Phi_\alpha} B))$$

is a net of continuous functions on the strip $0 \leqq \operatorname{Im}\{t\} \leqq 1$ which are holomorphic on the interior of the strip and whose derivatives are bounded uniformly in α and in t. Since

$$\lim_\alpha \varrho_{\Phi_\alpha}(A(\tau_t^{\Phi_\alpha} B)) = \varrho_\Phi(A(\tau_t^\Phi B))$$
$$\lim_\alpha \varrho_{\Phi_\alpha}(A(\tau_{t+i}^{\Phi_\alpha} B)) = \varrho_\Phi((\tau_t^\Phi B)A)$$

for all real t, this net converges pointwise to a function continuous and bounded in the closed strip, holomorphic on the interior of the strip, with the right boundary values, so 2. is proved.

4. Conservation of Entropy

Theorem 5. *Let $\Phi \varepsilon \mathfrak{B}_1$ and let ϱ be a translation-invariant state over $\mathfrak{A}$. For any $t \varepsilon R$, let the state ϱ_t over $\mathfrak{A}$ be defined by*

$$\varrho_t(A) = \varrho(\tau_t^\Phi A)\,.$$

Then, $S(\varrho_t) = S(\varrho)$ for all t.

Proof. By reversibility, it will be sufficient to show that $S(\varrho_t) \geqq S(\varrho)$, and, since S is upper semi-continuous, this will follow if we can show that ϱ_t can be approximated arbitrarily well by states with the same entropy as ϱ.

If a is a strictly positive integer, we let

$$\Lambda(a) = \{(n_1, \ldots, n_\nu) \in Z^\nu;\ -a < n_i \leqq a\}$$
$$N(a) = N(\Lambda(a)) = (2a)^\nu$$
$$\Gamma_a \quad = \{(2n_1 a, \ldots, 2n_\nu a);\ n_1, \ldots, n_\nu \in Z\}$$

and we let $x_1, x_2, \ldots$ be an enumeration of the elements of Γ_a. Define a one-parameter group of automorphisms ${}^a\tau_t^\Phi$ of $\mathfrak{A}$ by

$${}^a\tau_t^\Phi(A) = \lim_{N\to\infty} \exp\left\{\mathrm{it} \sum_{j=1}^N \tau_{x_j} U_\Phi(\Lambda(a))\right\} A \exp\left\{-\,\mathrm{it} \sum_{j=1}^N \tau_{x_j} U_\Phi(\Lambda(a))\right\}.$$

This one-parameter group of automorphisms corresponds to an interaction which differs from that defined by Φ only in that all interactions between translates of $\Lambda(a)$ by different elements of Γ_a are suppressed. Note that:

1. If $A \varepsilon \mathfrak{A}(\Lambda(a))$, ${}^a\tau_t^\Phi(A) = \exp\{\mathrm{it}\, U_\Phi(\Lambda(a))\} A \exp\{-\mathrm{it}\, U_\Phi(\Lambda(a))\}$.
2. If $x \varepsilon \Gamma_a$, $\quad \tau_x {}^a\tau_t^\Phi = {}^a\tau_t^\Phi \tau_x$.

Let

$$ {}^a\varrho_t(A) = \varrho({}^a\tau_t^\Phi(A)) $$

then ${}^a\varrho_t$ is a state over $\mathfrak{A}$ invariant under the subgroup Γ_a of Z^ν and its entropy is equal to that of ϱ. Therefore, if we define

$$ {}^a\bar{\varrho}_t(A) = \frac{1}{N(a)} \sum_{x \varepsilon \Lambda(a)} {}^a\varrho_t(\tau_x A) , $$

${}^a\bar{\varrho}_t$ is invariant under Z^ν and has the same entropy as ϱ. Taking into account the remarks at the beginning of the proof we see that all we have to prove is that

$$ \lim_{a \to \infty} {}^a\bar{\varrho}_t(A) = \varrho_t(A) $$

for all strictly local A in $\mathfrak{A}$.

By the translation invariance of ϱ,

$$ {}^a\bar{\varrho}_t(A) = \varrho\left(\frac{1}{N(a)} \sum_{x \varepsilon \Lambda(a)} \tau_{-x}\, {}^a\tau_t^\Phi\, \tau_x(A)\right) $$

so it will suffice to prove

$$ \lim_{a \to \infty} \frac{1}{N(a)} \sum_{x \varepsilon \Lambda(a)} \tau_{-x}\, {}^a\tau_t^\Phi\, \tau_x(A) = \tau_t^\Phi(A) . $$

Since A is strictly local, the terms in the sum on the left with $\tau_x(A) \notin \mathfrak{A}(\Lambda(a))$ become negligible as $a \to \infty$, so we can replace the left-hand side by:

$$ \lim_{a \to \infty} \frac{1}{N(a)} \sum_{x \varepsilon \Lambda(a)} \exp\{\mathrm{it}\, U_\Phi(\Lambda(a) - x)\} A \exp\{-\mathrm{it}\, U_\Phi(\Lambda(a) - x)\} . $$

Thus, to complete the proof it will suffice to prove the following assertion: For any $A \in \mathfrak{A}$, any t, and any $\varepsilon > 0$, there is a finite subset Λ of Z^ν such that, whenever $\Lambda' \supset \Lambda$,

$$ \|\exp\{\mathrm{it}\, U_\Phi(\Lambda')\} A \exp\{-\mathrm{it}\, U_\Phi(\Lambda')\} - \tau_t^\Phi(A)\| < \varepsilon . $$

This assertion is equivalent to the assertion that, for any t, any A, and any increasing sequence Λ_n of finite subsets of Z^ν whose union is all of Z^ν,

$$ \lim_{n \to \infty} \exp\{\mathrm{it}\, U_\Phi(\Lambda_n)\} A \exp\{-\mathrm{it}\, U_\Phi(\Lambda_n)\} = \tau_t^\Phi A . $$

For t small and A strictly local, this follows from the power series expansion for $\tau_t^\Phi(A)$. For t small and general A, the assertion follows since a sequence of isometries on a Banach space which converges strongly on a dense subset converges strongly everywhere. Finally, the assertion for general t is proved by remarking that, if a sequence of isometries on a Banach space converges strongly, the sequence of n^{th} powers converges strongly to the n^{th} power of the limit.

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