Integral Representations of Invariant States on B^* Algebras

O. LANFORD† AND D. RUELLE‡
State University of New York at Stony Brook, Stony Brook, New York

(Received 2 September 1966)

Let $\mathfrak{A}$ be a B^* algebra with a group G of automorphisms and K be the set of G-invariant states on $\mathfrak{A}$. We discuss conditions under which a G-invariant state has a unique integral representation in terms of extremal points of K, i.e., extremal invariant states.

1. INTRODUCTION AND NOTATIONS

LET $\mathfrak{A}$ be a B^* algebra, G a group, and τ a (group) homomorphism of G into the * automorphisms of $\mathfrak{A}$. If $\mathfrak{A}$ has an identity, the set of G-invariant states on $\mathfrak{A}$ is compact (for the w^* topology) and one may try to obtain an integral representation of G-invariant states in terms of extremal invariant states. If G is reduced to the identity, such an integral representation is unique if and only if $\mathfrak{A}$ is Abelian. It has, however, been remarked recently that uniqueness prevails under more general circumstances (see Refs. 1 and 2, and for further information, Refs. 3 and 4). The aim of this note is to discuss the general problem of existence and uniqueness of integral representations of invariant states, using Choquet's theory of integral representations on convex compact sets. While some of our results are best possible (in particular, the characterization of G-Abelian B^* algebras, Theorem 2.3), others could certainly be improved (see Sec. 4). Questions related to the existence of a topology on G are relevant for applications to physics, but are not discussed here.

If K is a metrizable compact (phase space) and G a group of homomorphisms of K (time evolution), it is known (see Ref. 5) that a measure on K, invariant under G, can be uniquely decomposed into ergodic measures, i.e., has an integral representation in terms of extremal invariant measures. In this note we obtain an extension of this result of ergodic theory to the noncommutative case (using an algebra of operators in Hilbert space instead of the algebra of continuous functions on a compact) and we weaken the metriza-

bility requirement. The physical problem we have in mind is that of statistical mechanics of an infinite system. An equilibrium state of such a system can be represented by a state ρ on a B^* algebra (e.g., the algebra of canonical commutation relations for a system of bosons), and we may assume invariance of ρ under some natural group G (e.g., the product of the Euclidean group and of the particle number gauge group). One can see that a decomposition of ρ into extremal G-invariant states corresponds to a decomposition into pure thermodynamic phases. Such a decomposition should thus be unique and the problem arises to study the conditions on a non-Abelian algebra and a group of automorphisms such that the invariant states have a unique integral representation in terms of extremal invariant states.

Throughout this note we use the following notations: $\mathfrak{A}$, a B^* algebra; G, a group; $\tau: g \rightarrow \tau_g$ a representation of G into the * automorphisms of $\mathfrak{A}$; $\mathfrak{A}'$, the dual of $\mathfrak{A}$ with the w^* topology; $E \subset \mathfrak{A}'$, the set of states on $\mathfrak{A}$ (if $\mathfrak{A}$ has an identity, E is compact); $\mathfrak{L}_G$, the subspace of $\mathfrak{A}$ generated by the elements $A - \tau_g A$ with $A \in \mathfrak{A}$, $g \in G$; $\mathfrak{L}_G^\perp$, the orthogonal complement of $\mathfrak{L}_G$ in $\mathfrak{A}'$; $E \cap \mathfrak{L}_G^\perp$, the set of G-invariant states.

If $\rho \in E$, we denote by $\mathfrak{H}_\rho$, the Hilbert space of the Gel'fand–Segal construction; π_ρ, the corresponding * homomorphism of $\mathfrak{A}$ into the bounded operators on $\mathfrak{H}_\rho$; $\Omega_\rho \in \mathfrak{H}_\rho$, the normalized vector, cyclic with respect to $\pi_\rho(\mathfrak{A})$ and such that $\rho(A) = (\Omega_\rho, \pi_\rho(A)\Omega_\rho)$ for all $A \in \mathfrak{A}$.

If $\rho \in E \cap \mathfrak{L}_G^\perp$ we denote by U_ρ, the unitary representation of G in $\mathfrak{H}_\rho$ such that $U_\rho(g)\Omega_\rho = \Omega_\rho$, $U_\rho(g)\pi_\rho(A)U_\rho(g^{-1}) = \pi_\rho(\tau_g A)$ for all $g \in G$, $A \in \mathfrak{A}$; P_ρ, the projection on the subspace of $\mathfrak{H}_\rho$ formed by the vectors invariant under $U_\rho(G)$.

2. G-ABELIAN ALGEBRAS

In Refs. 1 and 2, the group G was taken to be R^ν and it was assumed that if $A_1, A_2 \in \mathfrak{A}$ the commutator $[A_1, \tau_g A_2]$ vanishes when $g \rightarrow \infty$. A suitable generalization of this condition is the basis of our analysis; we formulate it first in a different manner.

¹ D. Ruelle, Commun. Math. Phys. 3, 133 (1966).
² D. Kastler and D. Robinson, Commun. Math. Phys. 3, 151 (1966).
³ S. Doplicher, D. Kastler, and D. Robinson, Commun. Math. Phys. 3, 1, (1966).
⁴ D. Robinson and D. Ruelle, "Extremal Invariant States," Institut des Hautes Etudes Scientifiques (1966).
⁵ R. Phelps, *Lectures on Choquet's Theorem* (D. Van Nostrand Company, Inc. Princeton, New Jersey, 1966).

Definition 2.1: $\mathfrak{A}$ is said to be *G*-Abelian if for all $\rho \in E \cap \mathcal{L}_G^\perp$ and $A_1, A_2 \in \mathfrak{A}$,

$$[P_\rho \pi_\rho(A_1) P_\rho, P_\rho \pi_\rho(A_2) P_\rho] = 0.$$

In other words the von Neumann algebra generated by $P_\rho \pi_\rho(\mathfrak{A}) P_\rho$ is Abelian.

Theorem 2.2 (Alaoglu-Birkhoff): Let $\{U_\alpha\}_{\alpha \in I}$ be a semigroup of contractions on a Hilbert space $\mathcal{K}$, i.e., a collection of operators such that

(1) $\|U_\alpha\| \le 1$ for all $\alpha \in I$

(2) For any $\alpha, \beta \in I, U_\alpha U_\beta = U_\gamma$ for some $\gamma \in I$.

Let P be the orthogonal projection onto the set of all vectors in $\mathcal{K}$ left invariant by all the U_α's. Then P is in the strong closure at the convex hull of $\{U_\alpha\}_{\alpha \in I}$.

This theorem is proved in Riesz–Nagy.[6] The theorem stated by Riesz and Nagy is slightly different from the one given above; what they do is to construct a net of convex linear combinations of the U_α's and show that it converges strongly. Although the fact that P is the strong limit of this net is not included in the statement of the theorem, it appears in the course of the proof.

Theorem 2.3: In order that $\mathfrak{A}$ be *G*-Abelian it is necessary and sufficient that, for all Hermitian A_1, $A_2 \in \mathfrak{A}$ and all $\rho \in E \cap \mathcal{L}_G^\perp$,

$$\inf_{A_1'} |\rho([A_1', A_2])| = 0,$$

where A_1' runs over the convex hull of $\{\tau_g A_1 : g \in G\}$.

In order that $\mathfrak{A}$ be *G*-Abelian, it is evidently necessary and sufficient that, for any $\rho \in E \cap \mathcal{L}_G^\perp, \Psi \in P_\rho \mathcal{K}_\rho$ with $\|\Psi\| = 1$, and A_1, A_2 Hermitian elements of the unit ball of $\mathfrak{A}$, we have

$$(\Psi, \pi_\rho(A_1) P_\rho \pi_\rho(A_2) \Psi) = (\Psi, \pi_\rho(A_2) P_\rho \pi_\rho(A_1) \Psi) \quad (*).$$

We prove first the sufficiency of the criterion stated in the proposition. Let $\epsilon > 0$; then by the preceding theorem, we can find positive numbers λ_i with $\sum_i \lambda_i = 1$ and elements g_i of G such that

$$\|(\sum \lambda_i U_\rho(g_i) - P_\rho) \pi_\rho(A_1) \Psi\| \le \tfrac{1}{2}\epsilon.$$

If we define

$$A_1' = \sum \lambda_i \tau_{g_i} A_1,$$

then both sides of (*) are unchanged if we replace

[6] F. Riesz and B. Sz.-Nagy, *Functional Analysis*, translated by L. Boron (Frederick Ungar Publishing Company, New York, 1955), Sec. 146.

A_1 by A_1', and we have

$$\|P_\rho \pi_\rho(A_1') \Psi - U_\rho(g) \pi_\rho(A_1') \Psi\|$$
$$= \|P_\rho \pi_\rho(A_1) \Psi - U_\rho(g) \pi_\rho(A_1') \Psi\|$$
$$= \|U_\rho(g)[P_\rho \pi_\rho(A_1) \Psi - \pi_\rho(A_1') \Psi]\| \le \tfrac{1}{2}\epsilon$$

for all $g \in G$.

Using this inequality, and the fact that A_1' is Hermitian, we get for any positive numbers λ_i' with $\sum_i \lambda_i' = 1$ and any $g_i' \in G$,

$$|(\Psi, \pi_\rho(A_1) P_\rho \pi_\rho(A_2) \Psi) - (\Psi, \pi_\rho(A_2) P_\rho \pi_\rho(A_1) \Psi)|$$
$$= |(\Psi, \pi_\rho(A_1') P_\rho \pi_\rho(A_2) \Psi) - (\Psi, \pi_\rho(A_2) P_\rho \pi_\rho(A_1') \Psi)|$$
$$\le 2 \cdot \sum_i \lambda_i' \|\pi_\rho(A_2) \Psi\| \cdot \|P_\rho \pi_\rho(A_1') \Psi - U_\rho(g_i') \pi_\rho(A_1') \Psi\|$$
$$\qquad + |(\Psi, \pi_\rho([\sum \lambda_i' \tau_{g_i'} A_1', A_2]) \Psi)|$$
$$\le \epsilon + |(\Psi, \pi_\rho([\sum \lambda_i' \tau_{g_i'} A_1', A_2]) \Psi)|.$$

But by hypothesis, $|(\Psi, \pi_\rho([\sum \lambda_i' \tau_{g_i'} A_1', A_2]) \Psi)|$ can be made arbitrarily small by an appropriate choice of λ_i' and g_i', so

$$|(\Psi, \pi_\rho(A_1) P_\rho \pi_\rho(A_2) \Psi) - (\Psi, \pi_\rho(A_2) P_\rho \pi_\rho(A_1) \Psi)| \le \epsilon.$$

Thus, (*) holds, so $\mathfrak{A}$ is *G*-Abelian.

Now we suppose that $\mathfrak{A}$ is *G*-Abelian, so (*) holds, and we let λ_i, g_i be as above. Then

$$\left| \left(\Psi, \pi_\rho \left(\left[\sum_i \lambda_i \tau_{g_i} A_1, A_2 \right] \right) \Psi \right) \right|$$
$$= \left| \left(\sum_i \lambda_i U_\rho(g_i) \pi_\rho(A_1) \Psi, \pi_\rho(A_2) \Psi \right) \right.$$
$$\left. - \left(\pi_\rho(A_2), \sum_i \lambda_i U_\rho(g_i) \pi_\rho(A_1) \Psi \right) \right|$$
$$\le 2 \cdot \|\pi_\rho(A_2) \Psi\| \cdot \left\| \left(\sum_i \lambda_i U_\rho(g_i) - P_\rho \right) \pi_\rho(A_1) \Psi \right\|$$
$$\quad + |(\Psi, \pi_\rho(A_1) P_\rho \pi_\rho(A_2) \Psi) - (\Psi, \pi_\rho(A_2) P_\rho \pi_\rho(A_1) \Psi)|$$
$$\le \epsilon,$$

so

$$\inf_{A_1' \in \text{convex hull of } \{\tau_g A_1\}} |(\Psi, \pi_\rho([A_1', A_2]) \Psi)| = 0,$$

so the criterion of the proposition holds.

Corollary 2.4: Let H be a subgroup of G. Then, if $\mathfrak{A}$ is *H*-Abelian, it is also *G*-Abelian.

We need only apply the criterion of the preceding proposition, observing that $\mathcal{L}_G^\perp$ is contained in $\mathcal{L}_H^\perp$ and that the convex hull of $\{\tau_g A_1 : g \in G\}$ contains the convex hull of $\{\tau_h A_1 : h \in H\}$.

Corollary 2.5: $\mathfrak{A}$ is *G*-Abelian whenever one of the following conditions is satisfied.

(i) For all $\rho \in E \cap \mathfrak{L}_{G}^{\perp}$ and self-adjoint $A_1, A_2 \in \mathfrak{A}$,

$$\inf_{g \in G} |\rho([A_1, \tau_g A_2])| = 0.$$

(ii) $\mathfrak{A}$ is Abelian.

(iii) $E \cap \mathfrak{L}_{G}^{\perp}$ is empty.

The usefulness of Definition 2.1 appears in the next two sections; we indicate here, however, the following result.

Proposition 2.6: If $\rho \in E \cap \mathfrak{L}_{G}^{\perp}$ and the von Neumann algebra $[P_\rho \pi_\rho(\mathfrak{A}) P_\rho]''$ generated by $P_\rho \pi_\rho(\mathfrak{A}) P_\rho$ is Abelian, then

$$P_\rho[P_\rho \pi_\rho(\mathfrak{A}) P_\rho]' = P_\rho[P_\rho \pi_\rho(\mathfrak{A}) P_\rho]''.$$

The vector Ω_ρ is cyclic for the restriction to $P_\rho \mathfrak{H}_\rho$ of $P_\rho[P_\rho \pi_\rho(\mathfrak{A}) P_\rho]''$; hence, if this von Neumann algebra is commutative, it is equal to its commutant (see Ref. 7, p. 89, Corollaire 2), namely to

$$P_\rho[P_\rho \pi_\rho(\mathfrak{A}) P_\rho]'$$

restricted to $P_\rho \mathfrak{H}_\rho$.

3. INTEGRAL REPRESENTATION OF G-INVARIANT STATES

In this and the next section, we use the theory of integral representations on convex compact sets (see Ref. 8). Let K be a convex compact set in a locally convex topological vector space. The unit mass at $\kappa \in K$ is denoted by δ_κ. We remind the reader that an order relation is defined on the positive measures of norm 1 on K by $\mu \prec \mu' \Leftrightarrow \mu(f) \leq \mu'(f)$ for all convex continuous f on K. A measure is called maximal if it is maximal for the order $\prec$, and K is said to be a simplex if every $\kappa \in K$ is the resultant of a *unique* maximal measure on K. In what follows we take $K = E \cap \mathfrak{L}_{G}^{\perp}$, where $\mathfrak{A}$ is assumed to have an identity. If $A \in \mathfrak{A}$, we denote by $\hat{A}$ the function on $E \cap \mathfrak{L}_{G}^{\perp}$ defined by $\hat{A}(\rho) = \rho(A)$.

Theorem 3.1: Let $\mathfrak{A}$ have an identity, $\rho \in E \cap \mathfrak{L}_{G}^{\perp}$, and let the von Neumann algebra generated by $P_\rho \pi_\rho(\mathfrak{A}) P_\rho$ be Abelian. Then, there exists a unique maximal measure μ_ρ on $E \cap \mathfrak{L}_{G}^{\perp}$ such that $\mu_\rho \succ \delta_\rho$ (i.e., μ_ρ has resultant ρ). The measure μ_ρ is determined by

$$\mu_\rho(\hat{A}_1 \cdots \hat{A}_l) = (\Omega_\rho, \pi_\rho(A_1) P_\rho \pi_\rho(A_2) P_\rho \cdots P_\rho \pi_\rho(A_l) \Omega_\rho). \quad (1)$$

Take $A_1, \cdots, A_l$ self-adjoint. Since the operators $P_\rho \pi_\rho(A_1) P_\rho, \cdots, P_\rho \pi_\rho(A_l) P_\rho$ commute, there exists a projection-valued measure F on R^l such that

$$P_\rho \pi_\rho(A_i) P_\rho = \int t_i \, dF(t_1, \cdots, t_l).$$

If $\mathfrak{F}$ is a complex polynomial of l variables, we have

$$|(\Omega_\rho, \mathfrak{F}(P_\rho \pi_\rho(A_1) P_\rho, \cdots, P_\rho \pi_\rho(A_l) P_\rho) \Omega_\rho)|$$
$$= \left| \left(\Omega_\rho, \int \mathfrak{F}(t_1, \cdots, t_l) \, dF(t_1, \cdots, t_l) \Omega_\rho \right) \right|$$
$$\leq \sup_{\|\Phi\|=1, P_\rho \Phi = \Phi} |\mathfrak{F}((\Phi, \pi_\rho(A_1)\Phi), \cdots, (\Phi, \pi_\rho(A_l)\Phi))|$$
$$\leq \sup_{\sigma \in E \cap \mathfrak{L}_{G}^{\perp}} |\mathfrak{F}(\sigma(A_1), \cdots, \sigma(A_l))|$$
$$= \sup_{\sigma \in E \cap \mathfrak{L}_{G}^{\perp}} |\mathfrak{F}(\hat{A}_l(\sigma), \cdots, \hat{A}_l(\sigma))|.$$

This shows that Eq. (1) defines a linear functional on the polynomials in the $\hat{A}$, which is continuous for the topology of uniform convergence on $E \cap \mathfrak{L}_{G}^{\perp}$. By the Stone–Weierstrass theorem, this functional extends uniquely to a measure μ_ρ on $E \cap \mathfrak{L}_{G}^{\perp}$, which is ≥ 0 and of norm 1.

Let $\rho_1, \cdots, \rho_m \in E \cap \mathfrak{L}_{G}^{\perp}$, $\lambda_1, \cdots, \lambda_m > 0$, $\sum \lambda_i = 1$ and $\rho = \sum \lambda_i \rho_i$. There exist (see Ref. 9, 2.5.1.) uniquely defined self-adjoint operators $T_i \in [\pi_\rho(\mathfrak{A})]'$ such that $0 \leq T_i \leq 1$ and for all $A \in \mathfrak{A}$.

$$\lambda_i \rho_i(A) = (T_i \Omega_\rho, \pi_\rho(A) T_i \Omega_\rho).$$

The T_i satisfy $\sum T_i^2 = 1$. If $g \in G$, we have

$$U(g) T_i U(g^{-1}) \in [\pi_\rho(\mathfrak{A})]',$$

the uniqueness of T_i and the fact that $\lambda_i \rho_i \in \mathfrak{L}_{G}^{\perp}$ then shows that $U(g) T_i U(g^{-1}) = T_i$, hence,

$$T_i \in [U(G)]', \quad [T_i, P_\rho] = 0.$$

By the uniqueness of the Gel'fand–Segal construction, we may identify $\mathfrak{H}_{\rho_i}$ with the closure of $\pi_\rho(\mathfrak{A}) T_i \Omega_\rho$, π_{ρ_i} with the restriction of π_ρ to $\mathfrak{H}_{\rho_i}$, and Ω_{ρ_i} with $\lambda_i^{-\frac{1}{2}} T_i \Omega_\rho$. Then U_{ρ_i} is identified with the restriction of U_ρ to $\mathfrak{H}_{\rho_i}$ and P_{ρ_i} with the restriction of P_ρ to $\mathfrak{H}_{\rho_i}$. In particular, $[P_{\rho_i} \pi_{\rho_i}(\mathfrak{A}) P_{\rho_i}]''$ is Abelian and μ_{ρ_i} is thus defined. We have

$$\mu_{\rho_i}(\hat{A}_1 \cdots \hat{A}_l) = (\Omega_{\rho_i}, \pi_\rho(A_1) P_{\rho_i} \cdots P_{\rho_i} \pi_{\rho_i}(A_l) \Omega_{\rho_i})$$
$$= \lambda_i^{-1}(T_i \Omega_\rho, \pi_\rho(A_1) P_\rho \cdots P_\rho \pi_\rho(A_l) T_i \Omega_\rho)$$
$$= \lambda_i^{-1}(\Omega_\rho, \pi_\rho(A_1) P_\rho \cdots P_\rho \pi_\rho(A_l) T_i^2 \Omega_\rho)$$

so that

$$\sum \lambda_i \mu_{\rho_i}(\hat{A}_1 \cdots \hat{A}_l) = \mu_\rho(\hat{A}_1 \cdots \hat{A}_l).$$

[7] J. Dixmier, *Les algèbres d'opérateurs dans l'Espace Hilbertien,* (Gauthier-Villars, Paris, 1957).

[8] G. Choquet and P. A. Meyer, Ann. Inst. Fourier 13, 139 (1963).

[9] J. Dixmier, *Les C*-Algèbres et leurs Représentations* (Gauthier-Villars, Paris, 1964).

Now let μ be a measure on $E \cap \mathcal{L}_G^\perp$ such that $\mu \succ \delta_\rho$. If $\phi \in \mathbb{C}(E \cap \mathcal{L}_G^\perp)$ and $\epsilon > 0$, one can find a measure μ' with finite support: $\mu' = \sum \lambda_i \delta_{\rho_i}$, $\lambda_i > 0$, $\rho_i \in E \cap \mathcal{L}_G^\perp$, such that $|\mu(\phi) - \mu'(\phi)| < \epsilon$ and $\sum \lambda_i \rho_i = \rho$ (see Ref. 10, p. 217, Prop. 3). If ϕ is convex we thus have

$$\mu(\phi) - \epsilon \leq \mu'(\phi) = \sum \lambda_i \delta_{\rho_i}(\phi)$$
$$\leq \sum \lambda_i \mu_{\rho_i}(\phi) = \mu_\rho(\phi),$$

hence $\mu_\rho \succ \mu$. Since μ is an arbitrary measure on $E \cap \mathcal{L}_G^\perp$ such that $\mu \succ \delta_\rho$, we see that μ_ρ is the unique maximal measure on $E \cap \mathcal{L}_G^\perp$ such that $\mu_\rho \succ \delta_\rho$ which concludes the proof of the theorem.

Corollary 3.2: If $\mathfrak{A}$ has an identity and is G-Abelian, then $E \cap \mathcal{L}_G^\perp$ is a simplex.

Remark 3.3: If $\mathfrak{A}$ is Abelian, the problem considered in this section reduces to that of decomposing an invariant measure on a compact set into ergodic measures (see Ref. 5, Sec. 10).

4. EXTREMAL G-INVARIANT STATES

Let $\mathcal{E}(E \cap \mathcal{L}_G^\perp)$ be the set of extremal points of $E \cap \mathcal{L}_G^\perp$, i.e., the extremal invariant states. The following statement characterizes the elements of $\mathcal{E}(E \cap \mathcal{L}_G^\perp)$.

Proposition 4.1: Let $\rho \in E \cap \mathcal{L}_G^\perp$. If $\mathfrak{A}$ is G-Abelian, the following conditions are equivalent:
  (i) $\rho \in \mathcal{E}(E \cap \mathcal{L}_G^\perp)$.
  (ii) The set $\pi_\rho(\mathfrak{A}) \cup U_\rho(G)$ is irreducible in $\mathfrak{H}_\rho$.
  (iii) P_ρ is one dimensional.

The simple proof is left to the reader. We remark only that the implications (i) $\Leftrightarrow$ (ii) $\Leftarrow$ (iii) do not make use of the assumption that $\mathfrak{A}$ is G-Abelian, and that (ii) $\Rightarrow$ (iii) follows from Proposition 2.6.

The measure μ_ρ of Theorem 3.1 is in the "good cases" carried by $\mathcal{E}(E \cap \mathcal{L}_G^\perp)$. This is so, for instance, if $\mathfrak{A}$ is (norm-)separable, because $E \cap \mathcal{L}_G^\perp$ is then metrizable (see Ref. 8, Corr. 14). We indicate now without proofs some more results in this direction.

Proposition 4.2: Let $\mathfrak{A}$ have an identity and $\mathfrak{B}$ be a self-adjoint subalgebra of $\mathfrak{A}$; define

$$\mathcal{F} = \{\sigma \in E: \text{The restriction of } \rho \text{ to } \mathfrak{B} \text{ has norm } 1\}.$$

Then,
  (i) $\mathcal{F}$ is a G_δ (a countable intersection of open subsets of E).
  (ii) If μ is a measure on E such that $\mu \geq 0$, $\mu(E) = 1$, and μ has resultant ρ, then

$$\rho \in \mathcal{F} \leftrightarrow \mu \text{ is carried by } \mathcal{F},$$

cf. Ref. 1, Theorem, Part 4.

Proposition 4.3: Let $(\mathfrak{A}_\alpha)$ be a countable family of sub-$B*$ algebras of $\mathfrak{A}$ such that $\bigcup_\alpha \mathfrak{A}_\alpha$ is dense in $\mathfrak{A}$. Let $\mathcal{C}_\alpha$ be a separable closed two-sided ideal of $\mathfrak{A}_\alpha$ for each α, and define

$$\mathcal{F}_\alpha = \{\sigma \in E: \text{the restriction of } \sigma$$
$$\text{to } \mathcal{C}_\alpha \text{ has norm } 1\}, \quad \mathcal{F} = \bigcap_\alpha \mathcal{F}_\alpha.$$

Then,
  (i)   If $\rho \in \mathcal{F}$, then $\mathfrak{H}_\rho$ is separable.
  (ii) There exists a sequence (A_i) of self-adjoint elements of $\mathfrak{A}$ such that if $\rho \in \mathcal{F}$ and $\sigma \in E$, then $\rho(A_i) \neq \sigma(A_i)$ for some i.
  (iii) If $\mathfrak{A}$ has an identity and is G-Abelian and if μ is a measure on $E \cap \mathcal{L}_G^\perp$ such that $\mu \geq 0$, $\mu(E \cap \mathcal{L}_G^\perp) = 1$ and μ has resultant $\rho \in \mathcal{F}$, then
  (μ maximal on $E \cap \mathcal{L}_G^\perp$)

$$\Leftrightarrow (\mu \text{ carried by } \mathcal{E}(E \cap \mathcal{L}_G^\perp)).$$

(i) and (ii) are easy, the proof of (iii) uses (ii), Corollary 3.2 and an argument in Ref. 1, Theorem, Part 5.

The usefulness of (iii) appears in statistical mechanics, where $\mathfrak{A}$ may not be norm separable but the states of interest satisfy a condition of the type $\rho \in \mathcal{F}$. One has then a unique decomposition $\rho \to \mu_\rho$ of ρ into extremal invariant states and those states are again in $\mathcal{F}$. For an explicit treatment see Ref. 11, in particular, the Appendix.

[10] N. Bourbaki, *Intégration* (Hermann et Cie., Paris, 1965), 2nd ed., Chaps. 1–4.

[11] D. Ruelle, "The States of Classical Statistical Mechanics," J. Math. Phys. (to be published).