# Analytic continuation of Dirichlet series with almost periodic coefficients

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**Abstract.** We consider Dirichlet series $\zeta_{g,\alpha}(s) = \sum_{n=1}^{\infty} g(n\alpha) e^{-\lambda_n s}$ for fixed irrational $\alpha$ and periodic functions $g$. We demonstrate that for Diophantine $\alpha$ and smooth $g$, the line $\text{Re}(s) = 0$ is a natural boundary in the Taylor series case $\lambda_n = n$, so that the unit circle is the maximal domain of holomorphy for the almost periodic Taylor series $\sum_{n=1}^{\infty} g(n\alpha) z^n$. We prove that a Dirichlet series $\zeta_{g,\alpha}(s) = \sum_{n=1}^{\infty} g(n\alpha)/n^s$ has an abscissa of convergence $\sigma_0 = 0$ if $g$ is odd and real analytic and $\alpha$ is Diophantine. We show that if $g$ is odd and has bounded variation and $\alpha$ is of bounded Diophantine type $r$, the abscissa of convergence $\sigma_0$ satisfies $\sigma_0 \leq 1 - 1/r$. Using a polylogarithm expansion, we prove that if $g$ is odd and real analytic and $\alpha$ is Diophantine, then the Dirichlet series $\zeta_{g,\alpha}(s)$ has an analytic continuation to the entire complex plane.

11M99, 30D99, 33E20 Dirichlet series, analytic continuation

## 1. Introduction

Let $g : \mathbb{R} \to \mathbb{C}$ be a piecewise continuous 1-periodic $L^2$ function with Fourier expansion $g(x) = \sum_{k=-\infty}^{\infty} c_k e^{2\pi i k x}$. Define the $\zeta$ function

$$\zeta_{g,\alpha}(s) = \sum_{n=1}^{\infty} \frac{g(n\alpha)}{n^s} .$$

For irrational $\alpha$, we call this a **Dirichlet series with almost periodic coefficients**. An example is the Clausen function, where $g(x) = \sin(2\pi x)$ or the poly-logarithm, where $g(x) = \exp(2\pi i x)$. Another example arises with $g(x) = x - \lfloor x + 1/2 \rfloor$, the signed distance from $x$ to the nearest integer. Obviously for $\text{Re}(s) > \sigma_0 > 1$, such a Dirichlet series converges uniformly to an analytic limit.

The case that $\alpha$ is rational is less interesting, as the following computation illustrates.

*If the function $g$ is odd and $\alpha = p/q$ is rational, the zeta function has an abscissa of convergence* 0 *and allows an analytic continuation to the entire plane.*

*Proof.* We can assume $\gcd(p,q) = 1$. Write

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{g(np/q)}{n^s} &= \sum_{\ell=1}^{q} \sum_{n=0}^{\infty} \frac{g((nq+\ell)(p/q))}{(nq+\ell)^s} \\ &= \frac{1}{q^s} \sum_{\ell=1}^{q} \sum_{n=0}^{\infty} \frac{g(n+\ell p/q)}{(n+\ell/q)^s} = \frac{1}{q^s} \sum_{\ell=1}^{q} g(\ell/q)\zeta(s,\ell/q) \; , \end{aligned}$$

where $\zeta(s,u) = \sum_{n=1}^{\infty} 1/(n+u)^s$ is the Hurwitz zeta function. Thus the periodic zeta function is a just a finite sum of Hurwitz zeta functions which individually allow a meromorphic continuation. Each Hurwitz zeta function is analytic everywhere except at 1, where it has a pole of residue 1: the series $\zeta(s,u) - 1/(s-1)$ allows an analytic extension to the plane. So, if $\sum_{n=1}^{q} g(n\alpha) = 0$, which is the case for example if $g$ is odd, then the periodic Dirichlet series has abscissa of convergence 0 and admits an analytic continuation to the plane. □

In particular, when $\alpha = 0$, the function $\zeta_{g,\alpha}$ is merely a multiple of the Riemann zeta function.

Another special case is $\lambda_n = n$, leading to Taylor series $f(z) = \sum_{n=1}^{\infty} a_n z^n$ with $z = e^{-s}$ and $a_n = g(n\alpha)$. Here too, the rational case is readily understood:

*If $\alpha = p/q$ is rational, the function $f(z) = \sum_{n=1}^{\infty} g(n\alpha) z^n$ has a meromorphic extension to the entire plane.*

*Proof.* If $\alpha = p/q$, define $h(z) = \sum_{n=1}^{q} g(n\alpha)z^n$. Then

$$f(z) = h(z)(1 + z^q + z^{2q} + \dots) = \frac{h(z)}{1 - z^q} \; .$$

The right hand side provides the meromorphic continuation of $f$. □

We usually assume the normalization $\int g \; dm = 0$ because we are interested in the growth of the random walk in the case $s = 0$, and because if $\int g \; dm \neq 0$, the abscissa of convergence is in general 1. Given a $g$ without this property, adding a constant we obtain $g_0 = g + c$ with average 0, and the corresponding function $\zeta_{g_0,\alpha}(s)$ differs from $\zeta_{g,\alpha}(s)$ by only a multiple of the Riemann zeta function on the domain where the series for $\zeta_{g,\alpha}$ is convergent. For $f(x) = a_0 + \sin(x)$ for example, where the Dirichlet series with $\lambda_n = \log(n)$ is a sum of the standard zeta function and the Clausen function. The abscissa of convergence is $\sigma_0 = 1$ except for $a_0 = 0$, where the abscissa drops to 0. By assuming $g$ to be an odd periodic function, we assure that $\int g(x) \; dx = 0$.

Zeta functions $\sum_{n=1}^{\infty} a_n e^{-\lambda_n s}$ can more generally be considered for any dynamically generated sequence $a_n = g(T^n x)$, where $T$ is a homeomorphism of a compact topological space $X$ and $g$ is a continuous function and $\lambda_n$ grows monotonically to $\infty$.

Random Taylor series associated with an ergodic transformation were considered in [5, 9]. The topic has also been explored in a probabilistic setup, where $a_n$ are independent random symmetric variables, in which case the line $\mathrm{Re}(s) = \sigma_0$ is a natural boundary [14]. Analytic continuation questions have also been studied for other functions: if the coefficients are generated by finite automata, a meromorphic continuation is possible [11]. In [1] Theorem 1.7 was shown recently that for any ergodic, non-deterministic process $a_n(\omega)$, the random Taylor series $\sum_n a_n(\omega) z^n$ has a natural boundary with probability one. In the same paper, there are results in the almost periodic case generalizing a result of Hecke [7]: if $g$ has finitely many discontinuities, and for at least one of them the left and right limit are not the same, then $\sum_n g(n\alpha) z^n$ has a natural boundary for all irrational $\alpha$.

In this paper we focus on Taylor series and ordinary Dirichlet series. We restrict ourselves to the case, where the dynamical system is an irrational rotation $x \mapsto x + \alpha$ on the circle. The minimality and strict ergodicity of the system will often make the question independent of the starting point $x \in X$ and allow techniques of Fourier analysis and circle maps. We are able to make statements if $\alpha$ is Diophantine.

Dirichlet series allow one to get information on the growth of the random walk $S_k = \sum_{n=1}^k g(T^n(x))$ for a $m$-measure preserving dynamical system $T : X \to X$ if $\int_X g(x)\, dm(x) = 0$. Birkhoff's ergodic theorem assures $S_k = o(k)$. Similarly as the law of the iterated logarithm refines the law of large numbers in probability theory, and Denjoy-Koksma type results provide further estimates on the growth rate in the case of irrational rotations, one can study the growth rate for more general dynamical systems.

In probability theory, if a $X_n$ are IID random variables with bounded variance and zero mean, then $\frac{1}{n} \sum_{n=1}^k X_n = 0$ almost surely by the strong law of large numbers. The growth rate of the partial sums is captured by the law of iterated logarithm $|\sum_{n=1}^k X_n| \leq C\sqrt{k \log\log(k)}$ which gives a better estimate on the growth of the partial sums. For random variables $X_n(\theta) = g(\theta + n\alpha)$ with odd $g$, on the probability space given by the circle with the usual Lebesgue measure, the random variables $X_n$ still have an identical distribution and zero mean and finite variance, but they are dependent. Even the correlation $\int_0^{2\pi} g(\theta) g(\theta + \alpha)\, d\theta$ is nonzero. For $g(x) = \sin(x)$ for example, where $X_n$ are random variables with variance 1 we have $\mathrm{Cov}[X_n, X_m] = \cos((n - m)\alpha)$. Denjoy-Koksma gives upper estimates on the growth rate similarly as the law of iterated logarithm did in the random case. Typically, we have $S_k \leq C \log(k)$. Almost periodic series $\sum_{k=1}^n g(k\alpha)$

were first studied seriously by Hecke [7] and interest continues until today. A recently popular example is the function $g(x) = (-1)^{\lfloor x \rfloor}$ which was picked up in [16] and which is covered by Denjoy-Koksma since $g$ is of bounded variation.

The relation with algebra is as follows:
if $S_k$ grows like $k^\beta$ then the abscissa of convergence of the ordinary Dirichlet series is smaller or equal to $\beta$. In other words, establishing bounds for the analyticity domain allows to get results on the abscissa of convergence which give upper bounds of the growth rate. Adapting the $\lambda_n$ to the situation allows one to explore different growth behavior. The algebraic concept of Dirichlet series helps to understand a dynamical concept.

Besides the relation with the dynamical growth of deterministic random walks, there is the motivation to **zeta-regularize** Fourier series. For $\text{Re}(s) > 1$, we have smooth 1-periodic functions

$$h_s(\alpha) = \sum_{n=1}^{\infty} g(n\alpha)/n^s \ .$$

Theorem 5.4 on analytic continuation shows that we can make sense of $h_s(\alpha)$ for almost all $\alpha$ even if $s$ is a general complex number. This is the analytic continuation of the series and function $\alpha \to h_s(\alpha)$ has a zeta-regularized Fourier series. In which function space this is possible needs further study. Finally, we would like to point out recent new relations between basic questions on analytic continuation with spectral theory and operator theory [1].

## 2. Almost periodic Taylor series

In this section, we look at the problem of natural boundaries for Taylor series where the coefficients $a_n$ are obtained from an irrational rotation $x \to x + \alpha$ on the circle. Taylor series with coefficients generated by general ergodic processes have been considered earlier, especially the case random Taylor series, where the coefficients are independent identically distributed random variables. However, the almost periodic case seems to have received little attention. We will see that arithmetic Diophantine conditions on $\alpha$ play a role.

A general Dirichlet series is of the form

$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n s} \ .$$

For $\lambda_n = \log(n)$ this is an ordinary Dirichlet series, while for $\lambda_n = n$, it is a Taylor series $\sum_n a_n z^n$ with $z = e^{-s}$. We primarily restrict our attention to these two cases.

We begin by considering the easier problem of Taylor series with almost periodic coefficients and examine the analytic continuation of such functions beyond the unit circle. Given a non-constant periodic function $g$, we can look at the problem of whether the Taylor series

$$f(z) = \sum_{n=1}^{\infty} g(n\alpha) z^n$$

can be analytically continued beyond the unit circle. Note that all these functions have radius of convergence 1 because $\limsup_n |g(n\alpha)|^{1/n} = 1$.

We have already seen in the introduction that if $\alpha = p/q$ is rational, the function $f$ has a meromorphic extension to the entire plane. There is another case where analytic continuation can be established immediately:

**Lemma 2.1.** *If $g$ is a trigonometric polynomial and $\alpha$ is arbitrary, then $f$ has a meromorphic extension to the entire plane.*

*Proof.* Since $g(x) = \sum_{n=-k}^{k} c_n e^{2\pi i n x}$, it is enough to verify this for $g(x) = e^{2\pi i n x}$, in which case the series sums to $f(z) = 1/(1 - e^{2\pi i n\alpha} z)$. □

On the other hand, if infinitely many of the Fourier coefficients for $g$ are nonzero, analytic continuation may not be possible.

A real number $\alpha$ for which there exist $C > 0$ and $r > 1$ satisfying

$$|\alpha - \frac{p}{q}| \geq \frac{C}{q^{1+r}}$$

for all rational $p/q$ is called **Diophantine of type** $r$. The set of real numbers of type $r$ has full Lebesgue measure for all $r > 1$. The intersection over all $r > 1$ of the numbers Diophantine of type $r$ is called the set of **Diophantine numbers**; it too is of full measure. For the convenience of the reader here is a proof of the last statement: write $||x||$ for the fractional part of $x$. The Diophantine condition of fixed type $r > 1$ reads $||q\alpha|| \geq \frac{C}{q^r}$. The open set $R_{q,C} = \{\alpha \mid |q\alpha| < Cq^{-r} \} \subset [0,1]$ has measure $C|q|^{-r}$ so that the set $R_C = \bigcup_{n \neq 0} R_{n,C}$ has finite measure $\leq AC$ where $A = \sum_{q>0} q^{-r} \leq 1 + 1/(r-1)$. Therefore $R = \bigcap_{C>0} R_C$ has measure smaller than $AC$ for any $C > 0$ and so measure zero. This means, the complement of $R = R_r$ has full measure for fixed $r > 1$. It is the set $R_r$ of Diophantine numbers of type $r > 1$. The intersection of all these sets is the intersection of countably many sets $R_{1+1/n}$ of measure 1, and so has full measure too. See [17, 8].

**Proposition 2.2.** *Fix $r > 1$. Assume $g$ is in $C^t$ for $t > r + 1$, and that all Fourier coefficients $c_k$ of $g$ are nonzero and that $\alpha$ is of Diophantine type $r$. Then the almost periodic Taylor series $f_{g,\alpha}(z) = \sum_{n=0}^{\infty} g(n\alpha) z^n$ can not be continued beyond the unit circle.*

*Proof.* Write

$$\begin{aligned} f_{g,\alpha}(z) &= \sum_{n=0}^{\infty} g(n\alpha)z^n = \sum_{n=0}^{\infty}\sum_{k=-\infty}^{\infty} c_k e^{2\pi i k n\alpha} z^n \\ &= \sum_{k=-\infty}^{\infty} c_k \sum_{n=0}^{\infty} e^{2\pi i k n \alpha} z^n = \sum_{k=-\infty}^{\infty} \frac{c_k}{1-e^{2\pi i k\alpha}z} . \end{aligned}$$

(Note that for $|z| < 1$ the Taylor series converges absolutely, and due to the smoothness assumption of $g$, the Fourier series converges absolutely: if $f \in C^t$, its Fourier coefficients satisfy $|c_k| \leq C/|k|^{t+1}$. (For non-integral $t$ the class $C^t$ can be defined by the decay of its Fourier series). It follows Cauchy's criterion (see e.g. [4]) that the double series converges absolutely and that we can change the order of summation. Fix now some $j$ and consider the radial limit

$$\begin{aligned} \lim_{s\to 1^-} f_{g,\alpha}(se^{2\pi i j\alpha}) &= \lim_{s\to 1^-} \sum_{k=-\infty}^{\infty} \frac{c_k}{1-se^{2\pi i k\alpha}e^{2\pi i j\alpha}} \\ &= \lim_{s\to 1^-} \left( \frac{c_{-j}}{1-s} + \sum_{\substack{k=-\infty \\ k\neq j}}^{\infty} \frac{c_k}{1-se^{2\pi i(k-j)\alpha}} \right) . \end{aligned}$$

The latter sum converges at $s = 1$, as by the Diophantine condition, the absolute value of the denominator $1 - e^{2\pi i(k-j)\alpha} = 1 - \cos(2\pi(k-j)\alpha) + i\sin(2\pi(k-j)\alpha)$ is bounded below by $|\sin(2\pi(k-j)\alpha)| \geq C/|j-k|^r$, while the numerator $c_k$ is bounded above by $A/|k|^{t+1}$ for some constant $A$ and $t > r+1$ by the differentiability assumption on $g$. Together

$$\left| \frac{c_k}{1-se^{2\pi i(k-j)\alpha}} \right| \leq \frac{A}{C} \frac{|j-k|^r}{k^{t+1}}$$

assures the convergence of the sum for $t > r+1$. Because the summation term $c_{-j}/(1-s)$ diverges for $s \to 1^-$, this radial limit is infinite. This happens for all $j$ because $c_{-j}$ is nonzero by assumption. It follows from Riesz lemma [21] that $f$ does not admit an analytic continuation to any larger set. □

*Remark* 2.3. This result is related to a construction of Goursat, which shows that for any domain $D$ in $\mathbb{C}$, there exists a function which has $D$ as a maximal domain of analyticity [21]. Classical examples of Taylor series with natural domain the unit disk include the lacunary Taylor series like $\sum_{j=1}^{\infty} z^{2^j}$ (or $\sum_{j=1}^{\infty} z^{j!}$ of Weierstrass, $\sum_{j=1}^{\infty} z^{j^2}$ of Kronecker or more generally with Hadamard for $\sum_j a_j z^{n_j}$ with $\inf_j n_{j+1}/n_j > 1$). As in Hecke's example, our examples are not analytic perturbations of lacunary series.

*Remark* 2.4. The function

$$f(z) = \sum_{k=-\infty}^{\infty} \frac{c_k}{1-e^{2\pi i k\alpha}z} = \sum_{k=-\infty}^{\infty} \frac{a_k}{z-z_k}$$

is defined also outside the unit circle. The subharmonic function $F(z) = \log|f(z)| = \int \log|z-w| \, dk(w)$ has a Riesz measure $dk = \Delta F$ supported on the unit circle. See [20].

*Remark* 2.5. Breuer-Simon [1] recently showed that if $g$ is a piecewise continuous function with finitely many discontinuities and at one of which the one-sided limits exist and are unequal, then for every irrational $\alpha$, the almost periodic Taylor series $\sum_{n=0}^{\infty} g(n\alpha) z^n$ has the unit circle as natural boundary. This includes Hecke's example $g(x) = x \bmod 1$. In the case that $g$ is a step function this is implied by the classical theorem of Szegő that power series with only finitely many distinct coefficients and which are not eventually periodic may not be extended beyond the unit disk [21]. Breuer-Simon also note that for a dense $G_\delta$ of sequences $a_n$ in the compact metric space $K^{\mathbf{N}}$, where $K$ is a compact subset of the complex plane, the unit circle is a natural boundary.

*Remark* 2.6. The condition that all Fourier coefficients of $g$ are nonzero may be relaxed to the assumption that the set of $e^{2\pi i k\alpha}$, where $k$ ranges over the indices of nonzero Fourier coefficients, is dense in $S^1$.

As the last remark may suggest, trigonometric polynomials are not the only functions whose associated series allow an analytic continuation beyond the unit circle.

**Proposition 2.7.** *Let $K \subset \{|z| = 1\}$ be an arbitrary closed set on the unit circle. There exists an almost periodic Taylor series which has an analytic continuation to $\mathbb{C} \setminus K$ but not to any point of $K$.*

*Proof.* Set

$$c_k = \begin{cases} 0 & \text{if } e^{2\pi i\alpha k} \in K, \\ 1/k! & \text{otherwise} \end{cases}$$

and let $g(x) = \sum_{k=-\infty}^{\infty} c_k e^{2\pi i k x}$, and let $\alpha$ be Diophantine of any type $r > 1$. Inside the unit circle we have

$$f_{g,\alpha}(z) = \sum_{k=-\infty}^{\infty} \frac{c_k}{1 - e^{2\pi i k\alpha} z} .$$

For any $j$ for which $e^{2\pi i j\alpha} \in K$, this sum converges uniformly on a closed ball around $z$ which does not intersect $K$ (since the denominators of the non-vanishing terms are uniformly bounded), and thus has an analytic neighborhood around such $e^{2\pi i j\alpha}$. Any point in $K$ lies in a compact neighborhood of such a point. On the other hand, by the arguments of the preceding lemma, analytic continuation is not possible in $K$ itself. □

## 3. Ordinary Dirichlet series

In this section we look at classical Dirichlet series $\sum_{n=1}^{\infty} a_n/n^s$. While in number theory, one usually considers situations where the $a_n$ are a multiplicative arithmetic sequence, this is not assumed here (though it is possible). Motivated by questions in dynamics like the growth rate estimates of the deterministic random walk $S_k = \sum_{n=1}^{k} g(n\alpha)$, we look at the case when $a_n = g(n\alpha)$ is an almost periodic sequence.

**Cahen's formula** for the abscissa of convergence of an ordinary Dirichlet series $\zeta_{g,\alpha}(s) = \sum_{n=1}^{\infty} a_n/n^s$ is

$$\sigma_0 = \limsup_k \frac{\log S_k}{\log k},$$

if $S_k = \sum_{n=1}^{k} g(n\alpha)$ does not converge [6]. We will compute the abscissa of convergence for two classes of functions and so derive bounds on the random walks $S_k$ which are stronger than those implied by the Denjoy-Koksma inequality.

The first situation applies to real analytic $g$, where we can invoke the cohomology theory of cocycles over irrational rotations which are important in dynamical systems systems theory (see e.g. [15, 2, 8] ). Let us explain the terminology: a 1-periodic function $f$ is also called a **cocycle**, if we look at the random walk $S_k = \sum_{n=1}^{k} f(n\alpha)$. The reason for the name is that one can look at the skew product systems $F : (x, v) \to (x + \alpha \bmod 1, v + f(x))$ on the cylinder $T \times R$ so that $F^k(x, v) = (x + k\alpha, S_k(x))$.
A function $f$ in some function space of periodic function is called a **coboundary** with respect to the irrational rotation $x \to x + \alpha$, if there exists an other periodic function $g$ in the same function space so that $f(x) = g(x + \alpha) - g(x)$. Coboundaries are interesting because their random walk is bounded: $S_k(x) = f(x) + f(x + \alpha) + \cdots + f(x + k\alpha) = g(x + (k + 1)\alpha) - g(x)$. The names "cocycle" and "coboundary" are not chosen by accident. An entire cohomology machinery has been developed in this framework, when considering $g(x) \to g(x + \alpha) - g(x)$ as an **exterior derivative** in one dimension. While this is well understood in the continuum, where the space of all smooth functions on the circle modulo the space of derivatives $g'$ is the real line (every function $\int f \, dx = 0$ is a derivative $f = g'$,) this is more subtle in the ergodic setup. There are cocycles $f$ of zero average, which are not discrete derivatives. The question very much depends on the function space and on Diophantine conditions of $\alpha$.

**Proposition 3.1.** *If $g$ is real analytic, $\int g(x) \, dx = 0$ and $\alpha$ is Diophantine, then the series for $\zeta_{g,\alpha}(s)$ converges and is analytic for $\mathrm{Re}(s) > 0$. In other words, the abscissa of convergence is $0$.*

*Proof.* Since $g_0 = 0$, $g = \sum_{n=1}^{\infty} g_n e^{2\pi i n x}$ is an additive coboundary: the Diophantine property of $\alpha$ implies that the real analytic function

$$h(x) = \sum_{n=1}^{\infty} g_n \frac{e^{2\pi i n x}}{(e^{2\pi i n \alpha} - 1)}$$

solves

$$g(x) = h(x+\alpha) - h(x) \; .$$

Because $S_k = \sum_{n=1}^{k} g(n\alpha)$ does not converge, but stays bounded in absolute value by $2||h||_\infty$, Cahen's formula immediately implies that $\sigma_0 = 0$.
Lets give a direct proof without Cahen's formula. For $0 < \mathrm{Re}(s) < 1$, we have

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{g(n\alpha)}{n^s} &= \sum_{n=1}^{\infty} \frac{h((n+1)\alpha)}{n^s} - \frac{h(n\alpha)}{n^s} \\
&= -h(\alpha) + \sum_{n=2}^{\infty} h(n\alpha) \left[ \frac{1}{(n-1)^s} - \frac{1}{n^s} \right] \\
&= -h(\alpha) + \sum_{n=2}^{\infty} h(n\alpha) \frac{n^s - (n-1)^s}{n^s (n-1)^s} \\
&\leq ||h|| \left( 1 + \sum_{n=2}^{\infty} \frac{n^s - (n-1)^s}{n^s (n-1)^s} \right) \; .
\end{aligned}$$

The sum is bounded for $1 > \mathrm{Re}(s) > 0$ because $|(n+1)^s - n^s| \leq |s n^{s-1}|$ so that $((n+1)^s - n^s)/((n+1)^s n^s) \leq |s|\, n^{-1}(n+1)^{-s} < |s|\, n^{-1-s}$. The function $\zeta_{g,\alpha}(s)$ is analytic in $\mathrm{Re}(s) > 0$ as the limit of a sequence of analytic functions which converge uniformly on a compact subset of the right half plane. The uniform convergence follows from Bohr's theorem (see [6]). □

If $g$ is a trigonometric polynomial, analytic continuation is possible for all irrational $\alpha$:

**Lemma 3.2.** *If $g$ is a trigonometric polynomial of period 1 with $\int_0^1 g(x)\, dx = 0$ and $\alpha$ is an arbitrary irrational number, then the abscissa of convergence of $\zeta_{g,\alpha}$ is 0.*

*Proof.* If $g$ is a trigonometric polynomial, then $g$ is a coboundary for **every** irrational $\alpha$ because $e^{2\pi i n x} = h(x+\alpha) - h(x)$ for $h(x) = e^{2\pi i n x}/(e^{2\pi i n x} - 1)$. In the case of the Clausen function for example and $s = 0$, we have $\sum_{k=0}^{n-1} \sin(2\pi k \alpha) = \mathrm{Im}(e^{2\pi i n \alpha} - 1)/(e^{2\pi i \alpha} - 1)$. It follows that the series $\zeta_{g,\alpha}(s)$ converges for all trigonometric polynomials $g$, for **all** irrational $\alpha \neq 0$ and all $\mathrm{Re}(s) > 0$. □

*Remark* 3.3. A special case is the zeta function

$$\sum_{n=1}^{\infty} \frac{e^{2\pi i n \alpha}}{n^s}$$

which can be written as $L(e^{2\pi i\alpha}, s)$, where $L(z,s) = \sum_{n=1}^\infty z^n/n^s$ is the **polylogarithm**. Integral representations like

$$L(z,s) = \frac{z}{\Gamma(s)} \int_0^1 [\log(1/t)]^{s-1} \frac{dt}{1-zt}$$

(see i.e. [18]) show the analytic continuation for $z \neq 1$ rsp. $\alpha \neq 0$. It follows that the function $\zeta_{g,\alpha}$ has an analytic continuation to the entire complex plane for all irrational $\alpha$ if $g$ is a trigonometric polynomial. We will use polylogarithms later again.

## 4. The bounded variation case

In this section, we discuss a situation where the smoothness assumptions on the function $g$ are relaxed. As long as $g$ is of bounded variation, some results may still be obtained.

The result in the last section had been valid if $\alpha$ satisfies some Diophantine condition and $g$ is real analytic. If the function $g$ is only required to be of bounded variation, then the abscissa of convergence can be estimated.

The **variation** of a function $f$ is $\sup_P \sum_i |f(x_{i+1}) - f(x_i)|$, where the supremum is taken over all partitions $P = \{x_1, \dots, x_n\}$ of $[0,1]$.
A real number is of **constant type** if the continued fraction approximation $p_n/q_n$ has the property that $q_{n+1}/q_n$ is bounded.

The Denjoy-Koksma inequality is treated in [8, 2]. Here is the exposition as found in [12].

**Lemma 4.1 (Jitomirskaja's formulation of Denjoy-Koksma).** *Assume $\alpha$ is Diophantine of type $r > 1$ and $g$ is of bounded variation and $\int_0^1 g(x)\, dx = 0$. Then $S_k = \sum_{n=1}^k g(n\alpha)$ satisfies*

$$|S_k| \leq C k^{1-1/r} \log(k) \mathrm{Var}(g) .$$

*If $\alpha$ is of constant type and $g$ is of bounded variation and $\int_0^1 g(x)\, dx = 0$, then $S_k \leq C \log(k)$.*

*Proof.* (See [12], Lemma 12). If $p/q$ is a periodic approximation of $\alpha$, then

$$|S_q| \leq \mathrm{Var}(f) .$$

To see this, divide the circle into $q$ intervals centered at the points $y_m = mp/q$. These intervals have length $1/q \pm O(1/q^2)$ and each interval contains exactly one point of the finite orbit $\{y_k = k\alpha\}_{k=1}^q$. Renumber the points so that $y_m$ is in $I_m$. By the intermediate value theorem, there exists a Riemann sum $\frac{1}{q}\sum_{i=0}^{q-1} f(x_i) = \int f(x)\, dx = 0$ for which every $x_i$ is in an interval $I_i$ (choosing the point $x_i = \min_{x \in I_i} f(x)$ gives an lower and $x_m = \max_{x \in I_m} f(x)$ gives an upper bound). If

$\sum_{j=0}^{q-1} f(y_j) - f(x_j) \leq \sum_{j=0}^{q-1} |f(y_j) - f(x_j)| + |f(x_j) - f(y_{j+1})| \leq \mathrm{Var}(f)$.
Now, if $q_m \leq k \leq q_{m+1}$ and $k = b_m q_m + b_{m-1} q_{m-1} + \cdots + b_1 q_1 + b_0$, then

$$S_k \leq (b_0 + \cdots + b_m)\mathrm{Var}(f) \leq \sum_{i=0}^{m} \frac{q_{i+1}}{q_i} \mathrm{Var}(f)$$

because $b_j \leq q_{j+1}/q_j$.
If $\alpha$ is of constant type then $\frac{q_{i+1}}{q_i}$ is bounded and $m < 2\log(k)/\log(2)$ implies $S_k \leq (2\log(k)/\log(2))\mathrm{Var}(f)$.
If $\alpha$ is Diophantine of type $r > 1$, then $||q\alpha|| \leq c/q^r$ for some $c > 0$ and $q_{i+1} \leq q_i^r/c$ which implies $q_{i+1}/q_i < q_{i+1}^{1-1/r}/c^{1/r}$ and so

$$|S_k| \leq (c^{-1/r} \sum_{i=1}^{m} q_i^{1-1/r} + \frac{k}{q_m})\mathrm{Var}(f) \leq (c^{-1/r} m q_m^{1-1/r} + \frac{k}{q_m})\mathrm{Var}(f) \ .$$

The general fact $m \leq 2\log(q_m)/\log(2) \leq 2\log(k)/\log(2)$ deals with the first term. The second term is estimated as follows: from $k \leq q_{m+1} \leq q_m^r/c$, we have $q_m \geq (ck)^{1/r}$ and $k/q_m \leq c^{-1/r} k^{1-1/r}$. □

*Remark* 4.2. One knows also $|S_n| \leq C\log(n)^{2+\epsilon}$ if the continued fraction expansion $[a_0, a_1, ...]$ of $\alpha$ satisfies $a_m < m^{1+\epsilon}$ eventually. See [3].

*Proposition* 4.3. *If $\alpha$ is Diophantine of type $r > 1$ and if $g$ is of bounded variation with $\int_0^1 g(x)\, dx = 0$, then the series for $\zeta_{g,\alpha}(s)$ has an abscissa of convergence $\sigma_0 \leq (1 - 1/r)$.*

*Proof.* The Denjoy-Koksma inequality Lemma 4.1 implies that for any $m > n$, the sum $S_{n,m} = \sum_{k=n}^{m} g(k\alpha)$ satisfies the estimate $S_{n,m} \leq C\log(m-n)\ (m-n)^{1-1/r}$, with $C = \mathrm{Var}(g)$. Cahen's formula for the abscissa of convergence gives

$$\limsup_{n\to\infty} \frac{\log S_{1,n}}{\log(n)} \leq 1 - \frac{1}{r} \ .$$

□

*Remark* 4.4. A weaker result could be obtained directly, without Cahen's formula. Choose $\ell_k \to \infty$ such that

$$U_k = \sum_{j=\ell_k}^{l_{k+1}} [g(j\alpha)/\ell_k^s] \leq C\log(\ell_{k+1} - \ell^k)(\ell_{k+1} - \ell^k)^{1-1/r}/\ell_k^s$$

$$V_k = \sum_{j=\ell_k}^{l_{k+1}} [\frac{g(j)}{\ell_k^s} - \frac{g(j)}{j^s}] \leq \frac{(\ell_{k+1} - \ell_k)^2}{l_{k+1}^s \ell_k^s} ||g||$$

are summable. Then

$$S_k = \sum_{j=\ell_k}^{l_{k+1}} g(j\alpha)/j^s \leq U_k + V_k$$

is summable.

To compare: if $g(T^n x)$ are independent, identically distributed random variables with mean 0 and finite variance $a$, then $\sum_{k=1}^n g(T^k x)$ grows by the law of the iterated logarithm like $a\sqrt{n \log\log(n)}$ and the zeta function converges with probability 1 for $\mathrm{Re}(s) > 1/2$. The following reformulation of the law of iterated logarithm follows directly from Cahen's formula:

*(Law of iterated logarithm) If $T$ is a Bernoulli shift such that $g(x) = x_0$ produces independent identically distributed random variables $g(T^n x)$ with finite nonzero variance, then the Dirichlet series $\zeta_{g,T}$ has the abscissa of convergence $1/2$ almost surely.*

We are not yet aware of any examples in the almost-periodic case where the abscissa of converges is strictly between 0 and 1.

## 5. Analytic continuation

In this section, we look at the problem to analytically continue an almost periodic zeta function. Taking a general function $g$ and an irrational rotation generalizes the situation of the Clausen function where $g(x) = \sin(2\pi x)$, which is the imaginary part of the polylog where $g(x) = \exp(2\pi i x)$. We will see that as in the polylog case, there is an analytic continuation of the zeta function to the entire complex plane.

The **Lerch transcendent** is defined as

$$L(z,s) = \sum_{n=0}^{\infty} z^n/(n+a)^s \; .$$

For $z = e^{2\pi i \alpha}$, we get the **Lerch zeta function**

$$L(\alpha,s) = \sum_{n=0}^{\infty} e^{2\pi i n \alpha}/(n+a)^s \; .$$

In the special case $a = 1$ we have the **polylogarithm** $L(z,s) = \sum_{n=1}^{\infty} z^n/n^s$. For $z = 1$ and general $a$, we have the **Hurwitz zeta function**, which becomes for $a = 1, z = 1$ the **Riemann zeta function**. The following two Lemmas are standard (see [10]).

*Lemma* 5.1. *For complex $a$ and $|z| = 1$, there is an integral representation*

$$L(z,s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1} e^{-at}}{1 - z e^{-t}} \, dt \; .$$

*For fixed $|z| = 1, z \neq 1$, this is analytic in $s$ for $\mathrm{Re}(s) > 1$. For fixed $\mathrm{Re}(s) > 1$, it is analytic in $z$ for $z \neq 1$.*

*Proof.* By expanding $1/(1 - ze^{-t}) = \sum_{n=0}^{\infty} z^n/e^{nt}$, the claim is equivalent to

$$\frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-at-nt}\, dt = \frac{1}{(n+a)^s} .$$

A substitution $u = (n+a)t, du = (n+a)dt$ changes the left hand side to

$$\frac{1}{\Gamma(s)} \int_0^\infty u^{s-1} e^{-u} \frac{1}{(n+a)^s}\, du .$$

Now use $\Gamma(s) = \int_0^\infty u^{s-1}e^{-u}\, du$ to see that this is $\frac{1}{(n+a)^s}$.

The improper integral is analytic in $s$ because $|1 - ze^{-t}|$ is $\geq \sin(\arg(z))$ for $\mathrm{Re}(z) \geq 0$ and $\geq 1$ for $\mathrm{Re}(z) \leq 0$ and for $\sigma = \mathrm{Re}(s) > 1$, we have

$$|L(z,s)| \leq \frac{1}{|\Gamma(s)|} \int_0^\infty |t^{\sigma-1}e^{-at}|\, dt \frac{1}{|1-z|} .$$

□

*Lemma* 5.2. *The Lerch transcendent has for fixed $|z| = 1, z \neq 1$ and $a > 0$ an analytic continuation to the entire s-plane. In every bounded region $G$ in the complex plane, there is a constant $C = C(G,a)$ such that $|L(z,s)| \leq C/|z-1|$ and $|(\partial_z)^n L(z,s)| \leq Cn!/|z-1|^{n+1}$.*

*Proof.* For any bounded region $G$ in the complex plane we can find a constant $C$ such that

$$|\int_0^\infty \frac{t^{s-1}e^{-at}}{1-ze^{-t}}\, dt| \quad \leq \quad (\int_0^\infty |t^{s-1}e^{-at}|\, dt) \max_t \frac{1}{|1-ze^{-t}|} \leq \frac{C}{|1-z|} .$$

Similarly, we can estimate $|\partial_z^n L(z,s)| \leq Cn!/|z-1|^n$ for any integer $n > 0$. The identity

$$L(z, s-1) = (a + z\partial_z)L(z,s) \tag{5.1}$$

allows us to define $L$ for $\mathrm{Re}(s) < 1$: first define $L$ in $0 < \mathrm{Re}(s) < 1$ by the recursion (5.1). Then use the identity (5.1) again to define it in the strip $-1 < \mathrm{Re}(s) < 0$, then in the strip $-2 < \mathrm{Re}(s) < 1$, etc. □

*Remark* 5.3. The Lerch transcendent is often written as a function of three variables:

$$\phi(x,a,s) = \sum_{n=0}^{\infty} e^{2\pi i n x}/(n+a)^s .$$

It satisfies the functional equation

$$\begin{aligned}(2\pi)^s \phi(x,a,1-s) \quad &= \quad \Gamma(s)\exp(2\pi i(s/4 - ax))\phi(x,-a,s) \\ &+ \quad \Gamma(s)\exp(2\pi i(-s/4 + a(1-x)))\phi(1-x,a,s)) .\end{aligned}$$

See [19]. Using the functional equation to do the analytic continuation is less obvious.

One of the main results in this paper is the following theorem:

*Theorem* 5.4. *For all Diophantine $\alpha$ and every real analytic periodic function $g$ satisfying $\int g(x)\ dx = 0$, the series $\zeta_{g,\alpha}(s) = \sum_{n=1}^{\infty} g(n\alpha)/n^s$ has an analytic continuation to the entire complex plane.*

*Proof.* The Fourier expansion of $g$ evaluated at $x = n\alpha$ gives

$$g(n\alpha) = \sum_{k=-\infty}^{\infty} c_k e^{2\pi i n k\alpha} \ .$$

It gives rise to a polylog expansion of $f = \zeta_{g,\alpha}$

$$f(s) = \sum_{k=-\infty}^{\infty} c_k L(e^{2\pi i k\alpha}, s)$$

with $a = 1$. Because $g$ is real analytic, there exists $\delta > 0$ such that $|c_k| \leq e^{-|k|\delta}$. [To see this, write the Fourier series as a Laurent series $\sum_k c_k z^k$ where $z = e^{2\pi i x}$. By the real analyticity assumption, it is convergent in a ring $\{\ e^{-\delta} \leq |z| \leq e^{\delta}\ \}$ for some $\delta > 0$. The sum $\sum_{k\geq 0} c_k z^k$ is a Taylor series convergent in $|z| \leq e^{\delta}$, the sum $\sum_{k>0} c_{-k} z^{-k}$ converges in $|z| > e^{-\delta}$. From the radius of convergence of the two Taylor series, we get $|c_k| \leq e^{-\delta|k|}$.] Since $k \neq 0$ and $\alpha$ is Diophantine, $|e^{2\pi i k\alpha} - 1| \geq c/|k|^r$ so that $L(e^{2\pi i k\alpha}, s) \leq |k|^r/c$ and

$$|f(s)| = \sum_{k=-\infty}^{\infty} |c_k||L(e^{2\pi i k\alpha}, s)| \leq \sum_{k=-\infty}^{\infty} e^{-|k|\delta} k^r/c < \infty \ .$$

□

## 6. A commutation formula

In this section we consider a procedure in which a Dirichlet series of the form above is considered as a periodic function in $\alpha$ for fixed $s$, which is then itself used to produce a new Dirichlet series in a second complex variable $t$. This process is symmetric in $s$ and $t$, providing another tool to study the questions of analytic continuation. In special cases these iterated series may take a variety of interesting forms.

For any periodic function $g$, the series

$$T_s(g) = \sum_{n=1}^{\infty} \frac{g(n\alpha)}{n^s}$$

produces for fixed $s$ in the region of convergence a new periodic function in $\alpha$. For fixed $\alpha$ it is a Dirichlet series in $s$. The Clausen function is $T(\sin 2\pi x)$ and the poly logarithm is $T(\exp(2\pi i x))$. We may then consider a new periodic function generated by this function, defined by $T_t(T_s(g))$. The following commutation formula can be useful to extend the domain, where Dirichlet series are defined:

*Lemma* 6.1 (Commutation formula). *For* $\mathrm{Re}(s) > 1$ *and* $\mathrm{Re}(t) > 1$, *we have*

$$T_s(T_t(g)) = T_t(T_s(g)) \ .$$

*Proof.* We have

$$T_s(g)(\alpha) = \sum_{n=1}^{\infty} \frac{g(n\alpha)}{n^s} \ ,$$

which we regard as a periodic function in $\alpha$. It is continuous in $\alpha$ if evaluated for fixed $\mathrm{Re}(s) > 1$. Then where our sum converges absolutely (which holds at least for $\mathrm{Re}(s), \mathrm{Re}(t)b > 1$),

$$\begin{aligned} T_t(T_s(g)) &= \sum_{m=1}^{\infty} \frac{T_s(g)(m\alpha)}{n^t} \\ &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{n^s m^t} g(mn\alpha) \\ &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{1}{n^s m^t} g(mn\alpha) = T_s(T_t(g)). \end{aligned}$$

□

In fact, the double sum can be expressed as a single sum using the **divisor sum function** $\sigma_t(k) = \sum_{m|k} m^t$:

$$\begin{aligned} T_s(T_t(g)) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{n^s m^t} g(mn\alpha) &= \sum_{k=1}^{\infty} \left( \left( \sum_{m|k} \frac{m^{t-s}}{k^t} \right) g(k\alpha) \right) \\ &= \sum_{k=1}^{\infty} \frac{\sigma_{t-s}(k)}{k^t} g(k\alpha) \ . \end{aligned} \quad (6.1)$$

In the notation of previous sections, we can formulate this as follows:
for $\mathrm{Re}(s), \mathrm{Re}(t) > 1$, set $g_s(\alpha) = \sum_{n=0}^{\infty} g(n\alpha)/n^s$, and we have

$$\zeta_{g_s,\alpha}(t) = \zeta_{g_t,\alpha}(s) \ .$$

For example, evaluating the almost periodic Dirichlet series for the periodic function $g_3(x) = \sum_{k=1}^{\infty} (1/k^3)\sin(2\pi kx)$ at $s = 5$ is the same as evaluating the almost periodic Dirichlet series of the periodic function $g_5(x) = \sum_{k=1}^{\infty} (1/k^5)\sin(kx)$ and evaluating it at $s = 3$. But since $g_3(x)$ is of bounded variation, the Dirichlet series $T(g_3)$ has an analytic continuation to all $\mathrm{Re}(s) > 0$ and $T_{0.5}(g_3)$ for example is defined if $\alpha$ is sufficiently Diophantine. The commutation formula allows us to define $T_3(g_{0.5}) = \zeta_{g_{0.5}}(3)$ as $\zeta_{g_3}(0.5)$, even so $g_{0.5}$ is not even in $L^2(T^1)$.

*Remark* 6.2. The commutation formula generalizes. The irrational rotation $x \mapsto x + \alpha$ on $X = S^1$ can be replaced by a general topological dynamical system on $X$.

In the particular case that $f$ is the Clausen function $g = T(\sin)$, the expression (6.1) is itself a Fourier series, whose coefficients are the divisor function $\sigma$. We have $g(x) = \sum_{n=1}^{\infty} \frac{1}{n^t} \sin(2\pi n x)$ and $t > s$: the value of $\zeta_{g,\alpha}(s)$ as a function of $\alpha$ is the Fourier transform on $l^2(\mathbb{Z})$ of the multiplicative arithmetic function $h(k) = \sigma_{t-s}(k)/k^s$. For $t = 2, s = 1$ for example, we get

$$\sum_n \frac{g(n\alpha)}{n} = \sum_{k=1}^{\infty} \frac{\sigma(k)}{k} \sin(2\pi k\alpha)$$

if $g$ is the function with Fourier coefficients $1/n^2$. In that case, the Fourier coefficients of the function has the multiplicative function $\sigma(n)/n$ (called the **index** of $n$) as coefficients. For $t = 1$, we have $\sum_{n=1}^{\infty} \frac{1}{n} \sin(2\pi n x) = 1/2 - (x \bmod 1)$ and for odd integer $t > 1$, the function

$$g_t(x) = \sum_{k=1}^{\infty} \sin(2\pi k x)/k^t$$

is a **Bernoulli polynomial**.

For $t = s$, we get

$$\sum_n \frac{g(n\alpha)}{n^s} = \sum_{k=1}^{\infty} \frac{d(k)}{k^s} \sin(2\pi k\alpha) ,$$

where $d(k)$ is the number of divisors of $k$. These sums converge absolutely for $\mathrm{Re}(s) > 1$. If $t$ is a positive odd integer, $g$ is a Bernoulli polynomial (e.g. for $t = 3$, we have $g(x) = (\pi^3/3)(x - 3x^2 + 2x^3)$). For $s = 2$ (and still $t = 3$), we have

$$\sum_n \frac{g(n\alpha)}{n^2} = \sum_{k=1}^{\infty} \frac{\sigma(k)}{k^2} \sin(2\pi k\alpha) .$$

The functions $f_{s,t}(\alpha) = T_s(T_t(\sin))$, regarded as periodic functions of $\alpha$, may be related by an identity of Ramanujan [23]. Applying Parseval's theorem to these Fourier series, one can deduce

$$\begin{aligned} \int_0^1 f_{s,t}(\alpha) f_{u,v}(\alpha)\, d\alpha &= \sum_{n=1}^{\infty} \frac{\sigma_{t-s}(n)}{n^t} \frac{\sigma_{v-u}(n)}{n^v} \\ &= \frac{\zeta(s+u)\zeta(s+v)\zeta(t+u)\zeta(t+v)}{\zeta(s+t+u+v)} . \end{aligned}$$

## 7. Unbounded variation

In this section we extend the results to a larger class of functions. It relates more to Denjoy-Koksma theory, which estimates the growth rate of the random walk $S_k = \sum_{n=1}^{k} g(n\alpha)$ for a given periodic function $g$ with zero mean and an irrational $\alpha$.

If $g$ fails to be of bounded variation, the previous results do not apply. Still, there can be boundedness for the Dirichlet series if $\mathrm{Re}(s) > 0$. The example $g(x) = \log|2 - 2\cos(2\pi x)| = 2\log|e^{2\pi ix} - 1|$ appears in the context of KAM theory and was the starting point of our investigations. The product $\prod_{k=1}^{n} |2\cos(2\pi k\alpha) - 2|$ is the determinant of a truncated diagonal matrix representing the Fourier transform of the Laplacian $L(f) = f(x+\alpha) - 2f(x) + f(x-\alpha)$ on $L^2(\mathbb{T})$.

The function $g(x)$ has mean 0 but it is not bounded and therefore has unbounded variation. Numerical experiments indicate however that at least for many $\alpha$ of constant type, $|\sum_{k=0}^{n-1} g(k\alpha)| \leq C\log(n)$. We can only show:

*Proposition* 7.1. *If $\alpha$ is Diophantine of type $r > 1$, then for $g(x) = \log|2 - 2\cos(2\pi x)|$,*

$$\sum_{n=0}^{k-1} g(n\alpha) \leq Ck^{1-1/r}(\log(k))^2$$

*and $\zeta_{g,\alpha}(s)$ converges for* $\mathrm{Re}(s) > 0$.

*Proof.* $\prod_{j=1}^{q-1} |e^{2\pi ij/q} - 1| = q$ because

$$\prod_{j=1}^{q-1} |e^{2\pi ij/q} - z| = \frac{z^q - 1}{z - 1} = \sum_{j=0}^{q-1} z^j$$

which gives $\prod_{j=1}^{q-1} |e^{2\pi ij/q} - 1| = q$. Define $\log^M(x) = \max(-M, \log(x))$. Now

$$\sum_{j=1}^{q-1} \log^M(|e^{2\pi ij\alpha} - 1|) \leq M\log(q)$$

for all $q$ and also for general $k$ by the classical Denjoy-Koksma inequality. Choose $M = 3\log(q)$. Then the set $Y_M = \{\log(|\sin(2\pi x)|) < -M\ \} = \{|\sin(2\pi x)| < e^{-M}\ \} \subset \{|x| < 2e^{-M} = 2q^{-3}\}$. The finite orbit $\{k\alpha\}_{k=1}^{q-1}$ never hits that set and the sum is the same when replacing $\log^M$ with the untruncated log. We get therefore

$$\sum_{j=1}^{q-1} \log(|e^{2\pi ij\alpha} - 1|) = \sum_{j=1}^{q-1} \log^M(|e^{2\pi ij\alpha} - 1|) \leq 3\log(q)^2\ .$$

The rest of the proof is the same as for the classical Denjoy-Koksma inequality. □

## 8. Questions

Finally, we look at some questions we encountered in this topic and which remain open.

We were able to get entire functions $\zeta_{g,\alpha}(s) = \sum_n g(n\alpha)/n^s$ for rational $\alpha$ and Diophantine $\alpha$. What happens for Liouville $\alpha$ if $g$ is not a trigonometric polynomial? Liouville means that there for every integer $m > 0$, there is $p, q$ with $q^m|\alpha - p/q| < 1$. What happens for more general $g$?

For every $g$ and $s$ we get a function $\alpha \to \zeta_{g,\alpha}(s)$. For

$$g(x) = \sum_{n=1}^{\infty} \frac{1}{n} \sin(2\pi n x)$$

and Diophantine $\alpha$, where

$$h(\alpha) = \zeta_{g,\alpha}(1) = \sum_{n=1}^{\infty} \frac{d(n)}{n} \sin(2\pi n\alpha) \ ,$$

we observe a self-similar nature of the graph. Is the Hausdorff dimension of the graph of $h$ not an integer?

Since many Dirichlet series in number theory allow an Euler product formula, it is natural to ask, whether the Dirichlet series considered here allow this. This is related to the question to find the intersection of almost periodic sequences which multiplicative arithmetic functions. This question was addressed in [22]. The multiplicative index function $\sigma(n)/n$ for example is almost periodic. But it is not in the rather narrow class of the almost periodic sequences, for which the hull of the sequence is a one dimensional circle.

One can look at the problem for more general dynamical systems. Here is an example: for periodic Dirichlet series generated by an ergodic translation on a two-dimensional torus with a vector $(\alpha, \beta)$, where $\alpha, \alpha/\beta$ are irrational, the series is

$$\sum_{n=1}^{\infty} \frac{g(n\alpha, n\beta)}{n^s} \ .$$

In the case $s = 0$, this leads to the Denjoy-Koksma type problem to estimate the growth rate of the random walk

$$\sum_{n=1}^{\infty} g(n\alpha, n\beta)$$

which is more difficult due to the lack of a natural continued fraction expansion in two dimensions. In a concrete example like $g(x, y) = \sin(2\pi xy)$, the question is, how fast the sum

$$S_k = \sum_{n=1}^{k} \sin(2\pi n^2 \gamma)$$

grows with irrational $\gamma = \alpha\beta$. Numerical experiments indicate subpolynomial growth that $S_k = O(\log(k)^2)$ would hold for Diophantine $\gamma$ and suggest the abscissa of convergence of the Dirichlet series

$$\zeta_{riemann}(s) = \sum_{n=1}^{\infty} \frac{\sin(2\pi n^2 \gamma)}{n^s}$$

is $\sigma_0 = 0$. This series is of some historical interest since Riemann knew in 1861 (at least according to Weierstrass [13]) that $h(\gamma) = \sum_{n=1}^{\infty} \frac{\sin(2\pi n^2 \gamma)}{n^2}$ is nowhere differentiable.

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