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Cellular automata with almost periodic initial conditions

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Abstract. Cellular automata are dynamical systems on the compact metric space of subshifts. They leave many classes of subshifts invariant. Here we show that cellular automata leave 'circle subshifts' invariant. These are the strictly ergodic subshifts of $\{0, 1\}^{\mathbb{Z}^d}$ obtained by a circle sequence $x_n = 1_J(n \cdot \alpha)$, where J is a finite union of half-open intervals. For such initial conditions, the evolution of the whole infinite configuration can be computed by evolving the finitely many parameters defining the set J. Moreover, many macroscopic quantities can be computed exactly for the infinite system. We illustrate that in one dimension by rule 18 and in two dimensions by the Game of Life. The ideas also apply to cellular automata acting on $\{0, \ldots, N-1\}^{\mathbb{Z}^d}$. This we illustrate by the HPP model, a lattice gas automaton with $N = 16$.

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1. Introduction

A one-dimensional cellular automaton is usually viewed as a rule ϕ that gives a time evolution to configurations $x \in A^{\mathbb{Z}}$, where A is a finite set, by

$$x_i^{t+1} = \phi(x_{i-r}^t, \ldots, x_i^t, \ldots, x_{i+r}^t)$$

where r is the range of the automaton. We will consider cellular automata as shift commuting maps on the space of shift-invariant subsets of $A^{\mathbb{Z}}$; i.e., as dynamical systems on a space of dynamical systems. This is the point of view taken in e.g. [24, 27]. It originates in topological dynamics (see e.g. [16]).

Cellular automata have attracted interest as models of 'self-organizing systems' [33] and as models of physical phenomena (see e.g. [34, 12]). Many papers report large scale simulations of cellular automata. The simulations often start from a random initial condition. To get results for a system of size L after t iterations there are two options: start with a system of size L and impose periodic boundary conditions or use free boundary conditions and start with a system of size $L + rt$. The aim is usually to sample the evolution of the automaton on the full shift $A^{\mathbb{Z}}$ and to study the properties of the attractor(s).

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We will consider here a situation where it is possible to compute *exactly* the time evolution of an infinite system. This rests on the observation that a certain class of subshifts, which we call *circle subshifts*, are invariant under cellular automata. In the case where $A = \{0, 1\}$, a circle subshift X_J is specified by a finite set of half-open intervals $J = \bigcup_{i=1}^{n}[a_i, b_i) \subset \mathbb{T}$ and an irrational number α (here and below $\mathbb{T} = \mathbb{R}\backslash\mathbb{Z}$ denotes the circle, or one-torus). The subshift X_J is the orbit closure of the sequence $x_n = 1_J(\theta + n\alpha)$; it is independent of the choice of $\theta \in \mathbb{T}$ (see section 3). A cellular automaton ϕ on $\{0, 1\}^{\mathbb{Z}}$ maps X_J to another circle subshift, to be denoted by $X_{\phi(J)}$. Clearly $\phi(J)$ can and often will consist of more intervals then J. The data defining $\phi(J)$ can be computed from J and α in finitely many steps. If we work in the ring $\mathbb{Q}[a, b, \alpha]$ then the vectors $\phi(a), \phi(b)$ defining the endpoints of the intervals of $\phi(J)$ are again in this ring and the computation can therefore be performed exactly in the sense that round-off errors do not play a role. In this way, a computer can evolve infinite aperiodic sequences under the cellular automaton storing and processing only finite data.

Note that by the unique ergodicity of circle subshifts, the density of $\lim_{N\to\infty} (2N + 1)^{-1}\sum_{k=-N}^{N} x_k$ of the sequence x exists and is independent of θ; it is given by the Lebesgue measure of J (i.e. $\sum_i b_i - a_i$). So this is an example of a macroscopic property that can be computed exactly for the infinite system after every iteration. Other examples will be discussed in section 3. There we also show how the correlation function and power spectrum can be computed.

It should be noted that in some respects cellular automata on circle subshifts behave very differently from cellular automata on the full shift $\{0, 1\}^{\mathbb{Z}}$. Take for example the automaton ϕ_{170} (we use Wolfram's [33] numbering for $r = 1$ automata), which is the left-shift. On a circle subshift, it induces the map $J \mapsto \phi(J) = J + \alpha$ which is non-mixing. The shift on $\{0, 1\}^{\mathbb{Z}}$, however, is mixing.

What we have said until now is not restricted to one-dimensional cellular automata and one can replace $\mathbb{Z}$ by $\mathbb{Z}^d$ to obtain d-dimensional cellular automata. One chooses d rationally independent rotations $\alpha = (\alpha_1, \dots \alpha_d)$ and defines the configuration by $x_n = 1_J(\theta + n \cdot \alpha)$ for $n \in \mathbb{Z}^d$. The orbit closure of x under translations gives a d-dimensional strictly ergodic subshift.

We can also treat circle subshifts of $\{0, \dots, N-1\}^{\mathbb{Z}^d}$. They are defined as follows. Let $J_0, \dots, J_{N-1}$ be finite unions of half-open intervals $[a, b)$ such that $\mathbb{T} = \bigcup_{k=0}^{N-1} J_k$ and $J_k \bigcap J_l = \emptyset$ for $k \neq l$. Then the orbit closure of $x_n = \sum_{k=0}^{N-1} k 1_{J_k}(\theta + n \cdot \alpha)$ is a strictly ergodic subshift $X_{J_0,\dots,J_{N-1}}$ of $\{0, \dots, N-1\}^{\mathbb{Z}^d}$. Any cellular automaton ϕ will map it to another circle subshift $X_{\phi(J_0),\dots,\phi(J_{N-1})}$. We illustrate this is section 5.2 with $N = 16$ to let the HPP automaton (a lattice gas cellular automaton) act on a circle subshift.

We have done some experiments with various cellular automata on circle subshifts. One quantity of interest is the number of intervals N_n in $\phi^n(J)$, which is a measure for the complexity of $X_{\phi^n(J)}$. Clearly, N_n is bounded if the orbit of the subshift under the cellular automaton is periodic. We find experimentally that N_n grows like n^a if the orbit is not periodic, where $a \leqslant 1$ for one-dimensional cellular automata and $a \leqslant 2$ for two-dimensional automata. We have no explanation for this phenomenon. We find it a little surprising that N_n does not grow exponentially.

The paper is organized as follows. Section 2 contains some general remarks about cellular automata as maps on subshifts. In section 3 the discussion narrows to circle subshifts. Section 4 considers rule 18 as an example. It contains a construction of aperiodic subshifts that are time periodic orbits for rule 18. Section 5 presents some experiments with higher dimensional cellular automata, the 'Game of Life' and lattice gas automata.

The paper concludes with a discussion of the computational advantages of circle subshifts. The last section also formulates some questions about cellular automata acting on circle subshifts.

2. Cellular automata on subshifts

Let A be a finite set and consider the space $\Omega := A^{\mathbb{Z}^d}$ with the product topology. The set A is called the alphabet, Ω the configuration space, and points in Ω are called configurations. Let $\mathbb{Z}^d$ act on Ω by translation: $(T_n x)_k = x_{n+k}$. A cellular automaton ϕ is a continuous map on $A^{\mathbb{Z}^d}$ such that $\phi \circ T_n = T_n \circ \phi$ for all $n \in \mathbb{Z}^d$. By the Curtis–Hedlund–Lyndon theorem (see e.g. [16]), there is for every cellular automaton ϕ a finite set $F \subset \mathbb{Z}^d$ such that $(\phi(x))_k$ only depends on $\{x_j\}_{j\in\{k+F\}}$.

A set $G \subset \Omega$ is called invariant if $T_n G = G$ for all $n \in \mathbb{Z}^d$. A compact invariant subset of Ω is called a subshift. The orbit of $x \in \Omega$ is the set $\mathrm{Orb}(x) = \{T_n x\}_{n\in\mathbb{Z}^d}$ and the orbit closure X_x of x is the closure of $\mathrm{Orb}(x)$ in Ω. So orbit closures are subshifts. A subshift X is called minimal if $X_x = X$ for all $x \in X$. A probability measure μ on (the Borel σ-algebra of) Ω or a subshift is called invariant if $\mu(G) = \mu(T_n G)$ for all measurable G and all $n \in \mathbb{Z}^d$. A subshift is called uniquely ergodic if there exists only one invariant probability measure on it; it is called strictly ergodic if it is uniquely ergodic and minimal.

A cellular automaton ϕ maps subshifts to subshifts. Thus the image $\phi(X)$ is what is called a topological factor of X. This implies that certain properties of subshifts are invariant under cellular automata. Examples are: topological transitivity (i.e., there is a dense orbit), minimality (i.e., every orbit is dense), unique ergodicity (see proposition 3.11 in [4]), being of finite type, and primality (there is no nontrivial factor).

Remarks.

(1) Prime subshifts exist [7]. If we apply a cellular automaton to a prime subshift, then every subshift in the orbit of the cellular automaton is topologically conjugated to the subshift one started with unless one reaches a fixed point consisting of a constant sequence.

(2) If X is uniquely ergodic with invariant measure m and has discrete spectrum G_0 (i.e., the eigenfunctions are dense in $\mathrm{L}^2(X, m)$, where m is the invariant measure), then $\phi^n(X)$ also has discrete spectrum G_n and G_n is a subgroup of G_0. The reason is that the measure theoretical factor $(\phi^n(X), \mathcal{B}, (\phi^n)^*(m))$ of $(X, \mathcal{A}, m)$ is isomorphic to $(X, \phi^{-n}(\mathcal{B}), m)$ and that the conditional expectation $E_m[f, \phi^{-n}(\mathcal{B})]$ of an eigenfunction $f \in L^2(X)$ is again an eigenfunction.

(3) A cellular automaton does not increase the topological entropy of a subshift. This observation, however, is of limited practical importance since for $d = 1$ subshifts generically (w.r.t. the Hausdorff metric) have topological entropy zero [31], whereas in higher dimensions the topological entropy is 'as a rule' infinite (cf [27]). The directional topological entropy in the direction $v \in \mathbb{Z}$ is defined as the topological entropy of T_v (cf [27]). Directional topological entropy does not increase either.

(4) The fact that cellular automata leave classes of subshifts invariant was our motivation to look at them. Call a minimal subshift X palindromic, if an element $x \in X$ (and hence all $x \in X$) contains arbitrary long palindromes. Cellular automata that commute with the involution $x_n \mapsto \overline{x}_n = x_{-n}$ map the set of palindromic subshifts into itself. Discrete Schrödinger operators with palindromic subshifts as potential have a generic set in their hull for which the spectrum is purely singular continuous [19]. Therefore, applying cellular automata to palindromic subshifts gives new classes of operators with singular continuous spectrum.

3. Cellular automata on circle subshifts

Denote by $\mathcal{J}$ the ring generated by half-open intervals $[a, b) \subset \mathbb{T}$, with the convention $[a, a) = \emptyset$. Every $J \in \mathcal{J}$ is a finite union of half-open intervals. Given $\alpha = (\alpha_1, \ldots, \alpha_d)$ and $\theta \in \mathbb{T}$, let $x_n = 1_J(\theta + n \cdot \alpha) \in \{0, 1\}^{\mathbb{Z}^d}$, where the dot denotes the Euclidean inner product. Assume that the α_i are rationally independent. Then the orbit closure of x is strictly ergodic, and independent of θ (see e.g. [18]). We call it a *circle subshift.*

Circle subshifts of $\{0, \ldots, N-1\}^{\mathbb{Z}^d}$ are defined analogously as the orbit closure of

$$x_n = f(\theta + n \cdot \alpha)$$

where f is a function taking values in $\{0, \ldots, N-1\}$ on half-open intervals that partition $\mathbb{T}$. In other words, $f = \sum_{k=0}^{N-1} k 1_{J_k}$, where each J_k is a finite union of half-open intervals $[a, b)$, $\bigcup_{k=0}^{N-1} J_k = \mathbb{T}$, and $J_k \bigcap J_l = \emptyset$ for $k \neq l$.

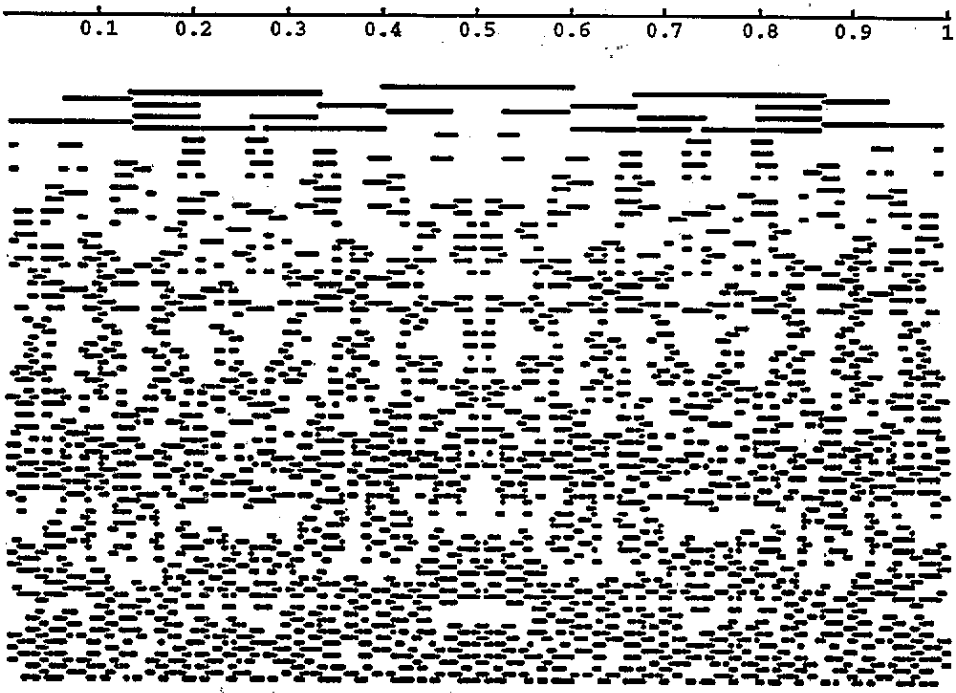

Figure 1. Evolution of an interval under rule 18 ($J = [0.4, 0.61)$, $\alpha = \sqrt{3}$).

Proposition 3.1. *Cellular automata leave circle subshifts invariant.*

Proof. Let X_J be a circle shift and ϕ a cellular automaton. We show how to find a finite collection of half-open intervals $\phi(J) \subset \mathbb{T}$ such that $\phi(X) = X_{\phi(X)}$.

Choose a $\theta \in \mathbb{T}$ and let $x_n = f(\theta + n \cdot \alpha)$. Let $F \subset \mathbb{Z}^d$ be a finite set such that $\phi(x)_n$ only depends on $\{x_j\}_{j \in \{n+F\}}$. Since ϕ commutes with translations, $\phi(x)_n = \phi(x)_m$ whenever x has the same pattern in $F + n$ and in $F + m$. There are finitely many different patterns P_i that can occur in a translate of F. For each P_i there is a $K_i \subset \mathbb{T}$ such that the pattern P_i occurs in $F + n$ if and only if $\theta + n \cdot \alpha \in K_i$. Each K_i is a finite union of half-open intervals because $\mathcal{J}$ is a ring closed under intersection and union. Indeed one has $K_i = \bigcap_{j\in F} f^{-1}(x_{n+j}) - j\cdot\alpha$ if P_i occurs in $n+F$. Hence $\phi(J) = \bigcup_i \{K_i \mid \phi = 1$ on $K_i\}$ □

Remarks.

(1.) This proof is constructive. We have implemented it as an algorithm in Mathematica [35] and all experiments we report here have been done with Mathematica. The program will be available in the MathSource server (mathsource@wri.com). Figure 1 shows the evolution of a circle subshift under rule 18.

(2.) The subshift is periodic if and only if $\alpha \in \mathbb{Q}^d$. A cellular automaton on a periodic subshift is the same as the cellular automaton with periodic boundary conditions with the same period.

(3.) A configuration $x \in \{0,1\}^{\mathbb{Z}^d}$ is called a blinker, if it is a periodic point of ϕ and has compact (and therefore finite) support $F = \{n \in \mathbb{Z}^d \mid x_n = 1\}$. If a cellular automaton ϕ has a blinker then we can construct aperiodic circle subshifts that are periodic orbits of ϕ. Let $G \supset F$ be a cube such that the configuration outside G does not effect the blinker in one iteration of ϕ. We can assume $0 \in F$. Choose $\theta \in \mathbb{T}$. There exists an $\epsilon > 0$ such that the intervals $I_n := [\theta + n\cdot\alpha, \theta + n\cdot\alpha + \epsilon)$ are disjoint for all $n \in G$. Now take $J := \bigcup_{n\in F} I_n$. Then the circle subshift defined by J consists of quasi-periodically spaced copies of the blinker, far enough apart so that the blinkers do not interact.

(4.) A configuration $x \in \{0,1\}^{\mathbb{Z}}$ is called a glider if it is a blinker for the cellular automaton $T_v\phi^n$ for some $v \in \mathbb{Z}\backslash\{0\}$ and $n \in \mathbb{N}$. The vector $v/n \in \mathbb{Q}$ is the velocity of the glider. Since $T_v\phi^n$ is a cellular automaton, it follows from the previous remark that there are circle subshifts for ϕ that consist of quasi-periodically spaced gliders.

(5.) If a cellular automaton ϕ of dimension $\geqslant 2$ contains a glider, then the topological entropy of ϕ is infinite. The reason is that one can then embed in it a (full) topological shift with an arbitrary large alphabet. The Game of Life therefore has infinite topological entropy and one has to use higher dimensional entropies [27] in order to get interesting quantities.

For the remainder of this section consider the case of $d = 1$.

Proposition 3.2. *Circle subshifts have purely discrete spectrum. If k is the largest integer such that $J = J + \frac{2\pi}{k} \pmod{2\pi}$, then the spectrum of X_J is $\{e^{2\pi ikn\alpha}\}_{n\in\mathbb{Z}}$.*

Proof. The measurable map $\psi : \mathbb{R}^1/(2\pi/k\mathbb{Z}) \to X, \theta \mapsto 1_J(\theta + n\alpha)$ conjugates $\theta \mapsto \theta + \alpha \pmod{2\pi/k}$ with the shift on X_J. The map ψ is injective: $\psi(x) = \psi(y)$ implies $1_J(\theta) = 1_{J+(x-y)}(\theta)$ for θ in a dense set and so $J = J + (x-y)$ with $x - y = 0 \pmod{2\pi/k}$.

By unique ergodicity, $\psi^*\nu_{Leb} = m$ so that the map ψ is measure preserving. The isomorphism between the two systems $\theta \mapsto \theta + k\alpha$ and the shift on X_J assures that the shift on X_J has also the spectrum $\{e^{2\pi ikn\alpha} \mid n \in \mathbb{Z}\}$. □

Note that k cannot decrease under a cellular automaton. If k increases then the cellular automaton creates additional symmetry.

Strict ergodicity implies that for every continuous function f on a circle subshift the average

$$\overline{f} := \lim_{N\to\infty} \frac{1}{2N+1} \sum_{k=-N}^{N} f(T_k x)$$

exists uniformly in $x \in X_J$ and is equal to $\int_X f\,\mathrm{d}m$, where m is the strictly ergodic probability measure on X (see e.g. [32], section 6.5). If $f(x) = f(x_{-l}, \ldots, x_l)$ then $\overline{f}$ can be calculated as follows. For each pattern p_i of length $2l+1$ that occurs in X_J, there is a corresponding $J_i \in \mathcal{J}$ (defined uniquely up to translation; cf the proof of proposition 3.1). If the Lebesgue measure of J_i is denoted by ρ_i then $\overline{f} = \sum_i f(p_i)\rho_i$. Thus we see that certain macroscopic properties of a circle subshift (the averages $\overline{f}$) can be computed exactly in finitely many steps. The simplest example is the density, which is the average of the function $f(x) = x_0$.

Other macroscopic quantities of interest are the correlation function $c(k) := \overline{x_k x_0}$ and its Fourier transform, the power spectrum. Note that both are independent of $x \in X_J$ by strict ergodicity. The power spectrum is a positive measure on $\mathbb{T}$. Li [23] has computed the power spectra of the $r = 1$ cellular automata starting from random initial conditions on a lattice of size 4096. Since the spectrum of X_J is discrete, the power spectrum is a purely discrete measure, i.e., a countable sum of weighted delta functions. The delta functions live at the points of the spectrum determined in proposition 3.2. The weights can be computed directly, without first computing the correlation function. Note that

$$m_\lambda := \lim_{N\to\infty} N^{-1} \sum_{n=l}^{l+N-1} x_n e^{-2\pi i n\lambda} = \int_{\mathbb{T}} 1_J(z) e^{-2\pi i \frac{\lambda}{\alpha} z}\, dz$$

uniformly in $k \in \mathbb{Z}$ because the sequence $\{n\lambda\}$ is well distributed for every irrational λ [22]. This uniformity suffices to prove that the weights are given by $|m_\lambda|^2$ (see [29], section IV.3.1, or [17], theorem 3.4). Thus the Fourier components of 1_J fully determine the power spectrum of X_J.

This also shows that circle subshifts can be used to get information about the attractor $\bigcap_{n>0} \phi^n(\{0,1\}^{\mathbb{Z}})$ of the full shift. Weak limit points of power spectra of $\phi^n(X_J)$ are power spectra of ergodic components of the attractor of the full shift.

4. A one-dimensional example: rule 18

The elementary automaton ϕ_{18} is one of the simplest nonlinear one-dimensional automata. It has been studied quite extensively, see, e.g. [10, 11, 24, 20, 21, 5, 28]. One of the interesting features of this automaton is that its evolution is linear on parts of the phase space. The nonlinear and interesting behaviour is the motion of the kinks, the boundaries between regions with linear motion. A sequence x has a kink at n, if for some $l \geqslant 0$, $[x_{n-l}, \ldots, x_{n+l+1}] = [1, 0, \ldots 0, 1] = [10^{2l}1]$. On the regions of linear motion, $\left(\phi_{18}^n(x)\right)_j = 0$ either for j even and n even ('type I' domains) or for j odd and n odd ('type II' domains) [10, 11].

Proposition 4.1. *The kink density is defined for any strictly ergodic subshift. For circle subshifts and irrational α, the kink density is strictly positive. The kink density does not increase under the evolution of ϕ_{18}.*

Proof. A sequence x contains a kink if and only if $x \in \bigcup_{n\in\mathbb{Z}} \bigcup_{l=1}^{\infty} U_{l,n}$, where $U_{l,n}$ are the cylinder sets $U_{l,n} = \{x \mid [x_{n-l}, \ldots, x_{n+l+1}] = [10^{2l}1]\}$. The kink density is $m(\bigcup_{n\in\mathbb{Z}} \bigcup_l U_{l,n})$ for all x in the subshift, by the unique ergodicity.

For irrational α, every circle sequence contains infinitely many kinks. Absence of kinks would imply that the orbit of some point of $\mathbb{T}$ under 2α would never hit J which is impossible since the rotation with angle 2α is also ergodic.

It is an observation of Grassberger [10, 11] that kinks can only annihilate and cannot be created. □

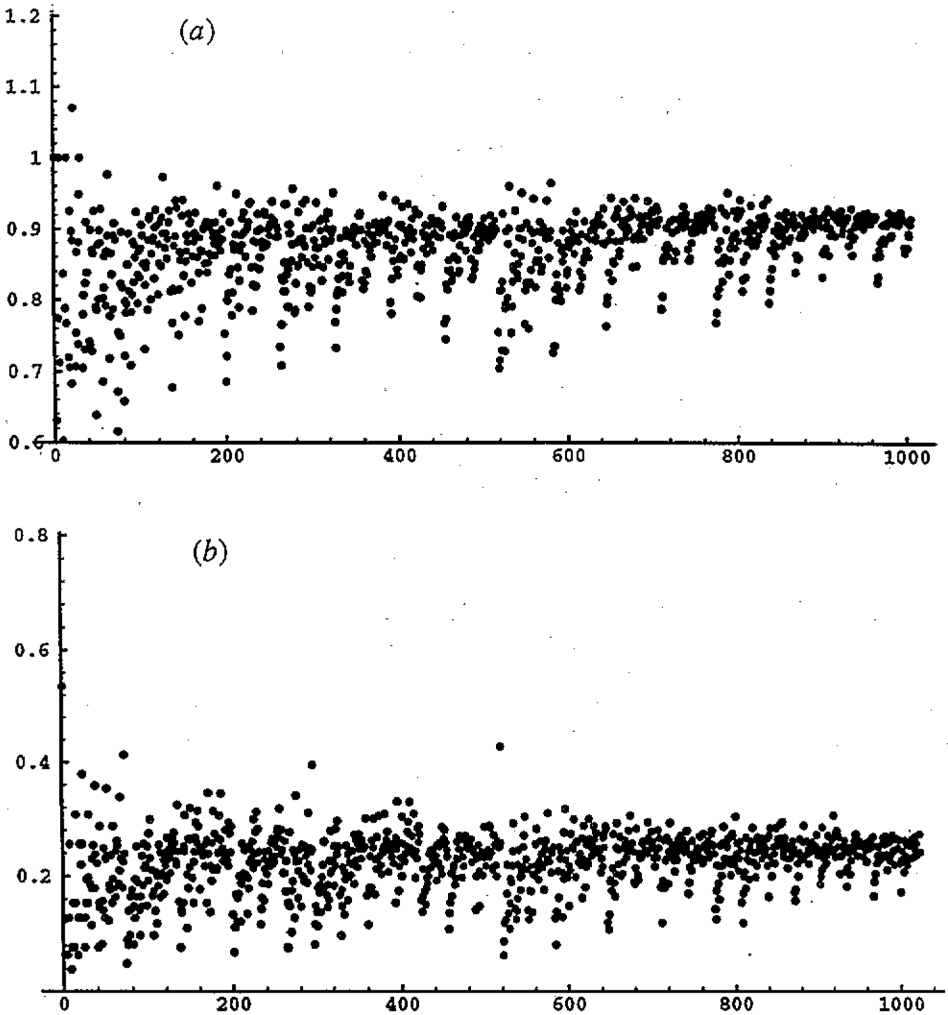

Figure 2. The evolution of (*a*) the number N_n of intervals plotted as $\log(N_n)/\log(n)$ and (*b*) the density under rule 18. The initial conditions are $J = [0.4, 0.7)$ and $\alpha = \sqrt{3}$.

Remarks.
(1) It follows from the monotonicity of the kink density that for a periodic orbit of ϕ_{18} the kink density is constant.
(2) If the kink density in a strictly ergodic subshift is constant under ϕ_{18}, then not even one kink is annihilated. For if one single kink were annihilated, then kinks on a set of positive density would be annihilated.

Recall that rule 18 is defined by $\phi_{18}(x)_j = 1$ if and only if $[x_{j-1}, x_j, x_{j+1}] = [100]$ or $[001]$. Figure 1 shows the evolution of a circle shift under rule 18 and Figure 2 shows how the number of intervals (*a*) and the density (*b*) evolve for the same circle shift. Note that the number of intervals grows slower than linearly. This is typical for all elementary one-dimensional cellular automata: we have never observed a growth that was faster than

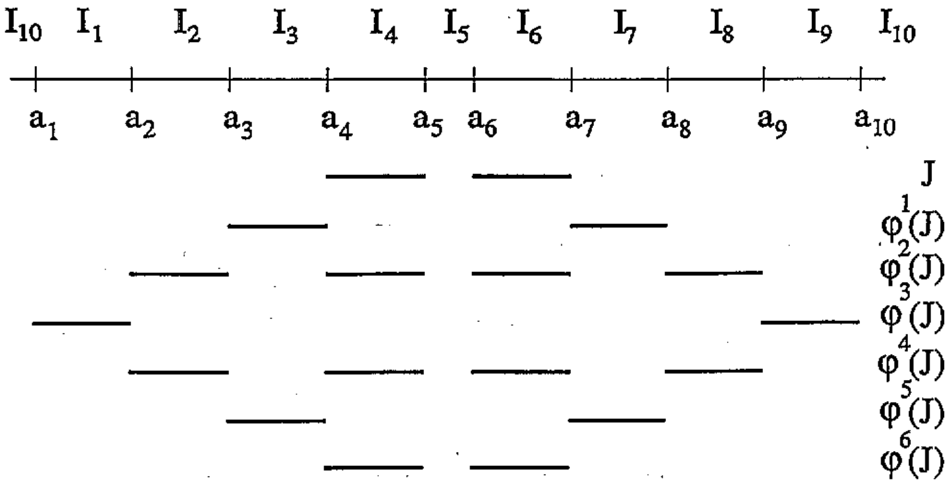

Figure 3. A cyclic aperiodic circle subshift for rule 18 for $l = 5$, $k = 6$ (see proof of proposition 4.2)

n. We have no explanation for this phenomenon. It is clear that the number of intervals is bounded if the time-evolution is periodic. We do not know whether the number of intervals can be bounded if the time evolution is not periodic.

We now construct orbits of ϕ_{18} which are aperiodic in space and periodic in time. For such configurations, the kink density remains constant. We use the word periodic for space periodicity and cyclic for time-periodicity.

Proposition 4.2. *For any cyclic point of ϕ_{18} with (space) $p = 2l$-periodic boundary condition and a reflection symmetry, there exists an aperiodic circle subshift which is a cycle of ϕ_{18} with the same time-period.*

Proof. Take α irrational with $\frac{1}{p-2} > \alpha > \frac{1}{p}$. For points a_i, $i = 1, \ldots, p$, let I_i denote the half-open interval in $\mathbb{T}$ starting at a_i. Choose the a_i such that the length $|I_i|$ of I_i is equal to α for $i \neq l, 2l$ and such that $|I_l| = |I_{2l}| = \beta := (1 - (p-2)\alpha)/2$.

Assume x is a p-periodic symmetric sequence (i.e., $x_n = x_{-n}$) that is not identically zero. The reflection symmetry implies that $x_0 = 0$ since $x_0 = 1$ would give $\phi^n(x)_0 = 0$ for all $n > 1$. Similarly, $x_l = 0$.

Define $J = \bigcup_{1 \leqslant i \leqslant 2l, x_i = 1} I_i$. We show now that the circle shift X_J is a k-cycle of ϕ_{18}. The set $\phi(J)$ is the set of points which are in exactly one of the two sets $J \pm \alpha$ and in no other of the sets $J, J \pm \alpha$. Now $I_1 \cap \phi(J) = I_{l+1} \cap \phi(J) = \emptyset$ since the intervals neighbouring I_l and I_{2l} have length $\alpha > \beta = |I_l| = |I_{2l}|$ and either both are subsets of J or both are disjoint from J by the symmetry of x. Any interval that is not I_l or I_{2l} or one of their neighbours is mapped into an other interval in $\{I_i\}_{i=1}^{2l}$. It then follows from the definition of rule 18 that $\phi_{18}^k(J) = J$. Figure 3 illustrates this for $l = 5$ and $k = 6$. □

Remarks

(1) Proposition 4.2 shows that rule 18 has uncountable many cycles in the space of all (circle) subshifts. We do not know whether for a given irrational α there are uncountably many circle subshifts that are cycles.

(2) The cyclic aperiodic subshifts constructed above are also attractors: e.g. a circle subshift

that differs from the one in the first line of the example in figure 3 by a small interval in I_5 is mapped under rule 18 to the circle subshift in the second line.
(3) Rule 18 does not have blinkers, so the procedure described in section 3 cannot be used to construct aperiodic cyclic circle subshifts for rule 18.

5. Higher dimensional cellular automata

5.1. The Game of Life

The most famous two dimensional automata and probably the best known automaton of all is Conway's Game of Life. Simulations with random initial conditions (giving probability p for a living cell) and lattices of sizes up to $L = 1100$ were performed in [30, 8]. In the time interval of order $[0, L^{1/2}]$, large fluctuations of the observables were measured, in the time interval from $[L^{1/2}, L^{4/3}]$, a scaling region with power law behaviour reaching a

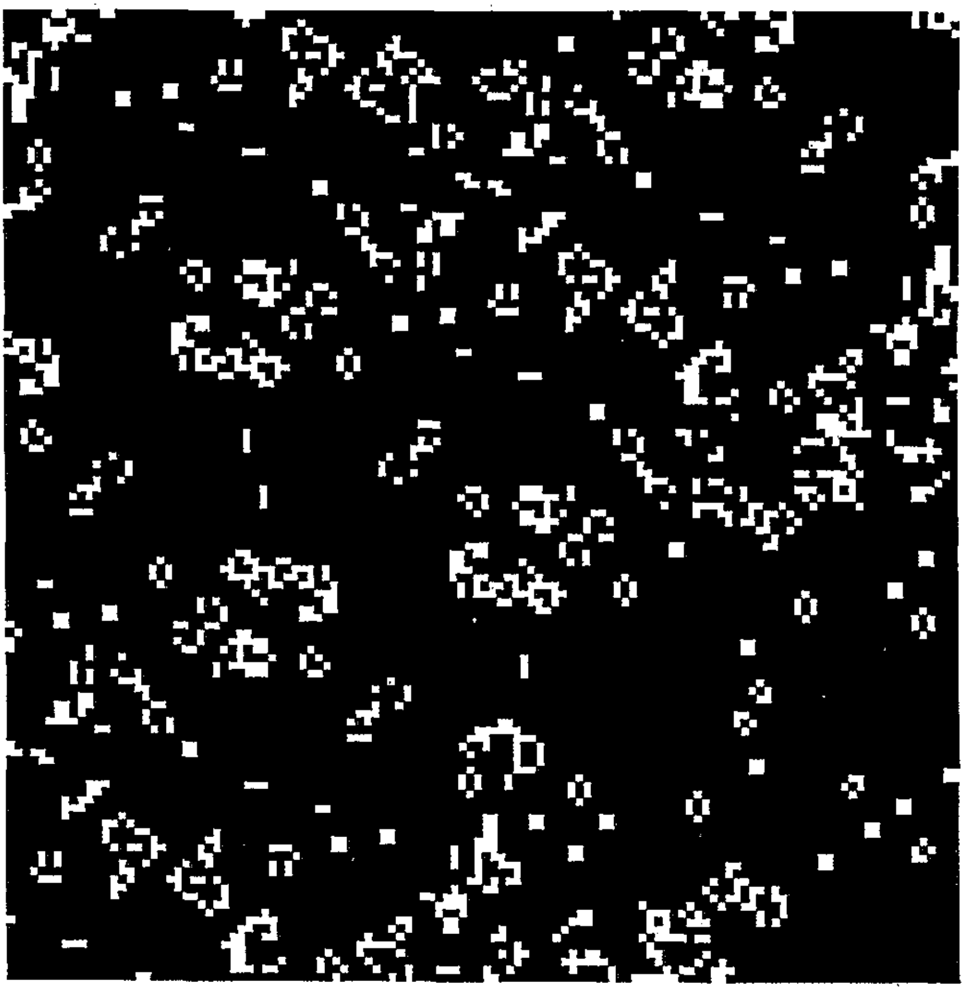

Figure 4. Part of the real infinite space configuration after 100 iterations of the Game of Life on the circle subshift with parameters $J = [0.3, 0.8)$, $\alpha_1 = \sqrt{3}$, $\alpha_2 = \sqrt{2}$. Living cells are white. $\phi_{\mathrm{Life}}^{100}(J)$ consists of 976 intervals.

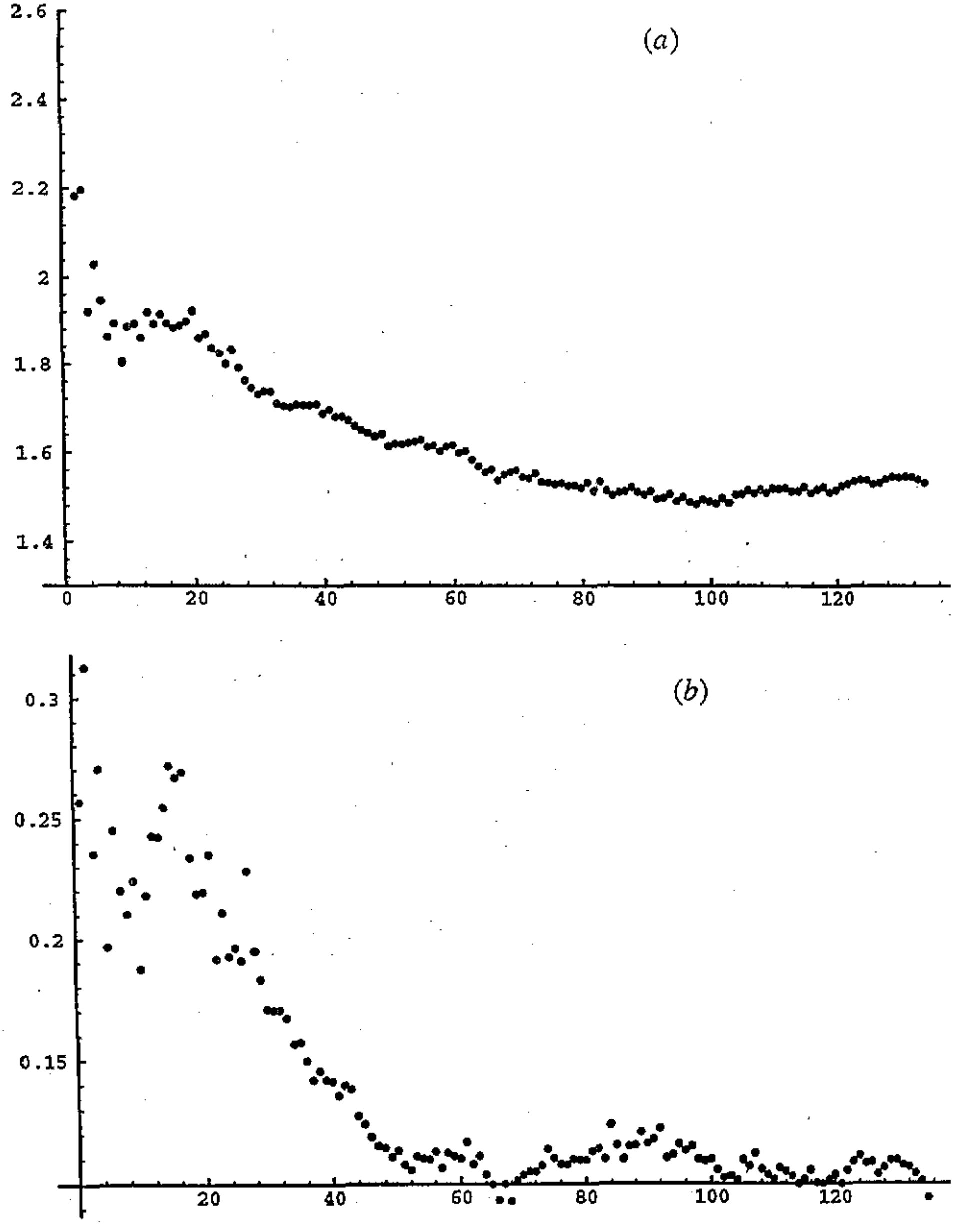

Figure 5. The evolution of (a) the number N_n of intervals plotted as $\log(N_n)/\log(n)$ and (b) the density under the Game of Life. Initial conditions were as in figure 4.

steady state region starting from $L^{4/3}$. Whatever boundary conditions has been taken in those experiments (probably periodic boundary conditions), they play a crucial role for the the number of time steps ($\sim$ 10 000) performed in those experiments. In [1] the Game of Life was claimed to show 'self-organized criticality', but this claim was dismissed in [2] as an artefact of small lattice sizes. It would be interesting to repeat these experiments with subshifts with irrational α, which eliminates dependence on boundary conditions, or with a sequence of rational α's to investigate the importance of the lattice size.

We made some runs with Life starting with a circle-shift initial condition given by one interval or two asymmetric intervals. Figure 4 shows a typical real-space configuration after 100 iterations; figure 5 shows the number of intervals *(a)* and the density *(b)* as a function af n for the same initial conditions. It seems that the number of intervals grows slower than quadratically if the evolution does not become time periodic.

5.2. Lattice gas automata

Lattice gas automata are cellular automata that model particles moving on a lattice. They can be used to study fluid flows. Perhaps the simplest lattice gas automaton is the HPP-model of Hardy, Pazzis and Pomeau [14, 13]. It is a deterministic, two-dimensional cellular

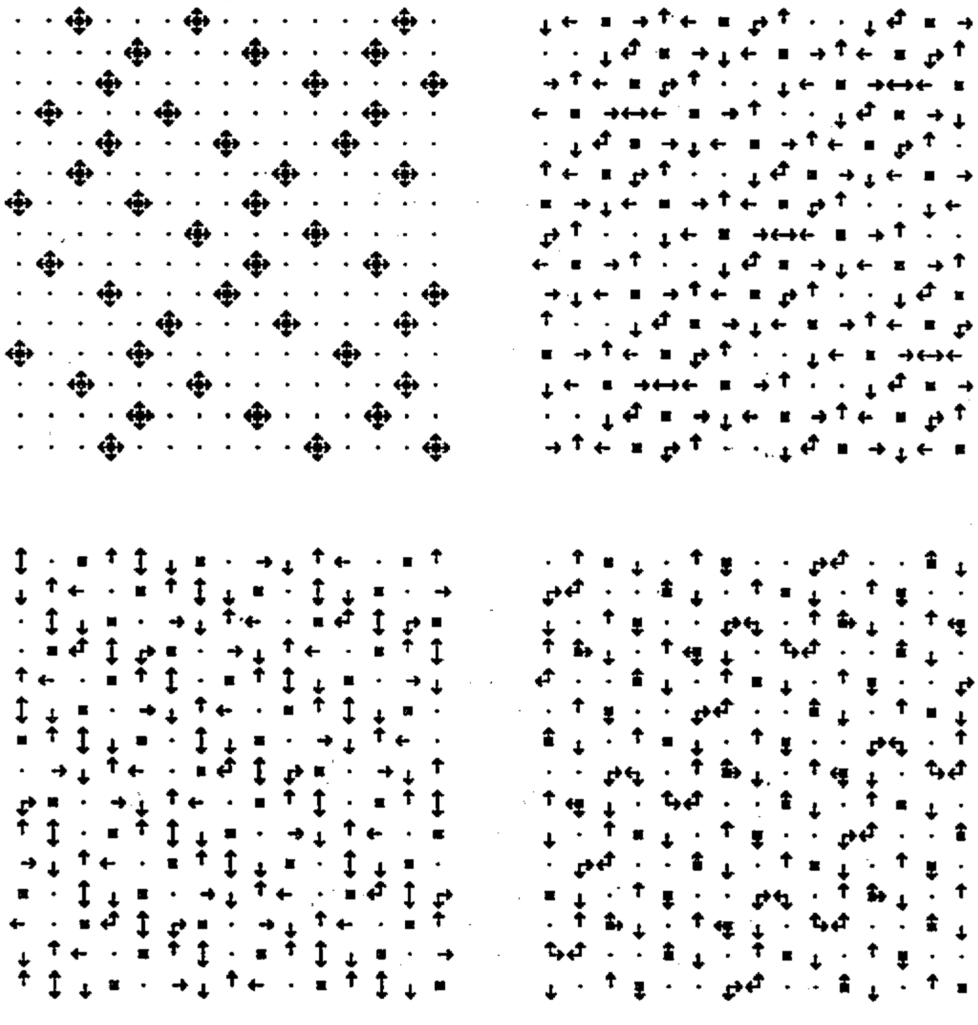

Figure 6. Part of the real space configuration of the HHP automaton with obstacles for the first 4 iterations on the circle subshift with parameters $f([0.3, 0.5)) = (1, 1, 1, 1, 1)$ (every obstacle has four particles) and $f(\mathbb{T} \setminus [0.3, 0.4)) = (0, 0, 0, 0, 0)$ (no particles nor obstacles), $\alpha_1 = \sqrt{3}$, $\alpha_2 = \sqrt{2}$. Note that these are snapshots of infinite configurations.

automaton. Numerically it shows relaxation to equilibrium. Another two-dimensional model of interest is the hexagonal model of Frisch, Hasslacher and Pomeau [6, 15]. This, however, is a non-deterministic cellular automaton. Three dimensional lattice gas automata have also been considered. For a review of lattice gas automata, see [15]. An extensive annotated bibliography of literature on lattice gas automata (deterministic and non-deterministic) is [9]. We mainly consider the HPP model.

In the HPP model particles move on $\mathbb{Z}^2$ along the directions of the coordinate axes. They all move at the same speed: one step per unit of time. The only interaction is when exactly two particles meet at a lattice point in a head-on collision (if three or four particles meet they do not interact). After a head-on collision, particles depart in opposite directions perpendicular to the axis along which they moved before the collision. There can be up to four particles on a lattice site, but at each lattice site there is at most one particle moving in each of the four directions $\pm e_1, \pm e_2$, where e_1, e_2 denote the standard orthonormal basis vectors in $\mathbb{Z}^2$.

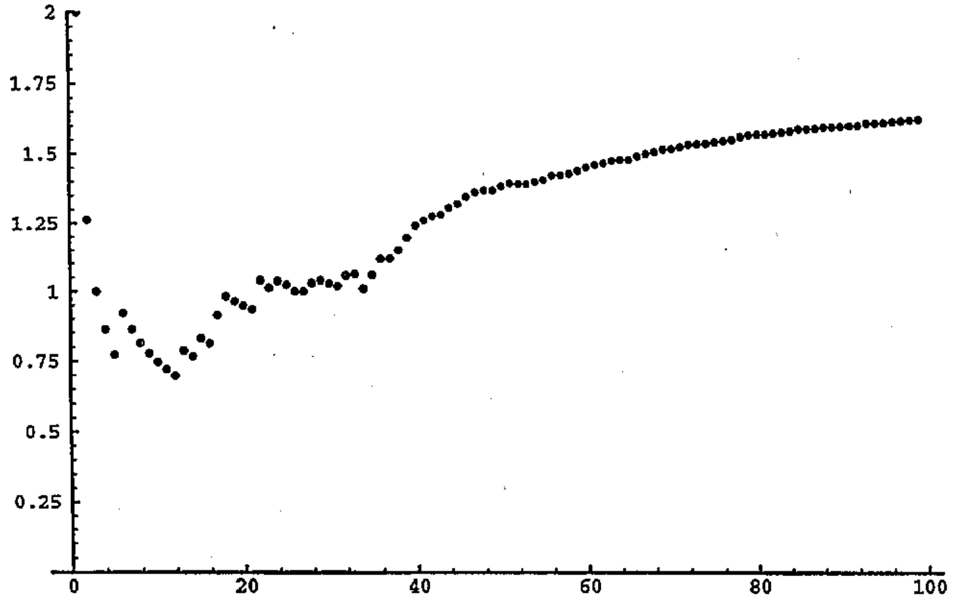

Figure 7. The growth of the number of intervals for the HPP automaton without obstacles. Initial conditions $f([0.3, 0.35)) = (1, 1, 1, 1)$ and $f(\mathbb{T} \setminus [0.3, 0.35)) = (0, 0, 0, 0)$, $\alpha_1 = \sqrt{3}$, $\alpha_2 = \sqrt{2}$.

The HPP model can be implemented as a cellular automaton acting on circle subshifts in the following way. There are 16 possible configurations at each point of the lattice, since one has to specify whether or not there is a particle moving in each of the directions $\pm e_1, \pm e_2$. This can be done by a vector $v = (v_1, v_2, v_3, v_4) \in \{0, 1\}^4$, where $v_i = 1$ if there is particle moving in, respectively the direction $e_1, e_2, -e_1, -e_2$. So the circle subshifts are now subshifts of the full shift over 16 symbols. Since $\phi(x)_n$ (the configuration at n after one time step) only depends on $x_{n\pm e_1}$ and $x_{n\pm e_2}$, one can write

$$\phi(x)_n := P\Big(x_{n-e_1}(1), x_{n-e_2}(2), x_{n+e_1}(3), x_{n+e_2}(4)\Big)$$

where P is the permutation of $\{0, 1\}^4$ that implements the interaction in that it exchanges $(1, 0, 1, 0)$ and $(0, 1, 0, 1)$. In words: to get the velocity distribution at the cell n in the next time step we let evolve the particles of the four nearest neighbours that have velocities towards n and then we take care of possible head-on collisions.

The model can obviously be extended to incorporate collisions with obstacles so that we simulate a fluid in an almost periodic porous medium. Figure 6 shows the first few steps of the evolution of part of the real space configuration defined by a circle subshift. The number of intervals grows slower than quadratically (see figure 7).

Automata with obstacles are of interest because lattice gas automata are used specifically to model fluid flow in complex geometries. For the implementation we take the alphabet $\{0, 1\}^5$, where the fifth bit describes whether or not there is an obstacle at that point. It is convenient to think an obstacle as a particle with zero velocity and infinite mass. The cellular automata rule is then changed so that particles are reflected at the obstacles.

The distribution of the obstacles in $\mathbb{Z}^2$ is determined by a finite union of half open intervals $J \in \mathcal{J}$ and the two irrational rotation $\alpha_i : \mathbb{T}^1 \to \mathbb{T}^1$. For every $\theta \in \mathbb{T}^1$ the obstacles are in

$$O(\theta) = \{n \in \mathbb{Z}^2 \mid \theta + n \cdot \alpha \pmod 1 \in J\} \subset \mathbb{Z}^2 .$$

A connected component of O is called a cluster.

Circle subshifts have been studied as a model of dependent percolation in [25, 26], for the case that J consists of one interval. It was found that (for given frequencies) there are three regimes (depending on the length of the interval): no infinite cluster, infinitely many infinite clusters (parallel 'strips') and a unique infinite cluster. If there is no infinite cluster for the complement of $O(\theta)$, all particles are trapped in 'chambers' and the time evolution becomes periodic.

Proposition 5.1. *If $\mathbb{Z}^2 \setminus O(\theta)$ contains no infinite cluster then the time evolution of the HPP cellular automaton is periodic.*

Proof. The motion of finitely many particles in a bounded chamber is periodic since the map ϕ is deterministic and because there are only finitely many states. Moreover, there is a bound on the possible periods depending on the size of the chamber. There is a global bound on the size of chambers (argument in lemma 4.3 of [26]). We have therefore only finitely many different chambers and hence finitely many periods. The smallest common multiple of all the periods for all chambers is the period of the cellular automaton. □

We have also implemented the hexagonal model (with and without obstacles), a three dimensional HPP model and a HPP model with mirrors (cf [3]). The programs will also be available at mathsource@wri.com.

6. Discussion and open problems

The advantages of performing computations of cellular automata on circle subshifts are the following. First, one can evolve an infinite lattice of configurations. Second, the density of the configuration and other quantities like for example the average momentum distribution of the fluid can be computed *exactly* in each time step. Finally, infinite aperiodic configurations can be evolved while storing only finitely many data. If we work over a ring over $\mathbb{Q}$ which can express the initial conditions as well as the irrational rotations, we have only to store finitely many bits to represent an aperiodic configuration.

A drawback of the interval algorithm is that the number of intervals can increase much during the calculation. Then the computations get so involved that only a few hundred or thousand time steps can be computed. This problem can be treated in two ways. The first possibility is to take α rational corresponding to periodic boundary conditions. Every cellular automaton with periodic boundary conditions can be treated with the interval algorithm and often the later is much faster because we have to treat less data. A second possibility is to smooth out the data as follows: in each step, every interval of size $\leqslant \epsilon$ is deleted and every gap of size $\leqslant \epsilon$ is filled. In the case of lattice gas automata, this has the disadvantage that momentum and particle conservation are violated.

It is clear that the smallest interval gives a bound on the number of intervals. In one dimensions, the continued fraction expansion of α can be used to estimate the number of iterations needed to create intervals of a given length. One can show that if $\alpha = (\sqrt{5}-1)/2$ and $J = [0, (\sqrt{5}-1)/2)$ then it takes more then 2×10^4 iterations to create an interval of length 10^{-10}. But the occurrence of small intervals depends on the Diophantine properties of α: for every ϵ there are α such that intervals of length ϵ can occur after any given number of steps.

Questions.

- It seems that the number of intervals grows polynomially like n^a for a cellular automaton (if the orbit is not periodic). In one dimension, we found growth rates below 1. We have no explanation for this phenomenon. It is not clear that there should always exist any definite growth rate at all. The number of intervals could conceivably grow exponentially. To decide this question, better experiments would be needed.
- For the one-dimensional cellular automaton of range 1, do there exist time and space aperiodic circle subshifts for which the number of intervals is uniformly bounded in time?
- Is there a definite decay rate of kinks under rule 18 acting on circle subshifts? Are there time-aperiodic orbits for which the kink density stays constant or does not tend to zero?
- Most cellular automata are not invertible on the full shift. But a non-invertible cellular automaton might become invertible when restricted to a class of subshifts like minimal subshifts or circle subshifts. It would be interesting to find (non-trivial) examples where this happens.

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