ISOSPECTRAL DEFORMATION OF DISCRETE RANDOM LAPLACIANS

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DISCRETE RANDOM LAPLACIANS

Let $T_1, T_2, \dots, T_d$ be commuting automorphisms of the probability space (X, μ). We call the $\mathbb{Z}^d$ action (X, T, μ) a *dynamical system* and write $T^n x = T_1^{n_1} T_2^{n_2} \dots T_d^{n_d}(x)$ for $n \in \mathbb{Z}^d$. Denote by $\mathcal{X}$ the *crossed product* of the von Neumann algebra $\mathcal{A} = L^\infty(X, M(N, \mathbb{C}))$ with the dynamical system (X, T, μ). The group $\mathbb{Z}^d$ acts on $\mathcal{A}$ by automorphisms $f \mapsto f(T^n)$ where $f(T^n)(x) = f(T^n x)$ and the algebra $\mathcal{X}$ is obtained by completing the algebra of all polynomials in the variables $\tau_1, \dots, \tau_d$ with coefficients in $\mathcal{A}$

$$K = \sum_{n \in F \subset \mathbb{Z}^d} K_n \tau^n, \ (KL)_n = \sum_{l+m=n} K_l L_m(T^l) \tau^n$$

with respect to the norm $|||K||| = |\ ||K(x)||\ |_\infty$, where $K(x)$ is the bounded linear operator on $l^2(\mathbb{Z}^d)$ defined by $(K(x)u)_n = \sum_m K_m(x) u_{n+m}$ and where $||\cdot||$ is the operator norm on $\mathcal{B}(l^2(\mathbb{Z}^d))$ and $|\cdot|_\infty$ the essential supremum norm. With the involution on $\mathcal{X}$ defined by

$$(\sum_n K_n \tau^n)^* = \sum_n K_n^*(T^{-n}) \tau^{-n}$$

$\mathcal{X}$ becomes a von Neumann algebra. It has the *trace*

$$\mathrm{tr}(K) = \int_X \mathrm{Tr}(L_0(x))\, d\mu(x) ,$$

where Tr denotes the usual trace on the matrix algebra $M(N, \mathbb{C})$. We call the elements in $\mathcal{X}$ *discrete random operators.*

In $\mathcal{X}$ lies the set of *discrete random Laplacians*

$$\mathcal{L} = \{L = a\tau + (a\tau)^* + b = \sum_{i=1}^d a_i \tau_i + (a_i \tau_i)^* + b\} ,$$

On Three Levels, Edited by M. Fannes *et al.*,
Plenum Press, New York, 1994

where we assume $b = b^*$ and a_i to be invertible. We call the vector $(a_1, \dots a_d)$ the *gauge potential part* and b the *scalar potential part* of the Laplacian. Discrete random Laplacians are by definition self-adjoint.

Given a normal element $K \in \mathcal{X}$. The functional calculus defines $f(K)$ for every function $f \in C(\sigma(K))$, where $\sigma(K)$ denotes the spectrum of K. According to Riesz's representation theorem, the bounded linear functional $f \mapsto \mathrm{tr}(f(K)) \in \mathbb{C}$ on $C(\sigma(K))$ defines a measure dk on $\sigma(K)$ which is called the *density of states* of K. It satisfies $\mathrm{tr}(f(K)) = \int f(E)\, dk(E)$ for all $f \in C(\sigma(K))$.

Given $L \in \mathcal{L}$ and $d \geq 2$. The *(multiplicative) curvature* of the Laplacian $L = a\tau + (a\tau)^* + b \in \mathcal{L}$ is defined as

$$F = \sum_{i<j} F_{ij}\, \tau_i \tau_j ,$$

where the value $F_{ij} = a_i a_j(T_i) a_i(T_j)^{-1} a_j^{-1}$ can be considered as the result of the parallel transport with the connection $a\tau$ around the *plaquette* $P_{ij}x = \{x, T_i x, T_i T_j x, T_j x\}$. The *additive curvature* given by

$$\sum_{ij} F_{ij}^{+} \tau_i \tau_j = \sum_{ij} [a_i \tau_i, a_j \tau_j] \tau_i \tau_j = \sum_{ij} (a_i a_j(T_i) - a_j a_i(T_j)) \tau_i \tau_j \ .$$

is vanishing if and only if $F_{ij} = 1$. We say in this case, a Laplacian has *zero curvature*. This means that parallel transport around each closed curve in the lattice $\mathbb{Z}^d$ gives the identity.

Examples of discrete random Laplacians are:

- *Random Jacobi operators*

If $d = N = 1$, we get random Jacobi matrices $L = a\tau + (a\tau)^* + b$. Special cases are *discrete random Schrödinger operators* $L = \tau + \tau^* + V$. Such operators have been studied much in the last years. (See [1], [2].)

- *Harper Laplacians*

Let $d = 2$ and $a_1, a_2 \in L^\infty(X, SU(N))$ satisfy $a_1(x) a_2(T_1 x) = e^{2\pi i \alpha} a_2(x) a_1(T_2 x) \cdot 1$, where α is an irrational number. In this case

$$L = a_1 \tau_1 + a_2 \tau_2 + (a_1 \tau_1)^* + (a_2 \tau_2)^*$$

is called a *Harper Laplacian.* It has by definition a constant curvature $e^{2\pi i \alpha}$ and $2\pi\alpha$ has a physical interpretation as a normalized magnetic flux. A special case of a Harper Hamiltonian is given by $(X, T_1, T_2, \mu) = (\mathbb{T}^1, x \mapsto x + \alpha, x \mapsto x, dx)$ with $a_1 = 1$, $a_2(x) = e^{2\pi i x}$, which leads to the Hofstadter case of the *discrete Mathieu operator* $L = \tau_1 + \tau_1^* + V$, where $V(x) = 2\cos(2\pi x)$. See [3] for recent results on the spectrum.

- *D'Alembert operator with Dirac operator*

Let $d = 4$. The operator $L = \sum_{i=1}^4 g_i(\tau_i + 2 + \tau_i^*)$ with $g_1 = -1, g_2 = g_3, g_4 = 1$ is a discrete version of the flat d'Alembert operator $\Box = \sum_{i=1}^4 g_i \frac{\partial^2}{(\partial x_i)^2}$. Define the Dirac operator $D = \sum_{i=1}^4 \gamma_i(\tau_i + \tau_i^*)$, where γ_i be the Dirac matrices, satisfying $\{\gamma_i, \gamma_j\} = 2 g_i \delta_{ij}$. The square

$$D^2 = \sum_{i=1}^4 g_i(\tau_i^2 + 2 + (\tau_i^*)^2)$$

is a Laplacian over the $\mathbb{Z}^d$-dynamical system $(X, T^2 = (T_1^2, T_2^2, \ldots, T_d^2), \mu)$.

• *Periodic Laplacians*
In the case $|X| = M < \infty$, the automorphisms T_i are just finite permutations of X. Ergodic T_i lead to periodic Laplacians which are mostly studied in the case $N = 1, a_i = 1, b = V$. If $d = N = 1$, we get periodic Jacobi matrices. The case $d = 2, N = 1$, $a_i = 1$ is the subject of the book [4].

We think about elements in $\mathcal{X}$ as discrete versions of *differential operators.* They can also be considered as the *Hamiltonian* of a quantum mechanical particle or describing the *geometry* of a discrete manifold. We think of the a_i as components of a *connection* or a *gauge potential* or a *one-form* and of the F_{ij} as the components of the *curvature* or a *gauge field* or a *two form*, always depending on geometrical, physical or algebraic preferences.

DISCRETE RANDOM DIRAC OPERATORS

The classical d'Alembert operator $\Box = \sum_{i=1}^4 g_i \frac{\partial^2}{(\partial x_i)^2}$ can be factorized as $L = D^2$ with the Dirac operator $D = \sum_i \gamma_i \delta_{x_i}$, where the Dirac matrices γ_i satisfy the anti-commutation relation $\{\gamma_i, \gamma_j\} = \pm 2 g_i \delta_{ij}$. Our aim is to construct a discrete random Dirac operator, which is by definition the square root of a discrete random Laplace operator L. For doing so, it is necessary to construct a new dynamical system from the given dynamical system (X, T, μ). We get a new probability space (Y, ν) by defining $Y = \bigcup_{I \subset \{1,\ldots,d\}} X_I$ to be the union of 2^d copies X_I of (X, μ) and letting ν be the normalized measure on Y, such that for $Z \subset X = X_I$, $\mu(Z) = 2^d \nu(Z)$. Define

$$S_i : X_I \to X_{I \Delta \{i\}},\ S_i(x) = \begin{cases} \mathrm{x}\ , i \notin I\ , \\ T_i(x)\ , i \in I\ . \end{cases}$$

$(Y, S_1, S_2, \ldots, S_d, \nu)$ is again a $\mathbb{Z}^d$ dynamical system and because $S_i^2(x) = T_i(x)$ for $x \in X_I$, we get the old system back by restricting S_i^2 on X_I. We call (Y, S, ν) a $2:1$ *integral extension* of (X, T, μ).

Define $\mathcal{Y}$ to be the crossed product of $\mathcal{B} = \mathcal{L}^\infty(\mathcal{Y}, \mathcal{M}(\mathcal{N}, \mathbb{C}))$ with the dynamical system (Y, S, ν). We will write elements in $\mathcal{Y}$ as $C = \sum_n C_n \sigma_n$, where $\sigma_i f = f(S_i)\sigma_i$. Let

$$\mathcal{D} = \{D = \sum_i d_i \sigma_i + (d_i \sigma_i)^* \mid d_i \in \mathcal{B}\} \subset \mathcal{Y}$$

be the set of discrete random Laplacians in $\mathcal{Y}$ which have zero scalar part. Define $\psi : \mathcal{Y} \to \mathcal{X}$ by $\psi(K)_n(x) = K_{2n}(x)$ for $x \in X = X_\emptyset$, where we use the notation $2n = (2n_1, \ldots, 2n_d)$. We say, an operator $D \in \mathcal{D}$ is a *discrete random Dirac operator*, if there exists $E \in \mathbb{C}$ such that $\psi(D^2 + E) \in \mathcal{L}$. In [5], we have constructed Dirac operators to every one-dimensional random Jacobi operator. In [6] we showed how the iteration of the factorization $L \mapsto D$ satisfying $L = D^2 + E$ leads to operators with spectra on Julia sets.

If $D = d\sigma + (d\sigma)^*$ is a Dirac operator to L, we can rewrite $\psi(D^2 + E) = L = a\tau + (a\tau^*) + b$ as

$$\begin{aligned} \{d_i \sigma_i, d_j \sigma_j\} &= \delta_{ij} 2 a_i\ , \\ \{d_i \sigma_i, (d_j \sigma_j)^*\} &= \delta_{ij} b_i\ , \end{aligned}$$

with $\sum_{i=1}^{d} b_i = b - E$. This implies

$$[a_i\tau_i, a_j\tau_j] = 0 , \quad (1)$$
$$[a_i\tau_i, (a_j\tau_j)^*] = \delta_{ij}c_i . \quad (2)$$

and the Laplacian L must have zero curvature.

ISOSPECTRAL TODA DEFORMATIONS OF DISCRETE LAPLACIANS

Define the cones $(\mathbb{Z}^d)^+ := \{n \in \mathbb{Z}^d \mid n_i > 0\}$ and $(\mathbb{Z}^d)^- = -(\mathbb{Z}^d)^+$ in the lattice $\mathbb{Z}^d$. We define in $\mathcal{X}$ the projections

$$K = \sum_n K_n\tau^n \mapsto K^{\pm} = \sum_{n\in(\mathbb{Z}^d)^{\pm}} K_n\tau^n$$

and denote the images of these projections in $\mathcal{X}$ by $\mathcal{X}^{\pm}$. We remark that if $K \in \mathcal{X}^+$ then $K^* \in \mathcal{X}^-$. Denote by $C^{\omega}(\mathbb{R})$ the set of all entire functions $\mathbb{C} \to \mathbb{C}$ which map $\mathbb{R}$ into itself and define the space of Hamiltonians

$$C^{\omega}(\mathcal{X}) = \{H : \mathcal{X} \to \mathbb{C} \mid K \mapsto H(K) = \mathrm{tr}(h(K)),\ h \in C^{\omega}(\mathbb{R})\} .$$

To such a Hamiltonian $H \in C^{\omega}(\mathcal{X})$ belongs the *Toda* differential equation

$$\dot{K} = [h'(K)^+ - h'(K)^-, K]$$

which gives a globally defined isospectral flow in $\mathcal{X}$. It is isospectral because $B = h'(K)^+ - h'(K)^-$ is a skew-symmetric operator. The *first Toda flow* is obtained with the Hamiltonian $H(L) = \mathrm{tr}(L^2/2)$. The random Toda flows do not leave $\mathcal{L}$ invariant unless $d = 1$. We have shown in [7] how one can linearize in the case $d = 1$ these infinite dimensional integrable dynamical systems. For finite $|X|$, the flows are then the classical periodic Toda lattices which can be linearized explicitly using algebraic geometry [8]. In more dimension, an isospectral Toda deformation in $\mathcal{L}$ is in general not possible. Already for finite $|X|$, there is a result of Mumford [9] which says that, generically, there exist no isospectral deformations of higher dimensional real Laplacians (N=1). In the complex case, it is however trivial to deform a self-adjoint Laplacian $L = \sum_i a_i\tau_i + (a_i\tau_i)^* + b \in \mathcal{L}$. Take any curve $g_t \in L^{\infty}(X, SU(N))$ satisfying $g(0) = 1$. Then $L(t) = g(t)Lg(t)^{-1}$ consists of self-adjoint isospectral Laplacians.

A SUFFICIENT CONDITION FOR ISOSPECTRAL DEFORMATIONS

Theorem 1 *If there exists $D \in \mathcal{D}$ and $E \in \mathbb{R}$ such that $L = \psi(D^2 + E)$, the orbit of the isospectral deformation*

$$\frac{d}{dt}L = [L^+ - L^-, L] =: [B, L]$$

stays in $\mathcal{L}$. If L is not a stationary point of this Toda flow, there exists a curve of isospectral Laplacians through L. The deformation can be written as

$$\dot{a}_i = a_i b(T_i) - b a_i ,$$
$$\dot{b} = \sum_{i=1}^{d} 2(a_i a_i^* - (a_i^* a_i)(T_i^{-1})) ,$$

The differential equation for $L = \psi(D^2 + E)$ is equivalent to

$$\dot{D} = [(D^2)^+ - (D^2)^-, D]$$

in $\mathcal{Y}$ and this is a decoupled system of one-dimensional random Volterra systems

$$\dot{d}_i = [d_i^2, d_i^*] = d_i \cdot (d_i d_i^*)(T_i) - (d_i^* d_i)(T_i^{-1}) \cdot d_i \,.$$

Proof. The differential equation $\frac{d}{dt}L = [B, L]$ has a solution in $\mathcal{X}$. We want to show that for an initial condition L having a factorization $L = \psi(D^2 + E)$, the solution stays in $\mathcal{L}$. It follows from Equation (1) and (2) that

$$\begin{aligned} [L^+ - L^-, L] &= [a\tau - (a\tau)^*, a\tau + (a\tau)^* + b] \\ &= [a\tau, b] - [(a\tau)^*, b] + [a\tau, (a\tau)^*] - [(a\tau)^*, a\tau] \end{aligned}$$

is in $\mathcal{L}$. This holds true as long as $L = \psi(D^2) + E$. To see that the factorization $L = \psi(D^2) + E$ is preserved by the flow we have to show that the Clifford structure

$$\begin{aligned} \{d_i\sigma_i, d_j\sigma_j\} &= \delta_{ij} 2a_i, \\ \{d_i\sigma_i, (d_j\sigma_j)^*\} &= \delta_{ij} b_i \,. \end{aligned}$$

stays invariant under the flow. The differential equation

$$\dot{D} = [(D^2)^+ - (D^2)^-, D]$$

in $\mathcal{Y}$ can be written as a decoupled set of Volterra systems

$$\dot{d}_i = [d_i^2, d_i^*] = d_i \cdot (d_i d_i^*)(T_i) - (d_i^* d_i)(T_i^{-1}) \cdot d_i \,.$$

We write from now on simply d instead of $d\sigma$ and d^* instead of $(d\sigma)^*$ in order to simplify the writings. With the differential equations

$$\dot{d}_i = [d_i^2, d_i^*] \,, \ \dot{d}_i^* = -[(d_i^*)^2, d_i] \,,$$

we get for $i \neq j$ using $\{d_i, d_j^*\} = 0$

$$\begin{aligned} \frac{d}{dt}\{d_i, d_j\} &= \dot{d}_i d_j + d_i \dot{d}_j + \dot{d}_j d_i + d_j \dot{d}_i \\ &= [d_i^2, d_i^*] d_j + d_i [d_j^2, d_j^*] + [d_j^2, d_j^*] d_i + d_j [d_i^2, d_i^*] \\ &= d_i^2 d_i^* d_j - d_i^* d_i^2 d_j + d_i d_j^2 d_j^* - d_i d_j^* d_j^2 + d_j^2 d_j^* d_i - d_j^* d_j^2 d_i + d_j d_i^2 d_i^* - d_j d_i^* d_i^2 \\ &= d_i^2 \{d_i^*, d_j\} + d_j^2 \{d_i, d_j^*\} - \{d_i^*, d_j\} d_i^2 - \{d_j^*, d_i\} d_j^2 = 0 \,. \end{aligned}$$

Similarly, using $\{d_i, d_j\} = \{d_i^*, d_j^*\} = 0$ for $i \neq j$, we get for $i \neq j$

$$\begin{aligned} \frac{d}{dt}\{d_i, d_j^*\} &= \dot{d}_i d_j^* + d_i \dot{d}_j^* + \dot{d}_j^* d_i + d_j^* \dot{d}_i \\ &= [d_i^2, d_i^*]\, d_j^* - d_i [(d_j^*)^2, d_j] - [(d_j^*)^2, d_j]\, d_i + d_j^* [d_i^2, d_i^*] \\ &= d_i^2 (d_i^* d_j^* + d_j^* d_i^*) - (d_j^*)^2 (d_i d_j + d_j d_i) - (d_i^* d_j^* + d_j^* d_i^*) d_i^2 + \\ &\quad (d_i d_j + d_j d_i)(d_j^*)^2 = 0 \,. \end{aligned}$$

□

Remarks.

- The condition in the proposition is always satisfied in one dimension, where every Laplacian can be deformed.

• The zero curvature conditions $[a_i\tau_i, a_j\tau_j] = [a_i\tau_i, (a_j\tau_j)^*] = 0$, $i \neq j$ for L do not imply in general that we can factor $L = D^2 + E$. We need that the zero curvature holds true also after deformation. This would need for example that the equation

$$\frac{d}{dt}[a_i\tau_i, (a_j\tau_j)^*] = 2[[a_i\tau_i, b], (a_j\tau_j)^*] = 0$$

is valid which implies that also a condition for b must be satisfied in order to have an isospectral deformation.

• Operators with unitary a_i and constant b are stationary points of the first Toda flow. Especially random Harper operators can not be deformed in this way.

• The necessary conditions

$$\begin{aligned} a_i a_j(T_i) &= a_j a_i(T_j) , \\ d_j a_i &= a_i(T_j) a_j(T_i), \quad i \neq j \end{aligned}$$

for isospectral deformations have only constant solutions if we require the a_i to be complex valued cocycles and if the maps T_i are ergodic. (To see this, we divide the first equation by the second one giving $a_j(T_i)^2 = a_j^2$ which implies that a_j is constant if T_i is ergodic.) We conclude that we need the entries of the Laplacian to be *matrix-valued* in order to obtain interesting isospectral deformations.

DISCRETE RANDOM PARTIAL DIFFERENCE EQUATIONS

Given a $\mathbb{Z}^d$ dynamical system (X, T, μ) and a *Lagrangian*

$$l(q_0, q_1, \ldots, q_d) = -\sum_{i=1}^{d} g_i \frac{(q_i - q_0)^2}{2} + \gamma \cdot V(q_0) ,$$

where $g_i \in \mathbb{R} \setminus \{0\}$ and $V \in C^2(\mathbb{T}^1, \mathbb{R})$ is a real-valued potential and γ is a real coupling constant. We define on the space $L^\infty(X, \mathbb{T}^1)$ the *functional*

$$\mathcal{S}(q) = \int_X l(q(x), q(T_1 x), q(T_2 x), \ldots, q(T_d x)) \, d\mu(x)$$

on $L^\infty(X, \mathbb{T}^1)$. A critical point q of this functional satisfies the *discrete random partial difference equation*

$$-\sum_{i=1}^{d} g_i(q(T_i) - 2q + q(T_i^{-1})) - \gamma \cdot V'(q) = \delta\mathcal{S}(q) = 0 .$$

The second variation at a critical point (the Hessian) is the random Laplacian

$$L = \sum_{i=1}^{d} a_i\tau_i + (a_i\tau_i)^* + b$$

with $a_i = -g_i$ and $b = \gamma \cdot V'' - \sum_{i=1}^{d} 2g_i$.

It is a nontrivial problem to find nontrivial critical points of the above functional. For large coupling constants γ, it is however quite easy using the anti-integrable limit idea of Aubry [10]. Denote by

$$\Sigma = \{\sigma \in \mathbb{T}^1 \mid V'(\sigma) = 0, \quad V''(\sigma) \text{ invertible}\}$$

the set of non-degenerate critical points of V.

Theorem 2 *Assume that Σ has at least 2 elements. Then there exists $\gamma_0 \in \mathbb{R}$ such that for coupling constants $|\gamma| > \gamma_0$, there exist nontrivial critical points of the functional $\mathcal{S}$.*

Proof. Write $\gamma = 1/\epsilon$ and take the equivalent functional

$$\mathcal{S}(q) = \int_X -\sum_{i=1}^{d} \epsilon g_i \frac{(q(T_i x) - q(x))^2}{2} + \gamma \cdot V(q(x))\, d\mu(x)\,.$$

In the limit $\epsilon = 0$ every function $q \in L^\infty(X, \Sigma)$ is a critical point. The Hessian at such a point is $L = V''(q)$ which is by definition of Σ invertible. Applying the implicit function theorem, there exists also a critical point q_ϵ for small $|\epsilon|$. □

We introduced the above functional $\mathcal{S}$ in [7] for the one-dimensional case, where the problem of finding critical points is equivalent to embed a factor of the dynamical system in a *monotone twist map* defined by the generating function l. In the case, when the dynamical system is the irrational rotation on $\mathbb{T}^1$, the functional $\mathcal{S}$ is called the Percival functional. In [11], we used the functional together with a theorem of Krieger and the anti-integrable limit of Aubry to show that every ergodic dynamical system with finite metric entropy can be embedded into a monotone twist map. For two-dimensional systems $d = 2$ with $a_1 = -a_2 = 1$ and $V(q) = \cos(q)$ a critical point satisfies a discrete version of the Sine-Gordon equation.

One can not only establish the existence of critical points but also prove that there exist uncountably many different critical points. This can be done with the help of a *generalized Morse index* at a critical point q which is defined to be the value of the integrated density of states $k(0) = \int_{-\infty}^{0} dk(E')$ of the Hessian $L = \delta^2(\mathcal{S})$ at the energy $E = 0$. This number is lying in the interval $[0, 1]$ and measures, how much of the spectrum of L is below 0. One could say that it measures the "dimension" of the infinite dimensional unstable manifold, which is passing through the critical point. If two critical points have a different index, then they must be essentially different in the sense that it is then excluded that one critical point is a translation of the other one.

Proposition 3 *If there exist two critical points σ_1, σ_2 of V with $V''(\sigma_1) > 0$ and $V''(\sigma_2) < 0$, then there exist uncountably many different critical points of $\mathcal{S}$ for $|\gamma|$ large enough. The Morse index at a critical point near $q \in L^\infty(X, \Sigma))$ is*

$$m(\{x \in X \mid V''(q(x)) < 0\})\,.$$

Proof. In the anti-integrable limit, the Hessian is an invertible diagonal operator and the density of states is a Dirac measure located on the set

$$\{V''(\sigma) \mid \exists Y(\sigma) \subset X,\ m(Y(\sigma)) > 0,\ \forall x \in Y(\sigma),\ \sigma = q(x) \in \Sigma\}\,.$$

Each point σ in this set has the mass $m(Y(\sigma))$. The integrated density of states at 0 is given by $m(\{x \in X \mid V''(q(x)) < 0\ \})$. Because the Hessian in the anti-integrable limit is invertible, the mass of the spectrum below 0 does not change under perturbations of the operator or (by the implicit function theorem) under perturbations of the critical point. This means that the Morse index is constant for critical points near the anti-integrable limit and it is given by the same formula. □

ZERO CURVATURE AND THE DENSITY OF STATES

We consider in this section Laplacians in $\mathcal{L}_{SU} = \{L \in \mathcal{L} \mid a_i \in L^\infty(X, SU(N)),\ b = 0\}$. Call

$$\mathcal{G} = \{G \in \mathcal{L} \mid G_0 \in L^\infty(X, SU(N)),\ G_n = 0,\ n \neq 0\}$$

the *group of $SU(N)$ gauge fields.* For $G \in \mathcal{G}$, the map

$$L \mapsto GLG^{-1}$$

on $\mathcal{L}_{SU}$ is called a *gauge transformation.* The $a_i\tau_i$ are transformed as

$$a_i\tau_i \mapsto Ga_iG(T_i)^{-1}\tau_i\ .$$

Gauge transformations leave the set of zero-curvature Laplacians invariant. Also the density of states is invariant under such transformations so that gauge transformations are isospectral deformations. We will just see that the density of states decides also, if the operator has zero curvature or not. In the special case of a Harper Hamiltonian $d = 2, N = 1$ with constant curvature, the density of states determines the curvature and so the normalized magnetic flux:

Proposition 4 *a) The operator $L \in \mathcal{L}_{SU}$ has zero curvature if and only if*

$$\mathrm{tr}(L^4) = \int_{\mathbb{R}} E^4\, dk(E) = NP_4(d)\ ,$$

where $P_4(d)$ is the number of closed paths of length 4 starting at $0 \in \mathbb{Z}^d$.
b) For a Harper operator $L = a_1\tau_1 + a_2\tau_2 + (a_1\tau_1)^* + (a_2\tau_2)^*$ with $N = 1$ satisfying

$$a_1a_2(T_1)a_1(T_2)^{-1}a_2^{-1} = e^{2\pi i\alpha}\ ,$$

the density of states determines the normalized magnetic flux $2\pi i\alpha$:

$$\mathrm{tr}(L^4) = \int_{\mathbb{R}} E^4\, dk(E) = 16 + 8\cos(2\pi i\alpha)\ .$$

c) Harper Hamiltonians with different $\cos(2\pi\alpha)$ are not isospectral.

Proof.
a) If we know the density of states, we can calculate

$$\mathrm{tr}(L^n) = \int E^n\, dk(E)\ .$$

The operator has zero curvature if and only if $a_ia_j(T_i)(a_i(T_j))^{-1}(a_j)^{-1} = 1$ and this is the case if and only if

$$\mathrm{Tr}(a_ia_j(T_i)(a_i(T_j))^{-1}(a_j)^{-1}) = N$$

for all i, j. Because the trace of an element a in $SU(N)$ is always $\leq N$ with equality in the case of zero curvature, this is equivalent to

$$\mathrm{tr}(L^4) = NP_4(d) \ ,$$

where $P_4(d)$ is the number of closed paths of length 4 in $\mathbb{Z}^d$ starting at one point.
b) L^4 contains 4 curvature terms like $a_1 a_2(T_1)(a_1(T_2)^{-1})^*(a_2^{-1})^*$ for each positively oriented plaquette at $x \in X$ and 4 curvature terms like $a_1 a_2(T_1 T_2^{-1})^* a_1(T_2^{-1})^* a_2(T_2^{-1})$ for each negatively oriented plaquette. There are additionally 16 constant summands 1 belonging to closed paths of length 4 which are not passing around a plaquette.
c) As a corollary of b), we obtain that Harper Laplacians with different $\cos(2\pi\alpha)$ are not isospectral because isospectral Laplacians have the same $\mathrm{tr}(L^4)$. □

The functional

$$S_{gauge}(L) = g \cdot (\mathrm{tr}(L^4) - NP_4(d))$$

on $\mathcal{L}_{SU}$ is the lattice action of pure lattice $SU(N)$ gauge field on a d dimensional infinite lattice. For finite X, the lattice is periodic and the functional is a finite sum. In general it is an averaged sum.

If we don't assume α to be constant for the Harper Hamiltonian, we can make the following remark about the relation between the curvature and the density of states.

Proposition 5 *Given $L \in \mathcal{L}_{SU}$. The curvature $F = a_1 a_2(T_1) a_1(T_2)^* a_2^*$ determines the density of states dk. On the other hand, the density of states dk does not determine the curvature F.*

Proof. If F is known, we can calculate $\mathrm{tr}(L^n)$ for each $n \in \mathbb{N}$ because $\mathrm{tr}(L^n)$ contains summands labeled by paths of length n and each path summand is the product of all the curvatures belonging to the plaquettes which are surrounded by this path. We can determine therefore also $\mathrm{tr}(\log(L - E))$ for $\mathrm{Im}(E) > 0$ and so the integrated density of states

$$k(E) = \mathrm{Im}(\mathrm{tr}(\log(L - E))) \ , \ E \in \mathbb{R}$$

which determines the density of states $dk = \frac{d}{dE} k(E) dE$. The curvature function $F \in L^\infty(X)$ and a translated function $F(T^n)$ belong both to the same density of states. It is thus in general not possible to determine dk from F. □

It can be seen in the same way that also for higher dimensional Laplacians $L = \sum_{i=1}^d a_i \tau_i + (a_i \tau_i)^*$ with $a_i \in L^\infty(X, SU(N))$, the curvature $F = \sum_{ij} F_{ij} \tau_i \tau_j$ determines the density of states.

QUESTIONS

We formulate some open points. We don't know if the existence of a Dirac operator also implies that higher Toda flows exist.

Is the sufficient condition in the Theorem 1 also necessary? In other words, can one factorize an operator if an isospectral deformation is possible?

We don't know if the property of factorization is a spectral property in the sense that the density of states decides whether there exists D with $\psi(D^2) + E = L$.

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