



0960-0779(95)00080-1

# Billiards in the $l^p$ Unit Balls of the Plane

M. JENG\* and O. KNILL

Division of Physics, Mathematics and Astronomy, Caltech, Pasadena, CA 91125, USA

(Accepted 16 July 1995)

**Abstract**—We study a one-parameter family of billiard maps  $T_p$  given by the convex tables  $|x|^p + |y|^p = 1$  for  $p \in [1, \infty]$ . We note that the topological entropy is positive if  $p \notin \{1, 2, \infty\}$ . We study the linear stability of some periodic orbits and observe that the stability of these orbits changes for  $p = 2$ . For  $p \notin \{1, 2, \infty\}$ , there exist elliptic periodic orbits suggesting that  $T_p$  is not ergodic for all  $p$ . We compute numerically the metric entropy of the maps  $T_p$  in the interval  $[1, 12]$  and the limiting behavior near  $p = 2$ .

## 1. INTRODUCTION

In this article, we study the one-parameter family of billiard maps  $T_p$  on the  $l^p$  unit ball in  $\mathbb{R}^2$ ,  $|x|^p + |y|^p = 1$ , for  $p \in [1, \infty]$ . In Fig. 1, we see examples of such tables.

For  $p = 1, 2$  or  $\infty$ , the map can be understood completely—for  $p = 1$ , or  $p = \infty$ , the table is a square, while for  $p = 2$ , the table is a circle. The family contains smooth

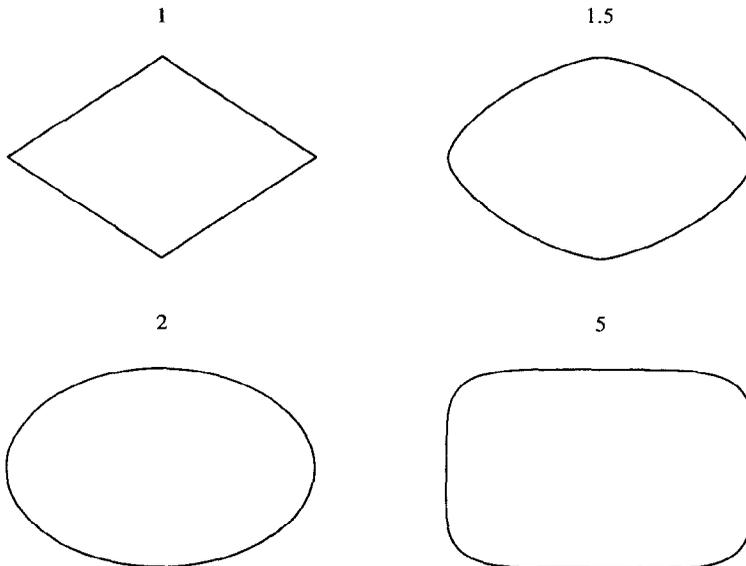


Fig. 1. The  $l^p$  unit ball  $|x|^p + |y|^p = 1$  in  $\mathbb{R}^2$  for four different parameters  $p$ .

\*Work supported by a SURF Undergraduate Research Fellowship.

algebraic curves for  $p = 2n$  with  $n \in \mathbf{N}$ . However, in general the curves are not differentiable. We prove that this family of billiard maps has positive topological entropies for the non-integrable cases  $p \neq 1, 2$ , and  $\infty$ , and we measure positive metric entropies, which indicate the coexistence of stable and random motions on a set of positive measure.

A few one-parameter families of billiard maps at convex tables have been studied already.

- *The Robnik billiard tables* [1]. These are given by the convex curves  $t \mapsto (\cos(t) + p \cos(2t), \sin(t) + p \sin(2t))$  for parameter values  $p \in [0, 1/4]$ . This one-parameter family of billiard maps contains the integrable case  $p = 0$ , where the table is a circle. It is known [2] that for  $p = 1/4$ , the map is not ergodic due to stable periodic orbits even though there are no invariant curves. Measurements show that the map has positive metric entropy for parameter values like  $\gamma = 0.2$ .
- *The Bunimovich stadium tables*. These are obtained by joining two unit half circles with two parallel straight lines of length  $p \in [0, \infty)$ . Again, the case  $p = 0$  gives the circle. It is known that this billiard has positive Lyapunov exponents almost everywhere for  $p > 0$  and Wojtkowsky [3] showed quantitatively that for sufficiently small  $p$ , the metric entropy is  $\geq C\sqrt{p}$ .
- The Benettin–Strelcyn Billiard [4, 5]. This is a one-parameter family of billiards where the curve consists of four arcs interpolating a circle.
- Other one-parameter families of billiard curves can be obtained by making the string construction on a strictly convex curve of length 1 [6]. For every string length  $p \in (1, \infty)$ , one obtains a billiard table. In the (limiting) case  $p = \infty$  the table is a circle.

In general, dynamical systems are neither integrable nor hyperbolic and one has to deal with maps exhibiting both stable and unstable behavior in the phase space. Coexistence of chaos and order seems to be the rule for smooth convex billiards: only the ellipses are known to be integrable and an old conjecture of Birkhoff states that this should be the only case (cf. [7, 8]). No uniformly hyperbolic convex billiard is known. However, the weaker condition of non-uniform hyperbolicity (positive Lyapunov exponent almost everywhere) can be achieved for convex billiards if one does not require the table to be smooth.

A routine investigation of a family of monotone twist maps can contain the following steps.

- Establish symmetries and smoothness of the map.
- Prove the existence or absence or destruction of invariant curves or estimate the size of regions without invariant curves [9].
- Investigate the (linear) stability of the periodic orbits in dependence of the parameter and find bifurcations. Numerically investigate the stability of some orbits.
- Determine the length spectra (= set of lengths of periodic orbits) and, related, the motion of the Dirichlet eigenvalues in dependence of the tables.
- Measure or estimate the metric entropy by numerical calculations of the Lyapunov exponents.
- Find horseshoes or homoclinic points in some cases.
- Investigate the limiting behavior near integrable maps.

We followed a part of this program in the case of our family given by the  $l^p$  unit balls. Except for the trivial integrable cases, we can prove that the topological entropy is positive, and measure positive metric entropies. Measurements of the metric entropy suggests that the Kolmogorov entropy  $p \mapsto h_m(T_p)$  behaves roughly like a piecewise convex function and that near  $p = 2$ , the entropy function  $p \mapsto h_m(T_p)/\log(|p - 2|)$  has a discontinuity in the first derivative.

## 2. SYMMETRY AND REGULARITY OF THE MAPS

Billiards is a dynamical system describing the motion of a particle inside a smooth closed curve  $\gamma$  moving inside the enclosed region in straight lines and bouncing off the wall according to the equal angle law of reflection (see [10] for a survey). Parametrizing the point of impact by a curve parameter  $s \in \mathbf{T} = \mathbf{R}/\mathbf{Z}$  and the angle of impact with  $\phi \in [0, \pi]$ , the Poincaré map  $(s, \phi) \mapsto (s', \phi')$  inherits all the information of the billiard flow. The map  $T: (s, \cos(\phi)) \rightarrow (s', \cos(\phi'))$  is a map on the annulus  $X = \mathbf{T} \times [-1, 1]$  preserving the Lebesgue area. The compact space  $X$  (also called the phase space) has the invariant boundary  $\mathbf{T} \times \{-1\}$  and  $\mathbf{T} \times \{1\}$ . The billiard map  $T$  has been studied first by Birkhoff and is a celebrated example of a monotone twist map, a class of conservative dynamical systems introduced by Poincaré. Every billiard map has the involutive symmetry  $S: (s, y) \mapsto (s, -y)$  making  $T$  reversible:  $T \circ S = S \circ T^{-1}$ .

We are studying the family of billiard maps  $T_p$  defined by the one-parameter family of curves  $\gamma_p: |x|^p + |y|^p = 1$ . Since the tables are symmetric by reflections at the  $x$  and  $y$  axes, this  $\mathbf{Z}_4$  symmetry is also present in the phase space  $X$ . Together with the involutive symmetry  $S$ , the whole phase space  $X$  has a dihedral symmetry. We can factor this symmetry out by identifying points  $(s_1, y_1), (s_2, y_2)$  if  $s_1 - s_2 = 0 \pmod{1/4}$  and  $y_1 = \pm y_2$ . In Fig. 2, we see numerical computations of orbits for some values of  $p$ .

We investigate first the continuity and smoothness of the maps  $T_p$  as well as of the map  $p \mapsto T_p$  from  $\mathbf{R}$  to a topological space of maps. Note that the billiard map in a convex domain is, in general, not continuous as the example of a general triangle shows. The billiard map  $T_p$  has at least the degree of smoothness of the normal map  $s \mapsto n(s)$  at the curve  $\gamma_p$ . By the  $\mathbf{Z}_4$  symmetry, we can restrict to the quadrant  $0 \leq x, y \leq 1$ , where the curve is given by  $x^p + y^p = 1$  and where the normal is  $n(s) = (px(s)^{p-1}, py(s)^{p-1})$ . The map  $T_p$  is  $C^\infty$  if  $(x(s), y(s))$  is away from the coordinate axes  $s \neq 0 \pmod{1/4}$ . Denote with  $C(X, X)$  the space of homeomorphisms of  $X$  with the topology given by the metric  $d_0(S_1, S_2) = \max_{x \in X} d(S_1(x), S_2(x))$ . Define for any compact set  $K \subset \text{int}(X)$  and  $S_i \in C^1(X, X)$

$$d_K(S_1, S_2) = \max_{x \in K} d(S'_1(x), S'_2(x))$$

and the topology on  $C^1(X, X)$  by  $S_n \rightarrow S$  if and only if  $d_K(S_n, S) \rightarrow 0$  for all  $j = 1, 2, \dots$  and all compact  $K \subset \text{int}(X)$ .

*Lemma 1.* For each  $p \in [1, \infty]$ , the twist map  $T_p$  is a homeomorphism of  $X$ .

For  $p = 2n, n \geq 1$ , the map is real analytic.

For  $p > 2$ , the map is  $C^1$ .

For  $1 < p < 2$ , the map is not differentiable.

The map  $p \in (1, \infty) \mapsto T_p \in C(X, X)$  is continuous. The map  $p \in (2, \infty) \mapsto T_p \in C^1(X, X)$  is continuous.

*Proof.* A billiard map in a convex curve  $\gamma$  is, in general, not a homeomorphism. In our case, the boundary has four symmetric points, where the curvature may be infinite. By imposing at these points the symmetric normal direction, we find for  $p \in (1, \infty)$  a continuous continuation of the normal direction field on the whole curve  $|x|^p + |y|^p = 1$ . In the limiting cases  $p = 1$  and  $p = \infty$ , the table is a square, and the billiard map can also be extended to the singular set in the phase space by defining in the four vertices of the square the normal direction pointing to the center of the table. For  $p = 2n$ , the curve is algebraic and so real analytic which implies that the map  $T_p$  is real analytic. Given a curve  $(t, f(t))$ , the curvature is given by

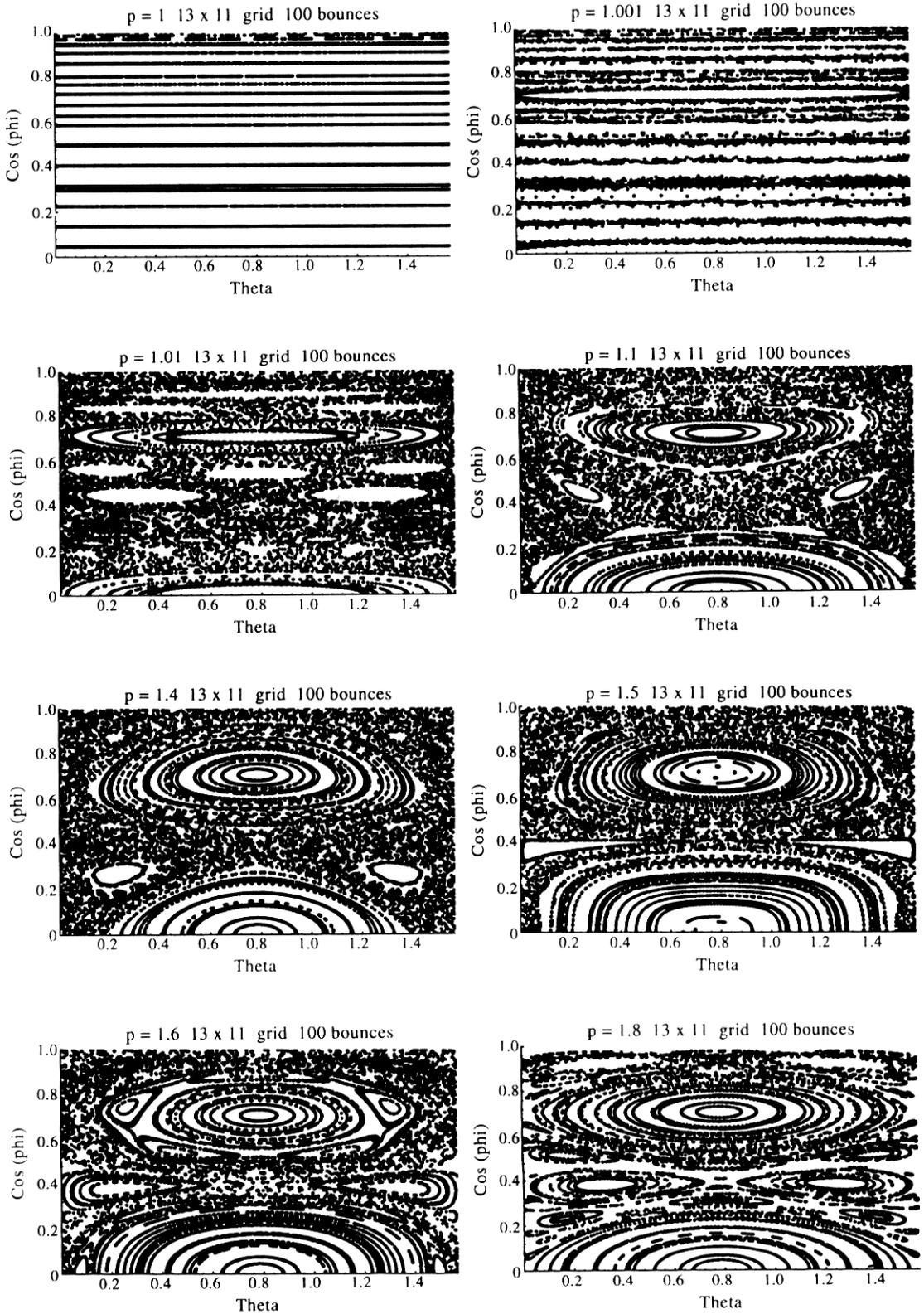


Fig. 2. Caption opposite.

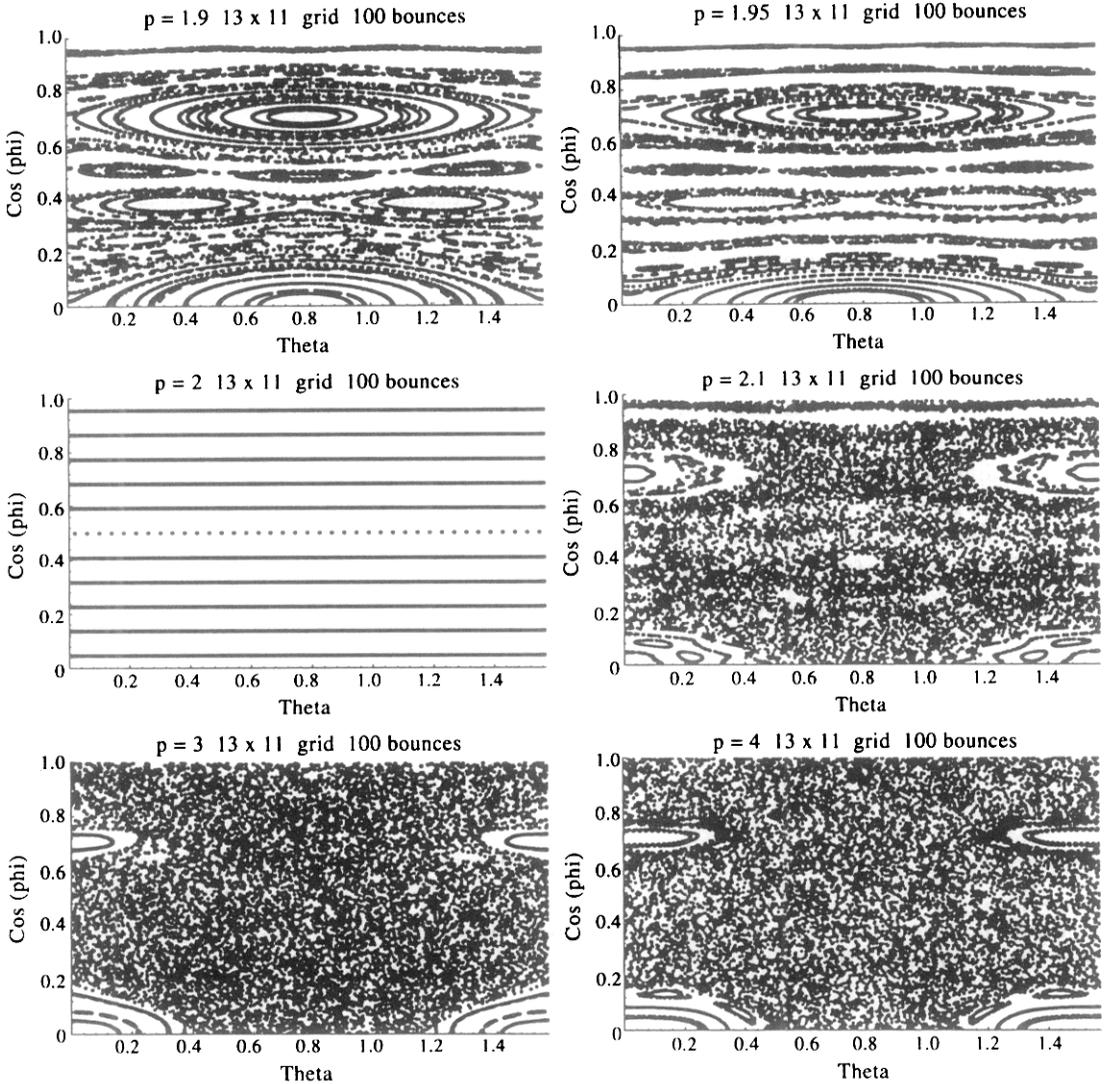


Fig. 2. Some orbits for different parameters  $p = 1.0$  up to  $p = 4.0$ . The pictures show the reduced phase space of the billiard map, there the dihedral symmetry has been factored out. The horizontal  $\phi$ -axis gives the place of impact from  $0$  to  $\pi/2$  and the vertical  $\cos(\phi)$  axis parametrizes the angle of impact. For each of the 14 parameter values we see  $13 \times 11$  orbits each with 100 iterations.

$$\kappa(t) = f''(t)(1 + f'(t)^2)^{-3/2}.$$

This means for  $f(t) = (1 - t^p)^{1/p}$

$$l(t) = \frac{(1 - p)t^{p-2}(1 - t^p)^{1/p-2}}{(1 + t^{2p-2}(1 - t^p)^{2/p-2})^{3/2}}$$

This is continuous as a function of  $t$  for  $p > 2$ , leading to a continuously differentiable map for  $p > 2$ . The curvature is unbounded in places for  $p < 2$ , so that for  $p < 2$ , the map is no longer continuously differentiable. However, for  $p \in (2, \infty)$ , the curvature depends continuously on  $p$  which shows that  $p \in (2, \infty) \mapsto T_p \in C^1(X, X)$  is continuous.

*Remark.* The map  $p \mapsto T_p \in C^1(X, X)$  is not continuous at  $p = 2$ , because for  $p = 2$ , the curvature at  $s = 0$  is 1 and for  $p > 2$ , the curvature at  $s = 0$  is vanishing. One can therefore

not apply KAM theory to conclude that for  $p > 2$ , near enough 2, the map  $T_p$  has invariant curves. Indeed, as we will see below, there are no invariant curves for  $p > 2$ .

### 3. NUMERICAL COMPUTATION OF ORBITS

In Fig. 2 we can see some orbits for different parameters  $p$ . The pictures show the reduced phase space of a billiard map, where the dihedral symmetry has been factored out. The horizontal  $\phi$ -axis gives the place of impact from 0 to  $\pi/2$  and the vertical  $\cos(\phi)$  axis parametrizes the angle of impact. For each of the 14 parameter values we see  $13 \times 11$  orbits each with 100 iterations.

### 4. PERIODIC ORBITS AND THEIR LINEAR STABILITY

According to Poincaré–Birkhoff's theorem, there exists for each pair  $(p, q)$  with  $p > 1$  and  $p, q \in \mathbf{Z}$  and  $0 < q/p < 1$  a periodic orbit of period  $p$  which winds around the table  $q$  times. These periodic orbits are called Birkhoff periodic orbits. In general, there exist many more orbits of period  $p$ . It is an open question whether the set of periodic orbits can form a set of positive Lebesgue measures in the phase space. There exist some results for small periods [11, 12]. We cannot decide this question for our case. Even if the set of periodic orbits forms a set of zero Lebesgue measures, they still generally have a significant influence on the global phase space picture. A stable periodic orbit keeps a whole neighborhood nearby stable, which prevents ergodicity. A first step in understanding the stability of the orbits is finding out whether they are linear stable. The stability of periodic orbits of small order can be decided by hand. We will treat orbits of periods 2 and 4.

*Proposition 2.* For any  $p \notin \{1, 2, \infty\}$ , there are four periodic orbits of period 2. They correspond to two fixed points  $x^{(1)}, x^{(2)}$  at  $\theta = 0$  and  $\theta = \pi/2$  in the reduced phase space. For  $p = 1, 2, \infty$ , there are whole curves of periodic orbits. For  $p \in (1, 2)$ , the orbit  $x^{(1)}$  is hyperbolic and the orbit  $x^{(2)}$  is elliptic. For  $p \in (2, \infty)$ ,  $x^{(1)}$  is elliptic and  $x^{(2)}$  is hyperbolic.

*Proof.* Look at an orbit of period 2 hitting the table at points  $P_1, P_2 = T(P_1)$  with curvatures  $\kappa_1$  and  $\kappa_2$  and let  $l$  be the distance between the two points. A computation of the trace  $2 - 4l(\kappa_1 + \kappa_2 - l\kappa_1\kappa_2)$  of the Jacobean of  $T^2$  (see Proposition 4 of [13] or Proposition 3 of [3]) shows that the orbit is elliptic if and only if  $l\kappa_1 < \kappa_1(\kappa_1 + \kappa_2)$  or  $(l\kappa_1 - 1)(l\kappa_2 - 1) > 0$ .

Consider a 2-orbit of length  $l$ , that meets the billiard table at points with radii of curvature  $R_1$  and  $R_2$ , with  $R_1 < R_2$ . If one of the inequalities  $l > R_1 + R_2$  or  $R_1 < l < R_2$  is satisfied, then the orbit is hyperbolic, while if either  $0 < l < R_1$  or  $R_2 < l < R_1 + R_2$  is satisfied, the orbit is elliptic. For our 2-orbits,  $R \equiv R_1 = R_2$ , so we get elliptic orbits when  $l < 2R$  and hyperbolic orbits when  $l > 2R$ .

We look first at the horizontal 2-orbit  $x^{(1)}$ . The vertical 2-orbit is obviously identical to the horizontal one. The horizontal orbit has length 2. The curvature at the endpoints is  $\infty, 1$ , and 0 for  $p < 2, p = 2$ , and  $p > 2$ , respectively. This gives radii of curvature of 0, 1, and  $\infty$  for  $p < 2, p = 2$ , and  $p > 2$ , respectively. For  $p < 2, l = 2$ , and  $R \equiv R_1 = R_2 = 0$ , so  $l > 2R$  and the orbit is hyperbolic. And for  $p > 2, l = 2$ , and  $R = \infty$ , so  $0 < l < R$ , and the orbit is elliptic. So the horizontal orbit is hyperbolic if  $p < 2$  and elliptic if  $p > 2$ . The diagonal orbit has  $l = 2^{1/2} \times 2 \times 2^{-1/p} = 2^{3p-2/2p}$ . The curvature at the endpoints is  $\kappa = (p-1)2^{2-p/2p}$ . The radius of curvature is then  $R = 2^{p-2/2p}/p-1$ . The orbit is elliptic if  $l < 2R$ , or

$$2^{3p-2/2p} < \frac{2}{p-1} 2^{p-2/2p}.$$

Multiplying both sides by  $2^{2-p/2p}$ , we get  $2 < 2/p - 1$ , which means  $p < 2$ . So the diagonal 2-orbit is elliptic if  $p < 2$ . Similarly, the diagonal 2-orbit is hyperbolic if  $p > 2$ .

*Proposition 3.* For any  $p \notin \{1, 2, \infty\}$ , there are two simple periodic orbits of period 4. They correspond to two fixed points  $x^{(3)}, x^{(4)}$  in the reduced phase space, and hit the table at  $\theta = 0$  and  $\theta = \pi/4$ , respectively.  $x^{(3)}$  is hyperbolic for  $p < 2$  and parabolic for  $p \geq 2$ .  $x^{(4)}$  is elliptic for  $p < 2$  and hyperbolic for  $p > 2$ .

*Proof.* We calculate the Hessian for  $x^{(3)}$ . Let  $\theta_i$  be the central angle for the  $i$ th point of the orbit,  $\phi_i$  be the angle between the path of the particle leaving the  $i$ th point and the directed tangent to the  $p$ -curve at that point.  $l_i$  gives the distance of the straight line connecting the points given by  $\theta_i$  and  $\theta_{i+1}$ , and  $\kappa_i$  is the curvature at the  $i$ th point. The Hessian is then

$$H = \begin{pmatrix} b_1 & a_1 & 0 & a_4 \\ a_1 & b_2 & a_2 & 0 \\ 0 & a_2 & b_3 & a_3 \\ a_4 & 0 & a_3 & b_4 \end{pmatrix},$$

with

$$a_i = \frac{\sin(\phi_i) \sin(\phi_{i+1})}{l_i}, \quad b_i = \sin^2(\phi_i) \left( \frac{1}{l_i} + \frac{1}{l_{i-1}} \right) - 2 \sin(\phi_i) \kappa_i.$$

In our case we have by symmetry  $a_i = a$  and all of the  $b_i = b$ . The determinant of the Hessian then simplifies to  $\det(H) = b^2(b^2 - 4a^2)$ . Define  $D \equiv |2 + \det(-H)/\prod_{i=1}^4 a_i|$ . The orbit is hyperbolic if  $D > 2$  and elliptic if  $D < 2$  (see [13]). For the symmetric 4-orbits we have

$$D = \left| 2 + \frac{b^2(b^2 - 4a^2)}{a^4} \right| = \left| 2 - 4 \left( \frac{b}{a} \right)^2 + \left( \frac{b}{a} \right)^4 \right|.$$

First look at the  $x^{(3)}$ -orbit. We get with  $\phi_i = \pi/4$ , and  $l_i = \sqrt{2}$ , so that

$$a = \frac{\sin^2\left(\frac{\pi}{4}\right)}{\sqrt{2}} = 2^{-3/2}, \quad b = \frac{1}{\sqrt{2}} - \sqrt{2}\kappa_i.$$

We have already calculated  $\kappa_i$  at these points, so we get

$$a = \frac{1}{2i\sqrt{2}} \quad b = \begin{cases} -\infty & \text{for } p < 2 \\ -\frac{1}{\sqrt{2}} & \text{for } p = 2 \\ \frac{1}{\sqrt{2}} & \text{for } p > 2 \end{cases}$$

$$D = \left| 2 - 4 \left( \frac{b}{a} \right)^2 + \left( \frac{b}{a} \right)^4 \right| = \begin{cases} \infty & \text{for } p < 2 \\ 2 & \text{for } p \geq 2 \end{cases}.$$

So the  $x^{(3)}$ -orbit is hyperbolic for  $p < 2$  and parabolic for  $p \geq 2$ . (Numerical experimentation suggests that the  $x^{(3)}$ -orbit is stable for  $p > 2$ ).

For the orbit  $x^{(4)}$ , all four points have  $l_i = 2 \times 2^{-1/p} = 2^{1-1/p}$ ,  $\kappa_i = (p - 1)2^{1/p-1/2}$ ,  $\phi_i = \pi/4$ , and  $\sin(\phi_i) = 1/\sqrt{2}$ . We then calculate

$$a = 2^{1/p-2}, b = -\left(p - \frac{3}{2}\right)2^{1/p}$$

and

$$D = |2 + 2^4(2p - 3)^2[(2p - 3)^2 - 1]|.$$

The  $x^{(4)}$ -orbit is elliptic if  $(2p - 3)^2 - 1 < 0$  and  $1 > 4(2p - 3)^2(1 - (2p - 3)^2)$ . So the orbit is elliptic if  $1 < p < 2$  and hyperbolic for  $p > 2$ . ■

**5. INVARIANT CURVES AND TOPOLOGICAL ENTROPY**

We have the following qualitative result about the topological entropy of the maps  $T_p$ .

*Proposition 4.* For  $p \in \{1, 2, \infty\}$ , the topological entropy  $h_{\text{top}}(T_p)$  is vanishing. For  $p \notin \{1, 2, \infty\}$ , the topological entropy of  $T_p$  is positive.

*Proof.* For  $p = 1$  or  $p = \infty$ , the curve  $\gamma_p$  is a square and  $T_p$  is integrable with integral  $I(s, y) = y + y'$ , where  $T_p(s, y) = (s', y')$ . The topological entropy is then zero. For  $p = 2$ , the curve  $\gamma_p$  is a circle and is integrable having the integral  $I(s, y) = y$ . On each invariant curve  $I = \text{const}$ , the map  $T_2$  is a rotation. The topological entropy is also zero. For  $p > 2$ , there exist points on the curve where the curvature is zero. A theorem of Mather [14] assures that there exists no homotopically nontrivial invariant curve. Angenent’s result [15] implies then that the topological entropy is nonvanishing. For  $p < 2$ , there exist points on the table where the curvature is unbounded. A theorem of Hubacher [16] assures that there exist no invariant curves near the boundary of  $X$ . Again, Angenent’s result leads to positive topological entropy. ■

**6. MEASUREMENTS OF THE KOLMOGOROV ENTROPY**

Since every billiard map  $T$  is a measure preserving map on a finite probability space, the Kolmogorov entropy  $h_m(T)$  (= metric entropy) is defined and finite. Pesin theory developed in [17] for  $C^2$ -diffeomorphisms and generalized by Katok and Strelcyn [18] to systems with singularities, states that positive metric entropy is equivalent with the fact that an iterate of the map  $T$  is a measure theoretically conjugated to a Bernoulli shift on a set of positive measure. In the case of a two-dimensional manifold  $X$  like in the billiard case, the Ruelle–Pesin formula

$$h_m(T) = \lim_{n \rightarrow \infty} \frac{1}{n} \int_x \log \|dT^n(x)\| dm(x)$$

reduces the calculation of the metric entropy  $h_m(T)$  to the determination of the averaged Lyapunov exponent. This formula is also true for maps with singularities satisfying the Katok–Strelcyn conditions in [18]. The billiard maps  $T_p$  studied here satisfy this condition.

Let us mention here also that the entropy  $h_m(T)$  of a monotone twist map is related to a determinant of an associated discrete Schrödinger operator  $L$  which is the Hessian of a natural variational problem associated to the dynamics. By the Thouless formula, one has  $h_m(T) = \log \det(|L|)$  (see [19]).

Positive Lyapunov exponents are, by Wojtkowsky’s theorem, equivalent to the existence

of an invariant cone bundle in the projective tangent bundle. For billiards, a formula of Wojtkowsky [3] gives a geometrical sufficient condition for the existence of an invariant cone bundle or using quadratic forms [20]. For a refinement of these methods see, for example, [21]. Unfortunately, no other mathematical tool is yet available to prove mathematically positive Lyapunov exponents for billiards. We computed them numerically.

In the case of the billiard map, there are coordinates (see [3]) in which the Jacobian of the billiard map can be put into the form

$$dT(x) = \begin{pmatrix} 1 & 0 \\ -\frac{1}{g(Tx)} & 1 \end{pmatrix} \begin{pmatrix} 1 & l(x) \\ 0 & 1 \end{pmatrix},$$

where  $g(x) \equiv \sin(\phi(x))/2\kappa(x)$ . Here  $\kappa$  is the curvature at an impact point,  $\phi$  is the angle of impact and  $l$  is the length of the trajectory between two successive impact points. In our numerical calculations, we took this as the Jacobian.

The first graph in Fig. 3 shows the entropy of the  $l^p$  billiard for  $1 \leq p \leq 2$ . The Kolmogorov entropy was measured for 200 values of  $p$  between 1 and 2. For each  $p$  the Kolmogorov entropy was found by calculating the Lyapunov exponent at each point of a  $17 \times 17$  grid of initial conditions. At each of these 289 points the Lyapunov exponent was estimated as the value of  $1/n \log \|dT^n(x)\|$  for  $n = 25,000$  and the average over the 289 initial conditions was plotted.

The graph in Fig. 4 shows the measured entropy of the  $l^p$  billiard for  $1 \leq p \leq 12$  (larger values of  $p$  introduce unreasonable numerical errors). We computed the Kolmogorov entropy for 200 values of  $p \in (1, 2)$  and 1000 values of  $p \in (2, 12)$ . We again measured each Kolmogorov entropy by determining the average Lyapunov exponent over a  $17 \times 17$  grid of initial conditions. The length of each orbit was  $n = 3461$ .

As expected, the Kolmogorov entropy tends towards 0 for  $p \rightarrow 1$  and for  $p \rightarrow 2$ . For  $1 \leq p \leq 2$ , we measured a maximum entropy of 0.28 at  $p = 1.23$ . For  $2 \leq p \leq 12$ , the entropy peaked around  $p = 4$  to  $p = 6$  with an entropy around 0.65.

Comparison between measurements of the entropy for orbits of length 3461 and orbits of

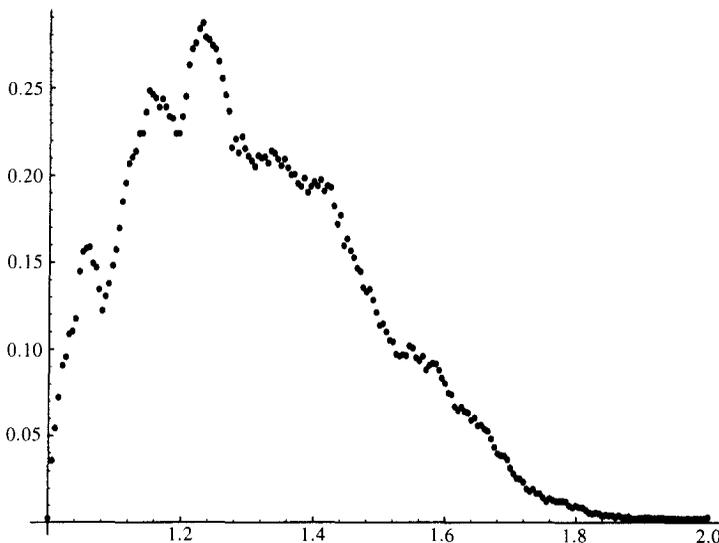


Fig. 3. The entropy of the  $l^p$  billiard for  $1 \leq p \leq 2$ . The Kolmogorov entropy was measured by determining the averaged Lyapunov exponent for 200  $p$ -values between 1 and 2.

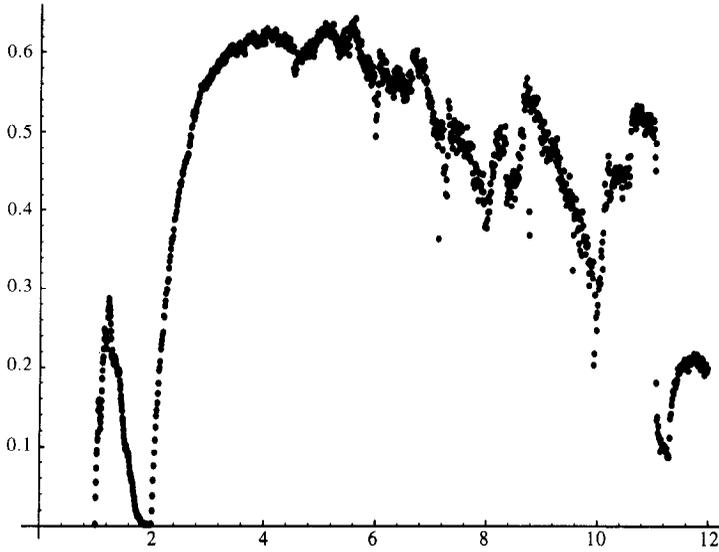


Fig. 4. The entropy of the  $l^p$  billiard for  $1 \leq p \leq 12$ .

length 25,000 gave differences smaller than 0.05. The graphs indicate that the entropy is piecewise monotone but that there are bumps.

We look now at the behavior of  $h_m(T_p)$  near  $p = 2$ . The measurements show that

$$h_m(T_{2+\epsilon}) \cong \alpha^{\text{sign}(\epsilon)} \epsilon \log(1/|\epsilon|)$$

with  $\alpha^- = 0$  and  $\alpha^+ \sim 0.5$ . We observe a phase transition at  $p = 2$ . The entropy grows with different growth rates for  $p > 2$  and  $p < 2$ .

We show the numerical values of  $h_m(p)/\log|p - 2|$  in Fig. 5. We computed the entropy for 400 different values of  $p$  between 1.8 and 2.2 using again a  $17 \times 17$  grid of initial points with 5000 each iterations so that for this run, 578 million billiard iterations had to be computed and for each orbit, the Jacobian  $dT^{5000}$  evaluated.

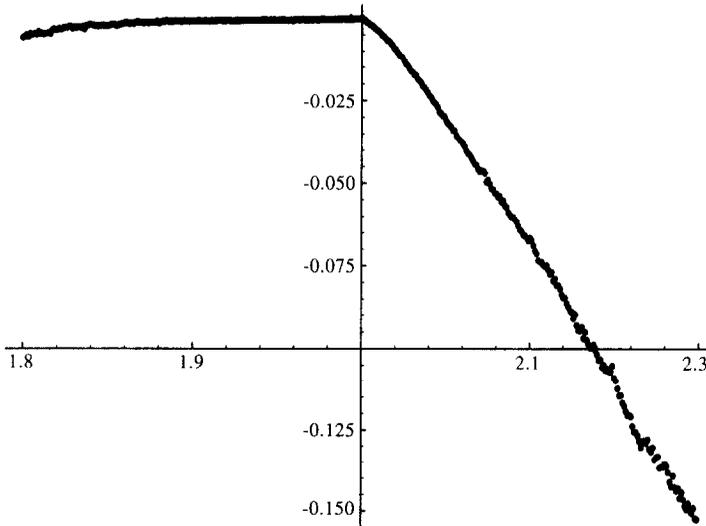


Fig. 5. Numerical values of  $h_m(p)/\log(|p - 2|)$ . We computed the entropy  $h_m(p)$  for 400 different values of  $p$  between 1.8 and 2.2.

## 7. DISCUSSION AND OPEN QUESTIONS

We could not yet give quantitative results about the topological entropy. This would be possible if we could show the existence of a horseshoe.

The measurements of the metric entropy indicate that  $p \mapsto h_m(T_p)$  is piecewise convex with a discrete set of sharp minima. We do not know the reason for this phenomenon but it reminds us of the breathing chaos measured in the case of a billiard in a gravitational field [22]. For our  $l^p$  billiards, we measured a local maximum around  $p = 4$ , which corresponds to the real analytic billiard table given by the algebraic curve  $x^4 + y^4 = 1$ . The behavior around  $p = 2$  is interesting.

We considered only linear stability and did not decide about the stability of the elliptic periodic orbits. Such computations are involved and have been done (computer assisted) in the case of the Robnik billiard [2]. Our experiments indicate that the linearly stable 2-periodic orbits are really stable and that the billiards  $T_p$  should, therefore, not be ergodic for all parameters  $p$ .

An interesting question is whether there exists a parameter value  $p \in (0, 1)$  with some homotopically nontrivial invariant curve. KAM theory does not give them and Hubacher's result excludes them only near the boundary of the phase space. The numerical experiments indicate however, that near  $p = 1$ , there might be some. One must say, however, that numerical computations near  $p = 1$  (and  $p = \infty$ ) become doubtful due to the fact that the curvature becomes huge in some places.

We did not determine the length spectra of the tables nor the Dirichlet eigenvalues. Since the  $l^p$  unit ball is a natural convex region in the plane, it would be interesting to know how Dirichlet data behave in dependence of  $p$ .

## REFERENCES

1. M. Robnik, Classical dynamics of a family of billiards with analytic boundaries, *J. Phys. A: Math. Gen.* **16**, 3971–3986 (1983).
2. A. Hayli, T. Dumont, J. M. Ollangier and J. M. Strelcyn, Quelques résultats nouveaux sur les billard de Robnik, *J. Phys. A: Math. Gen.* 2337–3249 (1987).
3. M. Wojtkowski, Principles for the design of billiards with nonvanishing Lyapunov exponents, *Commun. Math. Phys.* **105**, 391–414 (1986).
4. G. Benettin and J. Strelcyn, Numerical experiments on the free motion of a point mass moving in a plane convex region: Stochastic transition and entropy, *Phys. Rev. A* **17**, (1978).
5. M. Henon and J. Wisdom, The Benettin–Strelcyn oval billiard revisited, *Physica D* **8**, 157–169 (1983).
6. E. Gutkin and O. Knill, Billiards that share a common triangular caustics, Preprint (1994).
7. G. D. Birkhoff, On the periodic motions of dynamical systems, *Acta Math.* **50**, 359–379 (1927). See Collected Mathematical Papers, AMS, Providence, RI (1950).
8. M. Bialy, Convex billiards and a theorem of E. Hopf, *Math. Z.* **214**, 147–154 (1993).
9. E. Gutkin and A. Katok, Caustics for inner and outer billiards, submitted.
10. S. Tabatschnikov, Billiards, Preprint, Survey paper (1994).
11. L. Stojanov, An estimate from above of the number of periodic orbits for semi-dispersed billiards, *Commun. Math. Phys.* **124**, 217–227 (1989).
12. M. Rychlik, Periodic points of the billiard ball map in a convex domain, *J. Diff. Geom.* **30**, 191–205 (1989).
13. V. V. Kozlov and D. V. Treshchev, *Billiards*. Translations of Mathematical Monographs, 89, AMS (1991).
14. J. Mather, Glancing billiards. *Ergod. Th. Dynam. Systems* **2**, 397–403 (1982).
15. S. Angenent, A remark on the topological entropy and invariant circles of an area preserving twist map, in *Twist Mappings and their Applications*, edited by R. McGehee and K. Meyer, IMA Volumes in Mathematics, Vol. 44. (1992).
16. A. Hubacher, Instability of the boundary in the billiard ball problem, *Commun. Math. Phys.* **108**, 483–488 (1987).
17. Ya B. Pesin, Characteristic Lyapunov exponents and smooth ergodic theory, *Russian Math. Surveys* **32**, 55–114 (1977).
18. A. Katok and J. M. Strelcyn, *Invariant Manifolds, Entropy and Billiards: Smooth Maps with Singularities*, Lecture Notes in Mathematics, p. 1222. Springer, Berlin (1986).
19. O. Knill, Isospectral deformations of random Jacobi operators, *Commun. Math. Phys.* **151**, 403–426 (1993).

20. R. Markarian, Billiards with Pesin region of measure one, *Commun. Math. Phys.* **118**, 87–97 (1993).
21. V. J. Donnay, Using integrability to produce chaos: billiards with positive entropy, *Commun. Math. Phys.* **141**, 225–257 (1991).
22. P. Richter, H.-J. Scholz and A. Wittek, A breathing chaos, *Nonlinearity* **3**, 45–67 (1990).
23. V. F. Lazutkin, The existence of caustics for a billiard problem in a convex domain, *Math. Izvestija* **7**(1), 185–214, (1973).