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Positive Lyapunov exponents for a dense set of bounded measurable $SL(2, \mathbb{R})$-cocycles

OLIVER KNILL
Mathematikdepartement, ETH Zentrum, CH-8092 Zürich, Switzerland

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Abstract. Let T be an aperiodic automorphism of a standard probability space (X, m). We prove, that the set

$$\left\{A \in L^\infty(X, SL(2,\mathbb{R})) \,\middle|\, \lim_{n\to\infty} n^{-1} \log \|A(T^{n-1}x) \cdots A(Tx)A(x)\| > 0; \text{a.e.}\right\}$$

is dense in $L^\infty(X, SL(2,\mathbb{R}))$.

1. *Introduction*

Lyapunov exponents are useful in different domains of mathematics. In smooth ergodic theory one is interested in the ergodic behavior of smooth maps. The Lyapunov exponents can be used to determine, if a power of a given smooth map is equivalent to a Bernoulli automorphism on a set of positive measure. Moreover, with Pesin's formula the metric entropy can be expressed as a function of the Lyapunov exponents [**Pes 77, Kat 86**]. In the theory of discrete one-dimensional Schrödinger operators, Lyapunov exponents can decide, if the spectrum of such operators is not absolutely continuous [**Cyc 87**].

Thus, there is a strong enough motivation to calculate these numbers. It seems to be a matter of fact that calculating or even estimating the Lyapunov exponents is a difficult problem. Attempts in this direction have been made by several people: Wojtkowsky estimated the Lyapunov exponents under the assumption that an invariant cone bundle in the phase space exists [**Woj 86**]. The subharmonicity property of the Lyapunov exponents was used by Herman [**Her 83**] in order to estimate them for certain special mappings. In the case of random matrices, a criterion of Furstenberg [**Fur 63**], later generalized by Ledrappier [**Led 84**] applies. Chulaevsky has proved by analysing the skew product action, that the Lyapunov exponents are positive for certain cocycles arising in the theory of Schrödinger operators [**Chu 89**]. Young investigated certain random perturbations of cocycles [**You 86**]. She proved, that with this random noise the Lyapunov exponents depend continuously on the cocycle. Furthermore, they converge to the Lyapunov exponents of the unperturbed cocycle as the noise is reducing to zero. While analysing holomorphic parametrized cocycles Herman [**Her 83**] found a surprising result, which will be crucial for our work. (See § 2.4). Ruelle [**Rue 79a**] investigated the case in which a continuous cone bundle in the phase space is mapped into its interior. He found that in this case the Lyapunov exponents depend real analytically on the cocycle. Numerical experiments suggest that in the case of $SL(2,\mathbb{R})$-cocycles, positive Lyapunov exponents are quite frequent. Such cocycles arise in the case of smooth area-preserving mappings on a two-dimensional manifold or if one investigates one-dimensional discrete ergodic Schrödinger operators. Even for very special systems like the standard mapping of Chirikov on the two-dimensional torus, no estimates for the numerically measured Lyapunov exponents [**Par 86**] are available. There is still a wide gap between the numerical measurements and the effectively proved properties.

We want to investigate the Banach algebra $\mathscr{X}$ of all essentially bounded $M(2,\mathbb{R})$-cocycles over a given dynamical system. We are interested in the subset $\mathscr{P}$ of cocyles, where the upper and lower Lyapunov exponents are different almost everywhere. We show, that $\mathscr{P}$ is dense in $\mathscr{X}$ in the L^∞ topology, if the underlying dynamical system is aperiodic. In $\mathscr{X}$ lies the Banach manifold $\mathscr{A}$ of $SL(2,\mathbb{R})$-cocycles which forms a multiplicative group. Also here, if the dynamical system is aperiodic, $\mathscr{A}\cap\mathscr{P}$ is dense in $\mathscr{A}$. We find further, that an arbitrary small perturbation located on sets with arbitrary small measure can bring us in $\mathscr{P}$. This can provide some explanation for the fact, that one often obtains positive Lyapunov exponents when making numerical experiments.

We will further make a statement about circle-valued cocycles, which are here modeled as $SO(2,\mathbb{R})$-cocycles. The density result for positive Lyapunov exponents implies, that the subgroup of coboundaries is dense in the Abelian group of $SO(2,\mathbb{R})$-cocycles.

In § 2, we introduce the concepts and cite some known results, which are used in § 3 to prove our statements. In § 4, the results and some open problems are shortly discussed.

2. *Preparations*

2.1. *Matrix cocycles over a dynamical system.* A *dynamical system* (X, T, m) is a set X with a probability measure m and a measurable invertible map T on X which preserves the measure m. The probability space (x, m) is assumed to be a Lebesgue space. The dynamical system is *aperiodic*, if the set of periodic points $\{x\in X \mid \exists n\in\mathbb{N}$ with $T^n(x)=x\}$ has measure zero. If the dynamical system is aperiodic, there exists for every $n\in\mathbb{N}$, $n>0$ and every $\varepsilon>0$ a measurable set Y, such that $Y, T(Y),\ldots,T^{n-1}(Y)$ are pairwise disjoint and such that $m(Y_{\text{rest}})\le\varepsilon$ where $Y_{\text{rest}}=X\setminus\bigcup_{k=0}^{n-1}T^k(Y)$. This is *Rohlin's Lemma* (for a proof see [**Hal 56**]) and the set Y is called a *(n,ε)-Rohlin set.*

Denote by $M(2,\mathbb{R})$ the vector space of all real 2×2 matrices and with * matrix transposition. We will study the Banach space

$$\mathscr{X}=L^\infty(X, M(2,\mathbb{R}))=\{A: X\to M(2,\mathbb{R}) \mid A_{ij}\in L^\infty(X)\}$$

with norm

$$|||A|||=|\,\|A(\cdot)\|\,|_{L^\infty(X)}$$

where $\|\cdot\|$ denotes the usual operator norm for matrices. Define also

$$\mathscr{A}=L^\infty(X, SL(2,\mathbb{R}))$$

where $SL(2,\mathbb{R})$ is the group of 2×2 matrices with determinant 1. Take on $\mathscr{A}$ the induced topology from $\mathscr{X}$. Denote with $\circ$ matrix multiplication. With the multiplication

$$AB(x)=A(x)\circ B(x)$$

$\mathscr{X}$ is a Banach algebra. Name $A(T)$ the mapping $x\mapsto A(T(x))$. For $n>0$, we write

$$A^n=A(T^{n-1})A(T^{n-2})\cdots A$$

and $A^0=1$ where $1(x)$ is the identity matrix. With this notation, A satisfies the *cocycle-identity*

$$A^{n+m}=A^n(T^m)A^m$$

for $n, m\ge 0$. The mapping $(n, x)\mapsto A^n(x)$ is called a *matrix cocycle over the dynamical system* (X, T, m). With a slight abuse of language, we will just call the elements in $\mathscr{X}$ *matrix cocycles.* The Banach manifold $\mathscr{A}\subset\mathscr{X}$ is a multiplicative group. This group contains the commutative group

$$\mathscr{O}=L^\infty(X, SO(2,\mathbb{R}))$$

where $SO(2,\mathbb{R})=\{A\in SL(2,\mathbb{R}) \mid A^*A=1\}$.

2.2. *Lyapunov exponents and the multiplicative ergodic theorem.* According to the *multiplicative ergodic theorem of Oseledec* (see [**Rue 79**]), the limit

$$M(A):=\lim_{n\to\infty}((A^n)^*A^n)^{1/2n}$$

exists pointwise almost everywhere for an element $A\in\mathscr{X}$. Let

$$\exp(\lambda^-(A,x))\le\exp(\lambda^+(A,x))$$

be the eigenvalues of $M(A)(x)$. The numbers $\lambda^{+/-}(A,x)$ are called the *Lyapunov exponents* of A and they are measurable functions on X also possibly taking the value $-\infty$. We define

$$\mathscr{P}=\{A\in\mathscr{X} \mid \lambda^-(A,x)<\lambda^+(A,x), \text{ a.e.}\}$$

and call the elements in $\mathscr{P}$ *nonuniform partial hyperbolic.* For $A\in\mathscr{P}$, there exists (still according to the multiplicative ergodic theorem) a measurable mapping from X into the space of all one-dimensional subspaces of $\mathbb{R}^2$

$$x\mapsto W(x)$$

which is *coinvariant*

$$A(x)W(x)=W(T(x))$$

and such that for every $w\in W(x)$, $w\ne 0$, we have

$$\lambda^-(A,x)=\lim_{n\to\infty}n^{-1}\log|A^n(x)w|.$$

In the case $A \in \mathscr{A}$, we will refer to

$$\lambda(A, x) = \lambda^+(A, x) = -\lambda^-(A, x) \geq 0$$

as the *Lyapunov exponent.* It is for fixed $A \in \mathscr{A}$ a function in $L^\infty(X)$ and if $A \in \mathscr{P} \cap \mathscr{A}$, this function is positive almost everywhere. We shall call a cocycle in $\mathscr{A} \cap \mathscr{P}$ *nonuniform hyperbolic.* For $A \in \mathscr{A}$,

$$\lambda(A) = \lim_{n\to\infty} n^{-1} \int_X \log \|A^n(x)\| \, dm(x)$$

is the *integrated upper Lyapunov exponent.* The existence of this limit (taking possibly the value $-\infty$) can be seen also easily without knowledge of the multiplicative ergodic theorem, because the sequence

$$n^{-1} \int_X \log \|A^n(x)\| \, dm(x)$$

is monotonically decreasing. The integrated Lyapunov exponent is in the same way also defined for complex-valued matrix cocycles $A \in L^\infty(X, M(2, \mathbb{C}))$. The next lemma gives a formula for the integrated Lyapunov exponent. In **[Led 82]**, one can find a more general version of this lemma.

LEMMA 2.1. *If $A \in \mathscr{P} \cap \mathscr{A}$ and $w(x)$ is a unit vector in $W(x)$ then*

$$\lambda(A) = -\int_X \log |A(x)w(x)| \, dm(x).$$

Proof. Call

$$\psi(x) = -\log |A(x)w(x)|.$$

From the multiplicative ergodic theorem, we have

$$\begin{aligned}\lambda(A) &= -\lim_{n\to\infty} n^{-1} \int_X \log |A^n(x)w(x)| \, dm(x) \\ &= \lim_{n\to\infty} n^{-1} \int_X \sum_{i=0}^{n-1} \psi(T^i(x)) \, dm(x).\end{aligned}$$

Birkhoff's ergodic theorem gives now

$$\lambda(A) = \int_X \psi(x) \, dm(x) = -\int_X \log |A(x)w(x)| \, dm(x). \qquad \square$$

2.3. *Induced systems and derived cocycles.* If $Z \subset X$ is a measurable set of positive measure, one can define a new dynamical system (Z, T_Z, m_Z) as follows: denote by $n(x)$ the *return time* for an element $x \in Z$, which is $n(x) = \min\{n \geq 1 \mid T^n(x) \in Z\}$. Poincaré's recurrence theorem implies, that $n(x)$ is finite for almost all $x \in Z$. Now, $T_Z(x) = T^{n(x)}(x)$ is a measurable transformation of Z, which preserves the probability measure $m_Z = m(Z)^{-1}m$. The system (Z, T_Z, m_Z) is called *the induced system* constructed from (X, T, m) and Z. It is ergodic, if (X, T, m) is ergodic (see **[Cor 82]**).

The cocycle $A_Z(x) = A^{n(x)}(x)$ is called the *derived cocycle* of A over the system (Z, T_Z, m_Z). If Y is a (n, ε)-Rohlin set and $Y_{\text{rest}} = X \setminus \bigcup_{k=0}^{n-1} T^k(Y)$, then the return time of a point in $Z = Y \cup Y_{\text{rest}}$ is less or equal to n. This implies, that for $A \in \mathscr{X}$, the entries of A_Z are in $L^\infty(Z)$.

In the following Lemma 2.2, we citate a formula, which relates the integrated Lyapunov exponent of an induced system $\lambda(A_Z)$ with $\lambda(A)$. This formula is analogous to the formula of Abramov (see **[Den 76]**), which gives the metric entropy of an induced system from the entropy of the system. Lemma 2.2 is also stated in a slightly different form by Wojtkowsky **[Woj 85]**.

LEMMA 2.2. *If (X, T, m) is ergodic and $Z \subset X$ has positive measure, then*

$$\lambda(A_Z) \cdot m(Z) = \lambda(A).$$

Proof. Given $Z \subset X$ with $m(Z) > 0$, the return time $n(x) = \min\{n > 0 \mid T^n(x) \in Z\}$ of almost all $x \in Z$ is finite. Define for $x \in Z$

$$N_k(x) = \sum_{i=0}^{k-1} n((T_Z)^i(x)).$$

Because (X, T, m) is ergodic, we have for almost all $x \in Z$

$$\begin{aligned}\lambda(A_Z) &= \lim_{n\to\infty} n^{-1} \log \|(A_Z)^n(x)\| \\ &= \lim_{n\to\infty} n^{-1} \log \|A^{N_n(x)}(x)\| \\ &= \lim_{n\to\infty} \frac{N_n(x)}{n} \cdot \lim_{n\to\infty} N_n^{-1}(x) \log \|A^{N_n(x)}(x)\| \\ &= \lim_{n\to\infty} \frac{N_n(x)}{n} \cdot \lim_{k\to\infty} k^{-1} \log \|A^k(x)\| \\ &= \int_N n(x) \, dm_Z(x) \cdot \lambda(A).\end{aligned}$$

In the last equality, we have used Birkhoff's ergodic theorem applied to the system (Z, T_Z, m_Z) to obtain

$$\lim_{n\to\infty} N_m(x)/n = \int_Z n(x) \, dm_Z(x)$$

for almost all $x \in Z$. The recurrence lemma of Kac **[Cor 82]** gives $\int_Z n(x) \, dm_Z(x) = m(Z)^{-1}$. $\square$

We write

$$R(\phi) = \begin{pmatrix} \cos(\phi) & \sin(\phi) \\ -\sin(\phi) & \cos(\phi) \end{pmatrix}.$$

Given a cocycle $A \in \mathscr{A}$, we are interested in the parametrized cocycle $\beta \mapsto AR(\beta)$ where $\beta \in \mathbb{R}/(2\pi\mathbb{Z})$.

Let Z be a measurable subset of X with positive measure. We denote with χ_Z the characteristic function of Z. We will use later the following little technical lemma.

LEMMA 2.3. *For $A \in \mathcal{X}$ and $\beta \in \mathbb{R}$,* we have $(AR(\chi_Z \cdot \beta))_Z = A_Z R(\beta)$

Proof. Take $x \in Z$ and call $k = n_Z(x)$. By definition

$$A_Z(x)R(\beta) = A(T^{k-1}(x)) \circ \cdots \circ A(x) \circ R(\beta).$$

Because $T(x), T^2(x), \ldots, T^{k-1}(x)$ are not in Z, we have also

$$(AR(\chi_Z \cdot \beta))_Z(x) = A(T^{k-1}(x)) \circ \cdots \circ A(x) \circ R(\beta). \qquad \square$$

2.4. *A result of M. Herman.* We can write $A(x)$ in the *Iwasawa decomposition* $A(x) = D(x) \circ R(\phi(x))$ with

$$D(x) = \begin{pmatrix} c(x) & b(x) \\ 0 & c^{-1}(x) \end{pmatrix}$$

where, c, c^{-1}, and b are in $L^\infty(X)$ and $\phi : X \mapsto \mathbb{R}/(2\pi\mathbb{Z})$ is measurable. Crucial for us is the following result of Herman [**Her 83**], the proof of which we will repeat here.

PROPOSITION 2.4. (Herman.) $\int_0^{2\pi} \lambda(AR(\beta))\, d\beta \geq \int_X \log \sqrt{((c+c^{-1})^2 + b^2)/4}\, dm.$

Proof. Define the complex number $w = e^{i\beta}$ and the complex cocycle

$$B_w(x) = w \cdot e^{i\phi(x)} \cdot AR(\beta).$$

Because $|w \cdot e^{i\phi(x)}| = 1$ we have

$$\lambda(AR(\beta)) = \lambda(B_w).$$

One can write

$$B_w(x) = D(x) \circ (G + w^2 \cdot e^{2i\phi(x)}\bar{G})$$

where

$$G = \frac{1}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}.$$

We choose $r > 1$ and extend the definition of B_w from $\{|w| = 1\}$ to $\{|w| < r\}$. The mapping $w \mapsto B_w$ is a holomorphic mapping from $\{|w| < r\}$ to the Banach algebra $L^\infty(X, M(2, \mathbb{C}))$. We claim, that the mapping

$$w \mapsto \lambda(B_w)$$

is subharmonic.

Proof. For each $n \in \mathbb{N}$ and almost all $x \in X$, the mapping

$$w \mapsto b_n(w, x) = n^{-1} \log \|B_w^n(x)\|$$

is subharmonic on $\{|w| < r\}$, because $w \mapsto B_w^n(x)$ is analytic there. Define for $k \in \mathbb{N}$

$$a_{n,k}(w, x) = \max(b_n(w, x), -k).$$

From Fubini's theorem it follows, that also

$$a_{n,k}(w) = \int_X a_{n,k}(w, x)\, dm(x)$$

is subharmonic. The sequence $k \mapsto a_{n,k}$ is decreasing and therefore

$$a_n(w) = \inf_{k \in \mathbb{N}} \{a_{n,k}(w)\} = n^{-1} \int_X \log \|B_w^n(x)\|\, dm(x)$$

is also subharmonic. Finally, also

$$\lambda(B_w) = \inf_{n \in \mathbb{N}} \{a_n(w)\}$$

is subharmonic. We conclude

$$\int_0^{2\pi} \lambda(AR(\beta))\, d\beta = \int_{|w|=1} \lambda(B_w)\, dw \geq \lambda(B_0)$$

and calculate with

$$L = \begin{pmatrix} 1 & 0 \\ -i & 1 \end{pmatrix}$$

$$\begin{aligned} \lambda(B_0) &= \lambda(DG) \\ &= \lambda(L^{-1}DGL) \\ &= \int_X \log \sqrt{((c+c^{-1})^2 + b^2)/4}\, dm. \end{aligned} \qquad \square$$

Denote with ν the Lebesgue measure on $\mathbb{R}/(2\pi\mathbb{Z})$. Herman's proposition implies the following corollary:

COROLLARY 2.5. $\nu\{\beta \in \mathbb{R}/(2\pi\mathbb{Z}) \mid \lambda(AR(\beta)) = 0\} \leq 1/\lambda(A)$.

Proof. Because

$$\lambda(AR(\beta)) \leq \int_X \log \sqrt{(c+c^{-1})^2 + b^2)}\, dm$$

for all $\beta \in \mathbb{R}/(2\pi\mathbb{Z})$, we have

$$\begin{aligned} \nu\{\beta \in \mathbb{R}/(2\pi\mathbb{Z}) \mid \lambda(AR(\beta)) = 0\} &\leq 1 - \frac{\int_X \log \sqrt{((c+c^{-1})^2 + b^2)/4}\, dm}{\int_X \log \sqrt{((c+c^{-1})^2 + b^2)}\, dm} \\ &= \log(2) \Big/ \int_X \log \sqrt{((c+c^{-1})^2 + b^2)}\, dm \\ &\leq \log(2)/\lambda(A) \\ &\leq 1/\lambda(A). \end{aligned} \qquad \square$$

3. *Results*

3.1. *Density of nonuniform hyperbolicity.*

THEOREM 3.1. *If the dynamical system (X, T, m) is aperiodic, then $\mathcal{P} \cap \mathcal{A}$ is dense in $\mathcal{A}$.*

Proof. We assume without loss of generality, that the dynamical system is ergodic. In the general case, we can decompose the system in its ergodic components (see [**Den 76**]). The union of ergodic fibers, which are aperiodic, has measure 1. So it is enough to prove the statement for an ergodic aperiodic dynamical system.

Take an arbitrary $A \in \mathcal{A}$ and an $\varepsilon > 0$. We show, that there exists $B \in \mathcal{P} \cap \mathcal{A}$, with $|||B - A||| \leq \varepsilon$. Choose first a constant $\mu > 1$ such that

$$2|||A||| \cdot |\mu - \mu^{-1}| \leq \varepsilon/3. \tag{1}$$

Take next a $n \in \mathbb{N}$, $n > 0$ so big, that

$$8\pi|||A||| \cdot \frac{\mu + \mu^{-1}}{n \cdot \log(\mu)} \leq \varepsilon/3. \tag{2}$$

Take now a $(n, 1/n)$-Rohlin set Y and call $Z = Y \cup Y_{\text{rest}}$ where

$$Y_{\text{rest}} = \left(X \setminus \bigcup_{k=0}^{n-1} T^k(Y)\right).$$

Note, that

$$m(Z) = m(Y) + m(Y_{\text{rest}}) \leq 2/n.$$

We can easily find a cocycle $C \in \mathcal{A}$, with

$$|||C - A||| \leq \varepsilon/3 \tag{3}$$

such that $C_Z(z) \notin SO(2, \mathbb{R})$ for $z \in Z$ and so that additionally

$$|||C||| \leq 2|||A|||. \tag{4}$$

Looking at the induced system (Z, T_Z, m_Z) and the derived cocycle C_Z and applying Proposition 2.4, we see, that there exists a $\beta_0 \in \mathbb{R}/(2\pi\mathbb{Z})$, such that

$$\lambda(C_Z R(\beta_0)) > 0.$$

Using Lemma 2.3, we see, that the cocycle

$$D = CR(\chi_Z \beta_0)$$

satisfies $D_Z = C_Z R(\beta_0)$ and so

$$\lambda(D_Z) = \lambda(C_Z R(\beta_0)) > 0.$$

Now with Lemma 2.2

$$\lambda(D) = \lambda(D_Z) \cdot m(Z) > 0.$$

Because the dynamical system is assumed to be ergodic, the Lyapunov exponents of D are nonzero almost everywhere and there exists according to the multiplicative ergodic theorem a function $x \mapsto W(x)$, which is coinvariant: $A(x)\,W(x) = W(T(x))$. Call $u(x) \in [0, \pi)$ the modulo π unique angle, a unit vector $w(x) \in W(x)$ makes with the first basis vector in $\mathbb{R}^2$. This means, that the rotation $R(u(x))$ turns the first basis vector into the space $W(x)$. We use the notation

$$\text{Diag}\,(\mu^{-1}) = \begin{pmatrix} \mu^{-1} & 0 \\ 0 & \mu \end{pmatrix}.$$

The cocycle

$$E = R(u(T))\,\text{Diag}\,(\mu^{-1}) R(u(T))^{-1} D$$

has the same coinvariant direction $W(x)$ as D and if we take for $x \in X$ a unit vector $w(x) \in W(x)$, we can write using Lemma 2.1

$$\begin{aligned} \lambda(E) &= -\int_X \log|E(x)w(x)|\,dm(x) \\ &= -\int_X \log|\mu^{-1} D(x)w(x)|\,dm(x) \\ &= -\int_X \log|D(x)w(x)|\,dm(x) + \log(\mu) \\ &= \lambda(D) + \log(\mu). \end{aligned}$$

It follows with Lemma 2.2 and $m(Z) \leq 2/n$, that

$$\begin{aligned} \lambda(E_Z) &= \lambda(D_Z) + \frac{\log(\mu)}{m(Z)} \\ &\geq \lambda(D_Z) + \log(\mu) \cdot \frac{n}{2} \\ &\geq \log(\mu) \cdot \frac{n}{2}. \end{aligned}$$

Corollary 2.5 applied to the cocycle E_Z over the system (Z, T_Z, m_Z) implies, that

$$\nu\{\beta \in \mathbb{R}/(2\pi\mathbb{Z}) | \lambda(E_Z R(\beta)) = 0\} \leq \frac{2}{n \cdot \log(\mu)}.$$

We find therefore a $\beta_1 \in \mathbb{R}/(2\pi\mathbb{Z})$ with

$$\beta_1 \leq \frac{4\pi}{n \cdot \log(\mu)} \tag{5}$$

such that $\lambda(E_Z R(\beta_1 - \beta_0)) > 0$. But then

$$B = ER(\chi_Z(\beta_1 - \beta_0)) \in \mathcal{P} \cap \mathcal{A}$$

because of Lemma 2.2 and because Lemma 2.3 implies, that $B_Z = E_Z R(\beta_1 - \beta_0)$. We claim, that $|||B - A||| \leq \varepsilon$. To see this, we define

$$F = R(u(T))\,\text{Diag}\,(\mu^{-1}) R(u(T))^{-1} C.$$

The norm of F can be estimated as

$$|||F||| \leq |||C||| \cdot (\mu + \mu^{-1}). \tag{6}$$

Recall the definition of B

$$B = R(u(T))\,\text{Diag}\,(\mu^{-1}) R(u(T))^{-1} CR(\chi_Z \beta_1).$$

Using the inequalities (6), (4), (5) and (2), one gets

$$\begin{aligned} |||B - F||| &\leq |||F||| \cdot |\beta_1| \\ &\leq |||C||| \cdot (\mu + \mu^{-1}) \cdot |\beta_1| \\ &\leq 2|||A||| \cdot (\mu + \mu^{-1}) \cdot |\beta_1| \\ &\leq 8\pi|||A||| \cdot \frac{\mu + \mu^{-1}}{n \cdot \log(\mu)} \\ &\leq \varepsilon/3. \end{aligned}$$

Further, with the inequalities (1) and (4), we have the estimate

$$\begin{aligned}|||F-C||| &\le |||C|||\cdot|||1-R(u)\,\mathrm{Diag}\,(\mu^{-1})R(u)^{-1}|||\\ &= |||C|||\cdot|||R(u)[1-\mathrm{Diag}\,(\mu^{-1})]R(u)^{-1}|||\\ &= |||C|||\cdot|||\mathrm{Diag}\,(1-\mu^{-1})|||\\ &\le |||C|||\cdot|\mu-\mu^{-1}|\\ &\le 2|||A|||\cdot|\mu-\mu^{-1}|\\ &\le \varepsilon/3.\end{aligned}$$

From these two estimates and the inequality (3), the claim

$$|||B-A|||\le\varepsilon$$

follows. □

3.2. *Density of nonuniform partial hyperbolicity.*

Corollary 3.2. *If the dynamical system (X, T, m) is aperiodic, then $\mathcal{P}$ is dense in $\mathcal{X}$.*

Proof. Given an $\varepsilon\ge 0$ and an element $A\in\mathcal{X}$, we will show, that there exists a $B\in\mathcal{P}$, such that $|||A-B|||\le\varepsilon$. Note first, that the set

$$\{A\in\mathcal{X}\,|\,\exists\delta>0,\ \det(A(x))\ge\delta;\ \text{a.e.}\}$$

is dense in $\mathcal{X}$. Take an element C out of this set with

$$|||A-C|||<\varepsilon/2.$$

Then $D=C/\det(C)\in\mathcal{A}$. We can apply Theorem 3.1 to get an $E\in\mathcal{A}\cap\mathcal{P}$, with

$$|||D-E|||\le\frac{\varepsilon}{2}\Big/|\det(C)|_{L^\infty(X)}.$$

Call $B=E\cdot\det(C)$. Then

$$\begin{aligned}|||C-B||| &= |||D\cdot\det(C)-E\cdot\det(C)|||\\ &\le |||D-E|||\cdot|\det(C)|_{L^\infty(X)}\\ &\le\varepsilon/2\end{aligned}$$

and so the element $B\in\mathcal{P}$ with

$$|||A-B|||\le|||A-C|||+|||C-B|||\le\varepsilon$$

is found. □

3.3. *Perturbations located on sets of small measure.* The proof of Theorem 3.1 gives even more information, how one can perturb a given cocycle $A\in\mathcal{A}$, to get into $\mathcal{A}\cap\mathcal{P}$. The proof shows that there exist cocycles H_1 and H_2 arbitrarily near to 1, such that $B=H_1AH_2\in\mathcal{P}\cap\mathcal{A}$.

The Lyapunov exponents of $C:=H_1^{-1}BH_1(T^{-1})=AH_2H_1(T^{-1})$ are the same as the Lyapunov exponents of B, because for every $n\in\mathbb{N}$

$$H_1(T^n)^{-1}B^nH_1(T^{-1})=C^n.$$

We see that there exist cocycles $H\in\mathcal{A}$ arbitrary near the identity, such that $AH\in\mathcal{P}\cap\mathcal{A}$. In the same way, for $A\in\mathcal{X}$, there exist cocycles $H\in\mathcal{X}$ arbitrarily near the identity, such that $AH\in\mathcal{P}$.

Corollary 3.3. *Assume (X, T, m) is aperiodic. Given $\varepsilon>0$. For $A\in\mathcal{A}$, there exists a $B\in\mathcal{P}\cap\mathcal{A}$ (for $A\in\mathcal{X}$, there exists a $B\in\mathcal{P}$), such that, $|||B-A|||\le\varepsilon$ and $m\{x\in X\,|\,B(x)\ne A(x)\}\le\varepsilon$.*

Proof. Again we can assume that the dynamical system (X, T, m) is aperiodic ergodic, because if the result is true for all aperiodic ergodic fibers of a given system, it is also true for the system itself.

Given $A\in\mathcal{A}$ (the case $A\in\mathcal{X}$ follows the same lines) and given $\varepsilon\ge 0$. Take an $n\in\mathbb{N}$ with $2/n\le\varepsilon$ and choose a $(n, 1/n)$-Rohlin set $Y\subset X$ and define $Z=Y\cup(X\setminus\bigcup_{k=0}^{n-1}T^k(Y))$. The above remark concerning Theorem 3.1 applied to the induced cocycle A_Z says, that there exists a cocycle H over the dynamical system (Z, T_Z, m_Z), such that $A_ZH\in\mathcal{P}$ and

$$|||A_ZH-A_Z|||\le\varepsilon.$$

If we extend the cocycle H to X by setting it to 1 outside Z and define $B=AH$, we see with Lemma 2.2, that $B\in\mathcal{P}$, because $B_Z=A_ZH\in\mathcal{P}$.

The cocycle B is different from A only on Z. We have $m(Z)\le\varepsilon$ and also

$$|||B-A|||\le\varepsilon.$$ □

3.4. *An application to circle valued cocycles.* We say, $A, B\in\mathcal{A}$ are *cohomologous*, if there exists a $C\in\mathcal{A}$, such that $A=C(T)AC^{-1}$. What are the cohomology classes in $\mathcal{A}$? Cohomologous cocycles have the same Lyapunov exponents. So, the Lyapunov exponents could help to distinguish between different cohomology classes. One could ask further, what are the cohomology classes in the set $\{A\in\mathcal{A}\,|\,\lambda(A)=\lambda_0\}$, where λ_0 is a given nonnegative number. We will try to deal with this question.

Cocycles cohomologuous to 1 are called *coboundaries.* How big is the set of coboundaries? We have to restrict the problem still more in order to make a statement.

By a *circle valued cocycle,* we mean an element in $\mathcal{O}=L^\infty(X, SO(2,\mathbb{R}))$. The Banach manifold $\mathcal{O}$ is an Abelian subgroup of $\mathcal{A}$. We say, $A\in\mathcal{O}$ is a *coboundary* in $\mathcal{O}$, if there exists a $B\in\mathcal{O}$, such that $A=B(T)B^{-1}$. It is not difficult to see (and we will leave out the proof here), that $A\in\mathcal{O}$ is a coboundary in $\mathcal{A}$ if and only if it is a coboundary in $\mathcal{O}$.

There are no methods known with which to decide in general, if a given circle-valued cocycle is a coboundary or not. Even in the simplest cases, this remains a largely unsolved problem. There exist some analytic conditions in [**Bag 88**]. The problem has been studied intensively in the case, when the underlying dynamical system (X, T, m) is an irrational rotation of the circle [**Mer 85**] and where the cocycle takes only two values, because this has application in representations of Lie groups and in number theory. The concept is also important in the construction of ergodic skew products.

We call $\mathcal{C}$ the set of coboundaries in $\mathcal{O}$. They form a subgroup of $\mathcal{O}$. The Abelian group $\mathcal{H}^1=\mathcal{O}/\mathcal{C}$ is a *first cohomology group* of the dynamical system. How big is this group? A first step towards answering this question makes the following theorem.

THEOREM 3.4. *If the dynamical system (X, T, m) is aperiodic, $\mathscr{C}$ is dense in $\mathcal{O}$.*

Proof. Because $\mathcal{O} \subset \mathscr{A}$ and $\mathscr{P}$ is dense in $\mathscr{A}$, we can find for every $A \in \mathcal{O}$ a sequence $A_n \to A$ with $A_n \in \mathscr{P}$. As in the proof of Theorem 3.1, we can find for almost all $x \in X$ a rotation $R(u_n(x))$, which turns the first bases vector of $\mathbb{R}^2$ into the coinvariant space $W_n(x)$ of A_n. We can then write

$$A_n(x) = R(u_n(T(x)))C_n(x)R(u_n(x))^{-1}$$

where $C_n(x)$ is upper tridiagonal. Because $A_n \to A \in \mathcal{O}$, we must have $C_n \to 1$. Call

$$B_n(x) = R(u_n(T(x)))R(u_n(x))^{-1}.$$

Then $B_n \to A$ and $B_n \in \mathscr{C}$. □

4. *Discussion*

4.1. *Aperiodicity.* The condition of the aperiodicity of the dynamical system (X, T, m) is necessary in all the results. A periodic ergodic dynamical system is just a cyclic permutation of a finite set X. If the cardinality of X is N, and $A \in \mathscr{A}$ is given, then the Lyapunov exponents of A can be written as

$$\lambda^{+/-}(A, x) = N^{-1} \log |\mu^{+/-}|$$

where $\mu^{+/-}$ are the eigenvalues of A^N. The Lyapunov exponents are different, if and only if $|\mathrm{tr}\,(A^N)| > 2$. We see, that $\mathscr{P} \cap \mathscr{A}$ is an open set in $\mathscr{A}$ and $\mathscr{A} \backslash \mathscr{P}$ has nonempty interior.

Given $A \in \mathcal{O}$. If $A = B(T)B^{-1}$ is a coboundary, we have

$$A^N = B(T^N)B^{-1} = 1.$$

The necessary condition $A^N = 1$ for A to be a coboundary is also sufficient. Let $A_1, A_2, \ldots, A_N$ be the values A takes on $X = \{1, 2, \ldots, N\}$ and assume $A^N = 1$. Define $B_1 = 1$ and $B_k = A_1 A_2 \cdots A_{k-1}$ for $k = 2, \ldots, N$. Then $A_k = B_{k+1}B_k^{-1}$ for all $k = 1, \ldots, N$ and A is a coboundary. So, coboundaries form a $(N-1)$-dimensional manifold in the N-dimensional torus $\mathcal{O} = SO(2, \mathbb{R})^N$.

4.2. *Open problems.* We conjecture that $\mathscr{P}$ is generic in $\mathscr{A}$ or that $\mathscr{P}$ contains even an open dense subset.

Do there exist analogous results for $L^\infty(X, M(d, \mathbb{R}))$ and $L^\infty(X, M(d, \mathbb{C}))$ for $d > 2$?

Can one also find a density result in $\mathscr{X}^0 = C^0(X, M(2, \mathbb{R}))$ if T is a homeomorphism on a compact metric space?

Is it even possible to find results in $\mathscr{X}^r = C^r(X, M(2, \mathbb{R}))$ if T is a C^r-diffeomorphism on a C^r-manifold?

We still do not have more powerful methods to decide if $A \in \mathscr{P}$ or not. Are the Lyapunov exponents of the standard mapping positive on a set of positive measure? We conjecture that for a general compact connected topological group G, the set of coboundaries is dense in $L^\infty(X, G)$.

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