The upper Lyapunov exponent of Sl(2,R) cocycles: Discontinuity and the problem of positivity

Oliver Knill
Mathematikdepartement
ETH Zentrum
CH-8092 Zürich

Abstract

Let T be an aperiodic automorphism of a standard probability space (X,m). Let $\mathcal{P}$ be the subset of $\mathcal{A} = L^\infty(X, Sl(2,R))$ where the upper Lyapunov exponent is positive almost everywhere.
We prove that the set $\mathcal{P} \setminus int(\mathcal{P})$ is not empty. So, there are always points in $\mathcal{A}$ where the Lyapunov exponents are discontinuous.
We show further that the decision whether a given cocycle is in $\mathcal{P}$ is at least as hard as the following cohomology problem: Can a given measurable set $Z \subset X$ be represented as $Y \Delta T(Y)$ for a measurable set $Y \subset X$?

1 Introduction

We want to investigate the Banach manifold

$$\mathcal{A} = L^\infty(X, Sl(2,R))$$

of all measurable bounded $Sl(2,R)$-cocycles over a given aperiodic dynamical system (X,T,m). We are interested in the subset $\mathcal{P}$ of cocycles where the upper Lyapunov exponent

$$\lambda^+(A,x) = \lim_{n\to\infty} n^{-1} \log ||A^n(x)||$$

is positive almost everywhere.

We have shown [Kni 90] that $\mathcal{P}$ is dense in $\mathcal{A}$. This could give some explanation why one encounters so often positive Lyapunov exponents when making numerical simulations. Numerical experiments suggest also that the Lyapunov exponents behave irregular in dependence of parameters. We prove in this note that the set $\mathcal{P} \setminus int(\mathcal{P})$ is not empty. On this set the Lyapunov exponent is discontinuous. Discontinuity of Lyapunov exponents has been mentioned at different places (see [You 86]). The only published result we found is in [Joh 84] in the case of $sl(2,R)-$ cocycles over almost periodic flows. Johnson proved there that discontinuities can already occur when changing a real parameter of the cocycle. His situation is different from ours in that he has a special flow and special cocycles occurring in the theory of Schroedinger operators, where we have an aperiodic but else arbitrary discrete dynamical system.

The idea for producing examples where λ^+ is discontinuous, is to exchange the expanding and contracting directions of the cocycle. This idea is not new and has been used in [Kif 82] to give examples where the Lyapunov exponent of identically distributed independent random matrices depend discontinuously from the common distribution. Our situation is different, because Kifer changes the dynamical system and not the cocycle. We will see that the exchanging of expanding and contracting directions must be done carefully. It can happen that the exchanging is making stable a part of the unstable directions and unstable a part of the stable directions. This, we don't want. A cohomology condition for measurable sets will assure that the stable and unstable directions become indistinguishable. This will give zero Lyapunov exponents. For certain cocycles, which we call weak, we can make such an exchanging by small perturbations.

We mention now some results which concern the regularity of the Lyapunov exponent: Hölder continuity (and in some cases even C^∞ smoothness) of the Lyapunov exponent with respect to a real parameter has been shown by le Page [Pag 89] in the case of independent identically distributed random matrices.

Ruelle's ([Rue 79a]) results show that there is an open set in $\mathcal{A}$ where the Lyapunov exponent is real analytic. It is the set

$$\mathcal{S} = \{A \in \mathcal{A} |\ \exists C \in \mathcal{A}\ \ \exists \epsilon > 0,\ [C(T)AC^{-1}(x)]_{ij} \geq \epsilon\}$$

which is contained in $int(\mathcal{P})$. One could call $\mathcal{S}$ the uniform hyperbolic part of $\mathcal{A}$ (or the set with exponential dichotomy [Joh 86]) and $\mathcal{P} \setminus \mathcal{S}$ the nonuniform hyperbolic part. The elements in $\mathcal{P} \setminus int\mathcal{P}$ which will be constructed here are not uniform hyperbolic. But we will see, that we can choose such elements in the closure of $\mathcal{S}$.

A lot of unsuccessful efforts to find more powerful methods to prove positivity of the upper Lyapunov exponent of $Sl(2,R)-$ cocycles led us to believe that the question whether A is in $\mathcal{P}$ is difficult and subtle in general. We want to illustrate this by showing that the decision can be at least as hard as deciding whether a certain circle valued cocycle is a coboundary. The circle valued cocycles considered here have the range $\{1,-1\}$. The group $\mathcal{E}$ of such cocycles can be identified with the set of measurable subsets of X with group operation Δ. The elements in $Z\Delta T(Z)$ are called coboundaries and form a subgroup. We will prove that the positivity of the Lyapunov exponent of a cocycle can depend on the question whether a certain set is a coboundary or not. This question about coboundaries has been investigated in [Bag 88]. In the special case, when (X, T, m) is an irrational rotation on the circle and the sets considered are intervals the problem has been treated in ([Vee 69], [Mer 85]). Even in this reduced form, the coboundary problem is still not solved.

2 Preparations

A *dynamical system* (X, T, m) is a set X with a probability measure m and a measure preserving invertible map T on X. We assume that (X, m) is a Lebesgue space and that the dynamical system is ergodic. The later involves no loss of generality because the arguments can be applied to each ergodic fibre of the ergodic decomposition in general. The dynamical system is called *aperiodic* if the set of periodic points $\{x \in X | \exists n \in N\ with\ T^n(x) = x\}$ has measure zero.

Denote by $M(2,R)$ the vector space of all real 2×2 matrices equipped with the usual operator norm $||\cdot||$. In the Banach space

$$L^\infty(X, M(2,R)) = \{A : X \to M(2,R) |\ A_{ij} \in L^\infty(X)\}$$

with norm $|||A||| = |\,||A(\cdot)||\,|_{L^\infty(X)}$ lies the Banach manifold

$$\mathcal{A} = L^\infty(X, Sl(2,R))$$

where $Sl(2,R)$ is the group of 2×2 matrices with determinant 1. Take on $\mathcal{A}$ the induced topology from $L^\infty(X,M(2,R))$. Denote with $\circ$ matrix multiplication. With the multiplication $AB(x)=A(x)\circ B(x)$ the space $L^\infty(X,M(2,R))$ is a Banach algebra. Name A(T) the mapping $x\mapsto A(T(x))$. For $n>0$ we write

$$A^n = A(T^{n-1})\cdots A(T)A$$

and $A^0=1$ where $1(x)$ is the identity matrix. The mapping $(n,x)\mapsto A^n(x)$ is called a *matrix cocycle over the dynamical system (X,T,m)*. With a slight abuse of language we will call the elements in $\mathcal{A}$ *matrix cocycles* or simply cocycles.

Denote with * matrix transposition. According to the *multiplicative ergodic theorem of Oseledec* (see [Rue 79]) the limit

$$M(A)(x) := \lim_{n\to\infty}((A^n)^*(x)A^n(x))^{1/2n}$$

exists almost everywhere for $A\in\mathcal{A}$. Let

$$\exp(\lambda^-(A,x)) \le \exp(\lambda^+(A,x))$$

be the eigenvalues of $M(A)(x)$. The numbers $\lambda^{+/-}(A,x)$ are called the *Lyapunov exponents* of A. Because T is ergodic we write $\lambda^{+/-}(A)$ for the value, $\lambda^{+/-}(A,x)$ takes almost everywhere. Because $M(A)(x)$ has determinant 1 one has $-\lambda^-(A)=\lambda^+(A)$. We call $\lambda(A)=\lambda^+(A)$ *the Lyapunov exponent* of A and define

$$\mathcal{P}=\{A\in\mathcal{A}|\ \lambda(A)>0\}.$$

For $A\in\mathcal{P}$ there exist two measurable mappings $W^{+/-}$ from X into the projective space P^1 of all one dimensional subspaces of R^2 which satisfy

$$A(x)W^{+/-}(x)=W^{+/-}(T(x)).$$

$W^{+/-}(x)$ are the eigenspaces of $M(A)(x)$. Given $A\in\mathcal{A}$ we can define the *skew product action* $T\times A$ on the space $X\times P^1$:

$$T\times A:(x,W)\mapsto (T(x),A(x)W).$$

The projection π from $X\times P^1$ onto X defines a projection π^* of probability measures. We say, a probability measure μ on $X\times P^1$ *projects down* to $\pi^*\mu$. Ledrappier [Led 82] has found an addendum of the multiplicative ergodic theorem. We report here only a special case. For $W\in P^1$ we will always denote with w a unit vector in W.

Proposition 2.1 *a) If $A\in\mathcal{P}$ there exist exactly two ergodic $T\times A$ invariant probability measures $\mu^{+/-}$ on $X\times P^1$ which project down to m and one has*

$$\lambda^{+/-}(A)=\int_{X\times P^1}\log|A(x)w|d\mu^{+/-}(x,W).$$

The measures $\mu^{+/-}$ have their support on

$$X^{+/-}=\{(x,W^{+/-}(x))|x\in X\}.$$

b) For every ergodic $T\times A$ invariant probability measure μ which projects down to m

$$\lambda(A)=|\int_{X\times P^1}\log|A(x)w|d\mu(x,W)|.$$

Remark: Part a) of proposition 2.1 has been stated also in [Her 81] and in the case of cocycles with random noise in [You 86].

Let $Z \subset X$ be a measurable set of positive measure. A new dynamical system (Z,T_Z,m_Z) can be defined as follows: Poincaré's recurrence theorem implies that the *return time* $n(x) = min\{n \geq 1|T^n(x) \in Z\}$ is finite for almost all $x \in Z$. Now, $T_Z(x) = T^{n(x)}(x)$ is a measurable transformation of Z which preserves the probability measure $m_Z = m(Z)^{-1} \cdot m$. The system (Z,T_Z,m_Z) is called *the induced system* constructed from (X,T,m) and Z. It is ergodic if (X,T,m) is ergodic (see [Cor 82]).
The cocycle $A_Z(x) = A^{n(x)}(x)$ is called the *derived cocycle* of A over the system (Z,T_Z,m_Z). In the following lemma 2.2 we cite a formula which relates the Lyapunov exponent of an induced system $\lambda(A_Z)$ with $\lambda(A)$. This formula is analogues to the formula of Abramov (see [Den 76]) which gives the metric entropy of an induced system from the entropy of the system. Lemma 2.2 is also stated in a slightly different form by Wojtkowsky [Woj 85].

Lemma 2.2 (Wojtkowsky) *If* $m(Z) > 0$ *then* $\lambda(A_Z) \cdot m(Z) = \lambda(A)$.

Remark: Wojtkowsky gives the formula

$$\lambda(A_Z) = \int_Z n(x)dm_Z(x) \cdot \lambda(A).$$

The version given here follows with the recurrence lemma of Kac [Cor 82] which says $\int_Z n(x)dm_Z(x) = m(Z)^{-1}$.

3 Cocycles with values in $\{1,-1\}$

We denote with $\mathcal{E}$ the set of $\{1,-1\}$*-valued cocycles*

$$\mathcal{E} = \{A \in \mathcal{A}|\ A(x) \in \{1,-1\}\}.$$

To each $A \in \mathcal{E}$ we can associate a measurable set

$$\psi(A) = \{x \in X|A(x) = -1\}.$$

It is easy to see that

$$\psi(A)\Delta\psi(B) = \psi(AB)$$

where Δ denotes the symmetric difference. ψ is invertible. So the group $\mathcal{E}$ is isomorphic to the group of measurable sets in X with group operation Δ. We call a measurable set Y a *coboundary* if there exists a measurable set Z such that $Y = Z\Delta T(Z)$. Also $A \in \mathcal{E}$ is called a *coboundary* if $\psi(A)$ is a coboundary. We will use the notation $Y^c = X \setminus Y$. Given a cocycle $A \in \mathcal{E}$ we can build a skew product $T \times A$ on $X \times \{1,-1\}$ as follows:

$$T \times A : (x,u) \mapsto (T(x), A(x)u).$$

It leaves invariant the product measure $m \times \nu$ where ν is the measure $\nu(\{1\}) = \nu(\{-1\}) = 1/2$ on $\{1,-1\}$. One can see the skew product action $T \times A$ as follows: Take two copies of the dynamical system (X,T,m). The dynamics is then given on both copies as usual. Only when hitting the set $\psi(A)$ one jumps to the other system.
A necessary and sufficient condition under which the ergodicity of (X,T,m) implies the ergodicity of $(X \times \{1,-1\}, T \times A, m \times \nu)$ is given in the following result of Stepin [Ste 71]:

Proposition 3.1 (Stepin) *For $A \in \mathcal{E}$ the skew product $T \times A$ is ergodic if and only if A is not a coboundary.*

Proof: Assume $A = \psi^{-1}(Y)$ and $Y = Z\Delta T(Z)$. The set

$$Q = Z \times \{1\} \cup Z^c \times \{-1\}$$

is $T \times A$ invariant. Therefore $T \times A$ is not ergodic.
On the other hand, assume $A = \psi^{-1}(Y)$ and there exists a set $Q \subset X \times \{1, -1\}$, $0 < (m \times \nu)(Q) < 1$, which is $T \times A$ invariant. Let Z be defined by the equation

$$Q \cap (X \times \{1\}) = Z \times \{1\}.$$

One checks that $Y = Z\Delta T(Z)$. So, A is a coboundary. ■

Remark: There exists a generalization of the above result (as formulated in [Lem 89]): Take a compact abelian group G with Haar measure ν. A measurable map $A : X \to G$ is called a $G-$ *cocycle*. Such a cocycle defines a skew product $T \times A$ on $X \times G$:

$$(T \times A)(x, g) = (T(x), A(x)g)$$

which preserves the measure $m \times \nu$. The result is that $T \times A$ is ergodic if and only if for any nontrivial character $\chi \in \hat{G}$ the circle valued cocycle $x \mapsto \chi(A(x))$ is not a coboundary. The proof given in [Anz 51] in the case where G is the circle can be modified easily to prove the general result. As a special case, if G is the cyclic group of order 2, one gets the above result of Stepin.

Lemma 3.2 *A measurable set Y with $m(Y) > 0$ is a coboundary if and only if $(T_Y)^2$ is not ergodic.*

Proof: Assume first that Y is a coboundary $Y = Z\Delta T(Z)$. Call $Z_1 = Z \setminus T(Z)$ and $Z_2 = T(Z) \setminus Z$. Then $T_Y(Z_1) = Z_2$ and $T_Y(Z_2) = Z_1$ imply $(T_Y)^2(Z_1) = Z_1$. Therefore $(T_Y)^2$ is not ergodic because $0 < m(Z_1) < 1/2$.
If on the other hand $(T_Y)^2$ is not ergodic then $\exists Z \subset Y$ with $(T_Y)^2(Z) = Z$ and $0 < m(Z) < m(Y)$. We claim that $Y = Z\Delta T_Y(Z)$. Because

$$Z \cap T_Y(Z) = (T_Y)^2(Z) \cap T_Y(Z) = T_Y(T_Y(Z) \cap Z)$$

the ergodicity of T_Y implies that $Y = Z \cap T_Y(Z)$ or $m(Z \cap T_Y(Z)) = 0$. The first case implies $Y = Z$ which is not possible because of the assumption $m(Z) < m(Y)$. So $m(Z \cap T_Y(Z)) = 0$. The same argument with $Z' = Y \setminus Z$ implies

$$m(Z' \cap T_Y(Z')) = m(Y \setminus (Z \cup T_Y(Z))) = 0.$$

From $Y = Z \cup T_Y(Z)$ and $m(Z \cap T_Y(Z)) = 0$ we get $Y = Z\Delta T_Y(Z)$.
If $n(x)$ denotes the return time of a point $x \in Y$ to Y we define

$$U = \{T^k(x) | x \in Z, k = 0, \ldots, n(x) - 1\}.$$

Then

$$U\Delta T(U) = Z\Delta T_Y(Z) = Y$$

and Y is a coboundary. ■

We define on $\mathcal{E}$ the metric

$$d(A,B) = m(\{x \in X \mid A(x) \neq B(x)\}) = m(\psi(A)\Delta\psi(B))$$

which makes $\mathcal{E}$ to a topological group.

Proposition 3.3 *If the dynamical system is aperiodic then the set of coboundaries as well as its complement are both dense in $\mathcal{E}$ with respect to the metric d.*

Proof: It is known that the set of $A = \psi^{-1}(Y)$ such that $(T_Y)^2$ is not ergodic is dense in $\mathcal{E}$ (See [Fri 70] p. 125). Applying lemma 3.2 gives that coboundaries are dense.
It is known that the set of $A = \psi^{-1}(Y)$ such that T_Y is weakly mixing is dense in $\mathcal{E}$([Fri 70] p. 126). If T_Y is weakly mixing also $(T_Y)^2$ is weakly mixing ([Fur 81] p.83) and $(T_Y)^2$ must be ergodic. Apply again lemma 3.2. ■

Remarks:
1) In proposition 3.3 has entered the assumption that the probability space (X,m) is a Lebesgue space. There exists an automorphism of a probability space such that each measurable set Y is a coboundary. (See [Akc 65].)
2) Proposition 3.3 gives some indication that the decision whether a set is a coboundary or not might be subtle, especially when trying to deal with the question numerically.
3) Let us mention that for an ergodic periodic dynamical system (X,T,m) a set $Z \subset X$ is a coboundary if and only if the cardinality of Z is even. This follows quickly from the above lemma 3.2. Proposition 3.3 is no more true in the periodic case.
4) Of course the construction of coboundaries is very easy: Take a measurable set $Z \subset X$ and form the coboundary $Y = Z\Delta T(Z)$. On the other hand, we don't know of an easy construction of sets which are not coboundaries.

Lemma 3.4 *Assume $Z \subset Y \subset X$ with $m(Z) > 0$. Then, Z is a coboundary for T if and only if Z is a coboundary for T_Y.*

Proof: Because $(T_Y)_Z = T_Z$ we have also $((T_Y)_Z)^2 = (T_Z)^2$. The claim follows with lemma 3.2. ■

We will use the following corollary of the proposition 3.3:

Corollary 3.5 *For every $Y \subset X$ with $m(Y) > 0$ there exists $Z \subset Y$ which is not a coboundary.*

Proof: Look at the dynamical system (Y, T_Y, m_Y). If (X,T,m) is aperiodic the proposition 3.3 assures that there exists $Z \subset Y$ such that Z is not a coboundary for T_Y. This means with lemma 3.4 that Z is not a coboundary for T. If (X,T,m) is periodic, choose $Z \subset Y$ which consists of one element. This Z is not a coboundary because $(T_Z)^2$ is trivially ergodic. ■

4 Continuity and Discontinuity of the Lyapunov exponent

Computer experiments indicate that the Lyapunov exponent λ is discontinuous. But from the topological point of view we have a big set were λ is continuous. Recall that a subset of a topological space is called *generic* if it contains a countable intersection of open dense sets. The complement of a generic set is called *meager*.

Theorem 4.1 *The set $\{A \in \mathcal{A} \mid \lambda$ is continuous in $A\}$ is generic in $\mathcal{A}$.*
$\mathcal{P} \setminus int\mathcal{P}$ is meager.

Proof: We can write

$$\lambda(A) = \lim_{n\to\infty} \lambda_n(A)$$

with $\lambda_n(A) = n^{-1} \int \log ||A^n|| dm$. So, λ is the pointwise limit of continuous functions λ_n. A theorem of Baire (see [Hah 32] p.221) states that the set of continuity points of such a function is generic.
The set $\mathcal{P} \setminus int(\mathcal{P})$ is a subset of all the discontinuity points. It is therefore meager. ■

Definition: We say a cocycle $A \in \mathcal{P}$ *is weak on* $Y \subset X$ if the following three conditions are satisfied:
a) the return time to Y^c is unbounded,
b) $A(x) = 1$ for $x \in Y$,
c) $(1,0) \in W^+(x)$ and $(0,1) \in W^-(x)$ for $x \in Y$.
We call A *weak*, if $A \in \mathcal{P}$ and there exists $Y \subset X$ with $0 < m(Y) < 1$ such that A is weak on Y.

Lemma 4.2 *There exist weak cocycles if (X,T,m) is aperiodic.*

Proof: If the dynamical system is aperiodic there exists for every $n \in N, n > 0$ and every $\epsilon > 0$ a measurable set Z such that $Z, T(Z), \dots, T^{n-1}(Z)$ are pairwise disjoint and such that $m(\bigcup_{k=0}^{n-1} T^k(Z)) \geq 1 - \epsilon$. This is *Rohlin's lemma* (for a proof see [Hal 56]) and the set Z is called a $(n,\epsilon)-$ *Rohlin set.*
Define the set

$$Y = \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{n} T^k(Z_n)$$

where Z_n is a $(n2^n, 1/2)$-Rohlin set. Then $m(Y) \leq 1/2$ and the return time to Y^c is not bounded. Take a diagonal cocycle $D(x) = Diag(\mu(x), \mu^{-1}(x))$ with $\mu(x) = 1$ for $x \in Y$ and $\mu(x) = 2$ for $x \in Y^c$. This cocycle D is weak. ■

The main result in this section is:

Theorem 4.3 *$\mathcal{P} \setminus int(\mathcal{P})$ is not empty if and only if (X,T,m) is aperiodic.*

For the proof we will need another lemma. Call

$$R(\phi) = \begin{pmatrix} cos(\phi) & sin(\phi) \\ -sin(\phi) & cos(\phi) \end{pmatrix}$$

and denote with $\mathcal{X}_Z$ the characteristic function of a measurable set $Z \subset X$.

Lemma 4.4 *If $A \in \mathcal{P}$ is weak on Y and $Z \subset Y$ is not a coboundary then the cocycle*

$$B(x) = R(\pi/2 \cdot \mathcal{X}_Z(x)) A(x)$$

is in $\mathcal{A} \setminus \mathcal{P}$.

Proof: Given a cocycle A which is weak on $Y \subset X$. The two sets

$$X^{+/-} = \{(x, W^{+/-}(x)) | x \in X\} \subset X \times P^1$$

are invariant under the skew product action $T \times A$. We call $T^{+/-}$ the action of $T \times A$ restricted to $X^{+/-}$ and $\mu^{+/-}$ the two ergodic $T \times A$ invariant measures projecting down to m. The dynamical systems $(X^{+/-}, T^{+/-}, \mu^{+/-})$ are isomorphic to (X, T, m). Define

$$B(x) = R(\pi/2 \cdot \mathcal{X}_Z(x))A(x)$$

where $Z \subset Y$ is not a coboundary. The set $X^+ \cup X^-$ is invariant under $T \times B$ and $(\mu^+ + \mu^-)/2$ is an invariant measure of $T \times B$ which projects down to m. The system $(X^+ \cup X^-, T \times B, (\mu^+ + \mu^-)/2)$ is isomorphic to $(X \times \{1, -1\}, T \times \psi^{-1}(Z), m \times \nu)$ which we have met in the last section. Stepin's result implies that the measure $(\mu^+ + \mu^-)/2$ is an ergodic $T \times B$ invariant measure on $X \times P^1$. This gives then with proposition 2.1b)

$$\begin{aligned} \lambda(B) &= |\int \log |A(x)w| d\mu^+(x, W) + \int \log |A(x)w| d\mu^-(x, W)|/2 \\ &= |\lambda^+(A) + \lambda^-(A)|/2 = 0 \end{aligned}$$

■

Proof of theorem 4.3: Assume (X,T,m) is aperiodic. It is enough to show: If A is weak then $A \in \mathcal{P} \setminus int(\mathcal{P})$. With lemma 4.2 follows then that $\mathcal{P} \setminus int\mathcal{P}$ is not empty.
Let $A \in \mathcal{P}$ be weak on Y and let $\epsilon > 0$ be given. We will construct a $B \in \mathcal{A}$ such that $\lambda(B) = 0$ and $|||B - A||| \leq \epsilon$. Choose $V \subset Y^c$, such that $T(V), \dots, T^n(V)$ are disjoint from Y^c and $m(V) > 0$. This is possible because the return time to Y^c is not bounded. Then there exists with corollary 3.5 a set $Z \subset V$ which is not a coboundary. Define

$$U = X \setminus \bigcup_{k=1}^{n-1} T^k(Z)$$

and look at the induced system (U, T_U, m_U). Then A_U is weak over $Y \cap U$ and with lemma 3.4 follows that Z is not a coboundary for T_U, because it is not a coboundary for T. Application of lemma 4.4 gives that

$$C = R(\pi/2 \cdot \mathcal{X}_Z)A_U$$

has zero Lyapunov exponent. Define the cocycle

$$B(x) = R(\pi/(2n) \cdot \mathcal{X}_{U^c}(x))A(x).$$

We check that

$$B_U = C.$$

This gives with lemma 2.2 and $\lambda(C) = 0$ also $\lambda(B) = 0$. Further

$$|||B - A||| \leq |||A||| \cdot \pi/2n \leq \epsilon.$$

We have shown that a weak cocycle is in $\mathcal{P} \setminus int\mathcal{P}$.
If (X, T, m) is periodic then the Lyapunov exponent is continuous and so $\mathcal{P} = int\mathcal{P}$. ■

Remarks:
1) We say $A, B \in \mathcal{A}$ are *cohomologous in* $\mathcal{A}$ if there exists $C \in \mathcal{A}$, such that $C(T)AC^{-1} = A$. Cohomologous cocycles have the same Lyapunov exponents and if A is conjugated to a weak cocycle then it is also in $P \setminus int\mathcal{P}$.
2) It was surprising for us to find diagonal cocycles in $\mathcal{P} \setminus int\mathcal{P}$. We expected that the arbitrary closeness of stable and instable directions are responsible for the discontinuity of the Lyapunov exponent. This can also be the case as the following remark indicates.
3) Assume $A \in \mathcal{P} \setminus int(\mathcal{P})$ and $A_n \to A$ with $\lambda(A_n) = 0$. Because $\mathcal{P}$ is dense in $\mathcal{A}$ (see [Kni 90]) we can find $B_n \to A$ with $|||B_n - A_n||| \leq 1/n$ and $B_n \in \mathcal{P}$. If μ_n denotes a $T \times B_n$ invariant proability measure projecting down to m then μ_n converges weakly to $(\mu^+ + \mu^-)/2$ where $\mu^{+/-}$ are the $T \times A$ invariant ergodic measures projecting down to m. In some sense the stable and unstable directions of B_n come closer and closer together as n is increasing.

5 Difficulty of the decision whether the Lyapunov exponent is positive

There are only a few methods to decide whether $A \in \mathcal{P}$ or not. The only method which works for general dynamical systems is Wojtkowsky's cone method [Woj 85]. But there are many examples where one measures positive Lyapunov exponent numerically without being able to prove it. This suggests that the general problem is difficult. The next theorem could be one of the reasons for the subtlety.

Theorem 5.1 *Given a measurable set* $Y \subset X$ *with* $0 < m(Y) < 1$. *There exists* $A \in \mathcal{A}$, *such that* $B = R(\pi/2 \cdot \mathcal{X}_Y)A \in \mathcal{P}$ *if and only if* Y *is a coboundary.*

Proof: Given $Y \subset X$ with $0 < m(Y) < 1$ we build the Kakutani skyscraper over Y, which is a partition $X = \bigcup_{i \geq 1} Y_i$ where $Y_1 = Y$ and $Y_{n+1} = T(Y_n) \setminus Y$. We have $m(Y_2) > 0$ because $m(Y) < 1$. Define $U = Y_2$ and the diagonal cocycle $A(x) = Diag(2, 2^{-1})$ for $x \in U$ and $A(x) = 1$ else. The Lyapunov exponent of A is

$$\lambda(A) = \log(2) \cdot m(U) > 0.$$

Clearly $(1,0) \in W^+(x)$ and $(0,1) \in W^-(x)$. We denote with $\mu^{+/-}$ the two ergodic $T \times A$ invariant measures on $X \times P^1$ which project down to m and have their support on

$$X^{+/-} = \{(x, W^{+/-}(x)) | x \in X\}.$$

If Y is not a coboundary we conclude like in the proof of lemma 4.4 that $(\mu^+ + \mu^-)/2$ is an ergodic $T \times B$ invariant measure projecting down to m and so $\lambda(B) = 0$.
If Y is a coboundary, that is if $Y = Z \Delta T(Z)$, there is a $T \times A$ invariant set

$$Q = \{(x, W^+(x)) | \ x \in Z\} \cup \{(x, W^-(x)) | \ x \in Z^c\}.$$

This set Q carries an ergodic $T \times A$ invariant measure μ which projects onto the measure m. Because $U = Y_2$ is disjoint from Y either $U \subset Z \cap T(Z)$ or U is disjoint from $Z \cup T(Z)$. This implies $U \subset Z$ or $U \subset Z^c$ and we have either $\{(x, W^+(x)) | \ x \in U\} \subset Q$

or $\{(x, W^-(x))|\ x \in U\} \subset Q$. Because $A(x)$ is different from the identity matrix only on U and is there $Diag(2, 2^{-1})$ we have

$$\lambda(B) = |\int_Q \log|A(x)w|d\mu(x,W)| = \log(2) \cdot m(U) = \lambda(A) > 0.$$

■

If we would have an algorithm to find out whether a given cocycle $A \in \mathcal{A}$ is in $\mathcal{P}$ or not we could also find out for a measurable set $Z \subset X$ if Z is a coboundary or not. So, the cohomology problem in $\mathcal{E}$ exhibits already a difficulty for calculating or estimating the Lyapunov exponents.

Let us mention to the end some open questions:

1) Assume T is a homeomorphism of a compact metric space X leaving a Borel probability measure m invariant. Is the upper Lyapunov exponent continuous on

$$C(X, Sl(2,R)) = \{A : X \to Sl(2,R)|\ A \ continuous\}?$$

2) We believe that the cohomology problem in $\mathcal{E}$ is difficult. Is there a difficult mathematical problem which is embeddable in the cohomology problem for measurable sets ?
3) For which $r \geq 0$ is there a nonempty set in $\mathcal{A}$ such that the Lyapunov exponent is there r times but not $r+1$ times differentiable ?
4) Find discontinuities of the Lyapunov exponent λ on special curves through $\mathcal{A}$. In the theory of random Jacobi matrices [Cyc 87] one would like to know about regularity properties of λ on the curve

$$E \mapsto A_E = \begin{pmatrix} V+E & -1 \\ 1 & 0 \end{pmatrix}$$

where $V \in L^\infty(X,R)$. Johnson [Joh 84] has examples for discontinuities in the case of almost periodic Schroedinger operators. An other interesting curve would be the circle

$$\beta \mapsto AR(\beta) = A \begin{pmatrix} cos(\beta) & sin(\beta) \\ -sin(\beta) & cos(\beta) \end{pmatrix}.$$

Can we always find $A \in \mathcal{A}$ where $\beta \mapsto \lambda(AR(\beta))$ is not continuous ?
5) Is every $A \in \mathcal{P} \setminus int\mathcal{P}$ cohomologous to a weak cocycle ?

Bibliography

[Akc 65] M.A.Akcoglu,R.V.Chacon. *Generalized eigenvalues of automorphisms.* Proc.Amer.Math.Soc., 16:676-680, 1965.

[Anz 51] H.Anzai. *Ergodic Skew Product Transformations on the Torus.* Osaka Math. J., 3:83-99, 1951.

[Bag 88] L.Baggett. *On circle valued cocycles of an ergodic measure preserving transformation.* Israel J. Math., 61:29-38, 1988.

[Cor 82] I.P.Cornfeld,S.V.Fomin,Ya.G.Sinai. *Ergodic Theory.* Springer, New York, 1982

[Cyc 87] H.L.Cycon,R.G.Froese,W.Kirsch,B.Simon. *Schroedinger Operators.* Texts and Monographs in Physics, Springer, 1987

[Den 76] M.Denker,C.Grillenberg,K.Sigmund. *Ergodic Theory on Compact Spaces.* Lecture Notes in Math., No. 527, Springer 1976

[Fri 70] N.A.Friedman. *Introduction to Ergodic Theory.* Van Nostrand-Reinhold, Princeton, New York, 1970

[Fur 81] H.Fürstenberg. *Recurrence in Ergodic Theory and Combinatorial Number Theory.* Princeton University Press, Princeton, New Jersey, 1981

[Hah 32] H.Hahn. *Reelle Funktionen, 1.Teil, Punktfunktionen.* Akademische Verlagsgesellschaft, Leipzig, 1932

[Hal 56] P.Halmos. *Lectures on Ergodic Theory.* The mathematical society of Japan, 1956

[Her 81] M.R.Herman. *Construction d'un difféomorphisme minimal d'entropie topologique non nulle.* Ergodic Theory & Dynamical Systems,1:65-76, 1981

[Joh 84] R.A.Johnson. *Lyapunov numbers for the almost periodic Schroedinger equation.* Illinois J. Math., 28:397-419, 1984

[Joh 86] R.A.Johnson. *Exponential Dichotomy, Rotation Number and Linear Differential Operators with Bounded Coefficients.* J. Differential Equations, 61:54-78, 1986

[Kif 82] Y.Kifer. *Perturbations of Random Matrix Products.* Z.Wahrscheinlichkeitstheorie verw.Gebiete, 61:83-95, 1982

[Kni 90] O.Knill. *Positive Lyapunov exponents for a dense set of bounded measurable Sl(2,R) cocycles.* Submitted for publication

[Led 82] F.Ledrappier. *Quelques Proprietés des Exposants Caracteristiques.* Lecture Notes in Math., No. 1097, Springer, 1982

[Lem 89] M.Lemańczyk. *On the Weak Isomorphism of Stricly Ergodic Homeomorphisms.* Monatsh. Math., 108:39-46, 1989

[Mer 85] K.D.Merrill. *Cohomology of step functions under irrational rotations.* Israel J. Math.,52:320-340, 1985.

[Pag 89] E. Le Page. *Regularité du plus grand exposant caractéristique des produits de matrices aléatoires indépendantes et applications.* Ann. Inst. Henri Poincaré , 25:109-142, 1989

[Rue 79] D.Ruelle. *Ergodic theory of differentiable dynamical systems.* Publ.math. de l'IHES, 50:27-58, 1979

[Rue 79a] D.Ruelle. *Analycity Properties of the Characteristic Exponents of Random Matrix Products.* Advances in Mathematics, 32:68-80, 1979

[Ste 71] A.M.Stepin. *Cohomologies of automorphism groups of a Lebesque space.* Functional Anal. Appl., 5:167-168, 1971

[Vee 69] W.Veech. *Strict ergodicity in zero dimensional dynamical systems and the Kronecker-Weyl theorem mod 2.* Trans.Amer.Math.Soc.,140:1-33, 1969

[Woj 85] M.Wojtkowski. *Invariant families of cones and Lyapunov exponents.* Ergodic Theory & Dynamical Systems, 5:145-161, 1985.

[You 86] L.-S.Young. *Random perturbations of matrix cocycles.* Ergodic Theory & Dynamical Systems, 6:627-637, 1986.

Lecture Notes in Mathematics 1486

L. Arnold H. Crauel J.-P. Eckmann (Eds.)

Lyapunov Exponents

Proceedings, Oberwolfach 1990

§ 3. Мультипликативная эргодическая теорема. Формулировка и примеры

Теорема 1. *Если выполнено условие (5) § 2, то однопараметрическая группа измеримых изоморфизмов $\{\widetilde{T}^t\}$ правильна по Ляпунову при $t \to \pm\infty$.*

Из этой теоремы и теоремы § 1 вытекает.

Теорема 2. *Если выполнено условие (5) § 2, то для почти всех x по мере μ существуют точные характеристические показатели Ляпунова всех порядков, т. е.*

$$\chi_{\pm}(x, e^k; \mathfrak{n}) = \lim_{t\to\pm\infty} \frac{1}{|t|} \ln \lambda(e^k, \mathfrak{n}(t, x)),$$

где $\lambda(e^k, \mathfrak{n}(t, x))$ — коэффициент растяжения по k-мерному направлению e^k.

Springer-Verlag

L. Arnold H. Crauel J.-P. Eckmann (Eds.)

Lyapunov Exponents

Proceedings of a Conference held in
Oberwolfach, May 28 - June 2, 1990

Springer-Verlag
Berlin Heidelberg New York
London Paris Tokyo
Hong Kong Barcelona
Budapest

Editors

Ludwig Arnold
Institut für Dynamische Systeme
Universität Bremen
Postfach 330 440
W-2800 Bremen 33, Germany

Hans Crauel
Fachbereich 9 Mathematik
Universität des Saarlandes
W-6600 Saarbrücken 11, Germany

Jean-Pierre Eckmann
Département de Physique Théorique
Université de Genève
CH-1211 Genève 4, Switzerland

Front cover:
V. I. Oseledets: A multiplicative ergodic theorem.
Trudy Moskov. Mat. Obsc. 19 (1968), 179-210

Mathematics Subject Classification (1980):
Primary: 58F
Secondary: 34F, 35R60, 58G32, 60F, 60G, 60H, 70L05, 70K, 73H, 73K, 93D, 93E

ISBN 3-540-54662-6 Springer-Verlag Berlin Heidelberg New York
ISBN 0-387-54662-6 Springer-Verlag New York Berlin Heidelberg

Printed in Germany

Typesetting: Camera ready by author
Printing and binding: Druckhaus Beltz, Hemsbach/Bergstr.
46/3140-543210 - Printed on acid-free paper

Preface

These are the Proceedings of a conference on Lyapunov Exponents held at Oberwolfach May 28 – June 2, 1990. The volume contains an introductory survey and 26 original research papers, some of which have, in addition, survey character.

This conference was the second one on the subject of Lyapunov Exponents. The first one took place in Bremen in November 1984 and lead to the Proceedings volume Lecture Notes in Mathematics # 1186 (1986). Comparing those two volumes, one can realize pronounced shifts, particularly towards nonlinear and infinite-dimensional systems and engineering applications.

We would like to thank the 'Mathematisches Forschungsinstitut Oberwolfach' for letting us have the conference at this unique venue.

March 1991

Ludwig Arnold (Bremen) | Hans Crauel (Saarbrücken) | Jean-Pierre Eckmann (Genève)

Table of Contents

Chapter 3: Infinite-dimensional Random Dynamical Systems

Chapter 4: Deterministic Dynamical Systems

Chapter 5: Engineering Applications and Control Theory