

Random Schrödinger operators arising from lattice gauge fields. I. Existence and examples

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We consider new models of ergodic Schrödinger operators whose existence relies on a cohomological theorem of Feldman and Moore in ergodic theory. These operators generalize the Harper operator which describes the case of a constant magnetic field. An example is the case when the magnetic field is given by independent random variables attached to the lattice plaquettes. A generalization of the Feldman–Moore theorem by Lind to non-Abelian groups also allows us to consider Schrödinger operators obtained from non-Abelian lattice gauge fields. The existence result extends to more general graphs like to operators on tilings and to higher dimensions. We compute some moment expansions for the density of states. For example, for independent, identically and uniformly distributed magnetic fields, a model which has been studied at least since 1970, and whose existence can also be seen without involving the above-mentioned existence theorem, we show that the n th moment is the number of closed paths in the two-dimensional lattice starting at the origin for which the winding number vanishes at each plaquette point. This goes beyond the Brinkman–Rice self-retracing path approximation. Other examples are a higher dimensional example, a one-dimensional Anderson model which can be treated in this framework, as well as the Hofstadter model with constant magnetic field, where one averages over all possible magnetic fields. We also reprove a result of Jitomirskaya–Mandelstam stating that the deterministic Aharonov–Bohm model is a compact perturbation of the free Laplacian. © 1999 American Institute of Physics. [S0022-2488(99)03911-0]

I. INTRODUCTION

In this article, we consider a class of ergodic discrete Schrödinger operators which we call discrete random electromagnetic Laplacians. An example in two dimensions is the bounded ergodic self-adjoint operator $L = A + A^*$ on $l^2(\mathbf{Z}^2, \mathbf{C}^N)$, where

$$(Au)_n = A_1(n)u_{n+e_1} + A_2(n)u_{n+e_2}$$

and where the unitary matrices $A_i(n) \in U(N)$ have the property that the magnetic fields

$$B(n) = A_2(n)^* A_1(n+e_2)^* A_2(n+e_1) A_1(n)$$

on different plaquettes are identically distributed $U(N)$ -valued random variables with law μ (Fig. 1).

The question arises: Given a magnetic field B determined by an arbitrary stochastic process, can we find A such that $B = dA$? If the $B(n)$ are invariant under translation in one direction, the answer is no in general: For a measurable circle-valued map B , there is in general no measurable circle-valued map A such that $B = dA = A(T)A^{-1}$. In a probabilistic or ergodic theoretical setup, nontrivial cohomological constraints appear. Feldman and Moore noticed, however, in Ref. 1 that in dimensions $d=2$, these constraints are absent. This stays true even if the Abelian group $U(1)$ is

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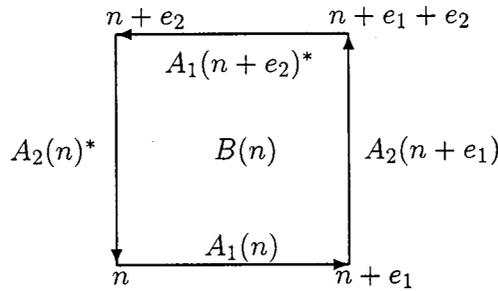


FIG. 1. The magnetic field $B=dA$ at a plaquette n is obtained by integrating up A along the boundary $B(n) = A_2(n) * A_1(n+e_2) * A_2(n+e_1) A_1(n)$.

replaced by a non-Abelian group like $U(N)$.² For Abelian groups, we generalized this theorem in Ref. 3 to higher dimensions. The existence theorem furthermore extends to discrete ergodic Laplacians on more general graphs like graphs defined by aperiodic tilings.

While in special cases the independent magnetic fields can be obtained directly by choosing the random variables $A_i(n)$ to be independent, this is not true in general. Already in the case of independent identically distributed magnetic fields, there are counterexamples: independent identically distributed random variables $A_i(n)$ lead in general to correlations between the magnetic field variables $B(n)$: If, for example, $A_i(n)$ take randomly the two values 1 and $e^{i\pi/4}$, then two adjacent plaquettes P_n, P_m cannot have magnetic fields $B(n)=1$ and $B(m)=-1$. In other words, vector potentials A , which give independent identically distributed magnetic fields B , are not independent in general.

There are several motivations to study such operators.

(1) Discrete magnetic Laplacians in two dimensions are tight binding approximations for the quantum mechanical model of an electron in the plane exposed to an ergodic magnetic field. This generalizes the Harper operator for which the magnetic field is constant in the plane. Independent identically distributed magnetic fields are models for which the spectral type⁴⁻⁶ is still unclear. One of the questions is whether such models have eigenvalues.⁷ Another problem is to describe the density of states for which approximations have been known since Ref. 8 and for which numerical investigations⁹⁻¹² have been done.

(2) Each random operator is an element of a C^* algebra (\mathcal{X}, tr) which is determined entirely by the field F . This generalizes the case when $B = F_{12} = e^{2\pi i \alpha}$ is constant and where \mathcal{X} is the rotation algebra.^{13,14} The relatively abstract Feldman–Moore theorem in ergodic theory allows us to so define subalgebras in the crossed products of $L^\infty(X)$ with a \mathbf{Z}^2 action which is generated by two unitaries U, V which have a prescribed commutator $UVU^{-1}V^{-1}$ in $L^\infty(X)$.

(3) In the partition function of one matrix models appears a van der Monde determinant (see, e.g., Ref. 15). Using the potential theoretical energy $I(L) = -\int \int \log|E-E'| dk(E) dk(E')$ of the density of states dk of an ergodic Schrödinger operator L one can define an infinite-dimensional van der Monde determinant $e^{-I(L)}$. This leads to the variational problem $L \mapsto e^{-I(L)}$ which is the topic of Ref. 16.

We now give an overview over the results of this paper. We first define electromagnetic Laplacians and observe that the Feldman–Moore–Lind results imply that ergodic magnetic Laplacians exist. The formalism can be considered as an ergodic version of differential forms. To a field $F=dA$ is attached a current $j=d^*F$ which is divergence-free $d^*j=0$. In two dimensions, not every current is given by a field. The equivalence classes of currents j , modulo currents of the form d^*F is the cohomology group $\mathcal{H}^1(U)$, a group with the cardinality of the continuum.³ However, in dimensions $d>2$, it is again a consequence of the triviality of higher dimensional cohomology groups that every one-form j is of the form $j=d^*F$.

In Sec. II, we compute moments of the density of states for independent identically distributed magnetic Laplacians in two dimensions. If the law of the magnetic field B is the Haar measure μ_{Haar} on $U(1)$, then the density of states is determined by a random walk in \mathbf{Z}^2 having the

following global geometrical constraints: the n th moment of the density of states, $\text{tr}(L^n)$ is the number of closed paths in \mathbf{Z}^2 which have length n and give zero winding number to every plaquette. The combinatorial problem to compute the moments of the density of states was considered first in Ref. 8, where the approximation was used that the paths should have no loops. This is now called the ‘‘Brinkman–Rice approximation.’’ The exact expression which we give in this paper for the number of paths is new and was not mentioned in Ref. 8 or in subsequent works on the problem that we are aware of. We show also that random magnetic fields with law μ_{Haar} can be generated by taking μ_{Haar} -distributed vector potentials, so that in this special case, Feldman–Moore’s existence theorem is not needed. We notice then that all the spectral properties of the operators in the Abelian as well as non-Abelian case depend only on the field $F=dA$ and not on the specific realization of the vector potential A . The explicit calculation of the moments of the density of states for independent identically distributed fields leads to an Aubry duality for the deformed operators $L_\lambda=A_1+A_1^*+\lambda(A_2+A_2^*)$: the density of states of L_λ is related to the density of states of $L_{1/\lambda}$ in the same way as for the Harper case.¹⁷

Some other examples follow in Sec. III. We review a result of Jitomirskaya and Mandelshtam¹⁸ stating that a change of the field on a finite set of cells is a compact perturbation of the operator. A special case is the magnetic Aharonov–Bohm operator with magnetic flux $B \in U(1)$ different from 1 only in one cell. This result stays true for aperiodic lattices like the Penrose lattice. It seems to be unknown, whether the Aharonov–Bohm perturbation from the free operator is trace class. Also the existence result generalizes to other periodic graphs or aperiodic tilings. We notice for example that to any measure μ on $U(1)$, there exists a measurable vector potential on a Penrose lattice such that the magnetic fields in the plaquettes are independent, identically distributed $U(1)$ -valued random variables with law μ . Because a Penrose graph is not a Cayley graph of a group, the more abstract setup of countable ergodic equivalence relations developed in Ref. 1 is needed.

II. NOTATION AND EXISTENCE

We consider first the two-dimensional case. Let (X, \mathcal{F}, m) be a probability space. Two commuting measure-preserving invertible transformations T_1, T_2 on X define a dynamical system with time \mathbf{Z}^2 . Let U be a Polish (=complete separable metrisable) group. Examples are subgroups of Lie groups like the unitary groups $U(N)$ or the special linear group $SL(N, \mathbf{C})$. A two-form $B \tau_{12}$ is defined by a measurable map $B \in \mathcal{U} = \mathcal{L}(X, U) = \{B | X \rightarrow U, \text{measurable}\}$. Two measurable U -valued maps $A_1, A_2 \in \mathcal{U}$ define a one-form or vector potential $A = A_1 \tau_1 + A_2 \tau_2$. Define the curvature of A as the two-form $dA \tau_{12}$ with

$$dA(x) = A_2^{-1}(x)A_1^{-1}(T_2x)A_2(T_1x)A_1(x).$$

Not every two-form B can be written as $B=dA$ with a one-form A . For example, if T_1 is the identity map and $T_2=T$ is ergodic, then not every measurable map $B \in \mathcal{U} = \mathcal{L}(X, U)$ can be written as $B=A^{-1}A(T)$ with $A \in \mathcal{U}$ because the cohomology group

$$\mathcal{H}^1(U) = \mathcal{U} / \{A \in \mathcal{U} | B = A^{-1}A(T)\}$$

of cocycles modulo coboundaries is nontrivial. We know that this group has the cardinality of the continuum.³ The following result of Feldman and Moore¹ was extended by Lind² to non-Abelian groups. A dynamical system given by a group T^g of automorphisms on (X, \mathcal{F}, m) is called free, if $m(\{T^g(x)=x\}) > 0$ implies $g=0$.

Theorem II.1 (Feldman–Moore–Lind): *Assume that the \mathbf{Z}^2 -dynamical system is free. Let U be a not necessarily Abelian Polish group. For any magnetic field distribution $B \tau_{12}$ with $B \in \mathcal{U}$, there is a vector potential $A = A_1 \tau_1 + A_2 \tau_2$, which satisfies $dA = B$.*

Example. A magnetic field B taking values in $\{1, -1\}$ is determined by the measurable set $Y = B^{-1}(-1) = \{x | B(x) = -1\}$. Feldman–Moore’s result implies that there exist two measurable sets Z_1, Z_2 such that

$$Y = Z_1 + T_1(Z_1) + Z_2 + T_2(Z_2),$$

where $+$ is the symmetric difference, the addition in the group \mathcal{F} .

Assume now that U is a subgroup of the unitary group $U(N)$ of $n \times n$ matrices. Given a one-form $A = A_1\tau_1 + A_2\tau_2$, we define a discrete self-adjoint random Schrödinger operator $L = A + A^*$ as follows: For almost all $x \in X$, consider the operator $L(x)$ on $l^2(\mathbf{Z}^2, \mathbf{C}^N)$ given by $(L(x)u) = (A(x) + A(x)^*)u$, where

$$(A(x)u)_n = A_1(x)u_{n+e_1} + A_2(x)u_{n+e_2}$$

and where $e_1 = (1,0), e_2 = (0,1)$ are the basis vectors in \mathbf{Z}^2 . We call $L = A + A^*$ a discrete random magnetic Laplacian. Such operators are discrete versions of the continuous operators $L = (\nabla - iA)^2$ (see, e.g., Ref. 5). We also call them “random magnetic Laplacians” if $U = U(1)$ or “random Yang–Mills Laplacians” if $U = U(N)$. Associated with L is a one-parameter family of operators $L = A_1 + A_1^* + \lambda(A_2 + A_2^*)$, $\lambda \in \mathbf{R}$ in which we will mainly concentrate on the case $\lambda = 1$. The field of a random Laplacian $L = A + A^*$ is defined for $d \geq 2$ as $F_{ij} = dA_{ij} = A_j^{-1}A_i(T_j)^{-1}A_j(T_i)A_i$. If $U = U(1)$, one says that the magnetic field $B = A_2^{-1}A_1(T_2)^{-1}A_2(T_1)A_1$ has the magnetic flux $\arg(B)$. The phases of L are the functions $\arg(A_i)$.

The operator L is not uniquely defined by the field B . With a zero-form $C \in \mathcal{U}$, the gauge transformed operator CLC^{-1} is also a discrete random Laplacian with the same magnetic field B but the gauge potential A has changed to $CAC^*\tau = \sum_i C_i A_i C_i(T_i)^*\tau_i$. The choice of the gauge is not the only source of nonuniqueness. The nontrivial nonuniqueness is measured by the moduli space of flat fields $\{(A_1, A_2) | dA = 0\} / \{A = dC\}$ which is for Abelian U as a group isomorphic to the first cohomology group $\mathcal{H}^1(U)$.

The above-mentioned definitions generalize to the higher dimensional case. Take d automorphisms T_1, \dots, T_d on a probability space (X, \mathcal{F}, m) . A one-form $A = \sum_{i=1}^d A_i \tau_i$ is given by d functions $A_i \in \mathcal{U} = \mathcal{L}(X, U(N))$ and defines a field

$$dA = F = \sum_{i < j} F_{ij} \tau_{ij},$$

where $F_{ij} = A_i A_j(T_i) A_j(T_i)^{-1} A_i^{-1}$. This gives random self-adjoint operator $L = A + A^*$ where each $L(x)$ acts on the Hilbert space $l^2(\mathbf{Z}^d, \mathbf{C}^N)$ by

$$(L(x)u) = \sum_{i=1}^d A_i(x)u + A_i^*u.$$

Theorem II.2 (Triviality of cohomology groups in higher dimensions): *If U is Abelian and \mathbf{Z}^d acts freely, then $dF = 0$ implies that there exists A such that $dA = F$.*

The proof of this theorem which was included in Ref. 19 is now the subject of a separate article.³ We call a random operator $L = A + A^*$ determined by F a random discrete electromagnetic Laplacian with field F . Examples are given in the following.

(1) A Harper operator is obtained when the field B takes a constant value in $U(N)$. Diagonalization reduces all questions to the case $N = 1$. For reviews see Refs. 20–22.

(2) An example of a quasiperiodic magnetic Laplacian is defined by the dynamical system $(U(1), T_1 : \theta \mapsto \theta + \alpha, T_2 : \theta \mapsto \theta + \beta, d\theta)$ and the magnetic field $B(n)(\theta) = \exp(i(\theta + n_1\alpha + n_2\beta)) \in U(1)$. The spectrum of the Laplacian is determined by the two real numbers α, β . Non-Abelian versions, where T_i are translations on the unitary group $U(N)$ are defined similarly. Note however that the vector potential (A_1, A_2) is only measurable and we cannot expect it to be almost periodic.

(3) An example of a limit periodic magnetic field is $B(n) = \sum_{j=0}^{\infty} b_j \cos(2\pi(n_1 2^{-j} + n_2 3^{-j}))$ with $\sum_j |b_j| < \infty$. As in the previous example, the operator L is not limit periodic and only the physically relevant field B is.

(4) Given a probability measure μ on $U(1)$ and independent, identically distributed random variables $B(n)$, $n \in \mathbf{Z}^2$ with law μ . Even so $B(n)$ are independent, the vector potential A is in general not given by independent identically distributed random variables.

(5) An example of an aperiodic, strictly ergodic field taking only finitely many values is $B(n) = 1 - 2 \cdot 1_{[0, \gamma)}(\theta + n_1 \alpha + n_2 \beta) \in \{-1, 1\}$, where γ, α, β are rationally independent.

(6) In all these examples, one obtains one-parameter families of deformed operators $L = A_1 + A_1^* + \lambda(A_2 + A_2^*)$ with $\lambda \in \mathbf{R}$. In the stationary independent magnetic field case the almost sure spectral properties of L depend only on λ and the field $B = dA$.

Higher dimensional cohomology groups^{23–25} are defined as follows (see Refs. 23–25).

Let $I = \{1, \dots, d\}$ and let \mathcal{T}_p be the set of sets $J = \{j_1 < j_2 < \dots < j_p\} \subset I$. Let \mathcal{C}^p be the set of maps $A: \mathcal{T}_p \rightarrow \mathcal{U}$ which becomes a group by pointwise addition. Extend this map to the set of all p -tuples $J = (j_1, j_2, \dots, j_p)$ with $j_k \in I$ by requiring $A_{\pi(J)} = \text{sign}(\pi) A_J$ for any permutation π of J . We write $A = \sum_J A_J \tau_J$. Define $d_p: \mathcal{C}^p \rightarrow \mathcal{C}^{p+1}$ by

$$d_p A = \sum_{i, J} (A_J(T_i) - A_J) \tau_{iJ}.$$

The kernel of d_p contains cocycles of degree p , whereas the image of d_{p-1} consists of coboundaries of degree p . Because for $A = \sum_J A_J \tau_J$,

$$d_p \circ d_{p-1} A = \sum_{i, j, J} [A_J(T_i T_j) - A_J(T_i) - A_J(T_j) + A_J] \tau_{i j J}$$

is both symmetric and antisymmetric in i, j , it must vanish and $d_p \circ d_{p-1} = 0$ gives rise to geometric cohomology groups $\mathcal{H}_{\text{geom}}^p(G, \mathcal{U}) = \ker(d_p) / \text{im}(d_{p-1})$.

Cohomology of currents. Given a \mathbf{Z}^d action on the group $\mathcal{U} = \mathcal{L}(X, U)$. A one-form $A = \sum_i A_i \tau_i \in \mathcal{C}^1$ defines an electromagnetic field $F = dA \in \mathcal{C}^2$ and so a current $j = d^* F = *d^* F \in \mathcal{C}^1$, where an asterisk (*) is the Hodge operation $*: \mathcal{C}^n \rightarrow \mathcal{C}^{d-n}, A_I \mapsto (-1)^{n(d-n)} A_{I^*}$, where $I^* = \{1, \dots, n\} \setminus I$. A current is defined even if the group \mathcal{U} is non-Abelian.

Proposition II.3: Assume $N \geq 1, d = 2$ or $N = 1, d \geq 2$. Every current $d^* F = j$ is divergence free: $d^* j = 0$.

Proof: If $d = 2$, the Hodge involution for one-forms is given by $A_1 \tau_1 + A_2 \tau_2 = (A_1, A_2) \mapsto (A_2, A_1^{-1})$. The divergence of j is given by

$$d^* j = *d^*(j_1, j_2) = (*d)(j_2, j_1^*) = j_1 j_2 (T_2)^* j_1^*(T_1) j_2.$$

If we plug in $j = (j_1, j_2) = d^* F = (F^* F(T_2), F(T_1)^* F)$, we get

$$d^* j = F^* F(T_2) F(T_2)^* F(T_1 T_2) F(T_2 T_1)^* F(T_1) F(T_1)^* F = 1.$$

In the Abelian case, $d^* j = 0$ follows in any dimension from $d^* d^* = 0$. □

One can ask whether every current j which is divergence free $d^* j = 0$ does come from a field F satisfying $d^* F = j$ (Fig. 2). The answer is “no” in two dimensions and “yes” in dimensions three or higher. There are uncountably many equivalence classes of currents in two dimensions because of the following Proposition.

Proposition II.4: Assume $d = 2$, let U be a Polish group and let G be a free \mathbf{Z}^2 action. The moduli space of all divergence free currents j modulo currents j coming from fields $j = d^* F$ is isomorphic to the first cohomology group $\mathcal{H}^1(U)$. On the other hand, for $d \geq 3$ and Abelian U , every divergence free current j is of the form $d^* F$.

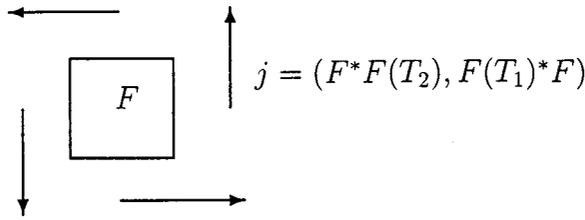


FIG. 2. Illustration of the current of a not necessarily Abelian Aharonov-Bohm field, where the field F is constant different from 1 on one plaquette only.

Proof: Assume first $d=2$. j is a cocycle if $j_2 j_1^*(T_1) = j_1^* j_2(T_2)$ and a coboundary if there exists a solution F of $j_1 = F(T_2)F^*$, $j_2 = FF(T_1)^*$. If j is a cocycle, then the Hodge dual $\tilde{j} = *j$ satisfies a zero curvature equation. Also, $d^*F = j$ if and only if \tilde{j} is a gradient $d(*F) = \tilde{j}$. The moduli space of zero curvature fields modulo gradient fields is $\mathcal{H}^1(U)$.

Assume now $d \geq 3$ and that U is Abelian. Given the one-form j , define the $(d-1)$ -form $\tilde{j} = *j$. Since $\mathcal{H}^{d-1}(U)$ is trivial for $d \geq 3$, there exists a $(d-2)$ -form \tilde{F} satisfying $d\tilde{F} = \tilde{j}$. Let $F = *\tilde{F}$. Then $d^*F = j$. □

Question. We do not know whether in dimensions $d \geq 3$, every current j can be written as d^*F with a field F satisfying additionally $dF=0$. If this were true and F were unique, an interesting class of higher dimensional operators $L = A + A^*$ were defined by taking independent identically distributed random variables j and taking A satisfying $d^*dA = j$.

III. GENERAL REMARKS ON THE SPECTRUM

If U is a subgroup of the unitary group $U(N)$, the Laplacian L is an element of a von Neumann algebra \mathcal{X} which is the crossed product of $\mathcal{A} = L^\infty(X, M(N, \mathbf{C}))$ with the \mathbf{Z}^d -action generated by automorphisms $f \mapsto f(T^n)$, where $f(T^n)(x) = f(T^n x)$. The algebra \mathcal{X} is obtained by completing the algebra of all polynomials in the variables τ_1, \dots, τ_d with coefficients in \mathcal{A} ,

$$K = \sum_{n \in F \subset \mathbf{Z}^d} K_n \tau^n, \quad (KL)_n = \sum_{l+m=n} K_l L_m(T^l) \tau^n$$

with respect to the norm $\|K\| = \| \|K(x)\| \|_\infty$. Here, $K(x)$ is the bounded linear operator on $l^2(\mathbf{Z}^d, \mathbf{C}^N)$ defined by $(K(x)u)(n) = \sum_m K_m(x)u(n+m)$ and $\|\cdot\|$ is the operator norm on $\mathcal{B}(l^2(\mathbf{Z}^d))$ and $\|\cdot\|_\infty$ the essential supremum norm. The involution in \mathcal{X} is $(\sum_n K_n \tau^n)^* = \sum_n K_n^*(T^{-n}) \tau^{-n}$ and the trace is $\text{tr}(K) = \int_X \text{Tr}(L_0(x)) d\mu(x)$, where Tr denotes the usual trace on the finite dimensional matrix algebra $M(N, \mathbf{C})$. This construction of Murray and von Neumann works in the same way, when \mathbf{Z}^d is replaced by a more general discrete group.

For any self-adjoint $L \in \mathcal{X}$, and if f is a continuous, bounded function on \mathbf{R} , the element $f(L) \in \mathcal{X}$ is defined through the functional calculus. The functional $f \mapsto \text{tr}(f(L))$ on $C(\mathbf{R})$ defines a measure on \mathbf{R} , called the density of states of L . If the \mathbf{Z}^2 action is ergodic, then the spectrum of $L(x)$ is constant almost everywhere and coincides with the support of the density of states.

The magnetic field does not determine L because of gauge ambiguity. However, all the information about the spectrum is determined from B only:

Proposition III.1: Assume $N \geq 1, d=2$ or $N=1, d \geq 2$. Given two one-forms A, \tilde{A} which satisfy $d\tilde{A} = dA = B$. Define $L = A + A^*, \tilde{L} = \tilde{A} + \tilde{A}^*$. There exists for every x a unitary operator $U(x)$ on $l^2(\mathbf{Z}^d, \mathbf{C}^N)$ such that $U^*(x)L(x)U(x) = \tilde{L}$. Especially, the density of states, the (in the ergodic case almost everywhere constant) spectral types and the spectrum depend only on the field B .

Proof: We consider first a setup, where measurability is discarded. Let $\mathcal{U} = U(N)^{\mathbf{Z}^2}$ be the set of possible fields $n \mapsto B(n)$. For every $B \in \mathcal{U}$, we can find a bounded self-adjoint operator $L = A + A^*$ on $l^2(\mathbf{Z}^2, \mathbf{C}^N)$ which has the field $B = dA$. The special gauge $A_2(n, m) = A_1(0, m) = 1$ for

$n, m \in \mathbf{Z}$, determines A and makes $L = L_B$ unique. For $N = 1$, a canonical gauge can also be defined for $d \geq 2$: let $E_i \subset \mathbf{Z}^d$ be the vector space spanned by (e_1, e_2, \dots, e_i) . Put $A_i = 1$ on E_1 , then on all lines orthogonal to E_1 in E_2 and inductively on all lines orthogonal to E_j in E_{j+1} . This determines A as can be seen by induction: first construct A on E_2 then on edges orthogonal to E_2 in E_3 , etc., always using $dA = B$. The condition $dB = 0$ assures that the definition is consistent. The map $B \mapsto L_B$ is continuous if \mathcal{U} has the product topology and $\mathcal{B}(l^2(\mathbf{Z}^d))$ has the strong operator topology. However a change of the magnetic field of one single plaquette changes L_B globally.

Take the diagonal operator $U_{n,m} = G(n) \delta_{n,m}$, where $G(n)A_i(n)G(n - e_i) = \tilde{A}_i$. This function $n \mapsto G(n)$ exists since any vector potential can be gauged to the canonical gauge.

For ergodic operators L , the density of states of L and \tilde{L} exists. We do not need the measurability of the conjugating operator $U(x)$ to get $L^n(x)_{00} = \tilde{L}^n(x)_{00}$. It follows that $\text{tr}(L^n) = \int_X \text{Tr}(L^n(x)_{00}) dm(x) = \int_X \text{Tr}(\tilde{L}^n(x)_{00}) dm(x) = \text{tr}(\tilde{L}^n)$.

Similarly, $(\phi, L^n(x)\phi) = (U(x)\phi, \tilde{L}^n(x)U(x)\phi)$ for every $\phi \in l^2(\mathbf{Z}^d, \mathbf{C}^N)$, so that also the spectral types are the same. □

Remark: The map $x \mapsto U(x)$ is not measurable in general. Examples with $T_1 = \text{Id}$ show this. In other words, while the operators $L(x)$ and $\tilde{L}(x)$ are conjugated in $\mathcal{B}(l^2(\mathbf{Z}^d, \mathbf{C}^N))$, the conjugation is in general not possible in the algebra \mathcal{X} .

Corollary III.2: Given a sequence of operators $L^{(n)} = A^{(n)} + (A^{(n)})^*$ and an operator $L = A + A^*$ defined over the same \mathbf{Z}^d action. Assume that the fields $B^{(n)} = dA^{(n)}$ converge to $B = dA$ in $L^\infty(X, \mathcal{U})^d$. If there exists an interval $I \subset \mathbf{R}$ such that $\sigma(L^{(n)}) \cap I = \emptyset, \forall n \in \mathbf{N}$ then also $\sigma(L) \cap I = \emptyset$.

Proof: In the canonical gauge, the operators converge pointwise in the strong operator topology and so in the resolvent sense. The claim follows from general principles (Ref. 26 Theorem VIII.24). □

IV. INDEPENDENT IDENTICALLY DISTRIBUTED MAGNETIC FIELDS

We concentrate in this paragraph on the case of magnetic Laplacians, where the magnetic fields $\{B(n)\}_{n \in \mathbf{Z}^2}$ are independent, identically distributed $U(1)$ -valued random variables B with law μ . This means that the probability that B takes a value in some interval $I \subset \mathbf{T}$ is $\mu(I)$. Denote by $\hat{\mu}_n = \int_{U(1)} z^n d\mu$ the n th moment of μ . The sequence $\{\hat{\mu}_n\}_n$ is the Fourier transform of the measure μ . Denote by Γ_n the set of oriented closed paths in \mathbf{Z}^2 of length n . Let $n(\gamma, P)$ be the winding number of the path γ with respect to the plaquette P . The moments of the density of states can be computed with a random walk expansion as it is used in statistical physics (see, e.g., Ref. 27).

Proposition IV.1: Let μ be a Borel measure on the circle $U(1)$. The n th moment of the density of states $\text{tr}(L^n)$ of an independent identically distributed magnetic Laplacian with law μ is

$$\sum_{\gamma \in \Gamma_n} \prod_P \left(\int_{U(1)} z^{n(\gamma, P)} d\mu(z) \right) = \sum_{\gamma \in \Gamma_n} \prod_P \hat{\mu}_{n(\gamma, P)}.$$

Proof: Given A satisfying $dA = B$. Write $\int_\gamma A$ for the product of the A_i along the path γ . The path γ encloses a region $\Omega(\gamma)$ which is a collection $\{P\}$ of plaquettes. We use the discrete Green formula $\int_\gamma A = \prod_P B^{n(\gamma, P)}$ to compute

$$\begin{aligned} \text{tr}(L^n) &= \sum_{\gamma \in \Gamma_n} \int_X \int_\gamma A(x) dm(x) \\ &= \sum_{\gamma \in \Gamma_n} \int_X \prod_P B_P(x)^{n(\gamma, P)} dm(x) \\ &= \sum_{\gamma \in \Gamma_n} \prod_P \int_X B_P(x)^{n(\gamma, P)} dm(x) \end{aligned}$$

Proof: In this case, $\hat{\mu}_k = \delta_{k,0}, \forall k \in \mathbf{Z}$. This implies $\Pi_P(\int_{U(1)} z^{n(\gamma,P)} d\mu(z)) = 0$ if there exists a plaquette P for which the path γ has positive winding number $n(\gamma,P)$. If the winding number is zero for all P , then $\Pi_P(\int_{U(1)} 1 d\mu(z)) = 1$. □

Remarks: (1) The number of closed paths γ of length n in \mathbf{Z}^2 for which $n(\gamma,P) = 0$ for all plaquettes P is in general strictly larger than the number of closed contractible paths of length n . A path can visit different plaquettes at different times without being contractible. See Fig. 3.

The additional paths not treated in the Brinkman–Rice approximation are so numerous that the radius of convergence for the Green function changes. Random walks with the stronger topological constraint of being contractible were investigated in Ref. 28.

(2) For the deformed operator $L = A_1 + A_1^* + \lambda(A_2 + A_2^*)$ with independent Haar distributed functions A_i , we get

$$\text{tr}(L^n) = \sum_{\gamma \in \Gamma_n^{(0)}} \lambda^{y(\gamma)},$$

where $\Gamma_n^{(0)}$ is the set of paths in Γ_n which give zero winding number to every plaquette and $y(\gamma)$ is the number of steps the path γ makes in the y direction.

Corollary IV.4: Let μ be the Haar measure on a finite cyclic subgroup \mathbf{Z}_p of $U(1)$. Then $\text{tr}(L^n)$ is the number of closed paths of length n beginning at $0 \in \mathbf{Z}^2$ for which the winding numbers satisfy $n(\gamma,P) = 0 \pmod p$ for all plaquettes P .

Proof: $\Pi_P(\int_{U(1)} z^{n(\gamma,P)} d\mu(z)) = 0$, if there exists a plaquette P which has a winding number $n(\gamma,P)$ which is not zero modulo p . □

Remarks: (1) The independence of the magnetic fields is essential in Proposition IV.1 and its Corollaries. Independent vector potentials would not be enough in general.

(2) The random walk expansion in Proposition IV.1 shows that $\text{tr}(L^n)$ is a polynomial in the infinite set of variables $\hat{\mu}_k, \hat{\mu}_k^{-1}$ with integer coefficients.

Let us illustrate the random walk expansion in the almost Mathieu–Harper case, where μ is a point measure so that the magnetic field is constant:

Corollary IV.5: If μ is the Dirac measure on $e^{2\pi i\alpha} \in U(1)$, then

$$\text{tr}(L^n) = \sum_{\gamma \in \Gamma_n} \prod_P e^{2\pi i\alpha n(\gamma,P)}.$$

(Especially, if μ is the Dirac measure on $1 \in U(1)$, then $\text{tr}(L^n) = ((2n)!)^2 (n!)^{-2}$ is the number of closed paths of length n beginning at $0 \in \mathbf{Z}^2$.) For the Harper magnetic Laplacian $L = A_1 + A_1^* + \lambda(A_2 + A_2^*)$,

$$\text{tr}(L^n) = \sum_{\gamma \in \Gamma_n} \lambda^{y(\gamma)} \prod_P e^{2\pi i\alpha n(\gamma,P)},$$

where $y(\gamma) \in 2\mathbf{N}$ is the number of steps of γ in the y direction.

For another illustration, let L be the Harper operator over the (nonergodic) integrable “twist map” $T:(x,y) \mapsto (x+y,y)$ on the two-dimensional torus \mathbf{T}^2 with invariant Lebesgue measure. We consider the potential $V(x,y) = 2 \cos(x)$. Call the random (now nonergodic) operator the “Hofstadter operator.” The density of states has zero Lebesgue measure.²⁹ The following illustration should be compared with the case of a random magnetic field:

Corollary IV.6: If L is the Hofstadter operator, then $\text{tr}(L^n)$ is the number of closed paths of length n beginning at $0 \in \mathbf{Z}^2$ for which the sum of all winding numbers $\sum_P n(\gamma,P)$ over all plaquettes P vanishes. □

Proof: $\int_{\mathbf{T}} \exp(\sum_P 2\pi i y n(\gamma,P)) dy = 1$ if and only if $\sum_P n(\gamma,P) = 0$.

Remarks: (1) Random walk expansions would work on any planar graph generalizing \mathbf{Z}^2 (see also Sec. VI). A solvable case is the Bethe lattice \mathbf{B}_{2h} with degree $2h$, where Kesten determined the number of closed paths of length n .³⁰ His calculation leads to $\text{tr}(L - E)^{-1} = (2h - 1)/(E/h$

$-1) + h\sqrt{E^2 - 4(2h-1)}$). The imaginary part of this divided by π is the density of states which has support on $[-2\sqrt{2h-1}, 2\sqrt{2h-1}]$. (For a modern calculation see Ref. 31). Since there are no closed loops on the Bethe lattice, there is no magnetic field and the density of states of any magnetic Laplacian on the Bethe lattice is the same.

(2) Corollary IV.5 shows that $\text{tr}(L^4) = 8 + 4(\hat{\mu}_0 + \hat{\mu}_0^{-1})$, so that discrete random magnetic Laplacians with different values of $\text{Re}(\hat{\mu}_0)$ have different density of states. Especially, almost Mathieu operators with different $\cos(\alpha)$ cannot be isospectral (see also Ref. 32).

(3) By changing the orientation of the plaquettes, it becomes obvious that the laws μ and $\mu(A) = \mu(A)$ give isospectral Laplacians.

In some cases, the theorem of Feldman–Moore is not needed for constructing the Laplacian.

Proposition IV.7: Let U be a compact subgroup of $U(1)$. If the law μ of the independent identically distributed U -valued random variables $\{A_i(n)\}_{n \in \mathbf{Z}, i=1,2}$ of the vector potential $A_1\tau_1 + A_2\tau_2$ is the Haar measure on U , then $B(n) = dA(n)$ are independent identically distributed U -valued random variables with law μ .

Proof: Given measurable subsets $Y_n \subset U$, $n \in \mathbf{Z}^2$ of positive measure, define $Z_n = B(n)^{-1}(Y_n) \subset X = U(\mathbf{Z}^2)$. Let $m = \mu(\mathbf{Z}^2)$ be the product measure on X . The claim is that

$$m(\bigcap_{n \in F} Z_n) = m(Z_k) \cdot m(\bigcap_{n \in F \setminus \{k\}} Z_n) \tag{1}$$

for any finite set $F \subset \mathbf{Z}^2$ and that the law of $B(n)$ is the Haar measure μ .

(i) $m(Z_n) = \mu(Y_n)$, for all $n \in \mathbf{Z}^2$.

Proof: A product of Haar distributed U -valued random variables is again Haar distributed because it must be U invariant. It follows that the law of $B(n) = A_2^*(n)A_1^*(n+1)A_2(n+1)A_1(n)$ is the Haar measure μ and therefore $m(Z_n) = \mu(Y_n)$.

(ii) For any finite set F of sets $\{Y_n\}_{n \in F} \subset U$ with $m(Y_n) > 0$, one has $m(\bigcap_{n \in F} B(n)^{-1}(Y_n)) > 0$.

Proof: We can realize one element in $\bigcap_{n \in F} B(n)^{-1}(Y_n)$ using the canonical gauge. There exists then an open neighborhood of this point in $U(\mathbf{Z}^2)$ which is in $\bigcap_{n \in F} B(n)^{-1}(Y_n)$. An open set has positive measure.

(iii) For $k \in F$, the measure $\tilde{\mu}(Y_k) = m(B(k)^{-1}(Y_k) | \bigcap_{n \in F \setminus \{k\}} Z_n)$ is equal to $\mu(Y_k) = m(Z_k) = m(B(k)^{-1}(Y_k))$.

Proof: By the uniqueness of the Haar measure, we have only to show that $\tilde{\mu}$ is translational invariant. By multiplying $A_1(k+l \cdot e_j), l=1, \dots, |F|$ with some constant $C = e^{2\pi i \alpha} \in U$, we change the field $B(k) \mapsto B(k)C$ without affecting $\{B(n)\}_{n \in F \setminus \{k\}}$. Therefore $\tilde{\mu}(Y_k) = \tilde{\mu}(Y_k + \alpha)$ and $\tilde{\mu} = \mu$.

Proof of the claim. By (ii), Eq. (1) can be written as

$$m(Z_k | \bigcap_{n \in F \setminus \{k\}} Z_n) = m(Z_k).$$

The left-hand side of this is by (iii) equal to $\tilde{\mu}(Y_k) = \mu(Y_k)$ and the right-hand side is by (i) also equal to $\mu(Y_k)$. □

Remarks: (1) There are other ways to get independent magnetic fields, if μ is the Haar measure: Define $A_2(n) = 1$ for all $n \in \mathbf{Z}^2$ and a family $\{A_1(n)\}_{n \in \mathbf{Z}^2}$ of independent Haar distributed random variables. An argument similar to the proof of Proposition IV.7 shows that $\{dA(n) = B(n)\}_{n \in \mathbf{Z}^2}$ are independent Haar distributed random variables.

(2) We do not know whether a generalization of Proposition IV.7 holds when U is non-Abelian.

(3) In dimensions $d > 2$, there is no hope to get a result analogous to Proposition IV.7, because there are then more plaquettes than bonds so that a single bond influences several plaquettes and prevents independent, identically distributed fields.

(4) Another open question is whether one has some or even pure point spectrum almost everywhere in the case of magnetic Laplacians with Haar distributed magnetic vector potentials. One would at least expect to have pure point spectrum for L_λ with λ large or small enough. For numerical calculations see Ref. 7.

(5) Proposition IV.7 shows that for those specific operators, there is more symmetry as in the Mathieu case. Aubry-duality goes deeper: the operators L_λ and $L_{1/\lambda}$ have the same spectral type because a multiplication of L_λ with $1/\lambda$ gives $L_{1/\lambda}$.

V. OTHER EXAMPLES

A. Laplacians in higher dimensions

We turn now to independent identically distributed magnetic Laplacians in higher dimensions. We restrict the discussion to the case $d=3$. As indicated already, we cannot realize independent, identically distributed electromagnetic fields F by a vector potential, since such fields do not satisfy the Maxwell equation $dF=0$, which is required if $F=dA$. Consider now time-dependent magnetic fields in the plane together with an electric field changing in time. Given a vector potential $A=(A_1, A_2, A_3) \in \mathcal{C}^1$, we think of A_1 as the electrostatic potential and of (A_2, A_3) as the magnetic vector potential. Then $dA=F$ is a three-dimensional field. $E_1=F_{12}$ and $E_2=F_{13}$ are the coordinates of an ‘‘electric’’ vector field in the plane and $B=F_{23}$ is a ‘‘magnetic’’ field in the plane. For fixed $k \in \mathbf{Z}$, denote by $L^{(k)}$ the magnetic Laplacian in the plane, given by the vector potential $(n, m) \mapsto (A_2(k, n, m), A_3(k, n, m))$. The operator $L^{(k)}$ is a two-dimensional magnetic Laplacian at time k .

The existence theorem in Ref. 3 assures that a field F satisfying $dF=0$ defines an electromagnetic Laplacian L determined by a one-form A satisfying $F=dA$. By prescribing the electric fields E_1, E_2 and the magnetic field $B^{(k_0)}$ at some time k_0 , the Maxwell equation $dF=0$ determines the whole field F .

The next proposition which follows from the central limit theorem for circle-valued random variables, emphasises why IID magnetic distributed operators with Haar distribution are natural.

Proposition V.1: *Let F be determined by the electric fields and the magnetic field at some time k_0 . Assume that the electric fields $\{E_1(n), E_2(n)\}_{n \in \mathbf{Z}^3}$ are independent identically distributed random variables with the same distribution μ which is not a Haar distribution of a subgroup of $U(1)$. Let $B(k_0, n)$, $n \in \mathbf{Z}^2$ be any set of random variables. Then the distribution of the magnetic field of the two-dimensional operators $L^{(k)}$ converges in law to the uniform Haar distribution of $U(1)$ for $|k| \rightarrow \infty$.*

Proof: The Maxwell equation $dF=0$ (which follows from $F=dA$), implies that

$$B^{(k+1)}(n) * = B^{(k)}(n) E_1^{(k)}(n + e_1) E_1^{(k)}(n) * E_2^{(k)}(n) E_2^{(k)}(n + e_1) *.$$

The proof of Proposition IV.7 shows that the random variables

$$\{C(n) = E_2^{(k_0)}(n + e_2) * E_2^{(k_0)}(n) E_1^{(k_0)}(n) * E_1^{(k_0)}(n + e_1)\}_{n \in \mathbf{Z}^2}$$

are all independent so that also $\{B^{(k_0 \pm l)}(n)\}_{n \in \mathbf{Z}^2}$ is obtained from $\{B^{(k)}(n)\}_{n \in \mathbf{Z}^2}$ by multiplying it with independent identically distributed random variables. The claim follows now from the central limit theorem for independent identically distributed $U(1)$ -valued random variables.³³ (On compact topological groups, the Haar measure plays the role of the Gaussian measure in \mathbf{R}) (Fig. 4). □

Proposition V.1 has the following interpretation: a time-dependent random electric field [which might be arbitrarily small but which does not take values in a subgroup of $U(1)$] turns an initially arbitrary magnetic field for time $|k| \rightarrow \infty$ into an independent identically distributed Haar distributed magnetic field.

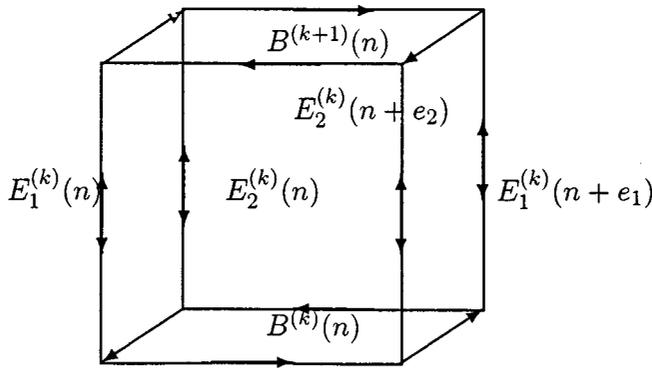


FIG. 4. The Maxwell equation $dF=0$ determines the magnetic field $B^{(k+1)}(n)^* = B^{(k)}(n)E_1^{(k)}(n + e_1)E_1^{(k)}(n)^*E_2^{(k)}(n)E_2^{(k)}(n + e_1)^*$ at time $(k + 1)$ from the magnetic field $B^{(k)}$ and the electric field $(E_1^{(k)}, E_2^{(k)})$ at time k .

B. One-dimensional operators

Take an electromagnetic Laplacian $L = A + A^*$ in d dimensions, where the electromagnetic field $dA = F$ has only electric components $F_{1k}(n) = E_k(n)$ which are constant in space ($\mathbf{Z}^d = \mathbf{Z} \oplus \mathbf{Z}^{d-1} = \text{space} \oplus \text{time}$) and depend therefore only on the first (=time) coordinate $n = n_1$. The restriction of L to the invariant Hilbert space of functions which are constant in space gives a one-dimensional operator $(Hu)_n = u_{n+1} + u_{n-1} + V(n)u_n$, where

$$V(n) = \sum_{k=1}^d E_k(n) + E_k(n)^* = \sum_{k=1}^d 2 \cos(\arg(E_k(n))).$$

Every one-dimensional operator can be written like this. The number of dimensions which are needed depends on the norm. Since $dF=0$, Feldman–Moore’s existence theorem shows that if V is an ergodic potential, then the equation $F = dA$ can be solved with a measurable vector potential A leading to an ergodic electromagnetic Laplacian. The one-dimensional potential $\sum_{k=1}^d 2 \cos(\arg(E_k(n)))$ is ergodic, if T_1 was ergodic.

Some Anderson models can be treated as random magnetic Laplacians and allow a combinatorial calculation of the density of states: given independent identically distributed random variables $V(n)$ $n \in \mathbf{Z}^d$ with law μ , define the $\mathcal{B}(l^2(\mathbf{Z}^d))$ -valued random variable $(Lu)_n = \sum_{|m-n|=1} u_m + V(n)u_n$ which is an Anderson model. By adding to each vertex of \mathbf{Z}^d an oriented loop, one obtains a new lattice \mathbf{L}^d . Denote by Γ_n the set of paths γ in \mathbf{L}^d which have length n . (Each loop has length 1 and we distinguish paths which pass in different directions through the loop) (Fig. 5).

Corollary V.2: (a) Given the discrete d -dimensional Anderson Schrödinger operator with independent identically distributed potential $V(n) = 2 \cos(\alpha(n))$, where $\alpha(n)$ are uniformly distributed in $[0, 2\pi]$. The n 'th moment of the density of states is the number of closed paths of length n in \mathbf{L}^d , for which every loop has vanishing winding number.

(b) If $V(n) = \pm 2$, where $V(n)$ are uniformly distributed in $\{0, 2\}$, the n 'th moment of the density of states is the number of closed paths of length n in \mathbf{L}^d for which every loop has an even winding number.

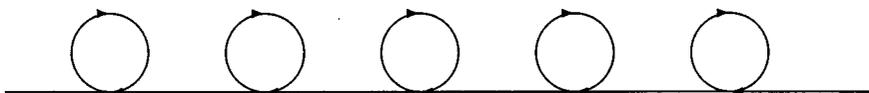


FIG. 5. The graph \mathbf{L} in the case $d=1$. At each vertex is attached an oriented loop.

Proof: Write $L = A + A^*$ as a $(d + 1)$ -dimensional magnetic Laplacian, where $A_j = 1, j = 1, \dots, d$ and $A_{d+1}(n) = \exp(i\alpha(n))$ are independent identically distributed $U(1)$ -valued random variables with uniform Haar distribution μ . This is equivalent to taking real-valued random variables $\alpha(n)$ with uniform distribution on $[0, 1]$ and to form the independent identically distributed potential $V(n) = 2 \cos(2\pi\alpha(n))$ which has an absolutely continuous law $4(2\pi)^{-1} \sqrt{1-x^2}$. As before, we compute with the random walk expansion

$$\text{tr}(L^n) = \sum_{\gamma \in \Gamma_n} \prod_P \hat{\mu}_{n(\gamma, P)}.$$

Since all nonzero moments of ν are zero, $\text{tr}(L^n)$ is the number of closed paths in the lattice \mathbf{L}^d which give in case (a) zero and in case (b) zero (mod 2) winding number to every loop. \square

Remark: Relations between two- and one-dimensional operators are prototyped by the Harper–Mathieu case $A_1 = \tau, A_2 = e^{2\pi i\alpha}$ which give the one-dimensional operator $\tau + \tau^* + 2 \cos(2\pi\alpha)$. For more examples with constant magnetic field, see Ref. 34. Other, not constant magnetic fields can be obtained as follows: let $A_1 \tau_1 = \tau$ be the unitary Koopman operator for a transformation T on a probability space Ω and let $A_2(x) = e^{2\pi i f(x)}$, where f is a $su(N)$ -valued random variable. Then $UV = VU e^{2\pi i(f(Tx) - f(x))}$ and we get a one-dimensional operator $L = \tau + \tau^* + 2 \cos(f(x))$ on $l^2(\mathbf{Z}, \mathbf{C}^N)$.

C. Deterministic Aharonov–Bohm Laplacians

It is illustrative to see what deterministic perturbations of the magnetic field does on the operator. We denote by L_F the d -dimensional Laplacian with field F in the special gauge.

Proposition V.3 (Jitomirskaya–Mandelstam Ref. 18): Assume U is Abelian. A change of $F \in U^{\mathbf{Z}^d}$ on a finite set of plaquettes leads to a compact perturbation L_F of the free Laplacian L_1 .

Proof: Assume first $d = 2$. If B is multiplied by $C \in U^{\mathbf{Z}^2}$ such that $C_n \neq 1$ only for finitely many n and $\prod_n C_n = 1$, we call $\tilde{B} = BC$ a zero flux perturbation of 1. It is enough to show the claim for a perturbation of the field B of one single plaquette. By construction, if \tilde{B} is a zero flux perturbation of B , then $L_{\tilde{B}}$ is a finite rank perturbation of L_B .

Let $L = L_B$ be the original operator and let $\tilde{L} = L_{\tilde{B}}$ be the operator belonging to \tilde{B} satisfying $\tilde{B}(n) = B(n)$ for all $n \in \mathbf{Z}^2$ except one n_0 , where $\tilde{B}(n_0) = B(n_0)C$ with $C = e^{i\alpha} \in U$. Define for each $k \in \mathbf{N}$ a zero flux perturbation B_k of B by changing \tilde{B} on k^2 plaquettes in a box of size $k \times k$ to $\tilde{B}_k C_k^{-1}$ with $C_k = e^{-i\alpha/n^2}$. Then, $L_{B_k} \rightarrow L_{\tilde{B}}$ in norm so that $L_{\tilde{B}}$ is a limit of finite rank operators L_{B_k} .

For general d , we can build any perturbation by composing finitely many perturbations lying in two-dimensional planes and for which the previous argument applies. \square

Remarks: (1) The Aharonov–Bohm operator [the situation when the field $B(n)$ is different from 1 exactly at one plaquette] shows that one has never a finite rank perturbation $L_B \rightarrow L_{\tilde{B}}$, if $B\tilde{B}^{-1}$ has compact support and nonzero flux. It would be interesting to know if the Aharonov–Bohm operator is a trace class perturbation of the free Laplacian.

(2) There is the following formula for the Fourier transform of the spectral measure $dk_l = dk_{e_l}$, where $e_l(n) = \delta_{ln}$ is a unit vector in $l^2(\mathbf{Z}^2)$:

$$\widehat{dk}_{ln} = \sum_{\gamma \in \Gamma_n} B^{n(\gamma)},$$

where $n(\gamma)$ is the winding number of the path with respect to a point in the plaquette, where B is different from 1.

(3) A similar argument shows that the Jitomirskaya–Mandelstam result is also true for some aperiodic tilings like the Penrose tiling.

(4) Beside the Abelian or non-Abelian Aharonov–Bohm operators (for which a complete spectral analysis is not yet done), other deterministic operators would be interesting to study. An example is a discrete version of the Iwatsuka operator L in $d=2$ (see Ref. 5), where the magnetic field B is translational invariant in one direction and asymptotically constant in the other direction. Then, L is a direct product of one-dimensional operators $(Lu)_n = u_{n+1} + u_{n-1} + \cos(n\alpha(n))u(n)$, where $\alpha(n) \rightarrow \alpha^\pm$ for constants α^\pm . If α^- or α^+ is rational, then also L has some absolutely continuous spectrum. If both α^\pm are irrational, Last's results²⁹ allow us to prove that L has no absolutely continuous spectrum. This is different from the continuous case, where the corresponding operator has purely absolutely continuous spectrum.

VI. MAGNETIC LAPLACIANS ON TILINGS AND OTHER LATTICES

A. Magnetic Laplacians on the triangular lattice

The triangular lattice is the Cayley graph of the group $G = \mathbf{Z}^2$ with the three generators $e_1, e_2, e_1 + e_2$. A situation with two different fluxes has been considered in Ref. 35 (see also Ref. 14). A magnetic field is a cocycle which assigns to each triangle $\Delta(g_1, g_2, g_3)$, $g_i \in \mathbf{Z}^2$ a group element in U . This cocycle is determined by the value of $B_d(n)$ on $\Delta(n, n+e_1, n+e_2)$ and $B_u(n)$ on $\Delta(n+e_1, n+e_1+e_2, n+e_2)$ for each $n \in \mathbf{Z}^2$. The two measurable maps $B_d, B_u \in L^\infty(X, U)$ and an ergodic \mathbf{Z}^2 action so determine the magnetic field.

Proposition VI.1: Every stationary $U(N)$ -valued field B on a triangular lattice in \mathbf{Z}^2 is given by a vector potential A so that $B = dA$. The spectral properties of L depend only on B . If $\{B(n)\}_{n \in \mathbf{Z}^2}$ are independent identically distributed random variables with Haar distribution on $U = U(1)$, then $\text{tr}(L^n)$ is the number of closed paths in the triangular lattice which give zero winding number to all triangles.

Proof: In order to get the vector potential A , we form $B(x) = B_u(x)B_d(x)$, which is the field on the quadratic plaquette $P(x)$. Feldman–Moore–Lind's theorem gives the existence of the first two coordinates (A_1, A_2) of the vector potential. We define then A_3 through $A_3A_2(T_1)A_1 = B_d$.

In the Abelian case, a second proof is obtained directly from the algebraic group cohomology for the group $G = \mathbf{Z}^2$ acting on $\mathcal{U} = \mathcal{L}(X, U)$: the magnetic field B with law μ is an algebraic 2-cocycle. Since the second cohomology group is trivial, it is of the form dA , where A is a one-form.

For Abelian U , the random walk expansion is done in the same way as for the square lattice by putting A_2 identically zero.

In order to see that all the spectral properties depend only on the field B , we take the same special gauge as in the square lattice case. \square

Remarks: (1) Discrete magnetic Laplacians on more general graphs with uniform magnetic field with values in $U(1)$ have been considered by Sunada.³⁶

(2) If the graph G is the Cayley graph of an infinite Abelian group with finitely many generators and $U \subset U(1)$, the magnetic Laplacians are elements in a hyperfinite von Neumann algebra \mathcal{X} . The second group cohomology vanishes and every algebraic cocycle B is of the form $B = dA$.

B. Magnetic Laplacians on aperiodic tilings

Aperiodic tilings in \mathbf{R}^2 define a plane graph and one can ask if it is possible to assign to the edges of the graph $U(1)$ random variables in such a way that the magnetic fields in the pieces of the tiling are independent identically distributed $U(1)$ -valued random variables. For simplicity, we consider only the case of the Penrose tiling with plaquettes built by Robinson triangles. The case when the plaquettes are Penrose rhombs can be reduced to that by multiplying the field values of the triangles building the rhomb.

Proposition VI.2: Given a measurable $U = U(1)$ -valued field distribution B on the Penrose lattice. There exists a vector potential A such that $dA = B$.

For a proof see Ref. 3. One can deduce from this:

Corollary VI.3: Given an independent identically distributed $U(1)$ -valued field B on the Robinson triangles of a Penrose tiling. There exists a measurable vector potential A on the edges of the Penrose graph such that $dA = B$.

Remarks: (1) For more general tilings, where all pieces of the tiling are composed of the same number k of triangles (which is the case in the Penrose tiling where each Penrose rhomb is a union of two Robinson triangles) we can also realize independent identically distributed magnetic field configurations, where the law $\mu = \nu \oplus \dots \oplus \nu$ is the k th convolution of a measure ν . This is for example the case if μ is the Haar measure on a closed subgroup of $U(1)$.

(2) The existence of the density of states of a magnetic Laplacian L on the tiling follows from the fact L in a finite type von Neumann algebra. Hof³⁷ has given a direct proof of the existence and proven that the density of states and spectrum is constant on the space of tilings.

(3) For independent identically distributed magnetic fields with Haar measure of $U(1)$, we get that $\text{tr}(L^n)$ is the number of closed paths in the tiling graph such that the winding number is zero for each tile. The computation of the density of states is already nontrivial for the free Laplacian with zero magnetic field. There are some numerical results about the random walk on Penrose lattice.³⁸

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