THE ADVENTURE OF TEACHING ALGEBRA

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Abstract. These are notes for a workshop given on January 20, 2017 at the Texas Graduate Center in Rio Grande Valley. After an introduction, we look at 4 units: A) Algebra notation, B) Problem ambiguities, C) Student misconceptions, D) Error sources. At the end, we include some remarks about role of creativity and Euler's textbook

Introduction

Having only taught sporadically on a high school level, my own teaching experience is mostly on an undergraduate college level so that I have still to learn. During more than 30 years of teaching in colleges, I noticed that algebra skills are pivotal for the success in calculus or linear algebra courses. I am under the impression that "algebra confidence" of incoming students has declined. And it is not a surprise: the reliance on more and more powerful computer assistants, the distraction from a richer and more connected technological world makes it harder to focus. Then there are pedagogical changes. There is more competition for example with other subjects, especially with calculus and computer science. These subjects are introduced earlier into the curriculum. So, it is maybe not only algebra alone that gets influenced by a broader range of subject menu. I don't think that the overall mathematics skills have diminished among incoming freshmen, in contrary, but that the knowledge has spread wider and thinner. We should welcome this but as teachers we have to be aware that it is now more likely that we encounter a student with some algebra gaps. With other parts of education it is similar. It is not that we read less, but reading in books has been augmented by more and more digital content in the form of articles, summaries, blogs or shifted even to audio and video content. We do less traditional sport because a lot of new sport disciplines have appeared. With algebra it is the same: our students have to learn programming skills early on for example or to be trained in statistics already in middle school. Other sciences like chemistry or physics experience the same challenges because exciting developments happen in other fields like engineering, astronomy or biology for example. I have experimented as a kid with chemistry lab and electronic kits made for teenagers; nowadays a student might play with Arduino boards or use a mini computer to build a robot, or then run a blog or youtube channel or play with 3D printers. So, the observation that algebra has become more challenging is probably just a consequence of the richer environments which sciences and education today can offer. Still, I believe that algebra remains an important pillar in our teaching. I hope to indicate also that is remains a great adventure, more so maybe even than in the past. I even believe that remains a frontier. I will say more to that at the end. [1] I'm looking forward to this workshop and exchange some ideas about the teaching of algebra.

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Key words and phrases. Mathematics education, Algebra.

[1] I myself work currently on some algebra in the theory of networks. One can calculate with networks in the same way as we calculate with numbers, but these new algebra explorations have only begun.

Igor Shafarevich once wrote [10]:

In the school mathematical education, algebra has the role of Cinderella and geometry the role of the Beloved Daughter. People would say "I have not chosen mathematics to be my profession but I will remember forever all the beauty of logical construction of geometry" but never anything similar concerning algebra.

Picture source: "Cinderella" by Hans Anker, 1920

Shafarevich then succeeds convincing the reader that algebra contains subdivisions of advanced mathematics like abstract algebra, number theory, combinatorics or even probability theory. And points out in the rest of the article that there is a lot of beauty in the subject.

As teachers, how can we make algebra more attractive? The common answer is to make it "more fun". A good example is the [6], where one can read in the preface: "I dedicate this book to you! Lets build the next generation of kick-ass gals who look algebra in the eye and say, "I shall be your master.' " By the way, I can only recommend this book. An other example are games, and smart phone applications and actual board or dice games. We will look at the creative aspect at the end. Maybe we can play with some of toys I brought with me. Naturally, there are also caveats with "fun activities". To see this, let me point out an interesting paradox. We live in a reverse world, where free time has become hard work and work time has become leisure:

A paradox of modern education.
School activities like homework or projects have to be a game and fun.
Hobby and free time activities like music or sport have to be competitive.

How can we make algebra more interesting while still do serious work. The book [7], written by a master of exposition, starts with concrete problems like:

Problem: What is the largest number you can write down with 3 letters 9. Is it 999 or $9 * 99$ or 99^9 or something else? What happens with 3 letters of 5 or 3 letters of 5 or 3 letters of 2? What happens with 4 digits of 2 etc. We start as a class to work on this question together.

To explore this problem, we are already in a quite tough environment, since we have to know about the basic operations. The most exciting of the 5 operations addition, subtraction, multiplication, division and exponentiation is definitely the 5th one. As addition and multiplication both have inverses, we should also include a sixth operation, the natural log. Perelman wisely chose an example, where one can explore all operations. The game of finding the maximum invites to **play** and compute. But like in sports, to play, we need also to train basic skills, maybe do some weight lifting to build up basic strength. The weight lifting here is done by doing lots of computations and remember the rules of the operations.

By the way, such problems have appeared in the news [13] in Germany just recently because of the problem

Problem: Which 3 digit number using the letters 2,3,6 is the largest?

The complaints were that this problem is too easy for 10th grade. Of course the answer is 632. But we can make this problem more interesting

Problem: What is the largest number we can form using the digits 2,3,6 and the basic 5 operations? [2].

FIGURE 1. The problem to find the largest number using 3 digits $2, 3, 6$ appeared in the press [13]. Many teachers complained that it is considered too easy (for 10th grade).

A) NOTATION

Abstract: When looking at the history of the development of algebra, the choice of notation has often been pivotal. How do we choose good notation? How do we find a good balance between detail and elegance? We will look at at few cases. The best way to explore this is with an example which has historical background as history illustrates the struggle to find good notation.

Before mathematicians were even writing mathematics, they used tokens. After having placed them for accounting reasons into pockets of clay, and writing a symbol for the content, one got rid of the tokens and used symbols. [1].
The first steps in mathematics was simply to keep track of numbers. It was numerical. Initial algebra had also connections with geometry. The area of a rectangle reflects the product of two lengths. It motivates to define multiplication as a commutative operation. This is not obvious and indeed, we had to learn that fundamentally, on a quantum level, the mathematics of the physical world is no more commutative.

Where does arithmetic end and algebra begin? The division point is difficult to make and also depends on the definition of what algebra is. If we think about algebra as a science to solve equations, then the subject is very old. If we associate it with a tool to use variables for manipulation, then this came only much later, with Vieta 1540-1603. An additional confusion is that mathematicians mean today with algebra mostly the theory of algebraic structures like groups, rings or fields. Problems which are best solved using variables (but were treated without) appeared much earlier in print. Cajori (p. 96) claims that the first arithmetic was printed in 1478 at Treviso in Northeastern Italy: It deals with the following algebra problem:

Problem: A courier travels from Rome to Venice in 7 days. An other courier starts at the same time and travels from Venice to Rome in 9 days. The distance from Venice to Rome is 250 miles. In how many days will the couriers meet?

We will work on this problem and see how useful it can be to use variables. As you can see on the worksheet, the solution of the problem is quite elaborate without the use of variables. We will see how we can do better than that textbook from 540 years ago.

Good notation needs taste. But it is also embedded in tradition. We don't want to break too much from tradition even if we believe to have found a better notation.

Problem: Do you recognize the formula $\alpha_1^2 + \Gamma^2 = \epsilon^2$?

Probably not. It is the Pythagoras formula but using strange notation.

Problem: Do you recognize the formula $E = mc^2$?

Yes, it is a formula from physics and E is the energy m is the mass and c is the speed of light. We have recognized it. The formula $y = mx^2$ for a parabola does not look that impressive, but it is the same formula of course. But it gives an other meaning as in the formula $E = mc^2$, the variable c is the speed of light which is just the constant. The formula $E = mc^2$ is closer to the equation $y = ax$ as it relates the quantity mass and energy.

With **basic arithmetic** we usually mean solving equations on the real number line. The expressions are built using **addition multiplication** and often also involve **exponentiation** as well as their inverses. When using variables, there is the difficulty to indicate which variable depends on the others.

Problem: What is the meaning of $y = x^2$?

If we look at y as an independent variable, then we deal with a parabola. We can also just think it as a substitution. Finally we can mean with it a rule, which maps a given number to a new number. How do we address this in a text? Computer languages have to deal with this. The equal sign is refined. There is :=, ==, = and they all mean completely different things. Additionally, we often write = even if we mean equivalence classes like having the same reminder when dividing by a fixed prime p.

Such things are even harder when we start doing calculus. An example is the topic of related rates, which deals with the chain rule. The chain rule is a difficult topic for textbook writers because variables become temporarily functions. More about this [2]. There are many jokes about variables. The most famous one is a drawing of a triangle with one variable labeled x. The teacher asks: "Find x". The student draws an arrow to the variable x and writes "Here it is".

In [11] the story of Wittgenstein's Sheep from [5] is told:

Problem: Teacher: 'Suppose x is the number of sheep in the problem', Student: 'But, Sir, suppose x is not the number of sheep.'

When John Littlewood asked the Cambridge philosopher Ludwig Wittgenstein, whether this was a profound philosophical joke, the later answered yes, that it was.

Here are some recommendations:

Tip: Use familiar notation.

Tip: Avoid indices $x_1^2 = x_2^2 + x_3^2$. Better: $x^2 = y^2 + z^2$.

Tip: Avoid decorations like $\hat{x}^2 = \tilde{x}^2 + \overline{x}^3$.

Tip: Avoid abbreviations if not needed, like: "find the LCM of 15 and 9"

Tip: Avoid ambiguities. (We have an entire section about that later)

Of course, there is always a balance. Variables can sometimes look like variables, sometimes like functions. The relationships are often hidden. In probability theory the confusion between variables and functions is even in the name. A **random variable** is nothing else than a function.

Tip: Honor traditions.

It is good to denote angles in geometry with Greek letters, vertices with capital letters and edges with lower case letters. Why? Because most textbooks make that assumption. Of course, one can break with traditions, and sometimes it is necessary, but there is always a risk of confusion. It has to be tried out.

B) Ambiguities

Abstract: As a teacher one is often not aware that a problem is not posed properly. This happens already in algebra which is one of the cleanest topics in mathematics. We will look at a few examples.

Order One of the first conventions one has to agree on is the **order of operations**. This is not so clear even if we use one operation only. For example:

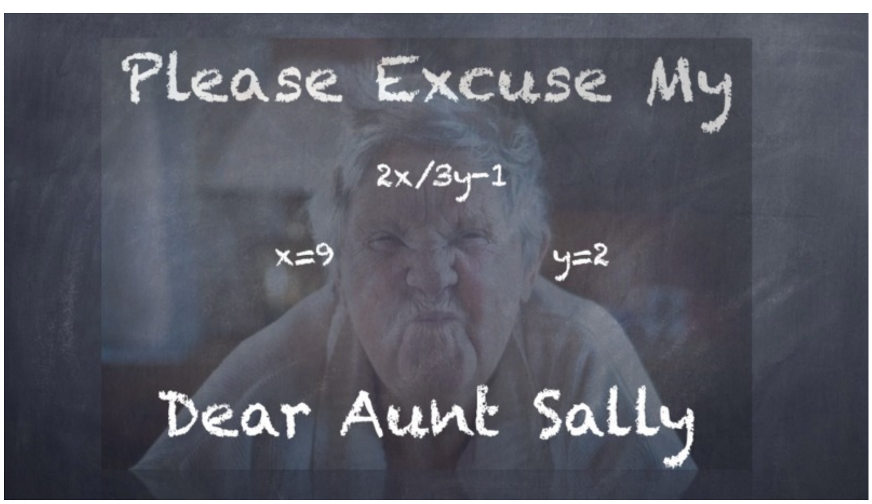
Problem: What is 2^{2^2}?

Problem: What is 2/2/2?

PEMDAS If different operations are involved, there is still confusion. There is a PEMDAS and PEDMAS and both are used. We will work on a few riddles together. The following question produced a lot of controversy [3]:

Problem: What is 2x/3y-1 if x=9 and y=2?

How can you remember PEMDAS. There are various sentences like "Please Eat Mom's Delicious Apple Strudels" or "Please excuse my dear aunt sally".



Problem: What is $x/3x$?

Noninvertibility. A common source for mistakes is the fact that x^2 is not bijective. Here is a silly example $5^2 = (-5)^2$ so that $5 = -5$ or $10 = 0$.

Spaces between letters. We often avoid the multiplication sign and this is ok to clarify formulas like $xy = z$. Sometimes, this can backfire.

Problem: $1\ 0 = 0$ shows $10 = 0$.

It can be particularly cruel in programming languages which allow a space for multiplication. And we all fall at least once into a trap because $x2$ and $2x$ mean completely different things because $2x$ is read as $2 * x$ and $x2$ is a variable.

Limits

Problem: What is 0^0.

To understand this, we need a bit of calculus. The limit 0^x for $x \to 0$ is because it is 0 for all $x > 0$. On the other hand, x^0 is 1 for all x. The correct answer is: it depends on how the limit x^y with $x, y \to 0$ is taken. With $x^x = e^{x \log(x)}$ the limit is 1.

Here is a silly proof that $1 = 0$.

Problem: Take the equation $x = 0$, multiply with $x - 1$ to get $x(x-1) = 0$. Now divide by x to obtain $x - 1 = 0$ or $x = 1$. We have shown $0 = 1$.

And here is an other one:

Problem: If $a = b$, then:

$$\begin{aligned} a^2 &= ab \\ a^2 - b^2 &= ab - b^2 \\ (a-b)(a+b) &= b(a-b) \\ a+b &= b \end{aligned}$$

substitute in the original $a = b$ equation to get $2a = a$ or $2 = 1$.

Problem: Take a triangle with given lengths. Compute an angle with the cos theorem, then compute the third angle with the sin theorem. The three angles don't always add up. The reason is the sin function is not bijective. The computer chooses the smaller angle.

Problem: Teacher: What is $11q - q$? Student: $10q$. Teacher: You're welcome.

Problem: Teacher: "What is $2k + k$". Student: 3000.

Problem: See [9].
$12 - 12 = 18 - 18$
$12 - 8 - 4 = 18 - 12 - 4$
$2(6 - 4 - 2) = 3(6 - 4 - 2)$
$2 = 3$.

Lets work together and see whether we can come up with a few more examples. The following section was not included in the original handout:

Noninvertibility. Here is an other source for ambiguities. It is due to the use of calculators and computers. It might come a bit as a surprise that even the best and most professional computing tools we have do not honor the distributative law.

Problem: Define $T(x) = 4x(1-x)$ and $S(x) = 4x - 4x^2$ and now apply the map 60 times to a number like 0.4. We get completely different answers. If you have access to a computer algebra system, try it out.

```
T[x_]:=4x(1-x); S[x_]:=4x-4x^2;
{Last[NestList[T,0.4,60]],Last[NestList[S,0.4,60]]}
```

Now try it out with fractions:

```
{Last[NestList[T,2/5,10]],Last[NestList[S,2/5,10]]}
```

In the first case, the 16 digit accuracy has already melted to 11 digits. The second computations gives still 0 (the fractions have 700000 digits!). Interesting: the second computation can no more be performed with fractions.

C) Misconceptions

Abstract: When grading assignments, one often stumbles upon misconceptions. Sometimes, one can predict them, sometimes, they come as a surprise. We look at a few examples.

Short cuts. When dealing with concrete algebraic expressions, one often tries to make shortcuts. These short cuts are often based on **wishful thinking**.

Problem: What went through the mind of the author writing $x/(x + y) = 1 + x/y$?

Unclear definitions. Similarly as in geometry, where notions like "polygon" or "angle" are often not precisely defined, it is also possible in algebra to fall into a trap.

Problem: Find all solutions to $x^3 = x$. Of course, we have $x = 1$.

But we have forgotten something else: Also $x = -1$ is a solution! Still we have not yet found all solutions. We can also have $x = 0$. The reason why we often forget negative solutions and 0 is that both concepts have been introduced relatively late historically. It took a long time until humanity saw the need for the number 0 and for negative numbers.

Problem: Find all solutions to $x^3 + x = 0$. Again we have $x = 0$ as a solution. Are these all? Yes, they are all real solutions. But we forgot the imaginary solutions $x = i$ and $x = -i$. Did we make a mistake. Not necessarily. The problem could implicitly have asked for real solutions. This can happen either in a larger context, or because the class has officially not been introduced to complex numbers.

Pushing definitions

Problem: Design your own proof of $1 = 2$ based on pushing definitions into realms, where they are not defined.

Here is an example of a misunderstanding due to lack of clarity:

Problem: "What happened to your girlfriend, the math student?" - "She cheated on me! Last night, I called her on the phone and she told me that she was in bed wrestling with three unknowns." [12]

Problem: We saw in a multi-variable calculus course once that the computation $51x^{50} - 51 = 0$ was simplified to $x^{50} = 0$. Can you figure out what went through the mind of the author?

Problem: Example [9]: $9/3 + (-16/4) = (9-16)/(3+4) = -7/7 = -1$ which is right despite the fact that the computation is nonsense.

Extrapolation.
There is a strong law of large numbers in probability theory which deals with averaging random variables. The **strong law of small numbers** is less well known. It is a phenomenon in number theory. Richard Guy formulated it so prominently that it is part of mathematical folklore: **"There aren't enough small numbers to meet the many demands made of them."** This leads to many surprising coincidences. Guy has a second law telling **"If two numbers look equal, it ain't necessarily so!"** We see that often when grading. Even so a student might have got to the right answer, it is not necessarily that the result is right. Here are some famous coincidences:

Problem: Fermat $2^{2^n} + 1$ is always prime.

From [9]:

Problem: $31, 331, 3331, 33331, 333331, 3333331, 33333331$ are all prime. They are all of the form $(10^n - 1)/3 - 2$.

Dyslexia This is illustrate this

Problem: What is $9x6$? Student: 54. What is $6 * 9$? Student: 45.

D) Error checking

Abstract: How do we run into errors? How do we detect errors in work? How do we prevent errors? How do we give feedback about errors? How do we learn from errors?

When watching students making errors, or if we watch ourselves doing errors, we see patterns. The best advise to learn about errors is to **know them**.

Tip: Know about pitfalls. Collect and treasure them!

Here are some examples:

- We don't understand the problem.
- We take short cuts.
- We do some wishful thinking.
- We transcribe the problem wrong.

Here is a way to make money:

Problem: From [9]: We all know that 1/4 dollar = 25 cents. By taking the square root we see that 1/2 dollar is 5 cents.

Square roots are often dangerous because square roots prevent a square from being moved away ....

Here is a variant

Problem: 1 Dollar = 100 Cents = $(10 \text{ Cents})^2$ = $(0.1 \text{ Dollars})^2$ = 0.01 Dollars = 1 Cent.

Obviously the confusion is the choice of units. This is a common confusion, led to failures to land on Mars and riddles thousands of spreadsheets in business. Here is a bit more advanced joke about the choice of units as it requires to know octal and decimal systems:

Problem: Why do mathematicians often confuse Christmas and Halloween? Because Oct 31 = Dec 25.

Problem: One of the earliest algebra problems is the "rule of three". The appear in Egypt, Mesopotamia, India, China, Islam and early Europe [4]. If $a/b = x/c$ then $x = ac/b$.

Problem: Assume one and a half cats eat one and a half mice in one and a half hours. Then four cats eat four mice in how many hours?

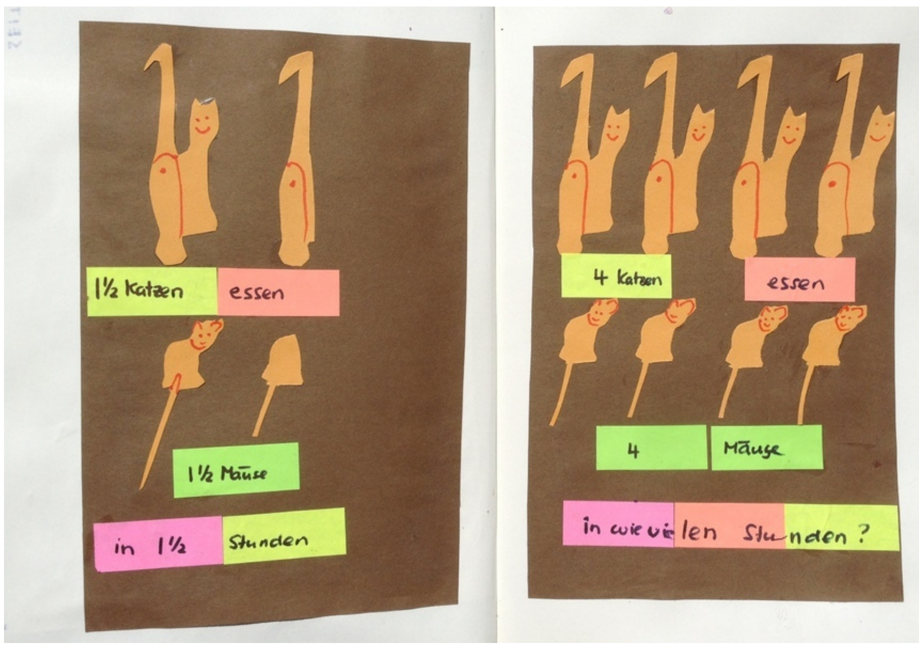


FIGURE 2. A problem from a diary visitor book of the Knill family house "Salmenfee" in the mountains. The house is a simple cottage without water and electricity, but that makes it charming.

Tip: Counter examples often appear when looking at extreme cases.

Problem: Since $n!$ is the product of the first n integers, we have $0! = 0$. But in reality we have $0! = 1$.

One can justify the definition in various way. There is a **Gamma function** $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt$ which satisfies $\Gamma(n) = (n-1)!$ (just do (n-1) times integration by parts) and $\Gamma(1) = 1$ so that $0! = 1$.

By the way, the **factorial function** is important in algebra too. Define the **Binomial coefficients**

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

It tells in how many ways we can choose k elements from n elements.

$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^n$

We say that the function $f(x) = (1+x)^n$ is the **generating function** of the Binomial coefficients.

Checking with examples

Tip: In order to check an expression, its good to try with examples.

Problem: Check the binomial formula from above in the case $n = 5$: compute $(1+x)^5$ as well as the sum of the 6 Binomial coefficients. (Pascal triangle!)

How to check? Multiple choice questions are a good choice. Conceptional problems are hard to code otherwise. Some can do it. Often it is a disaster.

Here is a tough problem:

Problem: $x^2 = 2^x$. What are the solutions?

We can try $x = 2$. How would we solve this? We have to write $2\log(x) = x\log(2)$ and get $x/\log(x) = 2/\log(2)$. Is 2 the only solution? We can not solve for x since this is a **transcendental equation**. Indeed, there is an other solution $x = 4$. Is there an other one? **Drawing the graph** can help but proving that there is no other one requires the analysis of the derivatives.

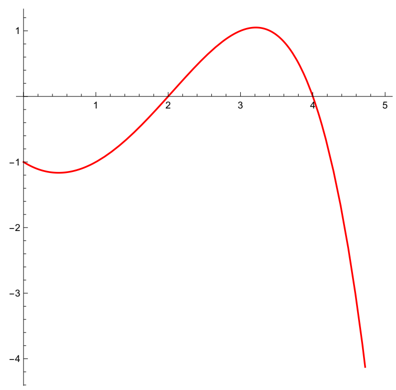


FIGURE 3. The graph of the function $x^2 - 2^x$. We see that there are two roots $x = 2$ and $x = 4$. They give us solutions to the equation $x^2 = 2^x$? Are these the only roots? How can we find a complete list of the roots?

This example illustrates a pitfall in which teachers can step easily.

Tip: Be careful with posing problems which are too difficult for the level!

How to give feedback? It is really very helpful as a teacher to know the possible pitfalls. This allows to act proactively and to know how to bend a false statement and make it write.

Problem: A student claims $1/9^{-1/2}$ is $-1/3$. What went wrong?

In that case, the confusion was the minus sign. One could first think that the confusion is a PEMDAS problem but $(1/9)^{-1/2} = 1/(9^{-1/2})$ gives 3 in both cases. The confusion comes from $1/3 = 3^{-1}$ is not equal to -3. The minus sign has fallen down. How do you give freeback to such an error?

Misunderstanding. A final story illustrates a very common source for errors:

Problem: There was a college student trying to earn some pocket money by going from house to house offering to do odd jobs. He explained this to a man who answered the door. ”How much will you charge to paint my porch?” asked the man. ”Forty dollars.” ”Fine” said the man, and gave the student the paint and brushes. Three hours later the paint-splattered lad knocked at the door again. ”All done!”, he says, and collects his money and adds ”by the way, that's not a Porsche, it's a Ferrari.”

Problem: Look at the following calendar of January

Su	*Mo*	*Tu*	*We*	*Th*	*Fr*	*Sa*
1	2	3	4	5	6	7
8	9	10	11	12	13	14
15	16	17	18	19	20	21
22	23	24	25	26	27	28
29	30	31				

to your friend and chose a date from a Sunday to Friday, then chose the date to the right, right, below and to the right-below, (like 6,7,13,14). Now ask your friend to compute their sum (like 40). You take the sum, subtract 16 and dividing by 4. You can tell now the date of your friend. Can you explain this?

Postfix: The role of creativity

One of the most important skills we need to have both as students and teacher is creativity. This also applies in teaching. One reason is that designing good problems needs creativity. It is not easy to be creative in a world as rich as we live in. The reason is that almost certainly we come up with something which has already been done. Its good to know that this is the rule and that one has to try very hard to eventually score at least in one part and have something really new.

Fortunately, there are techniques. Here are some tips. Some of them contradict each other, but it is not a paradox, if you apply the techniques at different times

Tip: Look at a lot of sources. Collect ideas, network around. Combine, mix, don't copy, modify, morph or take the opposite.

Tip: Contemplate, relax and let the inspiration come. Sometimes the ideas virtually come at night during sleep.

Tip: Start with a clean slate, fresh paper. Tabular rasa. The start: what is the most important thing?

Tip: Allow errors at first. Maybe it can be fixed. Most of the errors and sloppy starts can not be fixed but sometimes there is a way.

Tip: Start with a seed, let it grow. Get back to it after a while.

Tip: Let it happen. Sometimes it will fail, sometimes it will work. Most of the time it will fail.

When teaching algebra, we can also draw excitement from playing with it in our free time

Tip: Explore some unknown territory on your own.

There are some very books about creativity, like [8]. It advocates the so called **snowflake model** which can be summarized as follows:

> *Tolerate complexity, disorganization, and asymmetry. Enjoy the challenge of struggling through chaos toward resolution. Excel in problem solving skills. Find creative solutions and good answers. Develop a feel for questions which have potential to lead to discoveries. Mobility for perspectives on and approaches to problems. Think in opposites or contraries, metaphors and analogies and challenge assumptions. Be ready to take risks. Accept failure as part of the creative quest. Learn from mistakes. Work at the edge of competence. Scrutinize and judge own ideas. Seek criticism and advise. Put aside ego. Test ideas. Be motivated by its own sake, not for grades nor pay. Catalysis is enjoyment, satisfaction, and challenge for the work itself.*

I myself draw a lot of energy from my own mathematics explorations. My own research career has finished maybe 15 years ago, but this does not mean that doing research is over. Some important mathematics has been found by amateurs sleuths. I still like to think about new mathematics and explore new worlds, and as a teacher I can afford also not to care too much about what other mathematicians think or on a whim, change the subject. I care more about inspiring students (and other teachers!) Just these days, I was exploring an interesting new algebra of networks. Networks are a natural generalization of numbers.

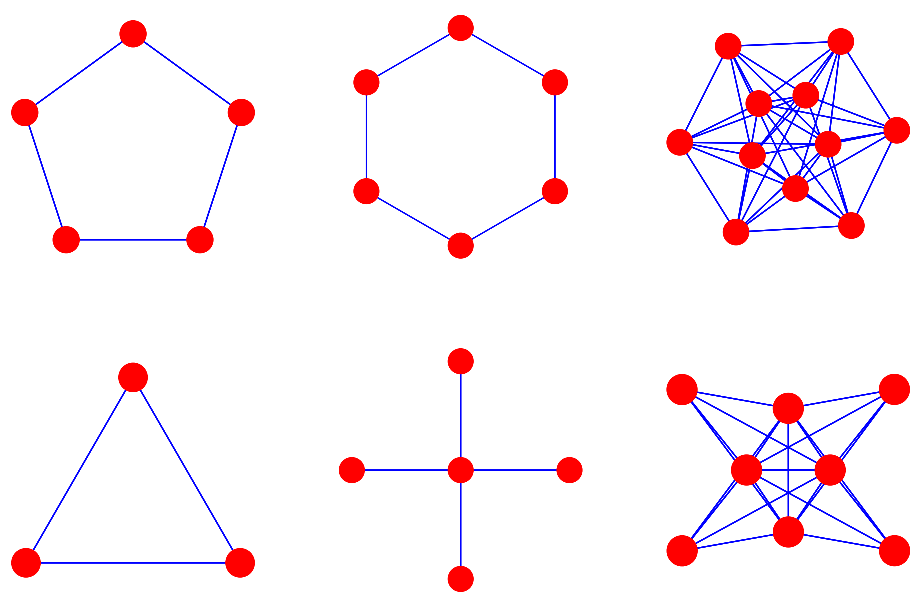


FIGURE 4. Multiplication of networks. What are the prime numbers?

Very early on, even before mathematicians were writing numbers, they counted with pebbles. Pebbles can also be seen as nodes of a network. Now, and that is what a kid do very early on, is that one can add pebbles to add. Now, a bit more advanced is to have these pebbles connect to each other. If you have such a networked group and an other networked group, we can add these two groups together and get two groups. This is the **addition of networks**. It generalizes counting and is not terribly interesting similarly as simple counting is not throw you off the chair. More is multiplication: take two networks. Now form again the union of the two groups, but only connect every member of the first group with a member of the second group. This operation is called the "join" in graph theory. It has nice topological properties which I skip here. The exploration of this arithmetic has an affinity with the mathematics of **Euclid**, who pursued both geometry as well as number theory. Indeed, networks come with a notion of "prime". A network is prime, if it can not be decomposed into a product of two other networks. One can for example see that if you have a network in which no triangles are present (groups of 3 which are all connected), then if the number of connections is prime and it is not a star graph, then the network is prime. The proof goes by showing that such a network needs to be a product of two networks without connections and that in that case the number of connections is a product $n * m$, where the first group has n and the second group has m members. The case when one network is a one-point graphs gives as a connection graph a star graph which is not prime. An example of a graph which factors is the **utility graph** $P_3 * P_3$ which has 9 edges. This graph has appeared early in geometry because of the riddle, whether the three utility providers Gas, Electricity, Water can serve 3 Houses A, B, C simultaneously without having their pipes crossing. The answer is no. You see that the prime factorization problem is related to a problem in geometry. Euclid would have liked it. Here are some Euclid like statements we call **Propositions** in honor of his book:

Problem: Euclid lemma: Every network can be factored into primes.

Hint: look at the clique number, the size of maximal cliques.
We don't know yet whether the factorization is unique but believe this to be the case. It took 3000 years from Euclid on to prove the uniqueness of prime factorization of numbers! This result is today called the **fundamental theorem of arithmetic**.

Problem: Euclid: There are infinitely many network primes of any dimension.

Hint: an easy solution. Check that the product of two graphs is connected. More sophisticated approach: Look at the volume, the number of maximal cliques. It multiplies under the multiplication of networks.

Appendix: Basic algebra skills

Here are some words of wisdom to pass along to students.

Tip: Fun and excitement can be built up quickly. But it evaporates also fast, if the skills are not practiced. So, Practice!

Here are just a few things I believe are absolutely pivotal and need to be mastered well so that one can have fun with algebra. There are enough resources on the web, spark notes, even videos etc to practice this:

Tip: **Know how to solve equations.**

Knowing the perils when taking square roots, knowing how to solve linear and quadratic equations blindly. Some teachers advocate to look for clever solutions. For success in calculus, it is also important to be able to do it. **To do it fast.** To find roots, relax and just use

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} .$$

Yes, mathematics is about beauty, but speed matters too. Elegance is impressive but if you get stuck all the time because of a stupid quadratic equation, the fun stops. Use your creativity for more interesting things! [3]

[3] I know that I disagree here with most educators but many students lose an entire grade due to the lack of ability to solve basic equations. Students do care about grades and most do not pursue the subject any further if they are repeatedly disappointed.

Tip: Master polynomial arithmetic including division.

This includes arithmetic of numbers and especially fractions! Maybe know a few things like $(x^2-4) = (x-2)(x+2)$ or $1+x^2+x^2+x^3+x^4 = (x^5-1)/(x-1)$. Division of polynomials can help to find roots faster. Know to simplify things like $(x^3+1)(x^3-1) = x^6-1$ well. Especially important is the ability to multiply polynomials (FOIL it out) and in particular, master and recognize expressions like $(x+y)^2 = x^2+2xy+y^2$. Know how to drag out parts from a square root or draw graphs of some basic functions like linear functions, parabolas and cubic equations. Know the effect on graphs if you change the parameters m, b in $y = mx+b$ or a, b, c in $y = ax^2+bx+c$.

Tip: Know thy logs and exponentials

Know the exponentiation rules! Blindly. If you get woken up at 4 AM after a long night party, you should still know the rules. There are not many. I write here log for the natural log and assume that the inside is positive $\log(\exp(x)) = x$ and $a^x = e^{x\log(a)}$ $\log(xy) = \log(x)+\log(y)$. Especially know the graphs of the exponential and log functions. Know that exp has no roots and that $\exp(0) = 1$ and $\log(1) = 0$. Don't mix that up.

Tip: Know thy trig functions

Especially the definition SOH-CAH-TOA. Know them well. It is only necessary to memorize two identities $\sin^2(x) = (1-\cos(2x))/2$ and $\cos^2(x) = (1+\cos(2x))/2$. Adding these identities gives $\cos^2(x)+\sin^2(x) = 1$ which is Pythagoras. Subtracting them gives $\cos^2(x)-\sin^2(x) = \cos(2x)$. Differentiating the later gives $2\cos(x)\sin(x) = \sin(2x)$. The functions $\cot(x) = 1/\tan(x)$ and $\sec(x) = 1/\cos(x)$ are less important but useful. From the inverse functions, $\arctan(x)$ is the most important one as $\arctan'(x) = 1/(1+x^2)$. The arctan usually takes values in $(-\pi/2, \pi/2)$. The arcsin and arccos are less frequent but know that arccos applied on $[-1,1]$ gives a value in $[0,\pi]$.

Tip: Know some theorems

Theorems are the backbone of mathematics. One definitely needs to know the **fundamental theorems:** the **fundamental theorem of algebra** goes a bit further than we did here but it assures that a polynomial of degree n has exactly n solutions, when counted with multiplicity. For example, $x^2+1 = 0$ has the two solutions $i, -i$. The equation $x^2+2x+1 = 0$ has the solution $x = -1$ with multiplicity 2. The **fundamental theorem of arithmetic** assures that one can factor every integer $n > 1$ uniquely into prime factors. The **fundamental theorem of trigonometry** is about a limit. It tells that $\lim_{x\to 0}\sin(x)/x = 1$. The **fundamental theorem of planimetry** is Pythagoras theorem assuring that in a right angle triangle the side lengths satisfy $a^2+b^2 = c^2$. Then maybe, even if you don't teach calculus, it is useful to know the **fundamental theorem of calculus** which tells $\int_0^x f'(t)\,dt = f(x)-f(0)$ and $(\int_0^x f(t)\,dt)' = f(x)$.

Tip: Read the masters

Last but not least, I have to recommend the algebra book of Euler (1707-1783). It was written around 1765 in German. In 1770, a Russian translation appeared, soon followed by the German edition. Johann Bernoulli translated it into French. It has been one of the most successful text books of all times and it is still readable today. It has been translated to English.

Figure 5. Euclid

Figure 6. Algebra book of Euler

Tip: Experiment and play

Most of the really beautiful mathematics has been developed by playing around. Archimedes drew circles in the sand. Gauss played with prime numbers. Euler made lots of experiments. To prepare for this workshop, I set up my own lab at home. I brought some of them to the workshop but it was of little success. Maybe the topic of fractions is a bit too elementary. But there are interesting problems to play with. Like: in how many ways can you write 1 as a sum of fractions of the form $1/k$ where k ranges in a given set. This is a problem from the theory of partitions. Ramanujan was a master in such topics and Ramanujan was also a master in playing around.

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FIGURE 7. Oliver's Fraction Lab at home.

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