TOP TEN ELEMENTARY ALGEBRA SKILLS

September 24, 2022

Oliver Knill, Harvard University

1) USING VARIABLES

X marks the spot

Look for the **unknown quantity** and label it as a variable. A popular letter is X.

Ana is 56 years old, and her father Bert is 80. How many years ago was Bert three times the age of Ana?

Ana is 56 years old, and her father Bert is 80. How many years ago was Bert three times the age of Ana?

X = number of years passed

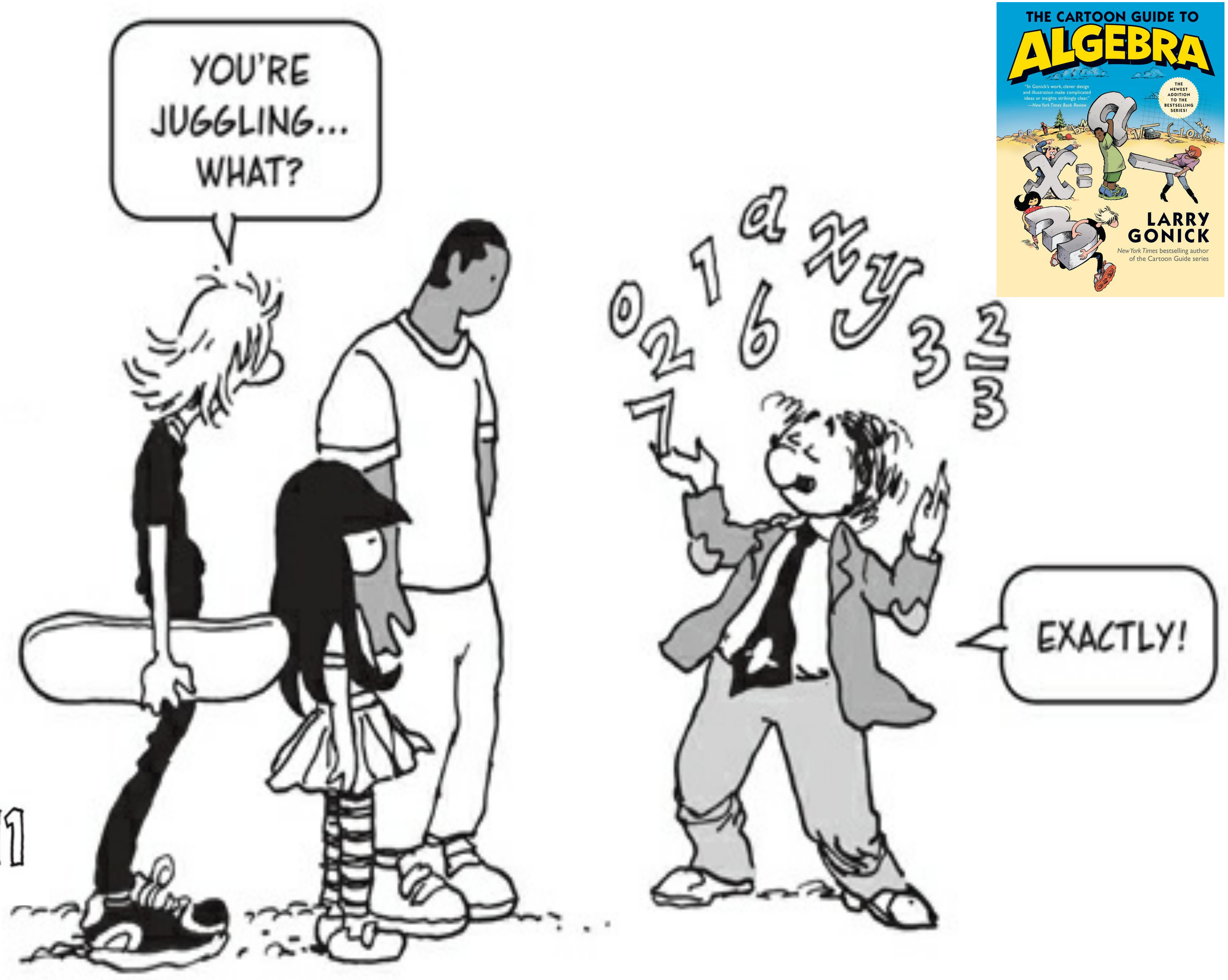
$$3(56-X) = 80-X$$

$$168-3X=80-X$$

Solve this for X

$$X=44$$

IN ALGEBRA, WE TREAT THAT THING CALLED "WHAT?" AS JUST ANOTHER NUMBER, TO BE HANDLED IN THE SAME WAY AS YOU WOULD TREAT 1 OR 2 OR 6. (BUT INSTEAD OF "WHAT?," WE'LL USUALLY WRITE x OR y OR SOME OTHER LETTER.)



$$2 \times \text{WHAT?} - 3 = 11$$

2) SEPARATING VARIABLES

Isolate the unknown

In order to solve for the unknown quantity X, **manipulate**the expressions until X is alone on one side.

Solve for x:

$$5x+1=9-3x$$

$$5x+1=9-3x$$

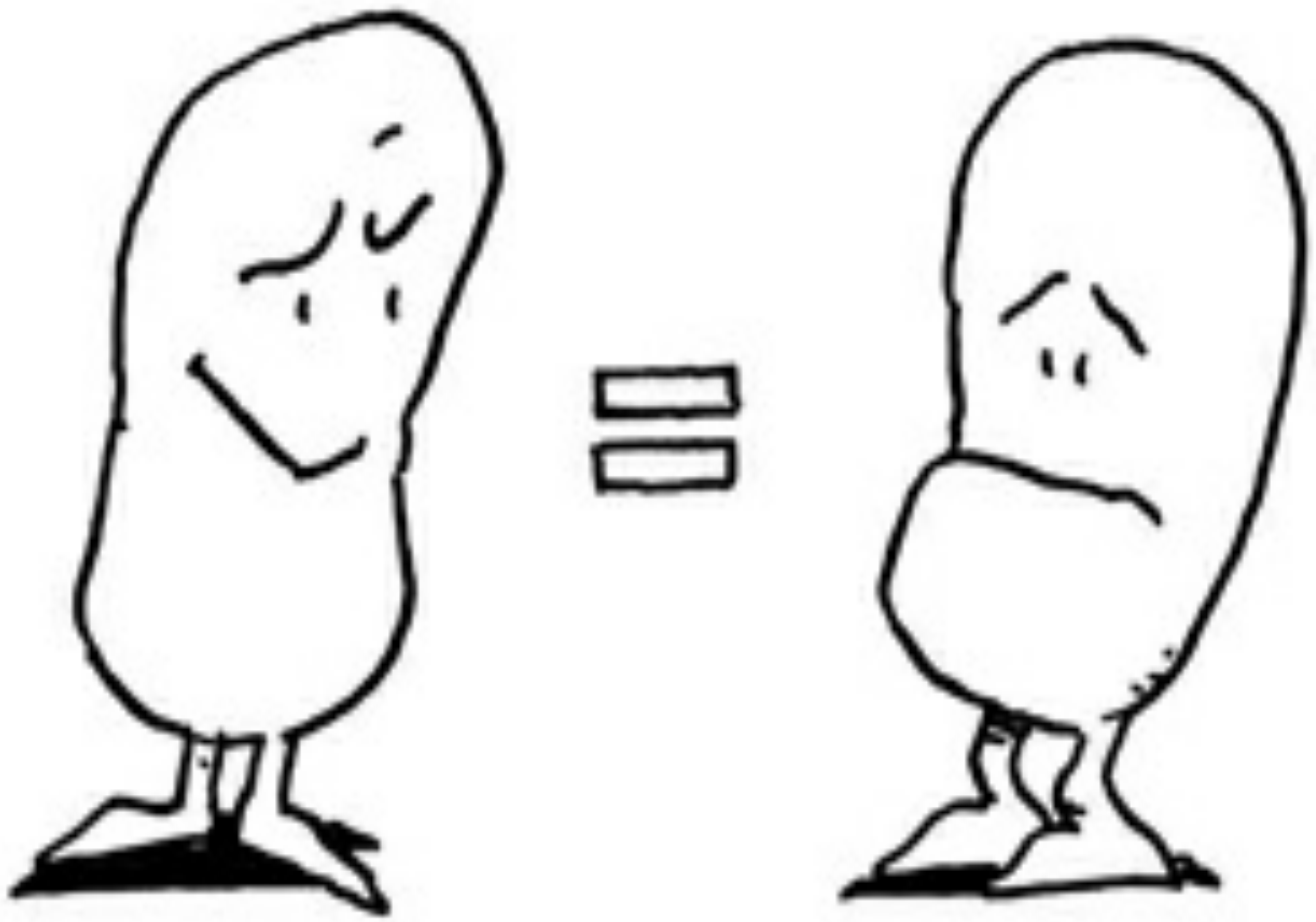
Bring all x variables to the left

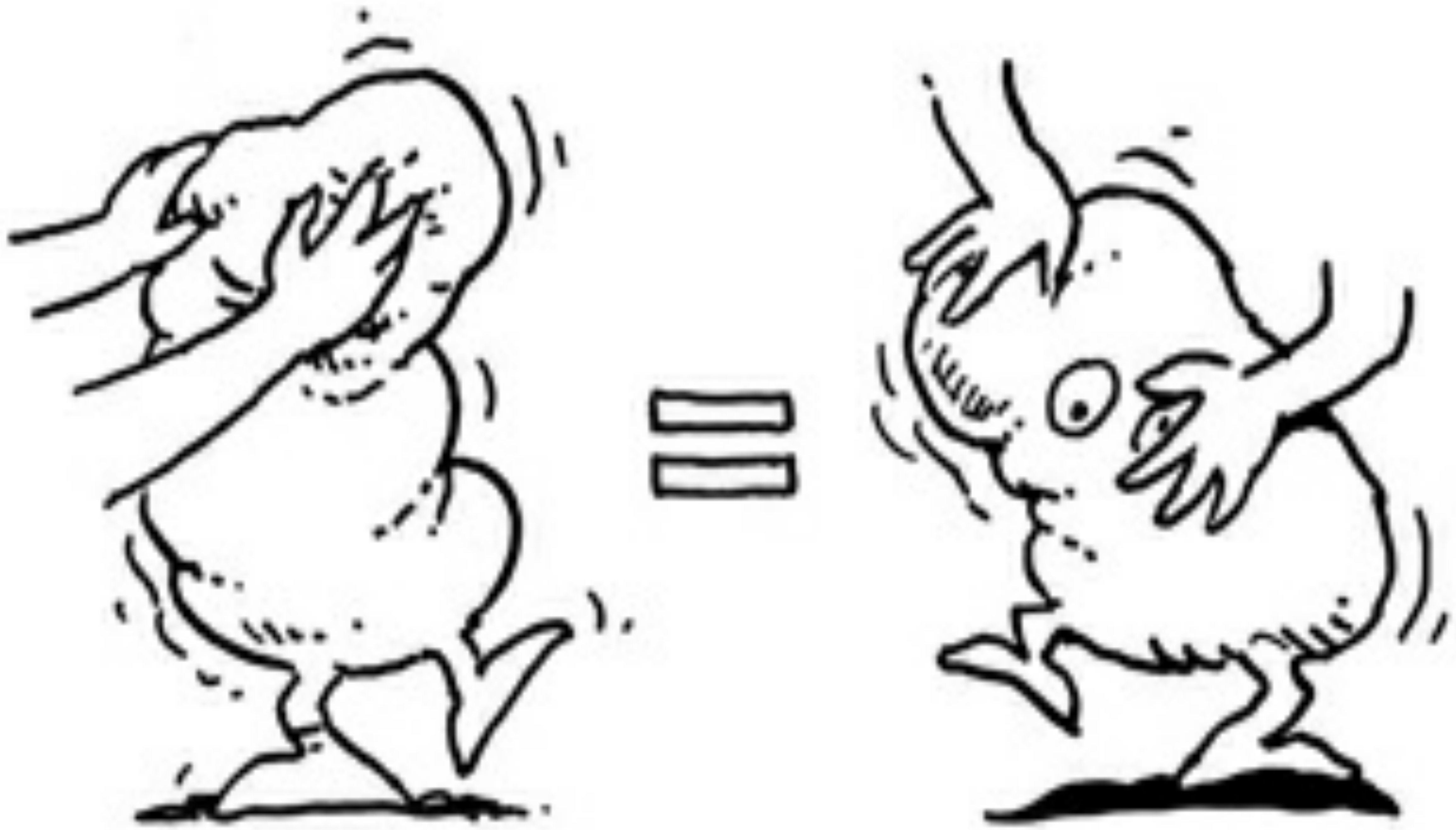
$$5x+3x = 9-1$$

Simplify

$$x = 1$$

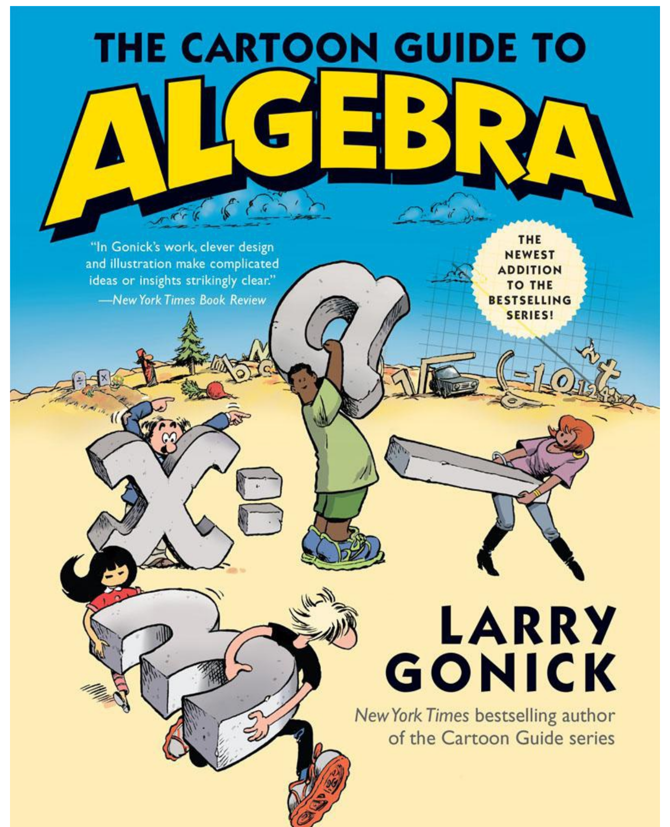
IN ALGEBRA, THE EQUATION COMES FIRST. AN EQUATION SAYS THAT ONE EXPRESSION EQUALS ANOTHER. THEN WE PUSH ITS EXPRESSIONS AROUND...





UNTIL THE ORIGINAL EXPRESSIONS ARE COMPLETELY GONE, AND THE UNKNOWN "WHAT?" OR x APPEARS ALONE ON ONE SIDE OF THE EQUATION, AND WE'RE LOOKING AT AN OLD-FASHIONED ARITHMETIC PROBLEM. THAT'S ALGEBRA!

$$x = \frac{3+3}{2}$$



3) ORDER OF OPERATIONS

PEMDAS

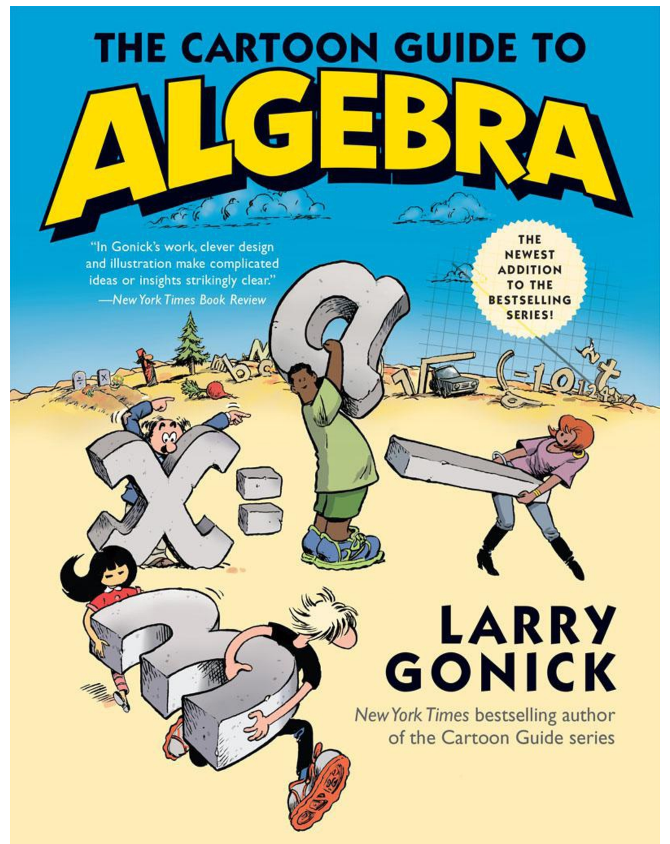
The algebra rule PEMDAS states that **exponentiation** is done before **multiplication and division** are performed before **addition and subtraction**. If an expression is ambiguous, **use brackets**. This is a century old recommendation which is valid until the present day.

You know x=4.
What is 8/2x+2?

You know x=4.
What is 8/2x+2?

Most Humans get 3.
Most Computers get 18
Some Humans get 8/10
Most Humans and Computers are right.
It depends on the order taken.


THERE ARE LAWS ABOUT THIS!
OKAY! FINE!


THE CARTOON GUIDE TO
ALGEBRA
LARRY GONICK
New York Times bestselling author of the Cartoon Guide series

AMBIGUOUS PEMDAS

OLIVER KNILL

Abstract. A source collection about the mathematical syntax in the ring $\mathbb{Z}$.

1. Pemdas

1.1. In the commutative **additive group** $(\mathbb{Z}, +, 0)$ one usually writes $-x$ for the inverse element. One has then $x + (-x) = 0$ or $x - x = 0$ for short. As it is convenient to enhance the addition $+$ with subtraction $-$, **associativity** obviously fails for subtraction: the two numbers $3 - (4 - 5)$ and $(3 - 4) - 5$ are not the same. In the first case, we need to place brackets.

1.2. The same applies in the **multiplicative group** $(\mathbb{Z} \setminus \{0\}, *, 1)$. The expression $3/4/5$ is only defined if we establish a rule how to read it. $(3/4)/5$ is not the same than $3/(4/5)$. The non-associativity of the division operation again forces to place brackets, even if the most common interpretation of $3/4/5$ is $3/20$.

1.3. When combining multiplication and division, brackets are even less enforced. We often write x/yz meaning $x/(yz)$ and do not understand it as $(x/y)z$. That brackets are required to avoid any confusion was noticed in [4] and reiterated in [2], the authority on mathematical syntax. See [5] for some history of notation. Even calculators do not have standards. A TI-82 or Casio calculator interprets $1/2x$ as $1/(2x)$ while the TI-83 interpret it as $(1/2)x$ [9].

1.4. To establish the order of operations in the ring $(\mathbb{Z}, +, *, 0, 1)$ one has established rules. The popular choice $PEMDAS$ $(=BEMDAS)$ orders the operation as **P**arenthesis (**B**rackets), **E**xponentiation, **M**ultiplication, **D**ivision, **A**ddition and **S**ubtraction. In the 21st century, discussions have flared up [3] and there still exists no agreement. Expressions like $3/ab$ with $a = 4, b = 7$ are usually interpreted as $3/28$ while a calculator evaluates it as $21/7$.

1.5. Since terms like a^b and especially x^2 appear frequently in formulas, exponentiation is included in the rules. The associativity fails like in $(3^3)^3 = 19683$ and $3^{(3^3)} = 7625597484987$ but no standard has even been formulated. The computer algebra system Mathematica starts from the right, while Excel or the computer algebra system Matlab reads from the left. Also for exponentiation, we need to put brackets.

1.6. Beside PEMDAS, the rule $BEDMAS$ $(=BODMAS)$ places division before multiplication. It is known in some programming languages. Most computer algebra implementations use $PE(MD)AS$ in which M and D are placed on the same footing and things are read left to right. For $3/2x/2$, a computer gives $3x/4$ while a human reader might read it as $3/(4x)$. There is no authority for a standard. Language is a social construct [10]. Therefore, we need to clarify or even over clarify.

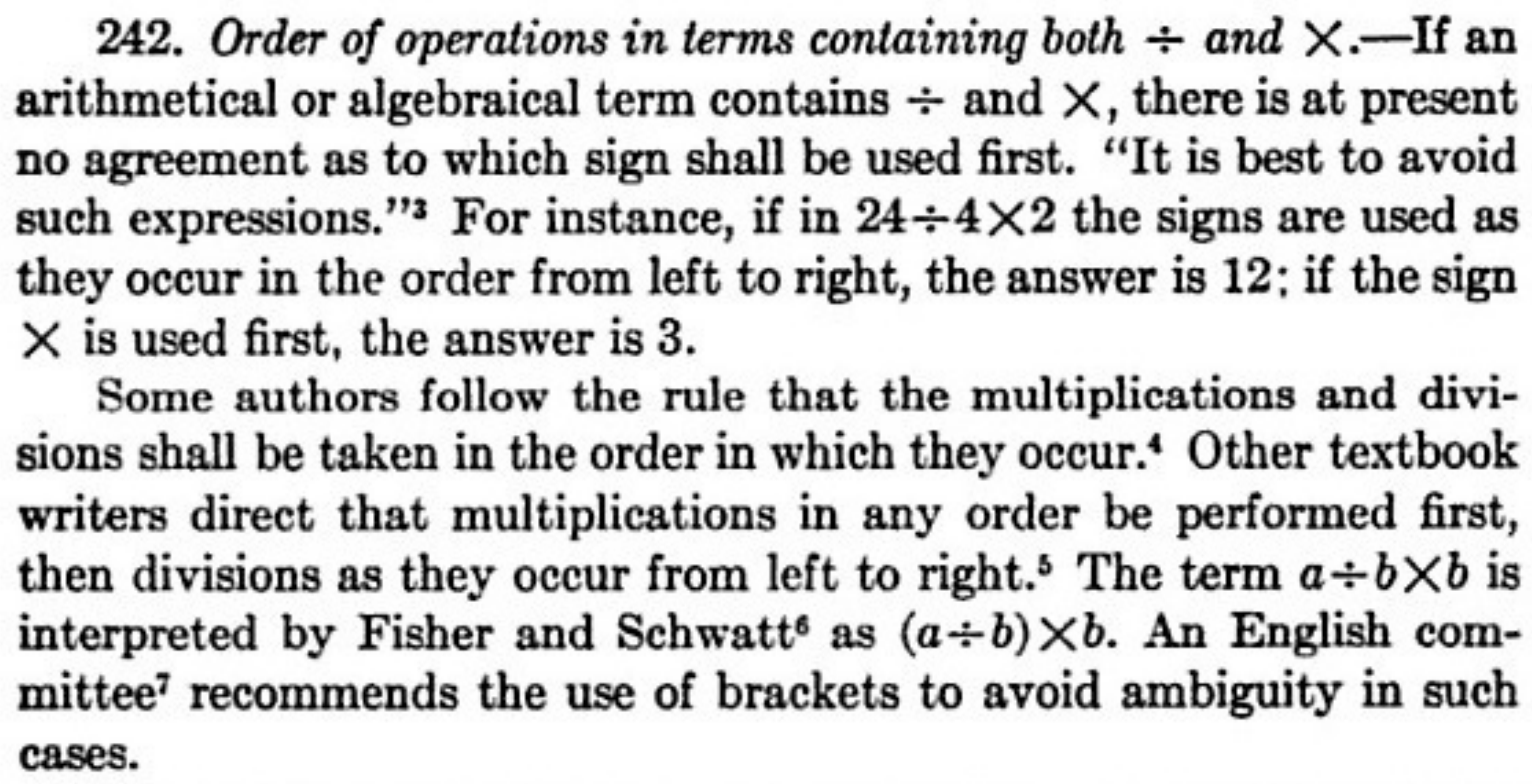
Date: February 28, 2021, Math E 320.
Key words and phrases. Order of Operation.

1.7. When deciding about the use of language, practice matters. When 12 million test subjects give the wrong answer to a simple arithmetic question [6], it is not a problem of the education level [7]. We also do not need to come up with an other standard or fix the issue [8]. The reality of teaching shows that even entrenched rules can be made more robust with more clarity. As a teacher one can observe that even stellar students often read non-ambiguous expressions $a/b + c$ as $a/(b + c)$. **Redundancy** in the form of brackets makes it clear. The article [3] from 2013 has hit the nail on the head.

1.8. Enforcing any of the standards like PE(MD)AS also would not only clash with the use of established literature it would be unrealistically complex to teach. When we read expressions, numbers like 2π are often seen as a unit. We might write $5/2\pi$ and mean $5/(2\pi)$ and not $(5/2)\pi$. We see terms like $pV/RT = n$ in thermodynamics and read it as $pV/(RT)$ rather than $(pV/R) * T$. We read $3m/4s$ as 3 meters per 4 seconds and not $(3/4)ms$. Usage also depends on school systems or programming languages. Use clarifying brackets.

1.9. To summarize: the syntax for writing rational and exponential expressions in a polynomial ring has a few places, where brackets are needed to avoid miscommunication. This is not a syntax problem which needs to be "solved". It is an issue to be aware of and which needs to be avoided. It is better to write expressions extra clearly, possibly even in a redundant way and anticipate any possible misunderstandings. These were the recommendations from almost 100 years ago [2] and things have not changed.

2. Sources


242. ***Order of operations in terms containing both ÷ and ×.***—If an arithmetical or algebraical term contains ÷ and ×, there is at present no agreement as to which sign shall be used first. "It is best to avoid such expressions."[3] For instance, if in 24÷4×2 the signs are used as they occur in the order from left to right, the answer is 12; if the sign × is used first, the answer is 3.

Some authors follow the rule that the multiplications and divisions shall be taken in the order in which they occur.[4] Other textbook writers direct that multiplications in any order be performed first, then divisions as they occur from left to right.[5] The term $a \div b \times b$ is interpreted by Fisher and Schwatt[6] as $(a \div b) \times b$. An English committee[7] recommends the use of brackets to avoid ambiguity in such cases.

Figure 1. From Cajori [2].

References

[1] AMS. Guide for Reviewers – Mathematical Reviews. June 6,1996, 1996.
[2] F. Cajori. *A history of Mathematical Notations.* The Open Court Company, London, 1928.
[3] T. Haelle. What is the answer to that stupid math problem on facebook. *Slate*, 2013.
[4] N.J. Lennes. Relating to the order of operations in algebra. *The American Mathematical Monthly*, 24:93–95, 1917.
[5] J. Mazur. *Enlightening Symbols, A short history of Mathematical notation and its hidden powers.* Princeton University Press, 2014.
[6] 20 Min. Millionen scheitern an dieser Mathe Gleichung. *20 Min*, 2019. Swiss newspaper.
[7] S. Strogatz. The math equation that tried to stump the internet. *New York Times*, 2019.
[8] J. Taff. Rethinking the order of operations. *Mathematics Teacher*, 111:126–131, 2017.
[9] Wikipedia. Order of operation. Accessed February 24, 2021, 2021.
[10] S. Wolfram. Mathematical notation: Past and future. Keynote at MathML conference 2000, 2000.

Department of Mathematics, Harvard University, Cambridge, MA, 02138

4) MULTIPLYING OUT

Know to FOIL

Multiplying out expressions requires the distributive law. One important case is (a+b)(c+d), where the acronym **FOIL** tells to add the product of the **F**irst, the product of the **O**uter, the product of the **I**nner and the produt of the **L**ast

$$(x+1)^2-(x-1)^2=4$$

Find x

$$(x+1)^2-(x-1)^2=4$$

Find x

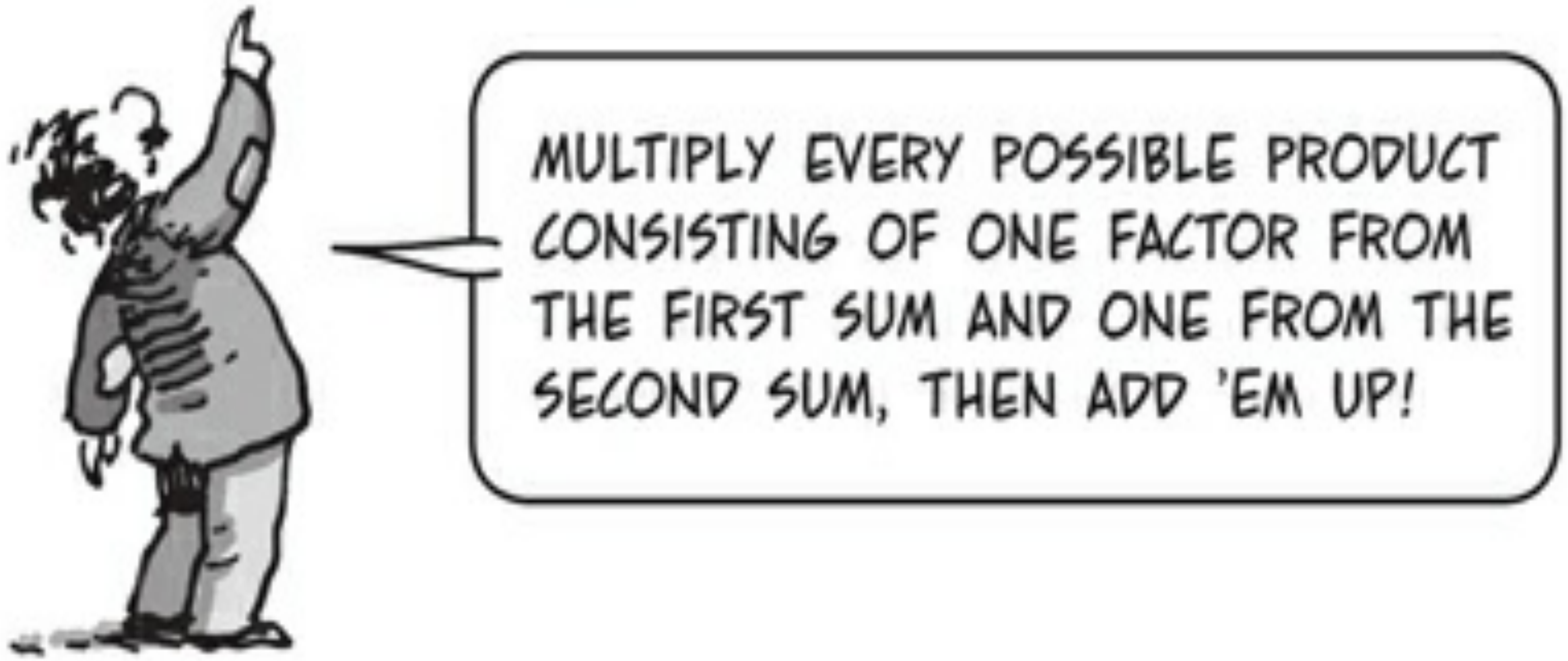
$$(x^2+2x+1)-(x^2-2x+1)= 4$$

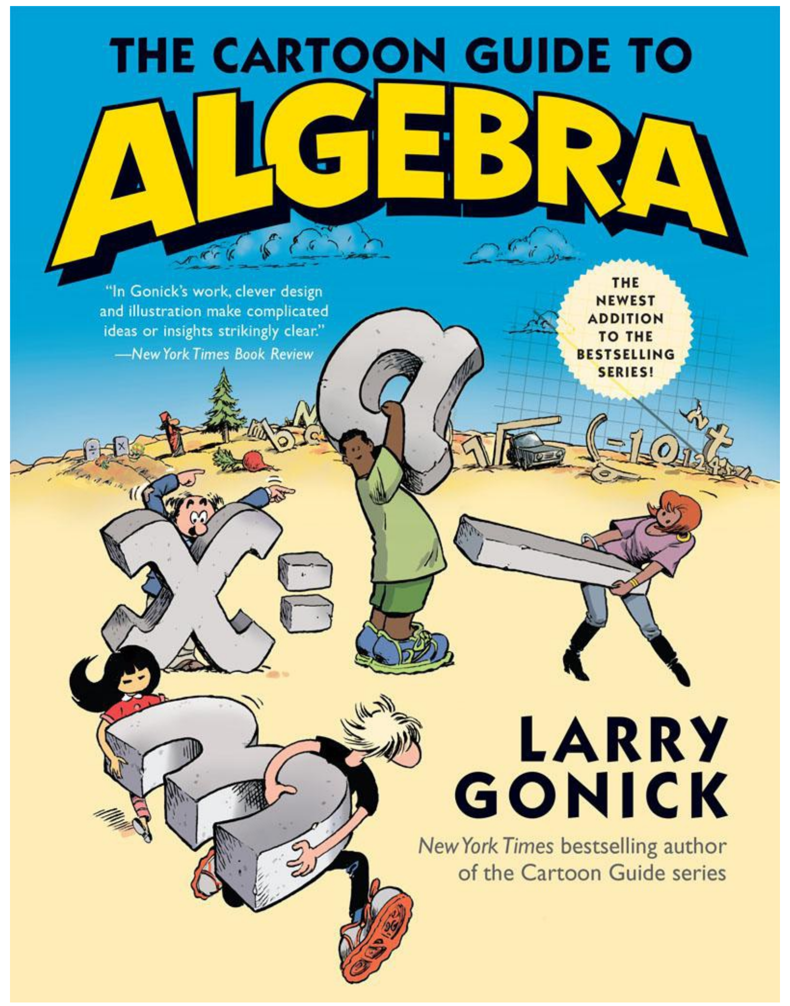
4x=4

x = 1

SO $a(c+d)+b(c+d) = (a+b)(c+d)$. WE ALSO KNOW THAT $a(c+d)+b(c+d) = ac+ad+bc+bd$. PUTTING THESE TWO TOGETHER GIVES US THE **EXPANSION OF $(a+b)(c+d)$:**

$$(a+b)(c+d) = ac+ad+bc+bd$$





5) FACTORING OUT

Divide and Conquer

To find roots of polynomial expressions, it can be helpful if one can **factor** them.

An example is

$$x^3 - 1 = (x^2 + x + 1)(x - 1)$$

Find all roots of

$$x^2 + 7x + 10 = 0$$

Find all roots of

$$x^2 + 7x + 10 = 0$$

Factor to get (x+5)(x+2)=0

Therefore,

x=-5 or x=-2.

The Roots of $(x-r)(x-s)$

ON PAGE 173, WE SAW HOW TO EXPAND $(x+r)(x+s)$. IF WE CHANGE THOSE PLUS SIGNS TO MINUSES, WE FIND THAT $(x-r)(x-s)$ EXPANDS IN MUCH THE SAME WAY.

$$(x-r)(x-s) = x^2 - rx - sx + (-r)(-s)$$

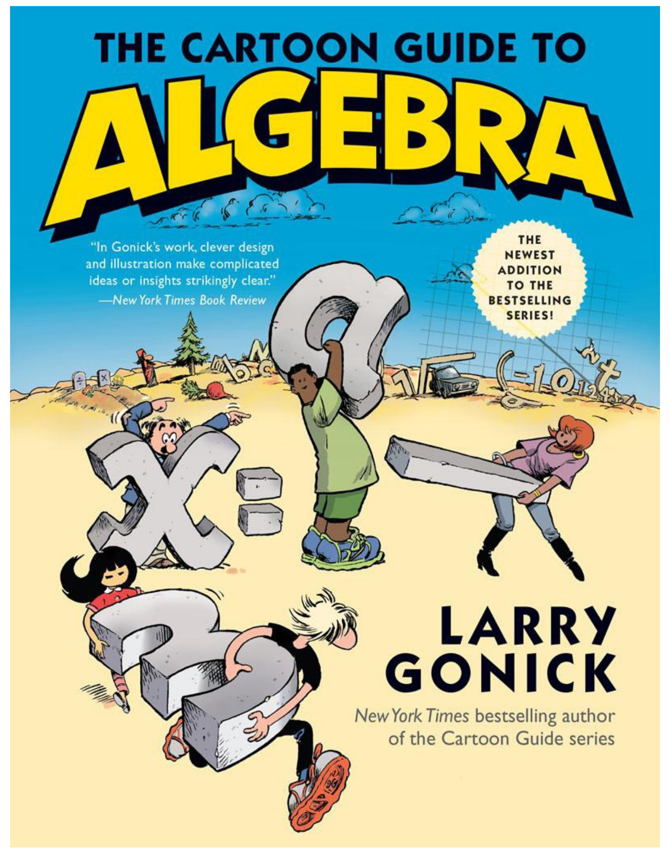
$$= x^2 - (r+s)x + rs$$

WE SAW ONE LIKE THIS IN EXAMPLE 5. HERE'S ANOTHER:

Example

$$(x-4)(x-7) = x^2 - (4+7)x + (4)(7)$$

$$= x^2 - 11x + 28$$



6) ADDING FRACTIONS

COMMON DENOMINATOR

To add fractions, we have to find a **common denominator** so that we can write the expression as one fraction.

Simplify

$$\frac{1}{1+x^2}+\frac{1}{1-x^2}$$

Simplify $\dfrac{1}{1+x^2} + \dfrac{1}{1-x^2}$

Make a common denominator and add. The result is

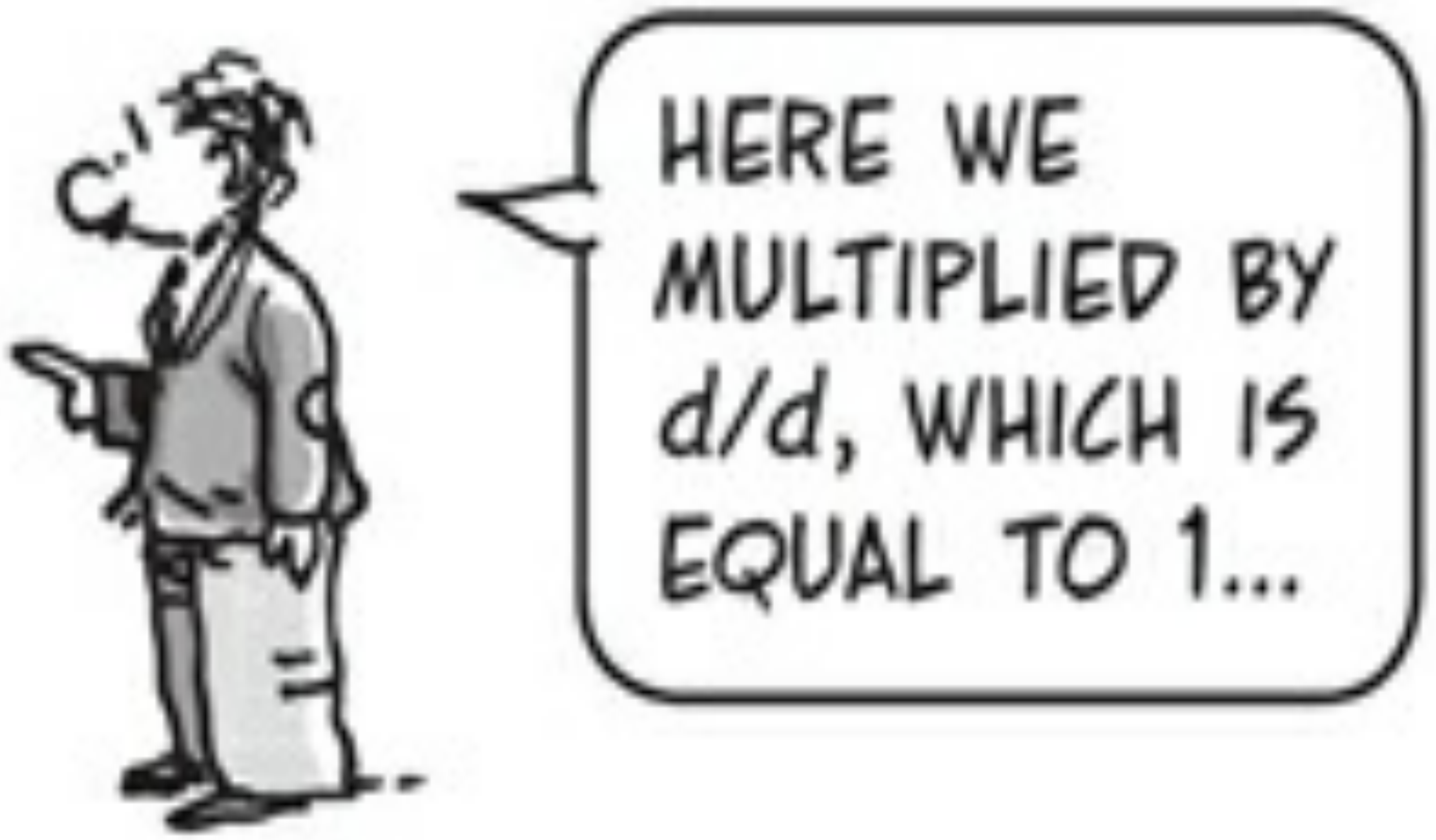
$$\frac{2}{1-x^4}$$

TO ADD

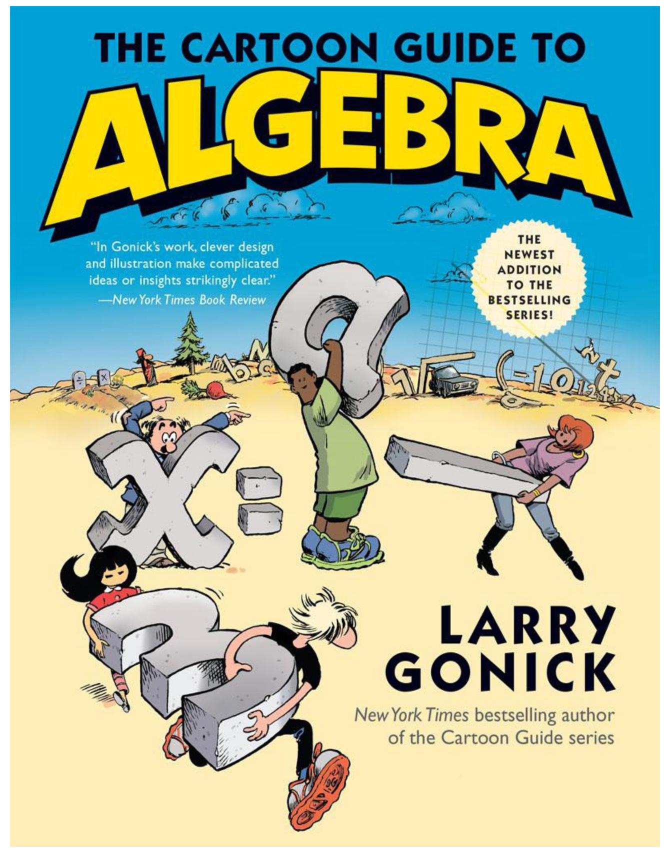
$$\frac{a}{b}+\frac{c}{d}$$

WE FOLLOW THE SAME STEPS AS WE DID WITH THIRDS AND FIFTHS. THE PRODUCT ***bd*** IS A COMMON DENOMINATOR, BECAUSE

$$\frac{a}{b}=\frac{ad}{bd}$$



$$\frac{c}{d}=\frac{bc}{bd}$$



7) MULTIPLYING POWERS

$$a^b \cdot a^c = a^{b+c}$$

If we multiply two groups of powers of a, the number of a factors adds. Like

2^3 2^5=(2*2*2) * (2*2*2*2*2) = 2^8

A) Is $2^6 + 2^4$ equal to 2^{10} ?

B) Is $6^2 \cdot 4^2$ equal to 24^2 ?

C) Is $2^6 \cdot 2^4$ equal to 2^{10} ?

A) Is $2^6 + 2^4$ equal to 2^{10} ?

B) Is $6^2 \cdot 4^2$ equal to 24^2 ?

C) Is $2^6 \cdot 2^4$ equal to 2^{10} ?

A) No

B) Yes

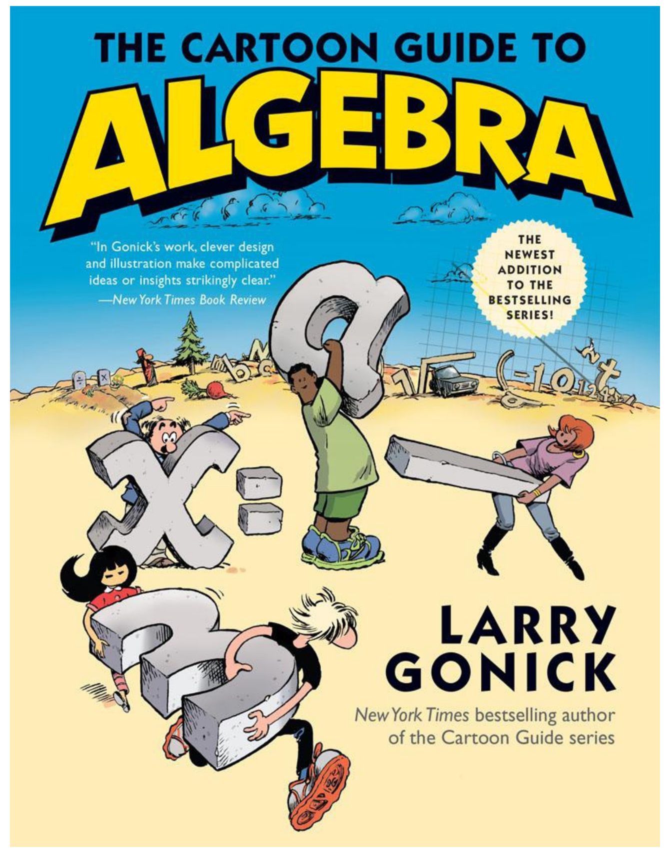
C) Yes

$$a^n a^m = a^{(n+m)}$$

$a^n a^m$ IS THE PRODUCT OF n FACTORS OF a MULTIPLIED BY m FACTORS OF a,

$$\underbrace{a \cdot a \cdot \ldots \cdot a}_{n} \cdot \underbrace{a \cdot a \cdot \ldots \cdot a}_{m}$$

MAKING A TOTAL OF $n+m$ FACTORS.



8) POWERS OF POWERS

$$(a^b)^c = a^{bc}$$

This power operation rule tells that building c groups with b elements gives us a group of bc elements.

$$(2^3)^2 - 2^{(3^2)} = ?$$

$$(2^3)^2 - 2^{(3^2)} = ?$$

$8^2 - 2^9 = 64 - 512 = -448$.

It is not zero.

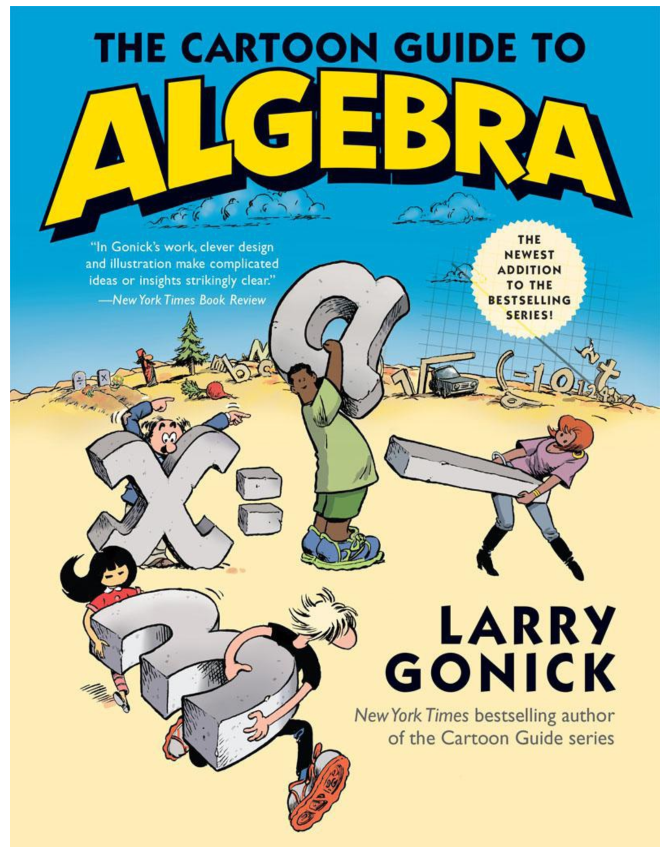
$$(ab)^n = a^n b^n$$

THIS COMES FROM THE COMMUTATIVE LAW.

$$(ab)^n = ab \cdot ab \cdot \ \dots \ \cdot ab$$

REARRANGING (OR "COMMUTING") THE ORDER GIVES

$$a \cdot a \cdot \dots \cdot a \cdot b \cdot b \cdot \dots \cdot b = a^n b^n$$



$$(a^n)^m = a^{nm}$$

WE CAN WRITE THE PRODUCT $(a^n)^m$

$$\left.\begin{matrix} a \cdot a \cdot \dots \cdot a \\ \cdot\, a \cdot a \cdot \dots \cdot a \\ \vdots \\ \cdot\, a \cdot a \cdot \dots \cdot a \end{matrix}\right\} m \text{ ROWS}$$

n FACTORS IN EACH ROW

THERE ARE nm FACTORS ALTOGETHER.

9) LINEAR EQUATIONS

$$mx+b=c$$

If m,b and y are given, we can solve for $x=(c-b)/m$. Equations of this type have been solved already by the Babylonians, without variables however!

For which x is

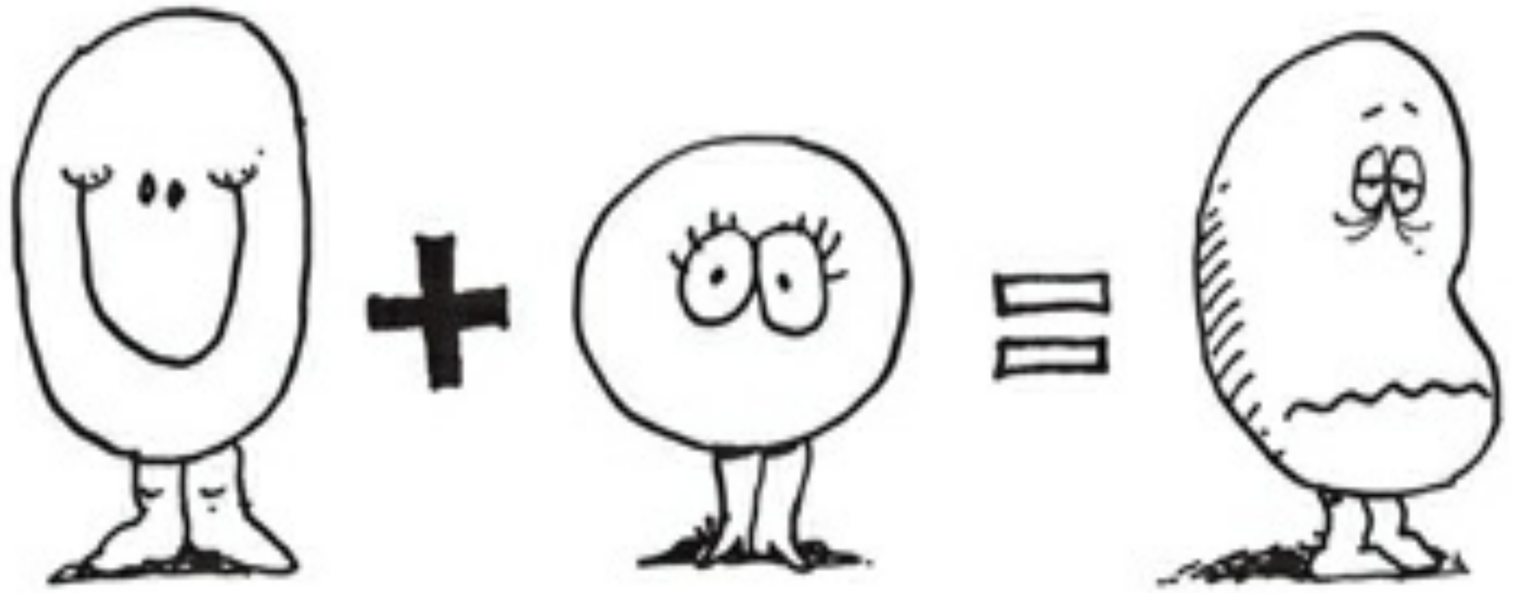
$7x-3=11$?

Solve $7x-3=11$

Move 3 on the other side and divide by 7. The result is $x=2$.

QUICK REBALANCING, or CALL THE MOVERS!

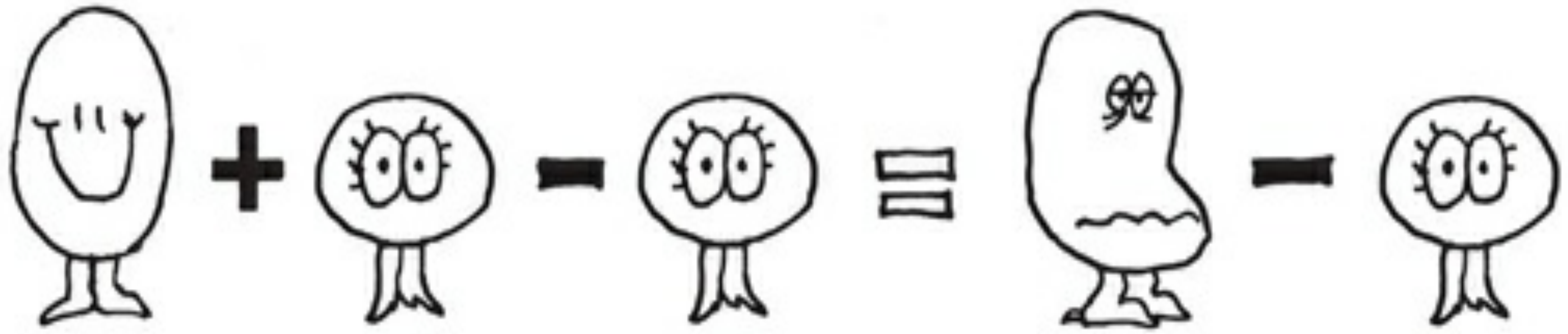
NOW I'M GOING SHOW YOU A QUICKER WAY TO REBALANCE EQUATIONS. LET'S IMAGINE ANY EQUATION WHERE ONE SIDE IS THE SUM OF TWO EXPRESSIONS.



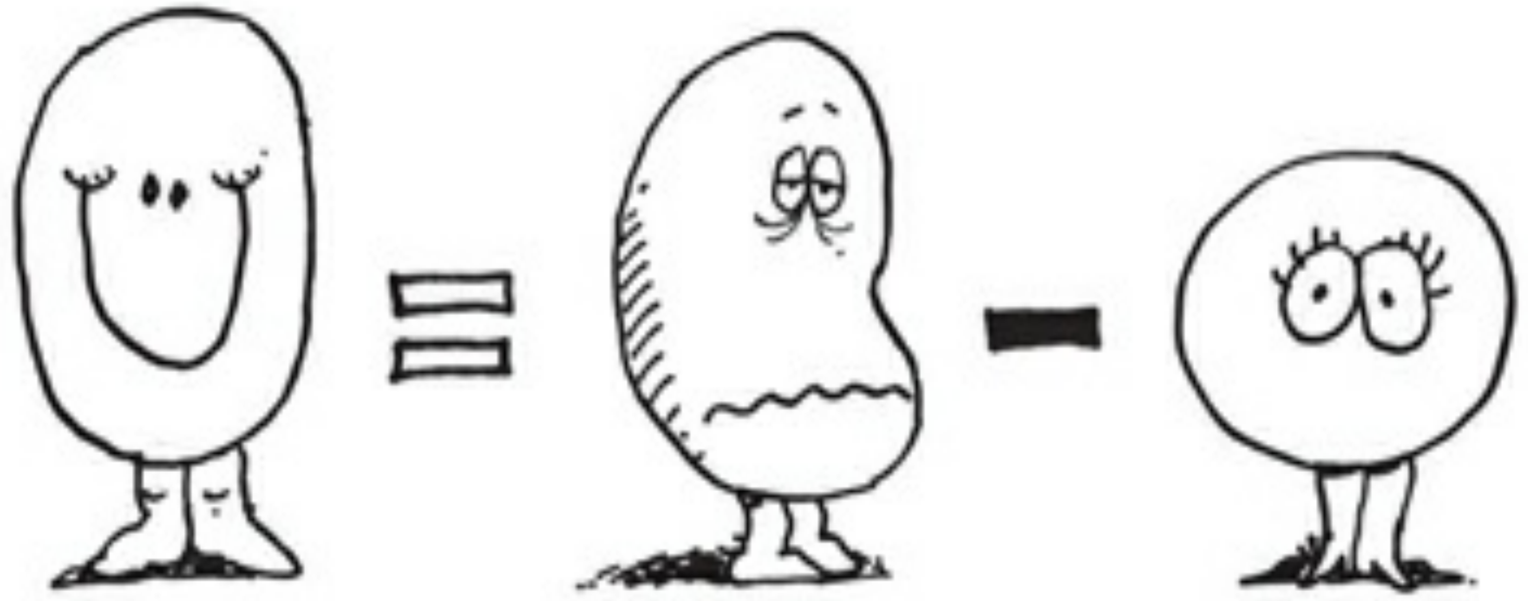
AND SUPPOSE WE WANT TO ELIMINATE

FROM THE LEFT.

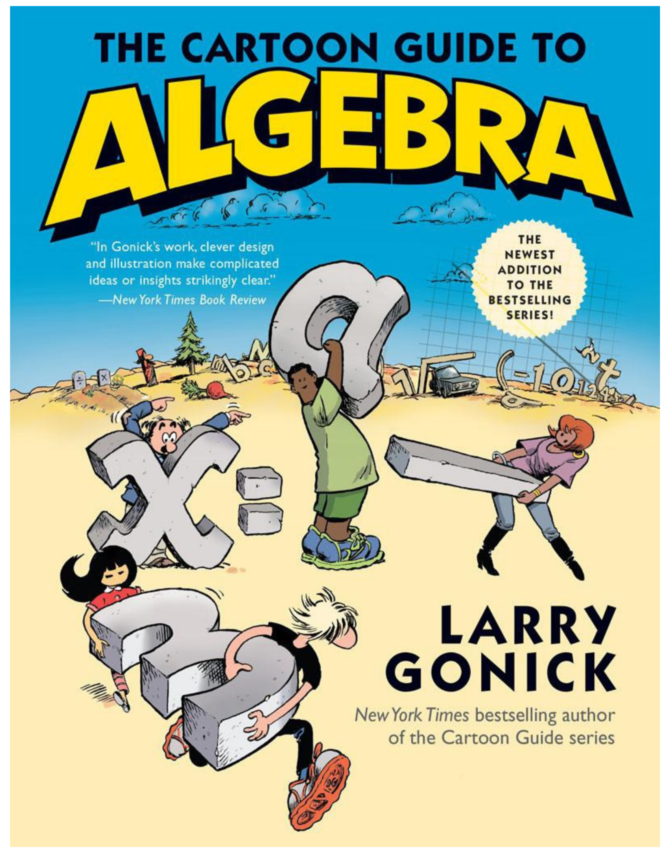
BY NOW, YOU KNOW THAT WE DO THIS BY SUBTRACTING THE EXPRESSION FROM BOTH SIDES. LET'S WRITE IT ON A LINE, RATHER THAN STACKED UP.



THE RESULT:



DO YOU SEE WHAT HAPPENS? THE EXPRESSION SEEMS TO JUMP FROM ONE SIDE TO THE OTHER, AND ITS SIGN CHANGES FROM PLUS TO MINUS!



10) QUADRATIC EQUATION

$$\frac{-b \pm \sqrt{b^2 - 4c}}{2}$$

To solve $x^2 + bx + c = 0$, just add $b^2/4$ on both sides to that $(x + b/2)^2 = b^2/4 - c$ allowing to solve for $x = \sqrt{b^2/4 - c} - b/2$

$$\frac{x}{x+1} = x - 1$$

Philipppo Calandri: "Una racolta di ragioni", 15th century

$$\frac{x}{x+1} = x - 1$$

$$x^2 + x - 1 = 0$$

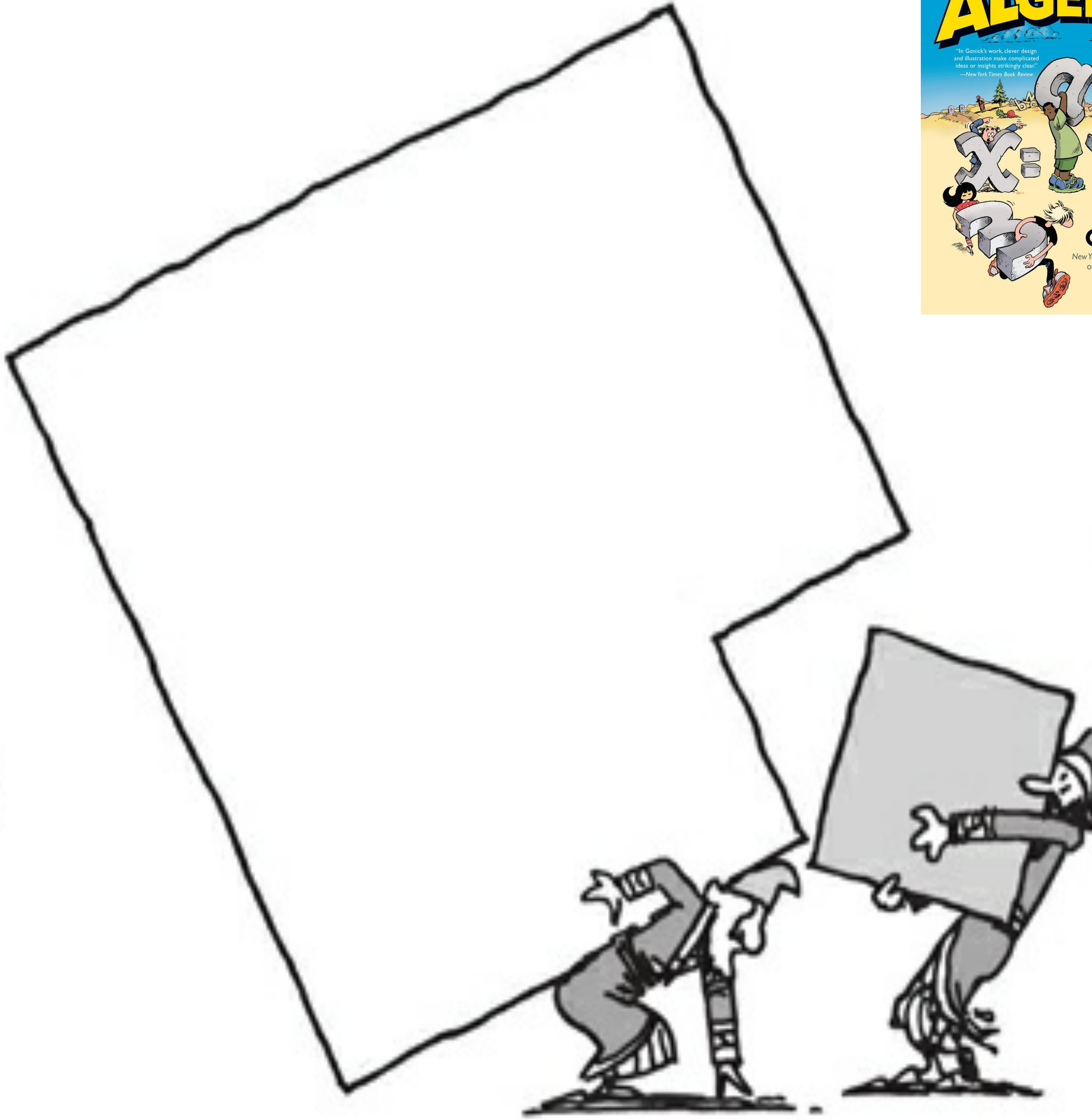
The quadratic equation gives

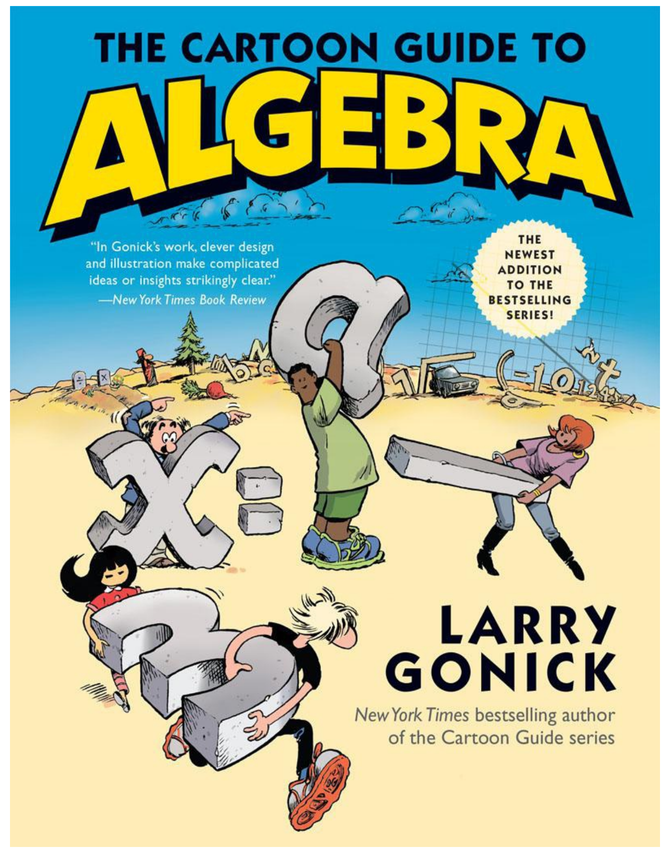
$$x = \frac{1 \pm \sqrt{5}}{2}$$

This is the golden ratio!

YOU MAY WONDER WHAT WE GET FROM SOLVING SUCH A SPECIAL EQUATION. HERE'S WHAT: IT TURNS OUT THAT WE CAN WRESTLE **EVERY QUADRATIC EQUATION** WITH LEADING COEFFICIENT 1 INTO THE FORM $(x+B)^2 = D$. THAT'S RIGHT! EVERY LAST ONE OF THEM. PERIOD. THIS BABYLONIAN TRICK IS CALLED...

Completing the SQUARE.

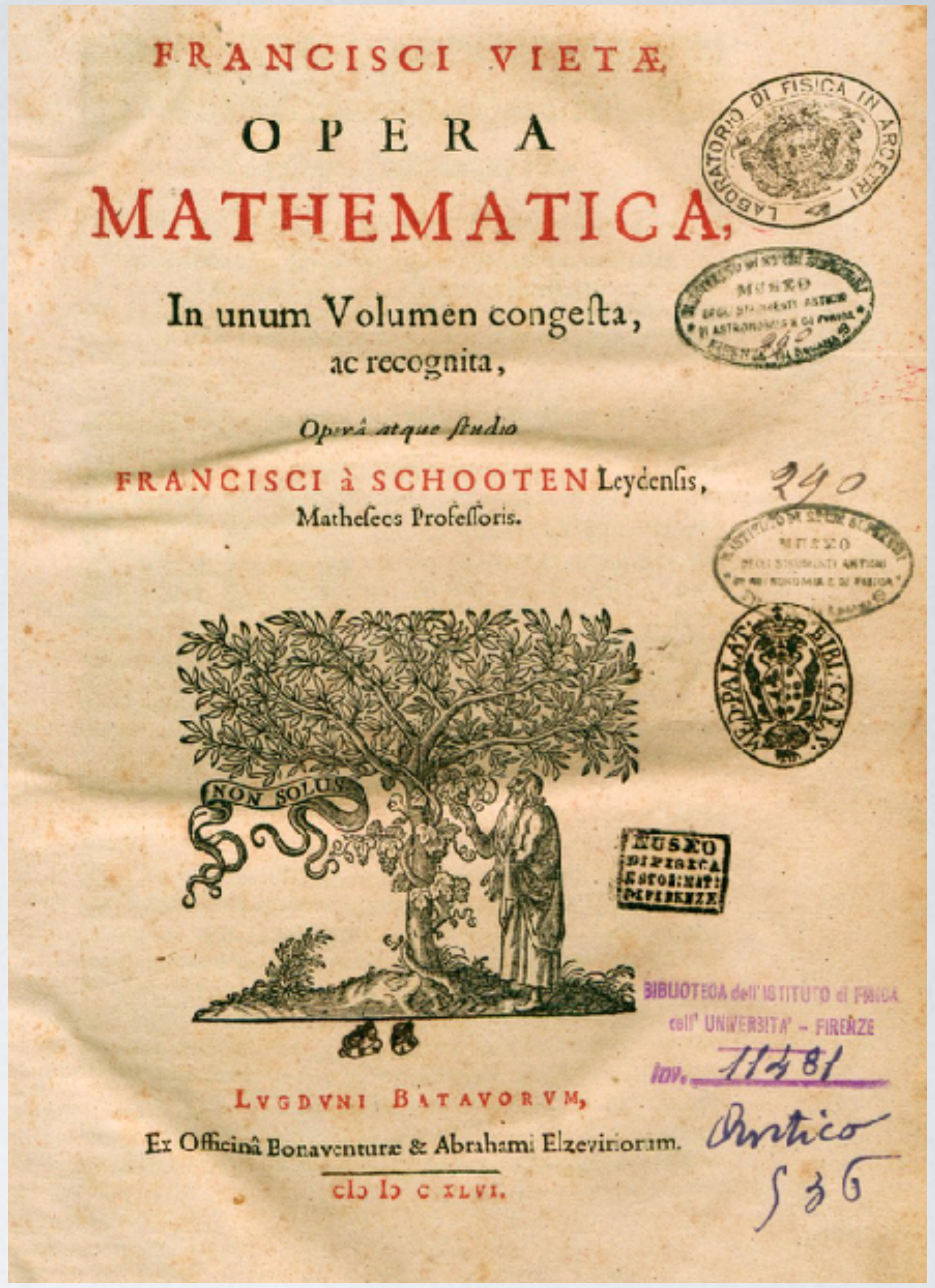





THE END

ABOUT THE SUBJECT

Elementary algebra was here understood in the sense that no functions appear, no geometry is used and no limits are involved. It is algebra historically going up to Francois Viete. While it usually is taught in middle school of secondary school curricula, it is pivotal in any college mathematics.


FRANCISCI VIETÆ

OPERA

MATHEMATICA,

In unum Volumen congesta, ac recognita,

Operâ atque studio

FRANCISCI à SCHOOTEN Leydensis, Matheseos Professoris.

NON SOLUS

LVGDVNI BATAVORVM,

Ex Officinâ Bonaventuræ & Abrahami Elzeviriorum.

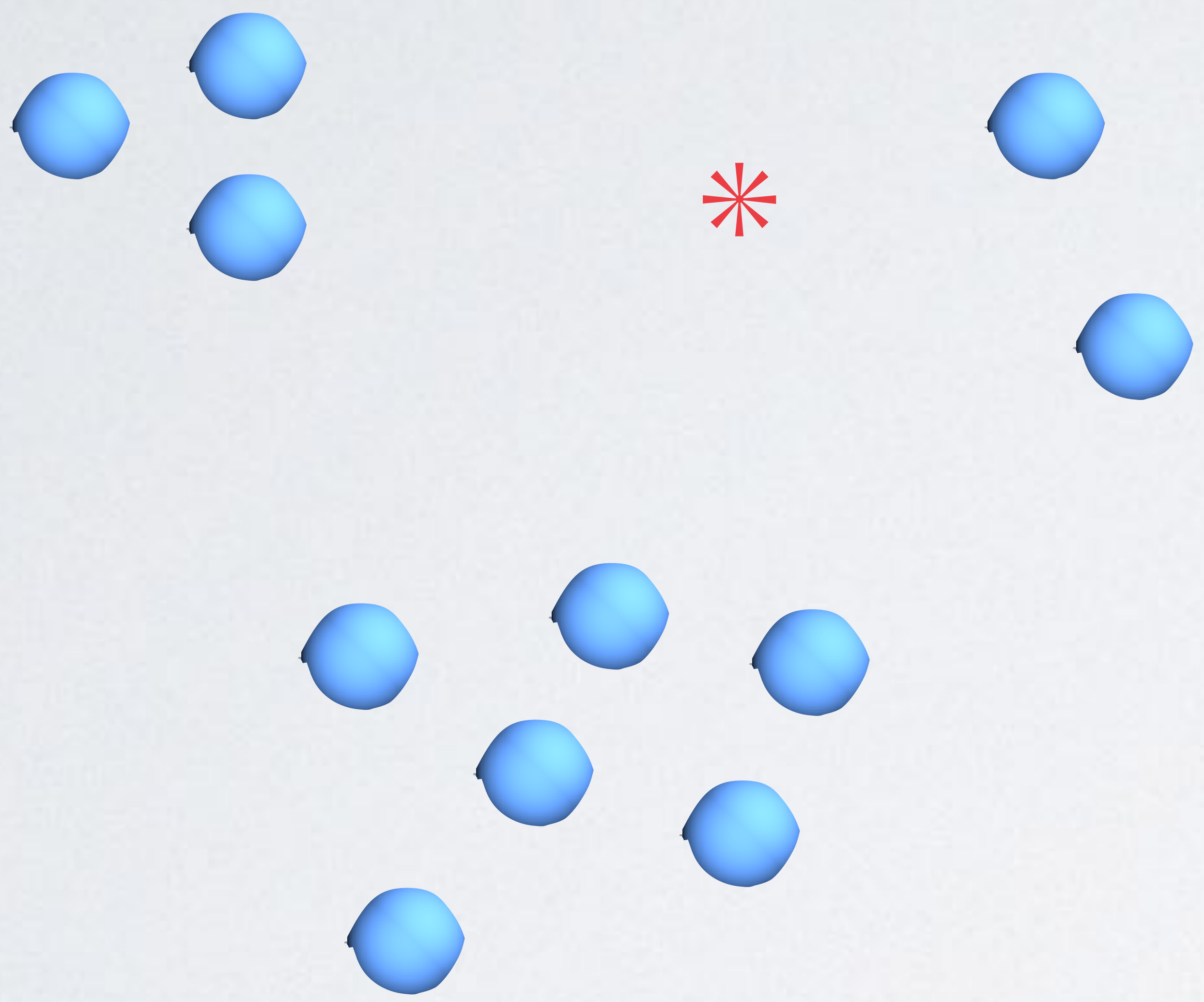
CIƆ IƆ C XLVI.

RELATIONS TO OTHER ROOTS

FLAVORS OF ALGEBRA

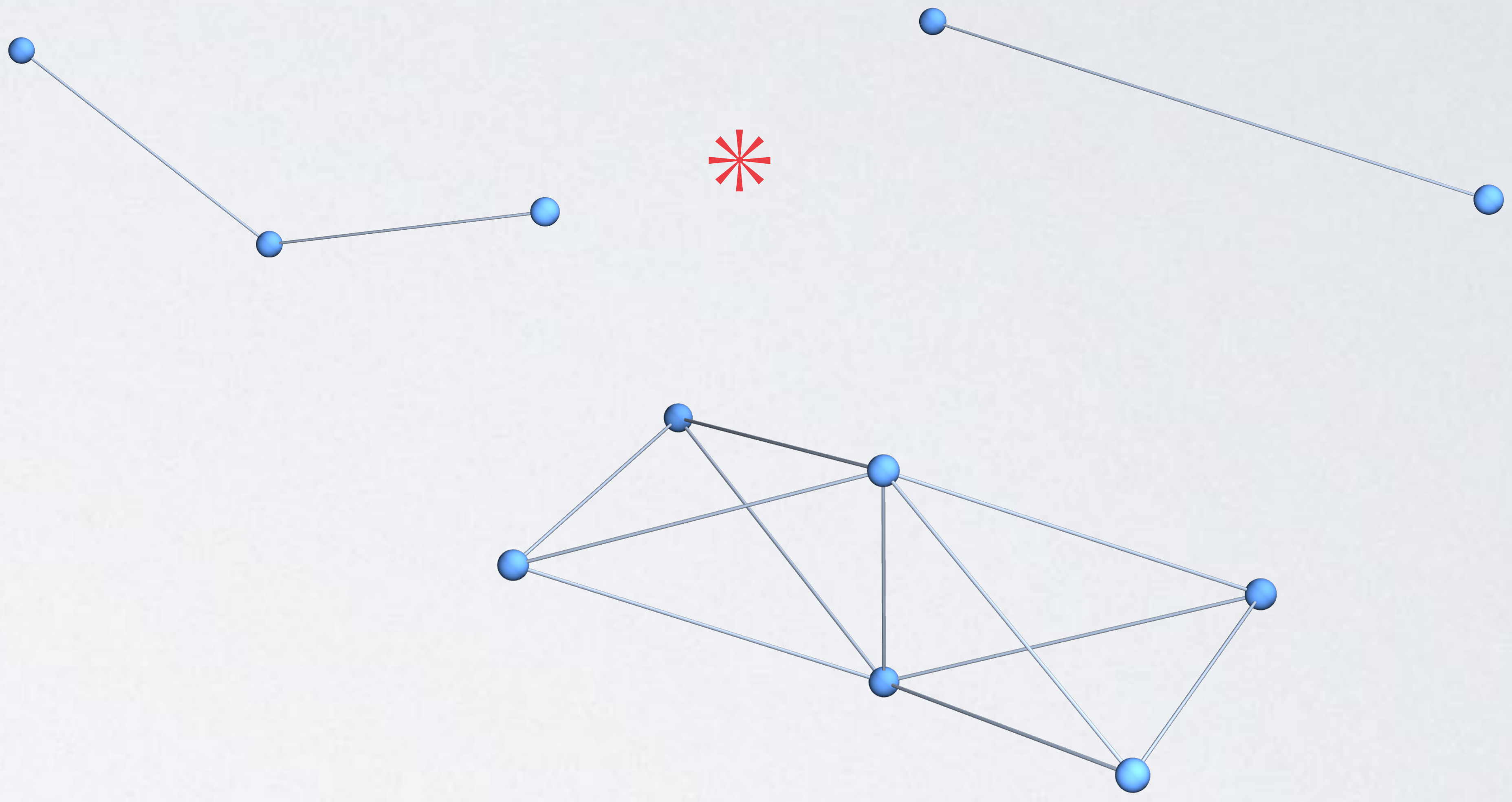
MY OWN WORK

There is a commutative algebra of geometric structures which goes beyond the algebra we know.



$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$

algebra of pebbles



$\mathcal{N} \subset \mathcal{Z} \subset \mathcal{Q} \subset \mathcal{R} \subset \mathcal{C}$

algebra of networks

rebel code
Extremal Graph Theory
DIFFERENTIAL GEOMETRY
Gravitation and Inertia
Graph Theory Applications
INTRODUCTION TO PROBABILITY
MIND PERFORMANCE
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Convex Polytopes
Grünbaum
GTM
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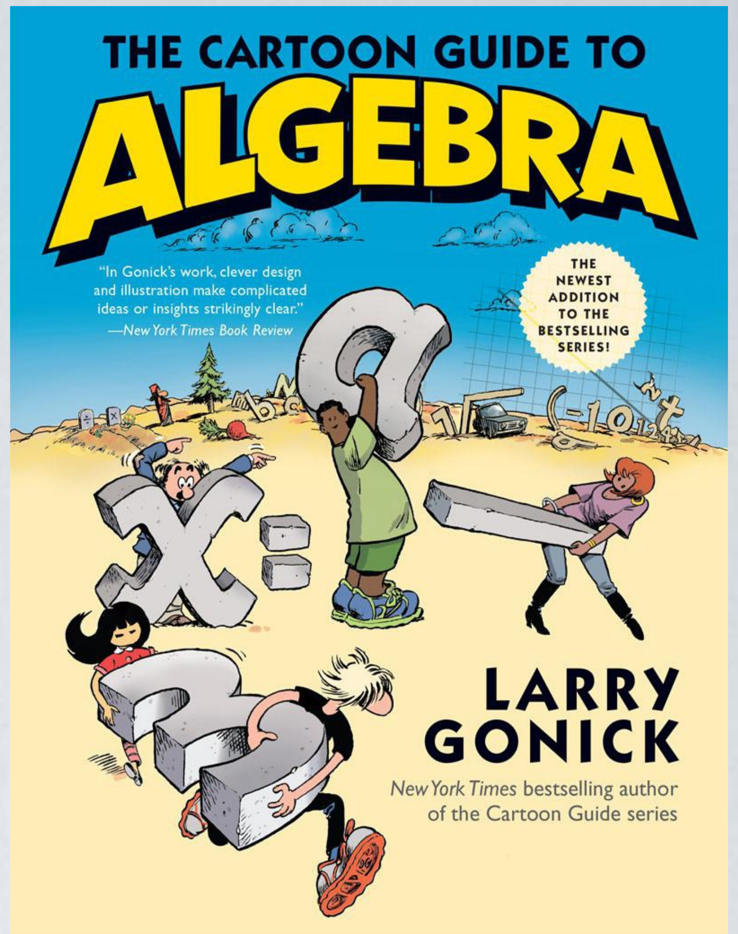
Introduction to Dynamical Systems
Algebra
Calculus of Variations
THE GEOMETRY
1/3
1/4
1/5
1/6
1/8
1/12


GEOMETRY
TYCHO & KEPLER
Shaping Space
The Physics BOOK
Statistical Mechanics
IAN STEWART
FEARLESS SYMMETRY
MEASURE AND INTEGRATION
Graph Theory and Complex Networks
TIME REBORN
THE HISTORY OF STATISTICS
Introduction to Numerical Analysis
MATHEMATICS
Calculus

ACKNOWLEDGEMENTS

As for literature, the **cartoon guide to algebra** of Larry Gonick is an excellent textbook. The samples borrowed here show this. Most elementary algebra textbooks are thousands of pages long. I myself learned about struggles in elementary algebra from thousands of college students. (I teach mathematics since 35 years and graded thousands of calculus and linear algebra exam papers during that time and see students work in class). I also learned a lot from 200 teachers during 12 years of teaching at the extension school as well as from my family: many in my closer family was involved at some point in teaching math.



ELEMENTARY ALGEBRA BOOKS