AMBIGUOUS PEMDAS

OLIVER KNILL

ABSTRACT. A source collection about the mathematical syntax in the ring $\mathbb{Z}$.

1. PEMDAS

1.1. In the commutative **additive group** $(\mathbb{Z}, +, 0)$ one usually writes $-x$ for the inverse element. One has then $x + (-x) = 0$ or $x - x = 0$ for short. As it is convenient to enhance the addition $+$ with subtraction $-$, **associativity** obviously fails for subtraction: the two numbers $3 - (4 - 5)$ and $(3 - 4) - 5$ are not the same. In the first case, we need to place brackets.

1.2. The same applies in the **multiplicative group** $(\mathbb{Z} \setminus \{0\}, *, 1)$. The expression $3/4/5$ is only defined if we establish a rule how to read it. $(3/4)/5$ is not the same than $3/(4/5)$. The non-associativity of the division operation again forces to place brackets, even if the most common interpretation of $3/4/5$ is $3/20$.

1.3. When combining multiplication and division, brackets are even less enforced. We often write x/yz meaning $x/(yz)$ and do not understand it as $(x/y)z$. That brackets are required to avoid any confusion was noticed in [4] and reiterated in [2], the authority on mathematical syntax. See [5] for some history of notation. Even calculators do not have standards. A TI-82 or Casio calculator interprets $1/2x$ as $1/(2x)$ while the TI-83 interpret it as $(1/2)x$ [9].

1.4. To establish the order of operations in the ring $(\mathbb{Z}, +, *, 0, 1)$ one has established rules. The popular choice $PEMDAS$ ($=BEMDAS$) orders the operation as **P**arenthesis (**B**rackets), **E**xponentiation, **M**ultiplication, **D**ivision, **A**ddition and **S**ubtraction. In the 21st century, discussions have flared up [3] and there still exists no agreement. Expressions like $3/ab$ with $a = 4, b = 7$ are usually interpreted as $3/28$ while a calculator evaluates it as $21/7$.

1.5. Since terms like a^b and especially x^2 appear frequently in formulas, exponentiation is included in the rules. The associativity fails like in $(3^3)^3 = 19683$ and $3^{(3^3)} = 7625597484987$ but no standard has even been formulated. The computer algebra system Mathematica starts from the right, while Excel or the computer algebra system Matlab reads from the left. Also for exponentiation, we need to put brackets.

1.6. Beside PEMDAS, the rule $BEDMAS$ ($=BODMAS$) places division before multiplication. It is known in some programming languages. Most computer algebra implementations use $PE(MD)AS$ in which M and D are placed on the same footing and things are read left to right. For $3/2x/2$, a computer gives $3x/4$ while a human reader might read it as $3/(4x)$. There is no authority for a standard. Language is a social construct [10]. Therefore, we need to clarify or even over clarify.

1.7. When deciding about the use of language, practice matters. When 12 million test subjects give the wrong answer to a simple arithmetic question [6], it is not a problem of the education level [7]. We also do not need to come up with an other standard or fix the issue [8]. The reality of teaching shows that even entrenched rules can be made more robust with more clarity. As a teacher one can observe that even stellar students often read non-ambiguous expressions $a/b + c$ as $a/(b + c)$. **Redundancy** in the form of brackets makes it clear. The article [3] from 2013 has hit the nail on the head.

1.8. Enforcing any of the standards like PE(MD)AS also would not only clash with the use of established literature it would be unrealistically complex to teach. When we read expressions, numbers like 2π are often seen as a unit. We might write $5/2\pi$ and mean $5/(2\pi)$ and not $(5/2)\pi$. We see terms like $pV/RT = n$ in thermodynamics and read it as $pV/(RT)$ rather than $(pV/R)*T$. We read $3m/4s$ as 3 meters per 4 seconds and not $(3/4)ms$. Usage also depends on school systems or programming languages. Use clarifying brackets.

1.9. To summarize: the syntax for writing rational and exponential expressions in a polynomial ring has a few places, where brackets are needed to avoid miscommunication. This is not a syntax problem which needs to be "solved". It is an issue to be aware of and which needs to be avoided. It is better to write expressions extra clearly, possibly even in a redundant way and anticipate any possible misunderstandings. These were the recommendations from almost 100 years ago [2] and things have not changed.

2. Sources

> **242.** *Order of operations in terms containing both* $\div$ *and* $\times$.—If an arithmetical or algebraical term contains $\div$ and $\times$, there is at present no agreement as to which sign shall be used first. "It is best to avoid such expressions."[3] For instance, if in $24 \div 4 \times 2$ the signs are used as they occur in the order from left to right, the answer is 12; if the sign $\times$ is used first, the answer is 3.
>
> Some authors follow the rule that the multiplications and divisions shall be taken in the order in which they occur.[4] Other textbook writers direct that multiplications in any order be performed first, then divisions as they occur from left to right.[5] The term $a \div b \times b$ is interpreted by Fisher and Schwatt[6] as $(a \div b) \times b$. An English committee[7] recommends the use of brackets to avoid ambiguity in such cases.

FIGURE 1. From Cajori [2].

QUESTIONS AND DISCUSSIONS. 95

multiplications and divisions are involved is not the one stated above, but the following:

All multiplications are to be performed first and the divisions next.

That is, $9a^2 \div 3a = 3a$ and not $3a^3$.

The multiplications may be taken in any order, but the divisions are to be taken in the order in which they occur from left to right.

That is, the associative law holds for the former but not for the latter.

Thus, $3 \times 5 \times 2 = (3 \times 5) \times 2$ or $= 3 \times (5 \times 2)$; but, $16 \div 4 \div 2 = (16 \div 4) \div 2$ and does not $= 16 \div (4 \div 2)$.

Compare the corresponding rules for addition and subtraction in § 1.

Mathematical Idioms. It might be agreed that, for the sake of simplicity and logical coherence, the past tense of the verb *to drink* should be *drinked*, but even so, English speaking people would continue to say *drank*, and not drinked. Precisely, for the same reason, all who know anything about the language of algebra regard $9a^2 \div 3a$ as equal to $3a$ and not $3a^3$, and, therefore, the rule just given is the correct one as determined by actual usage. When a mode of expression has become wide-spread, one may not change it at will. It is the business of the lexicographer and grammarian to record, not what he may think an expression should mean (no matter how far-fetched the usual or idiomatic usage may seem), but what it is *actually understood to mean by those who use it.* The language of algebra contains certain idioms and in formulating the grammar of this language we must note them. For example, that $9a^2 \div 3a$ is understood to mean $3a$ and not $3a^3$ is such an idiom. The matter is not logical but historical.

FIGURE 2. From Lennes [4].

What I want to talk about here today is some of the general principles that I've noticed in mathematical notation, and what those mean now and in the future.

This isn't really a problem about mathematics. It's really more a problem in linguistics. It's not about what mathematical notation could conceivably be like; it's about what mathematical notation as it's actually used is actually like—as it's emerged from history and presumably from the constraints of human cognition and so on.

And, in fact, I think mathematical notation is a pretty interesting example for the field of linguistics.

FIGURE 3. From Wolfram [10].
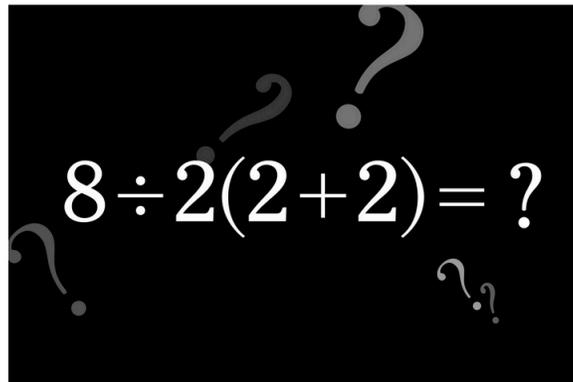
3

FIGURE 4.   From Haelle [3].

FIGURE 5.   From a 2019 article [7] in the New York Times.

No recommendation is made or implied about the font of italic type in which symbols for quantities are to be printed.

7.1.2 Subscripts

When, in a given context, different quantities have the same letter symbol or when, for one quantity, different applications or different values are of interest, a distinction can be made by use of subscripts.

The following principles for the printing of subscripts apply.

— A subscript that represents a physical quantity or a mathematical variable, such as a running number, is printed in italic (sloping) type.

— Other subscripts, such as those representing words or fixed numbers, are printed in roman (upright) type.

EXAMPLE

Italic subscripts		Roman subscripts	
C_p	(p: pressure)	C_g	(g: gas)
c_i	(i: running number)	c_3	(3: third)
$\Sigma_n\, a_n \omega_n$	(n: running number)	g_n	(n: normal)
F_x	(x: x-component)	μ_r	(r: relative)
g_{ik}	(i, k: running numbers)	S_m	(m: molar)
I_λ	(λ: wavelength)	$T_{1/2}$	(1/2: half)

NOTE   For a list of common subscripts, see IEC 60027-1.

7.1.3 Combination of symbols for quantities

When symbols for quantities are combined in a product of two or more quantities, this combination is indicated in one of the following ways:

$$ab,\ a\,b,\ a \cdot b,\ a \times b$$

NOTE 1   In some fields, e.g. vector algebra, distinction is made between $\boldsymbol{a} \cdot \boldsymbol{b}$ and $\boldsymbol{a} \times \boldsymbol{b}$.

Division of one quantity by another is indicated in one of the following ways:

$$\frac{a}{b},\ a/b,\ a\,b^{-1},\ a \cdot b^{-1}$$

Do not write ab^{-1} without a space between a and b^{-1}, as ab^{-1} could be misinterpreted as $(ab)^{-1}$.

NOTE 2   The solidus "/" can easily be confused with the italic upper-case "I" or the italic lower-case "l", in particular when sans serif fonts are used. The horizontal bar is often preferable to denote division.

These procedures can be extended to cases where the numerator or denominator or both are themselves products or quotients. In such a combination, a solidus (/) shall not be followed by a multiplication sign or a division sign on the same line unless parentheses are inserted to avoid any ambiguity.

EXAMPLE 1

$$\frac{ab}{c} = ab/c = ab\,c^{-1}$$

$$\frac{a/b}{c} = \frac{a}{bc} = (a/b)/c = a/(bc),\ \text{not } a/b/c$$

FIGURE 6.   ISO 80000-1:2009(E).

Formulas. You can help us to reduce printing costs by avoiding excessive or unnecessary quotation of complicated formulas. We linearize simple formulas, using the rule that multiplication indicated by juxtaposition is carried out before division. Thus, instead of the (coded) display:

```
$${1\over{2\pi i}}\int_\Gamma {f(t)\over (t-z)}dt$$
```

we might use

```
$(1/2\pi i)\int_\Gamma f(t)(t-z)^{-1}dt$.
```

Typesetting instructions. Formal copyediting of a hard-copy manuscript is the responsibility of our staff. However, it is helpful if the reviewer clarifies ambiguities by means of marginal notes (e.g., "one" (1) vs. "ell" (l), "zero" (0) vs. "oh" (O; in TeX math O), "union" ($\cup$) vs. the capital letter "you" (U), "wedge" ($\vee$) vs. "vee" (V), "wedge" ($\wedge$) vs. capital "lambda" (Λ), etc.).

FIGURE 7. From [1]: write $(1/2\pi i)\int_\Gamma f(z)dz$ rather than $\frac{1}{2\pi i}\int_\Gamma f(z)dz$.

Ihre Story, Ihre Informationen, Ihr Hinweis? **feedback@20minuten.ch**

Schaffst du es?   02. Mai 2019 21:00; *Akt: 02.05.2019 21:00*

Millionen scheitern an dieser Mathe-Gleichung

Eine scheinbar einfache Mathegleichung lässt gerade Menschen auf der ganzen Welt verzweifeln. Kannst du sie knacken?

Can You Solve This?

$$6 \div 2(1+2) =$$

ein aus *i*

Like

262

Share

Twittern

Mail

Fast 12 Millionen Leute haben sich an der vermeintlich einfachen Gleichung versucht und sind gescheitert.

Schaffst du es, auf Anhieb auf die richtige Lösung zu kommen? Schreib dein Ergebnis unten ins Kommentarfeld! Das korrekte Ergebnis erfährst du unten im Video, aber nicht schummeln! :-)

COMMUNITY-PUSH

FIGURE 8. From a Swiss newspaper article 2019: "12 Millions got this wrong".

OLIVER KNILL

References

[1] AMS. Guide for Reviewers – Mathematical Reviews. June 6,1996, 1996.

[2] F. Cajori. *A history of Mathematical Notations*. The Open Court Company, London, 1928.

[3] T. Haelle. What is the answer to that stupid math problem on facebook. *Slate*, 2013.

[4] N.J. Lennes. Relating to the order of operations in algebra. *The American Mathematical Monthly*, 24:93–95, 1917.

[5] J. Mazur. *Enlightening Symbols, A short history of Mathematical notation and its hidden powers*. Princeton University Press, 2014.

[6] 20 Min. Millionen scheitern an dieser Mathe Gleichung. *20 Min*, 2019. Swiss newspaper.

[7] S. Strogatz. The math equation that tried to stump the internet. *New York Times*, 2019.

[8] J. Taff. Rethinking the order of operations. *Mathematics Teacher*, 111:126–131, 2017.

[9] Wikipedia. Order of operation. Accessed February 24, 2021, 2021.

[10] S. Wolfram. Mathematical notation: Past and future. Keynote at MathML conference 2000, 2000.

Department of Mathematics, Harvard University, Cambridge, MA, 02138