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nking the Order of Operations

(or What *Is* the Matter with Dear Aunt Sally?)

Jason Taff

Consider the following expressions:

$$6 \cdot 7 + 3\sqrt{48+1} \text{ and } 9x^5 - 12x^3$$

Both are easily recognizable expressions from various levels of mathematics teaching. In the first, an expression that a student must simplify, the order in which the operations should be performed is traditionally taught using the well-known PEMDAS acronym (often rendered mnemonically as "Please Excuse My Dear Aunt Sally"). The second expression, perhaps a function to be graphed, requires an implicit understanding of the order of operations, but advanced algebra students, teachers, and mathematicians are unlikely to think about PEMDAS explicitly when parsing this expression. That is, the intuitions required to analyze the polynomial expression do not match the instructions given by PEMDAS. This strongly suggests that a more sophisticated concept of the order of operations is necessary once students reach an algebra course.

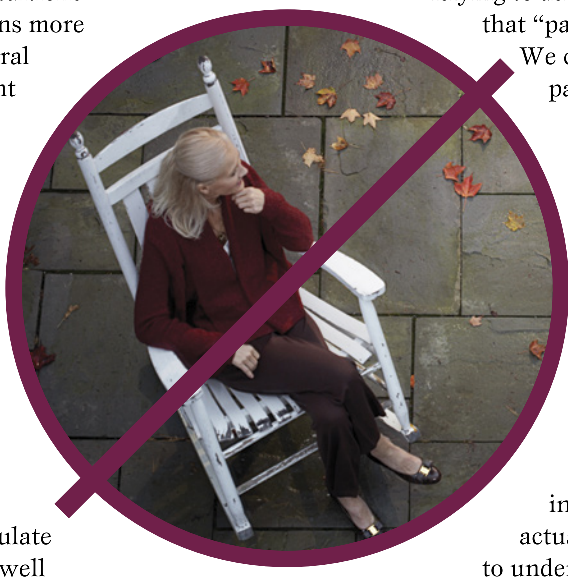
Shortcomings in the traditional PEMDAS approach to teaching the order of operations suggest a classroom-tested alternative, suitable for advanced students, that conforms more closely to the intuitions of experienced practitioners of mathematics.

In particular, the structures of those expressions are quite similar: Both expressions contain two terms, each comprising two factors, some of which themselves decompose into smaller subunits. Note that one would never say that in the polynomial, the cube in the second term "precedes" the coefficient multiplication in the first term. The structurally corresponding calculations in the numerical expression are 48 + 1 in the second term and 6 • 7 in the first. PEMDAS, however, suggests that the calculation of 48 + 1 should indeed precede 6 • 7, since 48 + 1 is contained in a grouping symbol. Are we prepared to teach our students that "precedence" is the relevant concept here?

During many years of teaching algebra, I have become increasingly aware of this discrepancy between our intuitions about expressions like these and the thought process suggested by the PEMDAS acronym. To expose students to more advanced intuitions about the order of operations, I have changed my approach to presenting and reviewing this concept. The analysis I suggest in this article was presented to advanced seventh-grade prealgebra students, but the approach is broadly applicable to older students, especially those who are reviewing the concept of the order of operations in preparation for thinking about more advanced concepts. The purpose of this article, then, is twofold. First, I will summarize some of the shortcomings of equating the order of operations concept with the PEMDAS procedure; my hope is that making these shortcomings explicit will help teachers at the high school level better understand their students' common misconceptions about the order of operations. Second, I will describe my alternative approach, which both matches our intuitions about the order of operations more closely and provides a natural place to introduce important vocabulary items to my students: *term* and *factor*.

SHORTCOMINGS OF PEMDAS

The more I taught the PEMDAS method, the more its shortcomings became apparent to me. It is worth noting that all students in my classes had been exposed to the order of operations before, and I carefully stipulate that PEMDAS might very well be an appropriate teaching technique in primary and middle school grades. At those stages, when many students are not developmentally ready to attach more abstract meaning to symbols, the goals are to produce an accurate calculation and to emphasize the need for having an ordering convention in the first place. From a more advanced point of view, though, the shortcomings become more troublesome. Note that space considerations limit the discussion of these shortcomings; the list below represents only some of the most egregious ones.



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Parentheses

Many textbooks point out that not all grouping is done with parentheses or similar bracketing symbols. None of the expressions below uses them to suggest grouping:

$$7\sqrt{a^2+b^2} \text{ or } \frac{7}{a^2+b^2} \text{ or } 7^{a+b}$$

Like the polynomial expression we began with, these more advanced expressions are subject to the order of operations convention, but PEMDAS does not quite seem like the right approach. Grouping here is suggested by the horizontal bars or merely by font size with no actual "grouping" symbols whatever. Replacing the *P* with a *G* for "grouping symbols" provides a superficial solution to this problem, but clearly an incomplete one. And furthermore, not all parentheses are intended to suggest grouping. In $16 + (-9)$ or $(x + 20) + (2x - 15) + (5x - 65) = 180$, the parentheses clarify the use of symbols or the thought process that went into creating the equation. Students who are trained to "do parentheses first" can quite legitimately be confused when those parentheses are not being used to indicate grouping.

Finally, as a pedagogical matter, it feels unsatisfying to ask students to memorize that "parentheses come first." We calculate expressions in parentheses first not because that is the convention (which is the case for multiplication preceding addition), but rather because that is a purpose of parentheses. As students become more adept at writing their own expressions and equations, they should begin to understand that grouping symbols are introduced onto the page by actual people who want readers to understand their intent. When parentheses are being used for grouping, we would prefer that our students recognize this because they fully understand the intended meaning of the parentheses, not because the letter *P* comes first in a memorized acronym. Indeed, memorizing that grouping symbols take precedence is barely needed, because most of those symbols have evolved to be so visually intuitive that defying them seems awkward, even for beginners.

Exponents

Putting the *E* before the *MDAS* certainly produces correct calculations. But this is more a fact about the scope of application of an exponent than about its place in the order of operations. That is, once students can understand and visualize that an exponent applies only to the single unit that it physically touches—whether that is an individual number or variable or a set of parentheses—the calculation of the product simply becomes another instance of multiplication. It therefore seems that a separate entry for exponents in an ordered list might not really be needed.

Grouping symbols are introduced by people who want readers to understand their intent.

Multiplication

The standard instructions for the *MD* part of *PEMDAS* are to calculate multiplication and division in order from left to right. Those instructions only really make sense, though, if the ÷ symbol is being used to indicate division, in which case a convention is required so that 30 ÷ 2 • 5 is unambiguous. If no division is present, asking students to multiply from left to right seems to contradict our lessons about how multiplication is commutative. For example, in such a calculation as 2 • 9 • 15, I would hope that my students would commute and multiply the 2 by the 15 first, rather than invoke PEMDAS and multiply left to right.

Division

In most courses, once the "order of operations" unit is finished, the ÷ symbol largely disappears, and the standard fraction bar takes its place. At this point, the location of the *D* in the order ceases to be an issue. This is because the basic purpose of the order of operations convention is to dictate the intended grouping of operations in the absence of explicit grouping symbols; and the fraction bar is unique among the symbols for basic operations in being simultaneously an operation and a grouping symbol. At higher levels of mathematics, then, most instances of division need not rely on the convention.

The more substantive situations in which multiplication and division are intermingled are such rules as the following:

$$\frac{a}{b}\bullet\frac{c}{d}=\frac{ac}{bd},$$

which can be seen as simply describing a situation in which the grouping function of the fraction bar may be circumvented: You can divide some numbers first and then multiply, or you can multiply other numbers first and then divide—they are equal, so take your pick! As before, an order of operations convention is implicit here, but PEMDAS does not quite make sense. Indeed, what "left to right" would mean on either side of the equals sign is not even clear.

Addition and Subtraction At the end of the order, we come to adding and subtracting. Although the truth is that the multiplication and division operations precede the addition and subtraction operations, the internal logic of the standard instructions ("addition and subtraction from left to right") suffers from a similar shortcoming as the *MD* instructions do: Once we become more skilled at simplifying expressions, we instinctively commute the terms of a sum, and when convenient, we reimagine subtraction as merely "addition of the opposite." All the terms are added simultaneously, really, and certainly not (necessarily) left to right.

PEMDAS as a Whole

When we teach our students to adhere to the PEMDAS order, I do not claim that we are teaching a mathematical falsehood. However, in my experience, the way the instruction is received by students can sometimes contain a subtle but highly misleading falsehood. Consider the following typical example of a dutiful, line-by-line adherence to PEMDAS:

$$\begin{gathered}
9 \bullet 4 + 4(50 - 3(2+8)) - 10^2 \\
9 \bullet 4 + 4(50 - 3(10)) - 10^2 \\
9 \bullet 4 + 4(50 - 30) - 10^2 \\
9 \bullet 4 + 4 \bullet 20 - 100 \\
36 + 4 \bullet 20 - 100 \\
36 + 80 - 100 \\
116 - 100 \\
16
\end{gathered}$$

Students who show work like this have likely internalized a principle something like, "Follow these rules, and you will get the right answer." This is, of course, formally correct. However, students who have risen to more advanced mathematics by trusting rote adherence to rules would not be stepping very far afield to internalize instead, "You must follow these rules to get the right answer." This is, of course, absolutely not correct, but I have become convinced that many students come into algebra classes believing it. Until being told otherwise, or discovering it for themselves, these students will be attempting to interpret increasingly difficult mathematics while carrying the fundamental misconception that it would be an error to square the 10 first. The approach detailed in the next section alleviates this misconception.

A MORE MATHEMATICAL APPROACH

I devised this new approach to try to bridge the divide between a calculation-based prealgebra approach and the more symbolic grasp of the order of operations necessary for algebra. Indeed, the approach emphasizes a precise use of vocabulary typically taught in algebra classes. Thus this approach may or may not be appropriate for students in younger grades (5–7) who are first learning the order of operations. For beginning algebra students, though, or for advanced algebra students reviewing basic material at a higher level, I am convinced that the approach can form a more mathematically sound basis for symbolic algebra than the traditional PEMDAS approach.

The key vocabulary items in question are *term* and *factor*, both of which are crucial to learning algebra, and both of which, in my experience, are often misused by students. As I discuss below, this new approach asks students to identify the structure of the terms and factors in an expression first, and in doing so, they naturally form a more precise intuition about the order of operations. Which brings me to the arguably most delightful part of this lesson: the acronym. As I drove to work on the day I planned to teach this new approach, I was trying to devise an acronym to replace PEMDAS. Imagine my surprise when I realized that the initials of "identify terms and factors first" literally contain my name: iTAFF! Indeed, my students were delighted by the thought that I seemed destined by my name to invent a new way to explain the order of operations.

The key vocabulary items *term* and *factor* are both often misusd by students.

For the purposes of this method, I explained that *terms* are expressions separated by a plus or a minus sign and that *factors* are expressions separated by a multiplication sign (or by implicit multiplication). Since the word *factor* does not typically refer to expressions involved in division, and since the timing of division is typically dictated by the grouping function of the fraction bar, we excluded division from our initial lessons. Students were easily able to incorporate use of fraction bars when they were reintroduced later. I did not treat exponents as a separate operation. Rather, we practiced visualizing an expression with an exponent simply as a list of identical factors.

The first goal of the iTAFF method is that students learn to identify the nested structure of the terms and factors. For example, $53 + 24 - 2 \cdot 8$ comprises three terms, the last of which comprises two factors. And $(x + 3y)^5$ comprises five factors, all identical and each comprising two terms, the second of which comprises two factors. The key instruction to achieve that goal became, "First, look for all pluses and minuses that are not contained in any grouping symbols." Those operations are the first separators of terms. Then, students were instructed to find all the factors within each term. They soon realized that one or more of those factors might itself be a complicated subexpression contained in a grouping symbol, which can be analyzed separately by starting the whole process over again, looking for pluses and minuses (within the subexpression) that are not contained in grouping symbols. This gave students the visual intuition that a complex expression within parentheses was merely a "separate problem," whose calculation could be handled independently of the larger-scale calculation. This intuition, in turn, reduced the intimidation some students felt in approaching expressions that are more complicated. I did not know what to expect when my students began that exercise, but with a little practice, they did not find this first parsing task especially difficult.

The second goal of the iTAFF method is for students to learn the actual order of calculation. And since they were already accustomed to putting addition and subtraction last in the order of operations, when I gave the second key instruction, they were not surprised: "We first look for those pluses and minuses not contained in grouping symbols because we want to be sure to perform them last, when they separate only individual numbers." Some students successfully began using a highlighter to illuminate those pluses and minuses to remind themselves not to "touch" those operations until later.

Multiplication of factors, on the other hand, is done as soon as the factors are known. With some

Find + and – signs not inside grouping symbols to separate terms.	$\underbrace{9 \cdot 4}_{term} + \underbrace{4(50 - 3(2+8))}_{term} - \underbrace{10^2}_{term}$
Identify the factors within each term (labeled with an *f* here). Note the treatment of the exponent.	$\underbrace{\overbrace{9}^{f} \cdot \overbrace{4}^{f}}_{term} + \underbrace{\overbrace{4}^{f}\overbrace{(50 - 3(2+8))}^{f}}_{term} - \underbrace{\overbrace{10}^{f} \cdot \overbrace{10}^{f}}_{term}$
Multiplication is performed (in no particular order) in the first and last terms, and we look for + and – signs not inside parentheses within the sub-expression.	$36 + \overbrace{4}^{f}\overbrace{\left(\underbrace{50}_{term} - \underbrace{3(2+8)}_{term}\right)}^{f} - 100$
Identify the factors within each term. But “don’t touch” the highlighted operations!	$36 + \overbrace{4}^{f}\overbrace{\left(\underbrace{50}_{term} - \underbrace{\overbrace{3}^{f}\overbrace{(2+8)}^{f}}_{term}\right)}^{f} - 100$
The + in (2+8) separates only individual numbers, so the addition can be performed.	$36 + \overbrace{4}^{f}\overbrace{\left(\underbrace{50}_{term} - \underbrace{\overbrace{3}^{f} \cdot \overbrace{10}^{f}}_{term}\right)}^{f} - 100$
Factors are known, so multiply.	$36 + \overbrace{4}^{f}\overbrace{(50 - 30)}^{f} - 100$
The minus in parentheses now separates only individual numbers, so subtract.	$36 + \overbrace{4}^{f} \cdot \overbrace{20}^{f} - 100$
Factors are known, so multiply.	$36 + 80 - 100$
+ and – now separate only individual numbers, so calculate using strategies previously studied.	16

Fig. 1 The iTAFF method is used to simplify the expression.

guidance, students quickly realized that calculations within any one particular term do not affect the calculations within another term, reinforcing the idea that requiring the PEMDAS order was too restrictive. As a practical matter, this means that students using the iTAFF method will work on each term individually, in any order they like, until the only operations left are the ones they identified in the first step. Only then will they perform the indicated addition and subtraction (a task that, presumably, they have already studied how to perform). As an example, in **figure 1**, I use the instructions of iTAFF to analyze and simplify the same expression calculated above. The detailed labeling in the table was presented on the board initially, but that was for instructional purposes only—students were not expected to be able to reproduce the labels. However, some students did find it useful to do so.

Once students became more fluent, adding square roots, absolute value signs, and fraction bars were a natural extension of the nesting and grouping concept. Students were comfortable calling them “grouping symbols that do something other than just grouping.”

Coming full circle here to an analysis of why PEMDAS has so many shortcomings to begin with is a worthwhile endeavor. We have two clues. First is the nested structure of mathematical expressions visible in the example above: terms inside factors inside terms, and so on. Second is the cyclic nature of iTAFF: Once a subexpression is identified as a term or a factor, the whole procedure starts over again on a smaller scale. Together, these clues lead to the conclusion that PEMDAS fares poorly because it never was an order of operations to begin with. It has always been a hierarchy of operations. PEMDAS tries to tell us what comes first or second or third, when the more relevant concept involves which operations more closely bind numbers and expressions into their nested structure; addition has to “wait” until the last step within each subexpression because multiplication binds expressions together more tightly than addition does. This notion can be further formalized by generating tree structures for mathematical expressions and by including function symbols (sines or logarithms) or integral and summation signs, but this is beyond the scope of this article.

The iTAFF procedure might seem to be a more complicated process than a strict application of PEMDAS. But that is because the goal is not simply to produce an accurate calculation, as it might have been on first studying the order of operations. Rather, the goals in more advanced courses are to encourage the precise use of *term* and *factor* and to use each order of operations problem as an opportunity to visualize expressions the way more experienced mathematics practitioners do—as a nested structure rather than a linear one. Even if calculating this way takes a little longer at first, there is value in the manner that the iTAFF procedure aligns students’ intuitions with our own much more closely than PEMDAS does. As is often the case, my own intuitions about the order of operations were clarified by having to address students’ misunderstandings. As a result, I no longer feel that teaching the PEMDAS procedure at the high school level adequately does the difficult job of teaching those intuitions. Teaching iTAFF has worked better for me.

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