Seminar in Analysis u. Zahlentheorie
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Ausblick: Γ Funkt., Dirichletreihen, Perron'sche Formel
Mittelwertsatz d. Dirichletreihen, ζ Funkt., Primzahlsatz
Dichtigkeitstheorem v. Polya.

Vortrag: 17.6 24.6 – 1.7 Dichtigkeitstheorem von Polya

Vorkenn.:

$$\left.\begin{array}{l}\Gamma(1)=1\\ n\Gamma(n)=\Gamma(n+1)\\ \frac{d^2}{dz^2}\log\Gamma(z)\geq 0\end{array}\right\}\Longrightarrow$$ axiom. eindeut. Entwickl. von $\Gamma(z)$

wollen solche Entwickl. auch für $\zeta(s)$

$$\zeta(s)=\prod_p\left(1-\frac{1}{p^s}\right)^{-1}\qquad \mathrm{Re}\, s>1$$

$$\zeta(2)=\frac{\pi^2}{6}\Longrightarrow \exists\ \infty \text{ viele Primzahlen}$$

$$\frac{1}{\zeta(s)}=\sum\frac{\mu(n)}{n^s}\qquad \mu_n=(-1)^k \quad k \text{ versch. Primz.}\qquad \mu(1)=1$$

$$\sum_{k=1}^{x-1}k^n=?\qquad \text{Bsp. } 1+2+\dots+(x-1)=\frac{x^2}{2}-\frac{x}{2}$$

$$1^2+2^2+\dots(x-1)^2=\frac{x^3}{3}-\frac{x^2}{2}+\frac{x}{6}$$

$$S_n(x)=\frac{x^{n+1}}{n+1}-\frac{x^n}{2}+B_1\frac{n}{1}\frac{x^{n-1}}{2}-B_2\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}\frac{x^{n-3}}{4}$$

$$+\dots+\begin{cases}(-1)^{\frac{n}{2}-1}B_{n/2}\,x & n \text{ gerade}\\ (-1)^{\frac{n-3}{2}}\,n\,B_{\frac{n-1}{2}}\,\frac{x^2}{2} & n \text{ ungerade}\end{cases}$$

$$B_k=\frac{2\cdot(2k)!\,\zeta(2k)}{(2\pi)^{2k}}\qquad k=1,2\dots\qquad \text{Bernoullizahl.}$$

$$\cot z=\frac{\cos z}{\sin z}=\frac{1}{z}-\frac{2}{\pi^2}\zeta(2)z-\dots\frac{2}{(2\pi)^{2n}}\zeta(2n)z^{2n-1}$$

$$\cot z=\frac{1}{z}+\sum_{n=1}^{\infty}\left(\frac{1}{z-n\pi}+\frac{1}{z+n\pi}\right)$$

$$=\frac{1}{z}-\frac{z}{3}-\frac{z^3}{45}-\frac{2z^5}{945}\dots\qquad |z|<\pi$$

$$\Longrightarrow\frac{\zeta(2n)}{\pi^{2n}}\ \text{rational}$$

Literat.: Norman Levinson "Gap and Density theorems" p 88 ff

G. Polya Untersuch. über Lücken u. Singul. von Potenzreihen
Mathm. Zeitschrift 29 S. 249

Das Dichtigkeitstheorem von Polya

Problem: Betrachten Dirichletreihen

$$f(z) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n z}$$

was kann man über die Singul. auf d. konv. abszisse aussagen?

Lemma

$\{z_n\} \subseteq \mathbb{C}$

$\lim_{n\to\infty} \frac{n}{z_n} = D$ Dichte

$\exists c \quad |z_n - z_m| \geq c\,|n-m|$

$$F(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{z_n^2}\right)$$

$(\exists c) F(re^{i\theta}) \leq c \cdot e^{\pi D r |\sin}$

$\forall \varepsilon > 0 \quad r \to \infty$

$F(re^{i\theta}) = O(e^{(\pi D r |\sin\theta| + \varepsilon r)})$

$1/F(re^{i\theta}) = O(e^{-\pi D r |\sin\theta| + \varepsilon r)})$ $\quad |re^{i\theta} - z_n| \geq \frac{1}{8} c$

$1/|F'(z_n)| = O(e^{\varepsilon |z_n|})$ $\quad n \to \infty$

Corollar

$\{\lambda_n\} \subseteq \mathbb{R}$

$\lim_{n\to\infty} \frac{n}{\lambda_n} = D$

$\exists c \quad \lambda_{n+1} - \lambda_n \geq c \quad \Rightarrow \quad *$

Satz

Dichtigkeitstheorem v. Polya

$$f(z) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n z}$$

$\lim_{n\to\infty} \frac{n}{\lambda_n} = D$

$\exists c \quad \lambda_{n+1} - \lambda_n \geq c$

$\Rightarrow$ Auf d. konv abszisse $\exists$ $\forall$ Intervallen mit Länge $\geq 2\pi D$ eine Singularität

Bew Satz 1

(unter Benutzung von Corollar, Phragmen Lindelöf. Annahme dass Dirichletreihe eine konv. Abszisse hositiv $x = x_0$

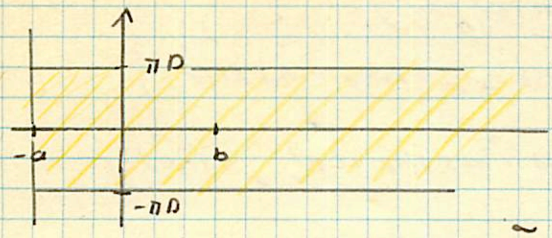
o.B.d.A konv.abszisse $x = 0$
sonst (konv.abszisse $x = x_0$) nehmen $\sum_{n=1}^{\infty} a_n e^{-\lambda_n(z - x_0)}$

$$= \sum_{n=1}^{\infty} \underbrace{a_n e^{+\lambda_n x_0}}_{a_n'} e^{-\lambda_n z}$$

nehmen an, $\exists$ Intervall I mit Länge $\geq 2\pi D$ ohne Singularität

o.B.d.A $I \supseteq [-\pi D, \pi D]$
sonst nehmen wir Funktion $\sum_{n=1}^{\infty} a_n e^{-\lambda_n z - d}$, was nichts an der Beh. ändert.

$f(z)$ besitze nun keine Singul. auf $x=0$ $|y| \leq B > \pi D$
$\exists\, a>0$, so dass $f(z)$ analytisch für $x \geq -a$ $|y| \leq B$



Definieren $H(\omega) := \int_{-\infty}^{\infty} f(z)\, e^{\omega z}\, dz$ $\omega = u+iv$, $u<0$ vorerst

wohldefiniert $\forall \omega$

$$\boxed{H(\omega)} = \int_{-\infty}^{\infty} f(z)\, e^{\omega z}\, dz \overset{(\text{Cauchy})}{=} \int_{-\infty}^{-a+iB} f(z)\, e^{\omega z}\, dz + \int_{-a+iB}^{\infty+iB} f(z)\, e^{\omega z}\, dz$$

$$= \int_{-\infty}^{-a+iB} f(z)\, e^{\omega z}\, dz + \int_{-a+iB}^{b+iB} f(z)\, e^{\omega z}\, dz + \int_{b+iB}^{\infty+iB} e^{\omega z}\Big(\sum_{k=1}^{\infty} a_k e^{-\lambda_k z}\Big) dz$$

$$\overset{\text{abs. konv.}}{=} \boxed{\int_{-\infty}^{-a+iB} f(z)\, e^{\omega z}\, dz + \int_{-a+iB}^{b+iB} f(z)\, e^{\omega z}\, dz - \sum_{k=1}^{\infty} a_k \frac{e^{(\omega-\lambda_k)(b+iB)}}{\omega-\lambda_k}} \qquad (1)$$

meromorphe Funktion

Definieren $G(\omega) = H(\omega)\cdot F(\omega)$

ist eine ganze Funktion

$$G(\omega) = H(\omega)\cdot F(\omega) = \underbrace{\prod_{l=1}^{\infty}\Big(1-\frac{\omega^2}{\lambda_l^2}\Big)}_{\text{ganz, dann } \leq \underbrace{\prod_{l=l_0}^{\infty}\left(1-\frac{\omega^2}{c^2n^2}\right)}_{\text{ganz}} \cdot \text{ganz}}\Big(\sum_{k=1}^{\infty} a_k \frac{e^{(\omega-\lambda_k)(b+iB)}}{\omega-\lambda_k} + \text{ganze}\Big)$$

l_0 so dass $\left|1-\frac{\omega^2}{c^2n^2}\right| \leq 1$

$$= \prod_{l\neq k}\Big(1-\frac{\omega^2}{\lambda_l^2}\Big)\sum_{k=1}^{\infty}(-a_k)\,\frac{\lambda_k+\omega}{} \cdot e^{(\omega-\lambda_k)(b+iB)}$$

$$\boxed{G(\lambda_n)} = +\sum_{k=1}^{\infty} a_k \prod_{\substack{l=1\\ l\neq n}}^{\infty}\Big(1-\frac{\lambda_n^2}{\lambda_l^2}\Big)\frac{(\lambda_n+\lambda_k)}{\lambda_n^2}\, e^{(\lambda_n-\lambda_k)(b+iB)}$$

$$= +a_n \prod_{\substack{l=1\\ l\neq n}}^{\infty}\Big(1-\frac{\lambda_n^2}{\lambda_l^2}\Big)\frac{2}{\lambda_n} \qquad (2)$$

$$= \boxed{-a_n F'(\lambda_n)} \quad \Big(= \sum_k \prod_{\substack{l=1\\ l\neq k}}^{\infty}\Big(1-\frac{\lambda_n^2}{\lambda_l^2}\Big)\frac{-2\lambda_n}{\lambda_k^2}$$

Schätzen nun H ab:

$\omega = \rho e^{i\gamma}$ $0<\gamma<\frac{\pi}{2}$

$$|H(\rho e^{i\gamma})| \leq e^{-a\rho\cos\gamma}\int_{-\infty}^{-a+iB}|f(z)|\,dz + e^{+b\rho\cos\gamma - B\rho\sin\gamma}\int_{-a+iB}^{b+iB}|f(z)|\,dz$$

$$+\, e^{b\rho\cos\gamma - B\rho\sin\gamma}\sum_{k=1}^{\infty}|a_k|\frac{e^{-b\lambda_k}}{\rho\sin\gamma}$$

noch unklar
b?

können auch Reflektierten Weg unter reeller Achse wählen

$$\Rightarrow H(\rho e^{\pm i\gamma}) = O\big(e^{-a\rho\cos\gamma} + e^{b\rho\cos\gamma - B\rho\sin\gamma}\big) \qquad \rho\to\infty$$

Es folgt aus dem Corollar d. Lemmas, also aus

$$\left.\begin{array}{l} F(re^{i\theta}) = O(e^{\pi D r|\sin\theta| + \varepsilon r}) \\ H(\varrho e^{\pm i\vartheta}) = O(e^{-a\varrho\cos\vartheta} + e^{b\varrho\cos\vartheta - B\varrho\sin\vartheta}) \end{array}\right\}$$

$$G(\varrho e^{\pm i\vartheta}) = O(e^{-a\varrho\cos\vartheta + \pi D\varrho\sin\vartheta + \varepsilon\varrho} + e^{b\varrho\cos\vartheta - (B-\pi D)\varrho\sin\vartheta + \varepsilon\varrho})$$

wählen nun $b = \frac{1}{2}(B-\pi D)\tan\vartheta$ $\qquad (B > \pi D)$

$$\Rightarrow B - \pi D = 2b\,\mathrm{ctn}\,\vartheta$$

$$G(\varrho e^{\pm i\vartheta}) = O(e^{-a\varrho\cos\vartheta + \pi D\varrho\sin\vartheta + \varepsilon\varrho} + e^{-b\varrho\cos\vartheta + \varepsilon\varrho})$$

$$G(\varrho e^{\pm i\vartheta}) = O(e^{-\frac{a}{2}\varrho\cos\vartheta + \pi D\varrho\sin\vartheta} + e^{-\frac{1}{2}b\varrho\cos\vartheta})$$ $\exists\, a$ ε bel. klein

- $D = 0$ $\quad \vartheta = \frac{1}{4}\pi$ $\quad \delta = \min(a,b)/4$

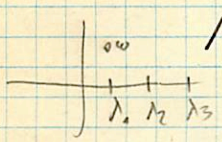
$$G(\varrho e^{\pm i\vartheta}) = O(e^{-\delta\varrho\cos\vartheta})$$

- $D > 0$ $\quad \vartheta = \tan^{-1}\frac{a}{4\pi D}$ $\quad \Rightarrow \quad \pi D\sin\vartheta = \frac{1}{4}a\cos\vartheta$

$$G(\varrho e^{\pm i\vartheta}) = O(e^{-\frac{1}{4}a\varrho\cos\vartheta} + e^{-\frac{1}{2}b\varrho\cos\vartheta})$$

$$G(\varrho e^{\pm i\vartheta}) = O(e^{-\delta\varrho\cos\vartheta})$$

(3) $\quad \delta > 0 \quad \Rightarrow \quad \boxed{e^{\delta\omega}G(\omega) = O(1)} \qquad |\omega| \to \infty \quad \arg\omega = \pm\vartheta$



$|\omega - \lambda_k| \geq \frac{1}{4}c \quad \Rightarrow \quad |H(\omega)| \leq A e^{A|\omega|} \qquad \exists A$

$\qquad\qquad\qquad\qquad |G(\omega)| \leq A e^{A(\omega)} \qquad \exists A$ aus Corollar

$G(\omega)$ ganz $\quad \Rightarrow \quad$ Maximum von $G(\omega)$ in $|\omega - \lambda_k| \leq \frac{1}{4}c$ wird angenommen in $|\omega - \lambda_k| = \frac{1}{4}c$

$$\Rightarrow \quad |G(\omega)| \leq A e^{A|\omega|} \quad \forall\,\omega$$

$$\Rightarrow \quad e^{\delta\omega}G(\omega) = O(e^{A|\omega|}) \quad \forall\,\omega$$

(3) $\qquad e^{\delta\omega}G(\omega) = O(1) \qquad \arg\omega = \pm\vartheta$

Phragmén Lindelöf $\qquad e^{\delta\omega}G(\omega) = O(1) \qquad |\arg\omega| \leq \vartheta$

speziell $\qquad G(u) = O(e^{-\delta u}) \qquad u \to \infty$

(2) $\qquad G(\lambda_n) = -a_n F'(\lambda_n)$

$$\Rightarrow \quad a_n = O\left(\frac{e^{-\delta\lambda_n}}{|F'(\lambda_n)|}\right)$$

Corollar $\qquad a_n = O(e^{-\delta\lambda_n + \varepsilon\lambda_n})$

da ε beliebig:

$$a_n = O(e^{-\delta\lambda_n/2})$$

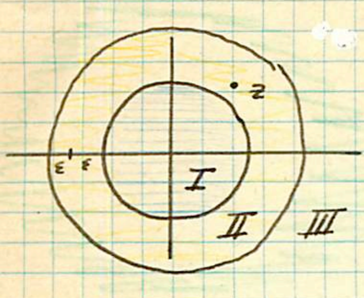
aber daraus würde folgen: Die Abszisse d. kon konverg. Abszisse von $f(s)$ ist bei $-\frac{1}{2}\delta$ oder links davon $\quad ⚡$

/ Cohen: $\quad \sigma_B = \overline{\lim_{n\to\infty}}\ \dfrac{\log|S_n|}{\lambda_n}$

$$S_n = \sum_{\nu=1}^{n} a_\nu$$

Unklar! $\quad$ Lindelöf, Cohen S. 89 $\lim_{\lambda\to\infty}$ ... S. 6

Bew. d. Lemmas $\qquad \varepsilon > 0$ fix $\qquad z$ fix



$z_n \in \mathrm{I} \qquad |z_n| \le (1-\varepsilon)|z|$

$z_n \in \mathrm{II} \qquad |z|(1-\varepsilon) < |z_n| < (1+\varepsilon)|z|$

$z_n \in \mathrm{III} \qquad |z_n| \ge (1+\varepsilon)|z|$

$$|F(z)| = \Big(\prod_{\mathrm{I}} \prod_{\mathrm{II}} \prod_{\mathrm{III}}\Big) \Big|1 - \frac{z^2}{z_n^2}\Big|$$

$\frac{n}{z_n} \to D \qquad \Longrightarrow \# z_n$ in II $< 2\varepsilon|z|(D+\delta) \qquad \delta > 0$, $|z|$ gen. gross

$$\Longrightarrow \prod_{\mathrm{II}} \Big|1 - \frac{z^2}{z_n^2}\Big| \le \prod_{\mathrm{II}} 3 \le 3^{2\varepsilon|z|(D+\delta)}$$

$$< e^{4\varepsilon|z|(D+\delta)} \qquad (1)$$

• nehmen von jetzt an an $\mathrm{Re}(z) > 0 \qquad$ o.B.d.A.

(2) $$\Big|\frac{1}{1 - z^2/z_n^2}\Big| = \frac{1}{|1+\frac{z}{z_n}||1-\frac{z}{z_n}|} \le \frac{1}{|1-\frac{z}{z_n}|} = \Big|\frac{z_n}{z-z_n}\Big| \qquad \text{f. genüg. grosse } n$$

$(\exists N) \qquad |z - z_n| \ge |z - z_N| \qquad$ (z_N mit minim. Abstand)

$$|z_N - z_n| \le |z - z_N| + |z_n - z| \le 2|z - z_n|$$

(3) $$\Longrightarrow \frac{1}{|z - z_n|} \le \frac{2}{|z_N - z_n|}$$

(2) (3) $$\to \Big|\frac{1}{1 - \frac{z^2}{z_n^2}}\Big| \le \frac{2|z_n|}{|z_N - z_n|}$$

• $\prod'_n$ Produkt über alle n ohne $n = N$

$$\prod_{\mathrm{II}}{}' \frac{1}{|1 - \frac{z^2}{z_n^2}|} \le \prod_{\mathrm{II}}{}' \frac{2|z_n|}{|z_N - z_n|}$$

• $M \qquad \# z_n$ in II

$|z_n - z_N| \ge c|n - N| \qquad$ (Vorauss.)

$$\prod_{\mathrm{II}}{}' \frac{1}{|1 - \frac{z^2}{z_n^2}|} \le 2^M \{(1+\varepsilon)|z|\}^M \prod_{\mathrm{II}}{}' \frac{1}{c|n-N|} \le \Big(\frac{3}{c}\Big)^M |z|^M \prod_{\mathrm{II}}{}' \frac{1}{|n-N|}$$

$$\prod_{\mathrm{II}}{}' |n - N| \ge \big(\underbrace{[\tfrac{1}{2}(M-1)]}_{\text{integer}}!\big)^2$$ da höchst. 2 Werte im Produkt gleich sind

$$\to \prod_{\mathrm{I}}{}' \frac{1}{|1 - \frac{z^2}{z_n^2}|} \le \Big(\frac{3}{c}\Big)^M |z|^M \Big/ \big([\tfrac{1}{2}(M-1)]!\big)^2$$

$\le \left(\frac{3}{c}\right)^M |z|^M e^{3M} e^{-M\log M} \qquad$ Stirling

$= \left(\frac{3}{c}\right)^M e^{3M} \cdot e^{M\log|z| - M\log M} \qquad n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$

$= \left(\frac{3}{c}\right)^M e^{3M} e^{|z| \frac{M}{|z|} \log \frac{|z|}{M}} \qquad n! \ge e^{M\log M} e^{2M}$

Weil $M \le 2\varepsilon|z|(D+\delta)$ und $x\log\frac{1}{x} < \sqrt{x}$ $x>0$ gilt

$$\frac{M}{|z|}\log\frac{|z|}{M} \le \left(\frac{M}{|z|}\right)^{1/2} < \frac{(2\varepsilon|z|(D+\delta))^{1/2}}{|z|^{1/2}} = (2\varepsilon(D+\delta))^{1/2}$$

$$\Rightarrow \prod_{II}{}' \frac{1}{\left|1-\frac{z^2}{z_n^2}\right|} \le \left(\frac{3}{c}\right)^{2\varepsilon|z|(D+\delta)} e^{10\varepsilon|z|(D+\delta)} e^{|z|(2\varepsilon(D+\delta))^{1/2}}$$

z in rechl. Halbebene $\Rightarrow \frac{1}{\left|1-\frac{z^2}{z_N^2}\right|} < \frac{2|z_N|}{|z-z_N|}$

$|z|$ gross genug $\Rightarrow |z_N| < 2|z| \Rightarrow \frac{1}{\left|1-\frac{z^2}{z_N^2}\right|} < \frac{4|z|}{|z-z_N|}$

Es gilt auch $\frac{4|z|}{|z-z_N|} \ge \frac{4|z|}{|z|+|z_N|} > \frac{4|z|}{3|z|} > 1$

$$\Rightarrow \prod_{II} \frac{1}{\left|1-\frac{z^2}{z_n^2}\right|} \le \frac{4|z|}{|z-z_N|} \prod_{II}{}' \frac{1}{\left|1-\frac{z^2}{z_n^2}\right|}$$

$$\Rightarrow \prod_{II} \frac{1}{\left|1-\frac{z^2}{z_n^2}\right|} \le \left(\frac{3}{c}\right)^{2\varepsilon|z|(D+\delta)} \frac{e^{10\varepsilon|z|(D+\delta)} e^{|z|(2\varepsilon(D+\delta))^{1/2}}}{|z-z_N|}$$

(1) $\prod_{I} \frac{\;}{\left|1-\frac{z^2}{z_n^2}\right|} < e^{4\varepsilon|z|(D+\delta)}$

$$\Rightarrow \exists A \quad |z-z_N|\, e^{-A|z|\varepsilon^{1/2}(D+\delta)^{1/2}} \le \prod_{II}\left|1-\frac{z^2}{z_n^2}\right| \le e^{A|z|\varepsilon^{1/2}(D+\delta)}$$

$D=0$ sei $\varepsilon=\frac{1}{2}$ $\Rightarrow |z-z_N|\, e^{-A|z|\delta^{1/2}} \le \prod_{II}\left|1-\frac{z^2}{z_n^2}\right| \le e^{A|z|\delta}$

$D>0$ $\delta=1$ $B=A(D+1)^{1/2}$ $|z-z_N|\, e^{-B|z|\varepsilon^{1/2}} \le \prod_{II}\left|1-\frac{z^2}{z_n^2}\right| \le e^{B|z|\varepsilon^{1/2}}$

(4) $|z-z_N|\, e^{-A|z|\delta^{1/2}} \le \prod_{II}\left|1-\frac{z^2}{z_n^2}\right| \le e^{A|z|\delta^{1/2}}$

haben jetzt Abschätzung in II
f. genug. grosse N_0 haben wir weil $\frac{n}{z_n} \to D$

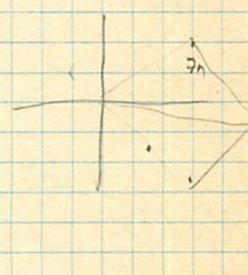
(5) $\left|\frac{D^2 z_n^2 - n^2}{z_n^2}\right| < \frac{1}{8}D^2\varepsilon^2$ $\quad n > N_0$

$\frac{|z|}{2} \le |z|$

$z_n \in I$ oder III $\quad |z_n| < 2|z|$

$|n^2 - z^2D^2| \ge |D^2 z_n^2 - z^2D^2| - |n^2 - D^2 z_n^2|$

$||z_n| - |z|| \ge \varepsilon|z|$



$$\Rightarrow |n^2 - z^2D^2| \ge D^2\varepsilon|z|\,|z_n+z| - \frac{1}{8}D^2\varepsilon^2|z_n|^2$$

$$\ge D^2\varepsilon|z|\,\left|\tfrac{1}{2}z\right| - \frac{1}{2}D^2\varepsilon^2|z|^2 = \frac{1}{2}D^2\varepsilon|z|^2(1-\varepsilon)$$

ε klein

(6) $\left|\frac{n^2-D^2z^2}{z^2}\right| \ge \frac{1}{4}D^2\varepsilon$

(5).(6) $$1-\varepsilon \le \left| 1 + \frac{z^2(z_n^2 D^2 - n^2)}{z_n^2(n^2 - D^2 z^2)} \right| \le 1+\varepsilon$$ $n \ge N_0$, $z_n / \le 2|z|$, $z_n \in I$ ~~II~~

$$1+ \frac{z^2(z_n^2 D^2-n^2)}{z_n^2(n^2-z^2D^2)} = \frac{1-\frac{z^2}{z_n^2}}{1-\frac{z^2D^2}{n^2}}$$

$$\Rightarrow \quad 1-\varepsilon \le \left| \frac{1-\frac{z^2}{z_n^2}}{1-\frac{z^2D^2}{n^2}} \right| \le 1+\varepsilon$$ (7) $n > N_0$

$$\exists C_1 \quad \frac{1}{C_1} \le \prod^{N_0} \left| \frac{1-z^2/z_n^2}{1-z^2D^2/n^2} \right| \le C_1$$ (8)

(7).(8) $\# z_n$ in $I \le 2|z| D$ grosse $|z|$

$$\Rightarrow \quad \frac{1}{C_1}(1-\varepsilon)^{2|z|D} \prod_I \left|1-\frac{z^2D^2}{n^2}\right| \le \prod_I \left|1-\frac{z^2}{z_n^2}\right| \le C_1(1+\varepsilon)^{2|z|D} \prod_I \left|1-\frac{z^2D^2}{n^2}\right|$$ (11)

$1+x < e^x \quad 1-x > e^{-2x} \quad x$ klein

$|z_n| > 2|z|$ und $\left|\frac{D^2 z_n^2 - n^2}{z_n^2}\right| \le \frac{1}{8} D^2 \varepsilon^2$

$$1-\frac{1}{8}\frac{D^2\varepsilon^2|z|^2}{n^2-|z|^2D^2} \le \left|1+\frac{z^2(z_n^2D^2-n^2)}{z_n^2(n^2-z^2D^2)}\right| \le 1+\frac{1}{8}\frac{D^2\varepsilon^2|z|^2}{n^2-|z|^2D^2}$$

$z_n \sim \frac{n}{D}$ $|z|$ gross $\quad |z_n| \ge 2|z|$ immer noch

$$\Rightarrow \quad 1-\frac{\varepsilon|z|^2D^2}{n^2} \le \left|1+\frac{z^2(z_n^2D^2-n^2)}{z_n^2(n^2-z^2D^2)}\right| \le 1+\frac{\varepsilon|z|^2D^2}{n^2}$$

damit und mit (7)

$$\rightarrow \quad (1-\varepsilon)^{2|z|D} \prod_{|z_n| \ge 2|z|} \left(1-\frac{\varepsilon|z|^2D^2}{n^2}\right) \le \prod_{II} \left| \frac{1-\frac{z^2}{z_n^2}}{1-D^2z^2/n^2} \right|$$

$$\le (1+\varepsilon)^{2|z|D} \prod_{|z_n| \ge 2|z|} \left(1+\frac{\varepsilon|z|^2D^2}{n^2}\right)$$ (9)

kleine x $\quad 1-x < e^x \quad 1-x > e^{-2x}$

$$\prod_{|z_n| \ge 2|z|} \left(1+\frac{\varepsilon|z|^2D^2}{n^2}\right) \le \exp\left(\varepsilon|z|^2D^2 \sum_{n \ge |z|D} \frac{1}{n^2}\right) \le e^{2\varepsilon|z|D}$$

weil $\sum_{n \ge |z|D} \frac{1}{n^2} \le \frac{1}{|z|D}$

$$\prod_{|z_n| \ge 2|z|} \left(1-\frac{\varepsilon|z|^2D^2}{n^2}\right) \ge \exp\left(-2\varepsilon|z|^2D^2 \sum_{n \ge |z|D} \frac{1}{n^2}\right) \ge e^{-4\varepsilon D|z|}$$

(9) $\rightarrow$ $$e^{-8\varepsilon D|z|} \le \prod_{II} \left| \frac{1-z^2/z_n^2}{1-z^2D^2/n^2} \right| \le e^{4\varepsilon D|z|}$$ (10)

(11) (10)

Anwendungen d. Dichtigkeitstheorems:

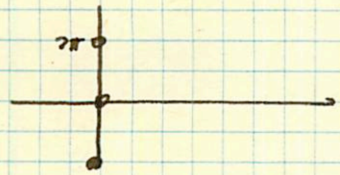
Betrachten $f(z) = \sum_{n=1}^{\infty} a_n e^{-n^{1+\varepsilon} z}$

$$\lim_{n\to\infty} \frac{n}{\lambda_n} = \lim_{n\to\infty} \frac{n}{n^{1+\varepsilon}} = 0$$

$\overset{DTP}{\Longrightarrow}$ Singularitäten liegen dicht auf konv. abszisse

$$f_1(z) = \sum_{n=0}^{\infty} e^{-nz} = \sum_{n=0}^{\infty} (e^{-z})^n = \frac{1}{1-e^{-z}}$$

analytische Fortsetzung von $f_1(z)$ auf ganze Ebene



$\lim_{n\to\infty} \frac{n}{\lambda_n} = 1 \overset{DT}{\Longrightarrow}$ mind. 1 Singul. in jedem Intervall $> 2\pi D$

$$f_2(z) = \sum_{n=0}^{\infty} e^{-n^2 z} = \cdot\, \theta(0, -\tfrac{z}{2}) - 1$$

$\lim_{n\to\infty} \frac{n}{n^2} = 0 \Rightarrow$

$\Rightarrow$ keine analyt. Fortsetzung möglich

$$f_3(z) = \sum_{n=0}^{\infty} z^{2^n} = \sum_{n=0}^{\infty} e^{2^n \log z} \qquad \lim \frac{n}{\lambda_n} = 0$$

keine analyt. Fortsetzung über Einheitskreis möglich

$$f(z) = \sum_{n=1}^{\infty} a_n z^n = \sum_{n=1}^{\infty} a_n e^{n \log z} = \sum a_n e^{-n\omega}$$

$\log z = -\omega \qquad \lambda_n = n$

$\mathrm{Re}\,\omega = -\log|z| \geq \lambda_0$

$|z| \leq e^{\lambda_0}$ konv. radius

$2\pi \geq \mathrm{Im}\,\omega_1 - \mathrm{Im}\,\omega_2 = \arg|z_2| - \arg|z_1|$

d.h. mind. eine Singularität auf konv.kreis

$f(z) = \sum_{n=1}^{\infty} a_n z^{n^2}$ Singul. dicht auf konvkreis

$f(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} = \sum_{n=1}^{\infty} e^{-\log n\, z}$ keine Aussage d. DT

Das Maximumprinzip

$\mathcal{H}(\Omega)$ Menge aller holomorphen Funktionen auf Ω

Satz (Maxim.prinzip)

Ω zush. beschr. offene Menge von $\mathbb{C}$
$f \in \mathcal{H}(\Omega)$ $\quad M := \sup\limits_{\xi \in \partial\Omega} \left(\overline{\lim\limits_{\substack{z \to \xi \\ z \in \Omega}}} |f(z)| \right)$

Falls f nicht const $\Rightarrow |f(z)| < M \quad \forall z \in \Omega$

Lemma 1 (Prinzip d. offen Abb)

Ω offen zush in $\mathbb{C}$ $\quad f \in \mathcal{H}(\Omega)$
f nicht konst $\quad U \subset \Omega$ offen $\Rightarrow f(U)$ offen

Lemma 2

Ω offen $\quad f \in \mathcal{H}(\Omega)$
U offen $\quad U \Subset \Omega$ $\qquad U \Subset \Omega :=$ U rel. komp. in Ω

$|f(z)| \leq \sup\limits_{w \in \partial U} |f(w)|$

Bew Satz $\ulcorner$ $M < \infty$

$|f(z)| \leq M \quad z \in \Omega$ — genügt wegen Lemmas

$\xi \in \partial\Omega \quad \exists U_\xi$ sodass $|f(z)| < M + \varepsilon \quad z \in U_\xi \cap \Omega$

$V := \bigcup\limits_{\xi \in \partial\Omega} U_\xi \cap \Omega \qquad K = \Omega - V \qquad K$ kompakt

$K \subset W \Subset \Omega \Rightarrow \partial W \subset V$

$|f(z)| \leq M \quad z \in \partial W$

$\xRightarrow{\text{Lemma 2}} \quad |f(z)| \leq M \quad z \in W$

$\Rightarrow |f(z)| \leq M \quad \forall z \in \Omega \qquad \lrcorner$

Lemma 1.1

Ω zush. offen in $\mathbb{C}$ $\quad f \in \mathcal{H}(\Omega)$
$\exists U \neq \emptyset \quad U \subset \Omega$ offen $\quad f|_U = 0$
$\Rightarrow f \equiv 0$ auf Ω

Lemma 1.2

Ω zush. offen in $\mathbb{C}$ $\quad f \in \mathcal{H}(\Omega)$
$Z_f := \{ z \in \Omega \mid f(z) = 0 \}$
$f \not\equiv 0 \Rightarrow Z_f$ diskret (abgeschl. isol. Punkte)

Bew. Lemma 1 $\qquad a \in \Omega$
Annahme $f(a) = 0$ $\quad$ o.E.d.A.
$D(a,r)$ mit $\overline{D(a,r)} \subset \Omega \quad f(z) \neq 0 \quad z \in \partial D$
$d := \inf \{ |f(z)| \mid |z-a| = r \}$

zeige: $w \notin f(\Omega) \Rightarrow |w| \geq \frac{d}{2}$ $\quad$ d. dann
$\{ |w| < \frac{d}{2} \} \subset f(\Omega)$

$\phi(z) := \frac{1}{f(z)-\omega}$ $\quad \omega \notin f(\Omega)$ $\quad \phi \in \mathcal{H}(\Omega)$

Lemma 2 $\quad |\phi(z)| \leq \sup_{|z-a|=r} |f(z)|$

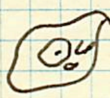
$|\phi(a)| = \frac{1}{|\omega|} \leq \sup_{|z-a|=r} |\phi(z)| = \frac{1}{\inf_{|z-a|=r} |f(z)-\omega|}$

1. Fall $|\omega| < d$ $\quad |f(z)-\omega| \geq |f(z)| - |\omega| \geq d - |\omega|$
auf $|z-a| = r$

$\frac{1}{|\omega|} \leq \frac{1}{d-|\omega|}$ $\quad |\omega| \geq d - |\omega| \Rightarrow |\omega| \geq \frac{d}{2}$

2. Fall $|\omega| \geq d \Rightarrow |\omega| \geq \frac{d}{2}$

also insges. $f(\Omega)$ Umgeb. von $f(a)$



Lemma 1.1 $-$ $f|U$ nicht konst.
$f(U)$ Umg. von U

~~Lemma 2.1~~

> Sei Ω offen in $\mathbb{C}$ $\quad v \in C^2(\Omega)$ reellwertig
> $\Delta u(x) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \geq 0 \quad z \in \Omega$
> $(\forall\, U \subset \Omega)\quad u(z) \leq \sup_{w \in \partial U} u(w)$

Bew Lemma 2

$\omega(z) = |f(z)|^2$
$\Delta\omega \overset{(?)}{=} 4|f(z)|^2 \quad f = x + iy$

$\Delta\omega = 4|f'(z)|^2$
$|f(z)|^2 \leq \sup_{w \in \partial U} |f(w)|^2 \quad f(z) \leq \sup_{w \in \partial U} f(w)$

Bew Lemma 2.1

Lemma 2.1.1

> I offen in $\mathbb{R}$
> $\phi \in C^2(\mathbb{R})$ $\quad \phi$ reellwertig
> $(\exists\, t_0 \in I)\quad \phi(t) \leq \phi(t_0) \quad \forall t \in I$
> $\Rightarrow \phi''(t_0) \leq 0$

$\ulcorner$

$\phi''(t_0) = \lim_{h\to 0} \frac{1}{h^2} \left(\phi(t_0+h) + \phi(t_0-h) - 2\phi(t_0) \right)$

$= \lim_{h\to 0} \frac{1}{h} \left(\underbrace{\frac{\phi(t_0+h) - \phi(t_0)}{h}}_{\phi'(t_0)} - \underbrace{\frac{\phi(t_0) - \phi(t_0-h)}{h}}_{\phi'(t_0-h)} \right)$

≤ 0

$\lrcorner$

Koroll. 2.1.2

> Ω offen in $\mathbb{R}^2$ $\quad u \in C^2(\mathbb{R}^2)$ reellwertig
> $\exists (x_0, y_0) \in \Omega \quad u(x,y) \leq u(x_0, y_0)$
> $\Longrightarrow \Delta(u(x_0, y_0)) \leq 0$

Bew Lemma 2.1 $\quad \Delta u(z) \geq 0 \quad z \in \Omega$

$\exists\ U \subset\subset \Omega \quad \exists\ z_0 \in \bar{U} \quad u(z_0) = \sup_{\omega \in \bar{U}} u(\omega)$

* ~~denn~~ denn würde u nicht ex.

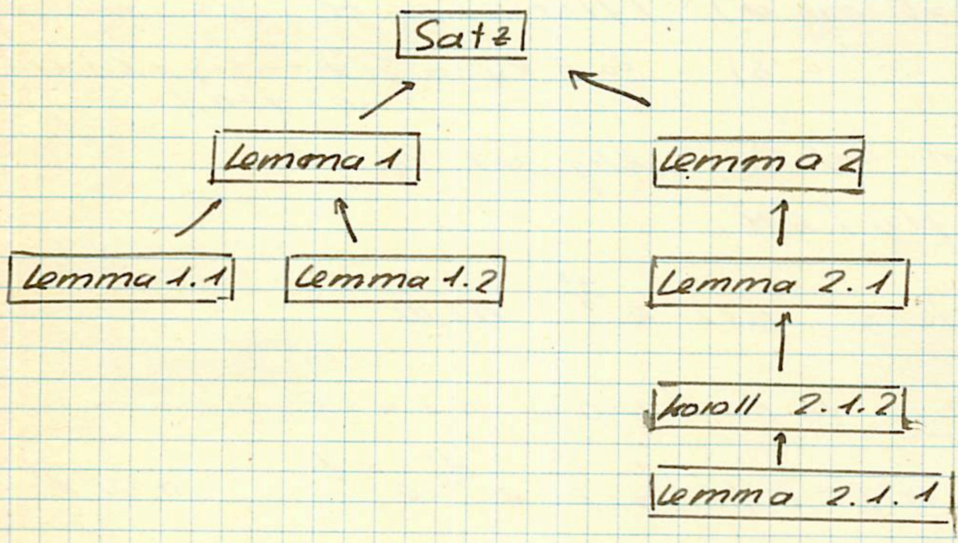
$\Longrightarrow z_0 \in U \quad u(\omega) \leq u(z_0) \quad \forall \omega \in U$

$\Delta u(z_0) \leq 0 \quad \Delta u \geq 0 \quad \varepsilon > 0$

$u_\varepsilon(z) < u(z) + \varepsilon |z|^2$

$\Delta u_\varepsilon(z) = \Delta u(z) + 4\varepsilon$

$\Longrightarrow u_\varepsilon(z) \leq \sup_{\omega \in \partial U} u_\varepsilon(\omega) \quad z \in U$



Phragmen - Lindelöf Prinzip

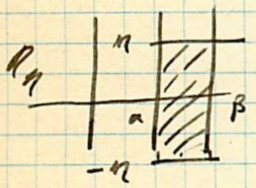
Satz 0 Max. Prinzip

Satz 1 Phragmen Lindelöf Prinzip

f analyt. in $D := \{z = x+iy \mid \alpha < x < \beta\}$
stetig auf ∂D
$|f(z)| \le M \quad \forall z \in \partial D$
f beschränkt auf $\bar{D}$
Beh $|f(z)| \le M \quad \forall z \in D$

1. Fall $\lim\limits_{y \to \pm\infty} f(x+iy) = 0 \quad \alpha \le x \le \beta$ glm.

$z_0 \in D$ belieb.



$\exists\ \eta > 0$ mit a) $|f(x+iy)| \le M \quad \forall x \ \ \alpha \le x \le \beta$

b) $z_0 \in R_\eta := \{ z = x+iy \in \mathbb{C} \mid \alpha < x < \beta, -\eta < y < \eta \}$

f anal. auf R_η stetig auf $\overline{R_\eta}$

$\Rightarrow |f(z)| \le M$

2. Fall $f_n(z) := f(z) \cdot e^{\frac{z^2}{n}} \quad n \in \mathbb{N}$

1. Schritt $\lim\limits_{y \to \pm\infty} f_n(x+iy) = 0 \quad \alpha \le x \le \beta$

$|f_n(z)| = |f(x+iy)| \, e^{\frac{\beta^2}{n}} e^{-\frac{y^2}{n}}$ o.Bd.A α^2

$\varepsilon > 0$

$y_0^2 := n \ln((d e^{\frac{\beta^2}{n}})^{-1} \varepsilon)$

da $\exists d > 0 \quad |f(z)| \le d \quad \forall z \in \bar{D}$

$y > y_0 \Rightarrow |f_n(x+iy)| < \varepsilon \quad \alpha \le x \le \beta$

2. Schritt $\exists c > 0 \quad |f_n(z)| \le M e^{\frac{c^2}{n}} \quad \forall z \in \partial D$

$|f_n(z)| \le |f(z)| \, e^{\frac{\beta^2}{n}} \quad \forall z \in \partial D$

3. Schritt $|f_n(z)| \le M e^{\frac{c^2}{n}} \quad \forall z \in D$ da f stetig auf

4. Schritt $\forall \varepsilon > 0 \ \exists n_0 \ \ n > n_0 \ \ |f_n(z) - f(z)| < \varepsilon \quad \forall z \in R_\eta$

$\varepsilon > 0$
$z \in R_\eta$ belieb

$|f_n(z) - f(z)| = |f(z)| \, |e^{\frac{z^2}{n}} - 1| \le d \, \underbrace{|e^{\frac{z^2}{n}} - 1|}_{< \frac{\varepsilon}{d}}$

5. Schritt $|f(z)| \le M \quad \forall z \in D$

$z \in D$ bel.

$\exists \eta$ mit $z \in R_\eta$, $|f_n(z)| \le M e^{\frac{c^2}{n}}$

$\lim_{n\to\infty} |f_n(z)| = \lim_{n\to\infty} |f_n(z)| = |f(z)| \le M$
sogar strikt

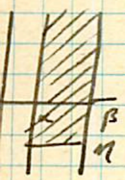
Satz 2

$f \in \mathcal{O}(D_\eta) \qquad D = \{z = x+iy \mid \alpha < x < y \quad y > \eta\}$

$\overline{\lim_{z\to\xi}}\ |f(z)| \le M \quad \forall \xi \in \partial R_\eta$

$\exists c > 0 \quad \exists \theta < 1 \qquad |f(z)| \le c \exp e^{\frac{\theta\pi|z|}{\beta-\alpha}} \quad \forall z \in D_\eta$

$\Longrightarrow |f(z)| \le M \quad \forall z \in D_\eta$



($f(z) = e^{e^{-iz}} \quad \alpha = -\frac{\pi}{2} \quad \beta = \frac{\pi}{2} \quad y = 0 \quad \theta = 1$

$|f(\pm\frac{\pi}{2} + iy)| = 1 \quad \forall y > 0 \qquad |f(iy)| = e^{e^y}$)

1. Fall $\quad \alpha = -\frac{\pi}{2} \quad \beta = \frac{\pi}{2} \quad \eta = 0 \quad \theta = 1$

$g(z) := f(z) \exp(-\sigma e^{-ikz})$
$\sigma > 0$
$0 < k < 1$

1. Schritt $\quad \exists c > 0 \quad |g(z)| \le c \cdot \exp\{e^{\theta|z|} - \sigma e^{k|y|\cos\frac{\pi k}{2}}\}$

$\forall z \in D$

$|g(z)| = |f(z)| \exp(\mathrm{Re}(-\sigma e^{-ikz}))$

$= |f(z)| \exp(\underbrace{-\sigma}_{>0} \underbrace{e^{ky}}_{>0} \underbrace{\cos kx}_{>0} \})$

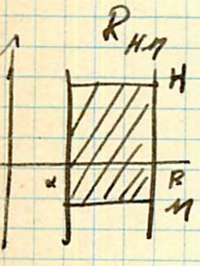
$\exists c > 0 \quad |f(z)| \le c \exp(e^{-\theta|z|})$ (Voraus.)

$|g(z)| \le c \exp\{e^{-\theta|z|} \sigma e^{ky} \cos\frac{k}{2}\pi\}$

2. Schritt $\quad |\lim_{y\to\infty} g(z)| = 0 \quad \alpha \le x \le \beta \quad$ glm

$e^{\theta|x+iy| - \sigma e^{ky}\cos\frac{k\pi}{2}} \le e^{\theta(y+\frac{\pi}{2})} \ldots \sigma e^{ky} \cos\frac{k\pi}{2}$

$\le e^{ky}\{e^{\overbrace{(\theta-k)}^{<0}y} e^{\theta\frac{\pi}{2}} - \underbrace{\sigma \cos\frac{k\pi}{2}}_{>0}\} \longrightarrow -\infty$



3. Schritt $\quad \exists H > 0 \quad \overline{\lim_{D\ni z\to\xi}} |g(z)| \le M \quad \forall \xi \in \partial R_{HM}$

$|g(z)| \le |f(z)|$

$\overline{\lim} |g(z)| \le \overline{\lim} |f(z)| \le M$ (2. Schritt u. Annahme)

4. Schritt $\quad |f(z)| \le M |e^{\sigma e^{-ikz}}| \quad \forall z \in R_\eta$

$z \in D_\eta$ beliebig

a) g analyt. in D_η
b) $\exists H > 0 \quad$ i) $z \in R_{HM}$
ii) $\overline{\lim_{z\to\xi}} |g(z)| \le M \ \forall \xi \in \partial R_{HM}$
c) beschränkt

Beh. wegen Maximumprinzip.

5. Schritt $\quad$ Beh. $\quad \sigma \to 0$
$|f(z)| \le \lim_{\sigma\to 0} M |e^{-\sigma e^{iy}}| = M$

2. Fall $\quad \alpha, \beta$ beliebig

$$g(z) := f(z) \exp\{-\sigma \exp(-ik \frac{\pi}{\beta-\alpha})$$

$$(z = \alpha + \frac{\beta-\alpha}{2})$$

Satz 3

$f \in \mathcal{O}(D_\eta) \quad (D_\eta$ wie in Satz 2)

$\exists\, c > 0 \quad \exists\, \theta < 1 \quad |f(z)| \leq c \cdot \exp(e^{\frac{\theta \pi |z|}{\beta-\alpha}} \quad \forall z \in$ y/m

f stetig auf ∂D_η

$\overline{\lim_{y \to \infty}} \; |f(x+iy)| \leq M \qquad x = \alpha$ und $x = \beta$

$\Longrightarrow \overline{\lim_{y \to \infty}} \; |f(x+iy)| \leq M \qquad \alpha \leq x \leq \beta$

1. Schritt $\quad f$ beschr. auf D_η

2. Schritt $\quad g(z) := f(z) \cdot \left| \underbrace{\frac{z}{z+hi}}_{<1} \right| \quad h > 0$

$|g(z)| < M + \varepsilon \quad \forall\, z \in D_{M+\varepsilon}$

3. Schritt $\quad |f(z)|\,(1-\varepsilon) \leq M + \varepsilon$

Die Γ-Funktion

axiom. Festlegung

1) $\Gamma(z)$ merom. Erweit. von $n!$

$\to \Gamma(z+1) = z\Gamma(z) \quad \Gamma(1) = 1$

2) logarithm. Konvexität von $\Gamma(x)$ d.h. $\ln\Gamma(x)$ konv

Def $$\boxed{\Gamma(z) := e^{-Cz} z^{-1} \prod_{n=1}^{\infty} \left(1+\frac{z}{n}\right)^{-1} e^{z/n}}$$

- $C = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k} - \ln n$ wohldefin. $\longrightarrow$ An III

- $\prod_{n=1}^{\infty} \left(1+\frac{z}{n}\right) e^{-\frac{z}{n}}$ ist ganz

$f_n(z) = \prod_{k=1}^{n} \left(1+\frac{z}{k}\right) e^{-\frac{z}{k}}$

$\log f_n(z) = \sum_{k=1}^{n} \underbrace{\log\left(1+\frac{z}{k}\right) - \frac{z}{k}}_{\frac{z}{k} + O\left(\left|\frac{z^2}{k^2}\right|\right)}$

$R > 0 \quad |z| < R \quad \sum_{k=1}^{\infty} O\left(\left|\frac{z^2}{k^2}\right|\right) < \infty$

d.h. $\log f_n(z) \xrightarrow{glm} f(z) \quad |z| \le R$

- $\Gamma(z+1) = z\cdot\Gamma(z) \qquad z \in \mathbb{C} - \{0, -1, \dots\}$

Betrachte $\frac{z\Gamma_n(z)}{\Gamma_n(z+1)} \xrightarrow{n\to\infty} 1$

$\Gamma_n(z) = \frac{e^{-Cz + \sum_{k=1}^{n} \frac{z}{k}}\, n!}{z \prod_{k=1}^{n}(z+k)}$

$\frac{z\Gamma_n(z)}{\Gamma_n(z+1)} = \frac{z\, e^{(-Cz+\sum_{k=1}^{n}\frac{z}{k})} n! \dots}{z(z+1)\dots(z+n) e^{\dots}}$

$= \frac{e^{\dots}(z+n+1)}{e^{-C+\sum\frac{1}{k}}}$

$= \frac{z+n+1}{n} \longrightarrow 1$

weil $\sum \frac{1}{k} = C + \ln n + \varepsilon(n)$

$e^{-C+\sum_{k=1}^{n}\frac{1}{k}} = e^{-\ln n + \varepsilon(n)}$

- Integraldarstellung $\int_0^\infty e^{-t} t^{z-1} dt \quad \operatorname{Re} z > 0$

Integral exist.: $\left|\int_0^\infty\right| \le \int_0^\infty |e^{-t} t^{z-1}| dt$

$\le \int_0^1 e^{-t} t^{\operatorname{Re}(z-1)} dt + \int_1^\infty e^{-t} |t^{z-1}| dt$

$\le \int_0^1 e^{-t} \frac{1}{t^{1-\operatorname{Re} z}} + \int_1^\infty e^{-t} \frac{1}{t^{1-\operatorname{Re} z}} dt$

$G_n(z) = \int_0^n \left(1-\frac{t}{n}\right)^n t^{z-1} dt \longrightarrow G(z) = \int_0^\infty e^{-t} t^{z-1} dt \quad \operatorname{Re} z > 0$

(i) $1 + \frac{t}{n} \le e^{\frac{t}{n}} \qquad 0 \le t \le n$

(ii) $1 - \frac{t}{n} \le e^{-\frac{t}{n}} \qquad 0 \le t \le n$

$G_n = \Gamma_n \qquad \frac{t}{n} := \tau$

Dirichlet reihen

Def $$\sum_{n=1}^{\infty} a_n e^{-\lambda_n \cdot s} \qquad \lambda_n \in \mathbb{R},\quad 0 \le \lambda_1 \le \lambda_2 \le \ldots \to \infty,\quad a_n \in \mathbb{C},\quad s \in \mathbb{C}$$

speziell $\lambda_n = n$ $\quad e^{-s} = x \quad \to \sum a_n x^n$

speziell $\lambda_n = \log n \quad a_n = 1 \quad \to \sum \frac{1}{n^s}$

Lemma

$$A(x) := \sum_{\lambda_n \le x} a_n$$

$x \ge \lambda_1 \quad \phi \in C^1[(0,\infty)]$

$$\Longrightarrow \sum_{\lambda_n \le \omega} a_n \phi(\lambda_n) = -\int_{\lambda_1}^{\omega} A(x)\phi'(x)\,dx + A(\omega$$

Falls $A(\omega)\phi(\omega) \to 0 \quad (\omega \to \infty)$

$$\longrightarrow \sum_{n=1}^{\infty} a_n \phi(\lambda_n) = -\int_{\lambda_1}^{\infty} A(x)\phi'(x)\,dx$$

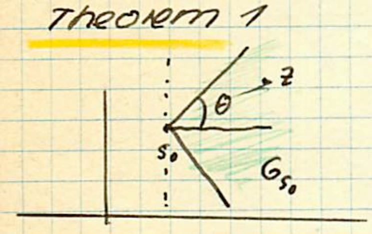
Γ

$$A(\omega)\phi(\omega) - \sum_{\lambda_n \le \omega} a_n \phi(\lambda_n) = \sum_{\lambda_n \le \omega} a_n(\phi(\omega) - \phi$$

$$= \sum_{\lambda_n \le \omega} \int_{\lambda_n}^{\omega} a_n \phi'(x)\,dx \quad - \int_{\lambda_1}^{\omega} a_1 \phi'(x)\,dx + \int_{\lambda_2}^{\omega} a_2 \phi'(x)$$

$$= \int_{\lambda_1}^{\omega} A(x)\phi'(x)\,dx \quad ⌟$$

Theorem 1



Falls $\sum_{n=1}^{\infty} a_n e^{-\lambda_n s}$ für $s = s_0 := \sigma_0 + i t_0$ konvergiert dann konvergiert sie in $G_{s_0,\theta} := \{ z \in \mathbb{C} \mid |\arg(z - s_0)| \le \theta < \frac{\pi}{2}$ für $0 < \theta \le \frac{\pi}{2} \}$ gleichmässig

Γ o.B.d.A $s_0 = 0$: $$\sum_{n \ge 1} a_n e^{-\lambda_n s} = \sum_{n \ge 1} \underbrace{a_n e^{-\lambda_n s_0}}_{b_n} e$$

sei $\nu \ge \mu + 1 \quad \nu, \mu \in \mathbb{N}$

$$\sum_{n=\mu+1}^{\nu} a_n e^{-\lambda_n \cdot s} = \sum_{n=1}^{\nu} a_n e^{-\lambda_n s} - \sum_{n=1}^{\mu} a_n e^{-\lambda_n s}$$

$$= + \int_{\lambda_1}^{\lambda_\nu} A(x) e^{-xs}(+s)\,dx + A(\lambda_\nu) e^{-\lambda_\nu \cdot s} + \int_{\lambda_1} e^{-xs}(-s)$$

$$- A(\lambda_\mu) e^{-\lambda_\mu s}$$

$$= s \cdot \int_{\lambda_\mu}^{\lambda_\nu} A(x) e^{-xs}\,dx + A(\lambda_\nu) e^{-\lambda_\nu \cdot s} - A(\lambda_\mu) e^{-\lambda_\mu \cdot}$$

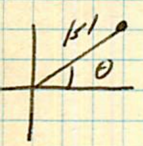
$$= s \int_{\lambda_\mu}^{\lambda_\nu} (A(x) - A(\lambda_\mu)) e^{-xs}\,dx + [A(\lambda_\nu) - A(\lambda_\mu)$$

$$\left|\sum a_n e^{-\lambda_n 0}\right| = \left|\sum_{n\geq 1} a_n\right| < \infty$$

$$\forall \varepsilon > 0 \quad \exists n_0 \in \mathbb{N} \quad |A(x) - A(\lambda_\mu)| < \varepsilon \quad \forall x > \lambda_\mu \geq n$$

$$\left|\sum_{n=\mu+1}^{\nu} a_n e^{-\lambda_n s}\right| \underset{s=\sigma+it}{\leq} |s|\,\varepsilon \int_{\lambda_\mu}^{\lambda_\nu} e^{-x\sigma}\,dx + \varepsilon e^{-\lambda_\nu \sigma}$$

$$\underset{\substack{\sigma \neq 0\\ \sigma > 0}}{\leq} |s|\,\varepsilon \, \frac{1}{\sigma} \underbrace{\left(-e^{-\lambda_\nu \sigma} + e^{-\lambda_\mu \sigma}\right)}_{\leq 2} + \varepsilon \underbrace{e^{-\lambda_\nu \sigma}}_{\leq 1}$$



$$< 2\varepsilon\left(\frac{|s|}{\sigma} + 1\right) \leq 2\varepsilon\left|\frac{1}{\cos\theta} + 1\right|$$

$\lrcorner$

Satz 2

Falls $\sum_{n\geq 1} a_n e^{-\lambda_n s}$ für $s = s_0$ konv.
$\longrightarrow$ konv. für $\operatorname{Re}(s) > \sigma_0$ u. glm in jedem beschränkten abgeschl. Gebiet, das in der Halbebene $\sigma > \sigma_0$ enthalten ist

Satz 3

Falls $\sum a_n e^{-\lambda_n s}$ für $s = s_0$ konv.
$\sum a_n e^{-\lambda_n s} = f(s)$ dann $f(s) \to f(s_0)$ $(s \to s_0)$
entlang dem Weg in $G_{s_0,\theta}$

$\ulcorner$ Also $\lim_{s\to s_0} \lim_{m\to\infty} \sum_{n=1}^{m} a_n e^{-\lambda_n s}$

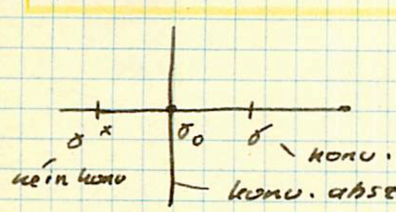
$$= \lim_{m\to\infty} \sum_{n=1}^{m} \lim_{s\to s_0} a_n e^{-\lambda_n s} = \lim_{m\to\infty} \sum_{n=1}^{m} a_n e^{-\lambda_n s_0}$$

$$= f(s_0) \quad \lrcorner$$

Satz 4

$f(s) \in A(G_{s,\theta})$ falls $\sum a_n e^{-\lambda_n s}$ in s_0 konv.

Def



konv. abszisse $s = \sigma_0$ konv. linie

Bsp

$\sum n^{-s}$ $\quad \sigma = 1 + \delta$ glm. konv. $\quad$ konv absz. 1
$\sum n^{-s}(-1)^n$ $\quad$ konv absz. 0
$\sum n!\, n^{-s}$ $\quad$ mit konv. absz. ∞

Satz 5

$\sum a_n e^{-\lambda_n s}$ konv. für $s = 0$ und für unendl. viele Werte von s $f(s) = 0$
$s \in \{ s \in \mathbb{C} \mid \sigma > \varepsilon > 0,\ |\arg(s)| \leq \theta < \frac{\pi}{2}$
$\Longrightarrow a_n = 0 \ \forall n \quad f(s) \equiv 0$

Identsatz $:\Rightarrow$ kein HP $\Rightarrow$ $s_n \to \infty$

(s_n) mit $\sigma_n < \sigma_{n+1}$

$$g(s) := e^{\lambda_1 s} f(s) = a_1 + \underbrace{\sum a_n e^{-(\lambda_n - \lambda_1)s}}_{\to 0 \quad s_n \to \infty}$$

$g(s) \to a_1 \quad s \to \infty$ glm konv.

$g(s_n) = 0 \quad f(s_n) = 0 \quad \forall n \quad \Rightarrow a_1 = 0$

$\dots \quad \forall n \quad a_n = 0$ $\perp$

$\to$ Dirichletentwicklung eindeutig

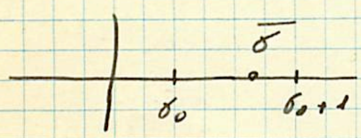
Was passiert für $\sigma = \varepsilon = 0$?

$$g(s) := \prod_{n=1}^{\infty} \left(1 - \tfrac{3}{2} e^{-ns}\right) \qquad Re(s) > 0$$

NS $\quad s_n = \frac{1}{n}\left(\log \frac{3}{2} + \underset{k \neq 0}{2\pi i k}\right) \longrightarrow 0$

entspr. absolut konv. abszisse $\bar{\sigma}$

$\sum a_n n^{-s} \qquad \sigma_0 < \infty \qquad \bar{\sigma} - \sigma_0 \leq 1$



Reihe konv. für $> \sigma_0 + 1$

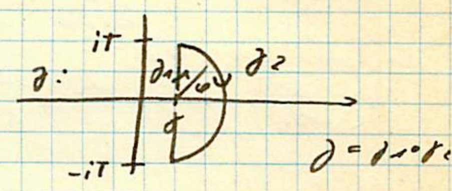
denn $\frac{1}{n^{\sigma_0 + 1 + \varepsilon}}$

a_n reell positiv $\Rightarrow$ in $s = \sigma_0$ Singularität

Perron'sche Formel

Lemma 1 $\boxed{\begin{array}{l} \sigma > 0 \quad u \in \mathbb{R} \implies \\ \dfrac{1}{2\pi i} \displaystyle\int_{\sigma - i\infty}^{\sigma + i\infty} \frac{e^{us}}{s}\, ds = \begin{cases} 1 & u > 0 \\ 0 & u < 0 \\ \frac{1}{2} & u = 0 \end{cases} \\ \text{wo } \displaystyle\int_{\sigma - i\infty}^{\sigma + i\infty} = \lim_{T \to \infty} \int_{\sigma - iT}^{\sigma + iT} \quad (T > 0 \quad T \in \mathbb{R}) \end{array}}$

Γ 1. Fall $u = 0$ $\quad \dfrac{1}{2\pi i} \displaystyle\int_{\sigma - i\infty}^{\sigma + i\infty} \frac{1}{s}\, ds$ $\qquad \gamma:$



$\gamma = \gamma_1 + \gamma_2$

$$\int_\gamma \frac{1}{s}\, ds = 0 \implies \int_{\gamma_1} = -\int_{\gamma_2} = \int_{-\gamma_2}$$

$$\gamma_2 : \varphi \mapsto Te^{i\varphi} + \sigma \qquad \varphi \in \left[-\tfrac{\pi}{2}, +\tfrac{\pi}{2}\right]$$

$$\int_{-\gamma_2} \frac{1}{s}\, ds = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{Te^{i\varphi} i\, d\varphi}{\sigma + Te^{i\varphi}} = \int_{-\frac{\pi}{2}}^{\pi/2} \Big(\underbrace{\frac{Te^{i\varphi} \cdot i}{\sigma + Te^{i\varphi}} - i}_{\frac{-\sigma \cdot i}{\sigma + Te^{i\varphi}}} \Big) d\varphi + \pi i$$

$$|\ldots| \le \pi \cdot \frac{\sigma}{|T| - \sigma} \xrightarrow[T \to \infty]{} 0$$

$$\implies \int_{-\gamma_2} \frac{1}{s}\, ds = \pi i, \quad \frac{1}{2\pi i} \int_{\gamma_1} \frac{1}{s}\, ds = \frac{1}{2}$$

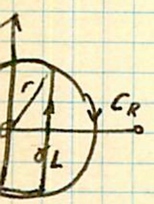
2. Fall $u > 0 \lor u < 0$

$$\int_{\sigma - iT}^{\sigma + iT} e^{us} \frac{1}{s}\, ds = \underbrace{\frac{e^{us}}{us} \Big|_{\sigma - iT}^{\sigma + iT}}_{I} + \underbrace{\int_{\sigma - iT}^{\sigma + iT} \frac{e^{us}}{us^2}\, ds}_{II}$$

(I): $|e^{u(\sigma + iT)}| = e^{u\sigma} = |e^{u(\sigma - iT)}| \qquad |s| > T$

$$\left| \frac{e^{u(\sigma + iT)}}{u(\sigma + iT)} - \frac{e^{u(\sigma - iT)}}{u(\sigma - iT)} \right| \le \left| \frac{e^{u(\sigma + iT)}}{u(\sigma + iT)} \right| + \left| \frac{e^{u(\sigma - iT)}}{u(\sigma - iT)} \right|$$

$$\le \frac{2e^{u\sigma}}{|u| T} \xrightarrow[T \to \infty]{} 0$$



(II) (a) $u < 0$: $L + C_R$
(b) $u > 0$: $L + C_L$

$u < 0$: $\left| \dfrac{e^{us}}{s^2} \right| \le \dfrac{e^{u\sigma}}{r^2}$ für $\operatorname{Re} s > \sigma$

$u > 0$: $\left| \dfrac{e^{us}}{s^2} \right| \le \dfrac{e^{u\sigma}}{r^2}$ $\qquad \operatorname{Re} s < \sigma$

also $\left| \displaystyle\int_{C_{R,L}} \frac{e^{us}}{us^2}\, ds \right| \le 2\pi r \dfrac{e^{u\sigma}}{|u| \cdot r^2} \xrightarrow[r \to \infty]{} 0$

Cauchy: (a) $u < 0$ $\displaystyle\int = -\int_{C_R} \to 0 \quad r \to \infty$

also $\displaystyle\int_{\sigma - i\infty}^{\sigma + i\infty} = 0$

② $u>0 \quad r\to\infty \quad \int_{L_1} = -\int_{C_R} + 2\pi i$

dann $\mathrm{Res}\left(\frac{e^{us}}{us^2}, s=0\right) = \lim_{s\to 0}\frac{d}{ds}\, s^2 \frac{e^{us}}{us^2}$

Also $\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{e^{us}}{us^2} = 1 \quad u>0$ ⌋

Falls $a>0$: $\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \frac{e^{(\lambda-\lambda_n)\cdot s}}{s}\,ds \begin{cases} 0 & \lambda<\lambda_n \\ 1 & \lambda>\lambda_n \\ \frac{1}{2} & \lambda=\lambda_n \end{cases}$

$\frac{1}{2\pi i}\sum_{n=1}^{\infty} a_n \int_{a-i\infty}^{a+i\infty} \frac{e^{(\lambda-\lambda_n)s}}{s}\,ds = \sum_{\lambda_n\le\lambda}{}' a_n \qquad \lambda_n=\lambda \Rightarrow a_n' = \frac{a_n}{2}$

Perr. Form.

$= \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} f(s)\frac{e^{\lambda s}}{s}\,ds \qquad f(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s}$

Lemma 2

Sei DR $\sum a_n e^{-\lambda_n s}$ konv. für $s=\beta$ reell

Dann gilt

$\sum_{\nu=1}^{n} a_\nu e^{-\lambda_\nu\cdot s} = o(|t|)$ für $s=\sigma+it$

glm für $\sigma \ge \beta+\varepsilon > \beta$ und

(d.h. $\delta>0\ \varepsilon>0\ \exists t_0 \ \left|\frac{1}{t}\sum_{\nu=1}^{n} a_\nu e^{-\lambda_\nu s}\right| < \delta \quad \sigma\ge\beta \ |t|>t_0$

speziell: $n=\infty$: $f(s) = o(|t|)$ glm für $\sigma\ge\beta+\varepsilon>\beta$

Γ o.E.d.A $\beta=0$ denn $\ulcorner \sum a_n e^{-\lambda_n s} = \sum \underbrace{a_n e^{-\lambda_n\beta}}_{:=b_n} \cdot e^{-\lambda_n\underbrace{(s-\beta)}_{:=s'}}$

$= \sum b_n e^{-\lambda_n s'}$ ⌋

① $1<N<n$:

$\sum_{\nu=1}^{n} a_n e^{-\lambda_\nu s} = \left(\sum_{\nu=1}^{N} + \sum_{\nu=N+1}^{n}\right) a_\nu \underbrace{e^{-\lambda_\nu s}}_{:=\phi(\lambda_\nu)} \cancel{= \phi(\lambda_\nu)}$

(→ lemma letztes Kapitel)

$= \sum_{\nu=1}^{N} \ldots + \underbrace{A(\lambda_n)}_{:=\sum_{\lambda_\nu\le\lambda_n} a_\nu}\phi(\lambda_n) - A(\lambda_N)\phi(\lambda_N)$

$= \sum_{\nu=1}^{N} \ldots - \int_{\lambda_1}^{\lambda_n} A(x)\,\phi'(x)\,dx - \int_{\lambda_1}^{\lambda_N} A(x)\,\phi'(x)\,dx$

$+ A(\lambda_n)\phi(\lambda_n) - A(\lambda_N)\phi(\lambda_N)$.

$= \sum_{\nu=1}^{N} \ldots + s\int_{\lambda_N}^{\lambda_n} A(x)e^{-xs}\,dx + A(\lambda_n)\phi(\lambda_n) - A(\lambda_N)\phi(\lambda_N)$

$\left|\sum_{\lambda_n\le\omega} a_n\phi(\lambda_n)\right| \le \left|\sum_{\nu=1}^{N}\ldots\right| + |s|\cdot\int_{\lambda_N}^{\lambda_n} |A(x)|\, e^{-x\sigma}\,dx + |A(\lambda_n)|\cdot\underbrace{e^{-\lambda_n\sigma}}_{\le 1} \le C\cdot N + |s|\cdot\int_{\lambda_N}^{\lambda_n} e^{-x\sigma}\,dx + 2C_1$

$\le \int_{\lambda_N}^{\infty} e^{-x\sigma} = \frac{1}{\sigma}e^{\ldots}$

(Bem: $\sum_{\nu\ge 1} a_\nu e^{-\lambda_\nu s}$ konvergent $\sigma>0$, also

$\exists c>0$ so, dass $|a_\nu e^{-\lambda_\nu s}| < c \quad \forall \nu\ge 1$

~~Also~~ $|A(x)| < C_1 \quad \forall x\ge 0$)

$\le C_2(N+1) + |s|\, 2C_1 \frac{1}{\sigma}\underbrace{e^{-\lambda_N\sigma}}_{\le e^{-\lambda_N\varepsilon}} \le C_3 N + 2C_1\underbrace{\frac{|s|}{\sigma}}_{\le 1+\ldots} e^{-\lambda_N\varepsilon}$

$\leq C_3 N + C_3 \left(1 + \frac{|t|}{\varepsilon}\right) e^{-\lambda_N \varepsilon}$

② $N \geq n$ $\quad \left| \sum_{\nu=1}^{n} \right| \leq c \cdot n \leq c N$, abhängig von $|t|$

$N = N(|t|) \to \infty$ langsam. abs.

$\Rightarrow \sum a_\nu e^{-\lambda_\nu s} = o(|t|)$

Perronsche Formel

$\sigma_0 \in \mathbb{R}$ Konv. abszisse d. DR $\sum a_n e^{-\lambda_n s} = f(s)$, Summenfunkt. $f(s)$

$\rightarrow \sigma > \sigma_0 \quad \sigma > 0$

$$\boxed{\sum_{\lambda_n \leq \omega}{}' a_n = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{f(s)}{s} e^{\omega s} \, ds}$$

Γ $\quad \lambda_n \leq \omega < \lambda_{n+1}$:

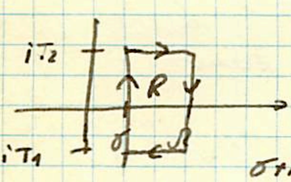
$$g(s) := e^{\omega s} \left\{ f(s) - \sum_{\nu=1}^{n} a_\nu e^{-\lambda_\nu s} \right\}$$

$$= \sum_{\nu=n+1}^{\infty} a_\nu e^{-(\lambda_\nu - \omega) \cdot s} =: \sum_{\nu \geq 1} b_\nu e^{-\mu_\nu \cdot s}$$

$(b_\nu = a_{n+\nu}$, $\mu_\nu = \lambda_{n+\nu} - \omega)$ $\Rightarrow 0 < \mu_1 < \mu_2 < \dots \to \infty$

Lemma ① $\Rightarrow \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{g(s)}{s} \, ds$

$$= \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{f(s) e^{\omega s}}{s} \, ds - \underbrace{\frac{1}{2\pi i} \int_{\sigma - i\infty} \sum_{\nu}^{n} a_\nu \frac{e^{(\omega - \lambda_\nu) s}}{s} \, ds}_{(L1) = \sum_{\lambda_n \leq \omega}' a_n}$$



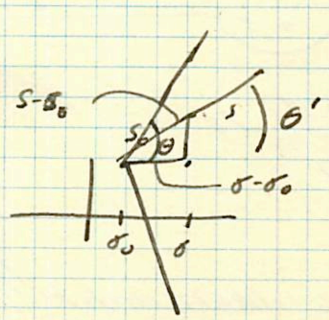
Cauchy $\quad \frac{1}{2\pi i} \int_{\partial R} \frac{g(s)}{s} \, ds = 0$

$$\Rightarrow \frac{1}{2\pi i} \int_{\sigma - iT_1}^{\sigma + iT_2} \dots = \frac{1}{2\pi i} \left[\int_{\sigma - iT_1}^{\Omega - iT_1} + \int_{\Omega - iT_1}^{\Omega + iT_2} + \int_{\Omega + iT_2}^{\sigma + iT_2} \right] = iT_1 + I_2 + I_3$$

T_1, T_2 fest $\quad \Omega \to \infty \quad I_2 \to 0$: $\quad \Gamma \quad g(s) = \sum \underbrace{b_\nu e^{-\mu_\nu s_0}}_{a_\nu} \cdot \underbrace{e^{-\mu_\nu (s - s_0)}}_{\phi(\mu_\nu)}$

$= \sum_{\nu \leq \infty} a_\nu \phi(\mu_\nu)$

$\underset{\text{v.L.}}{=} (s - s_0) \int_{\mu_1}^{\infty} A(x) e^{-x(s - s_0)} \, dx + A(\infty) \phi$



$$|g(s)| \leq |s - s_0| \int_{\mu_1}^{\infty} |A(x)| e^{-x(\sigma - \sigma_0)} \, dx$$

$$= \frac{|s - s_0|}{\sigma - \sigma_0} \underbrace{|A(x)| \, e^{-\mu_1(\sigma - \sigma_0)}}_{\leq k}$$

$$\leq \underbrace{\frac{|s - s_0|}{\sigma - \sigma_0}}_{= \frac{1}{\cos\theta'} \leq \frac{1}{\cos\theta}} \cdot k = c(\theta)$$

$|g(s)| \leq c(\theta) \quad (|\theta| < \frac{\pi}{2})$

$$|I_2| = \left| \frac{1}{2\pi i} \int_{\Omega - iT_1}^{\Omega + iT_2} \frac{g(s)}{s} \, ds \right|$$

$$\leq \frac{1}{2\pi} \int_{\Omega - iT_1}^{\Omega + iT_2} \frac{c}{\underbrace{|s|}_{\geq \Omega}} \, ds$$

$$< \frac{c}{2\pi} \cdot \frac{T_2 + T_1}{\Omega} \to 0 \quad \Omega \to \infty$$

$I_1, I_3 \to 0$

Funktionalgleichung d. Riemannschen Zetafunktion

$$\zeta(s) = \sum_{n\geq 1} \left(\frac{1}{n}\right)^s \qquad \operatorname{Re} s > 1$$

$$|\zeta(s)| \leq \sum_{n\geq 1} \left|\left(\frac{1}{n}\right)^s\right| \leq \sum_{n \geq 1} \left(\frac{1}{n}\right)^\sigma \qquad \sigma = \operatorname{Re} s$$

Satz

$$\boxed{\zeta(s)\cdot\Gamma(s) = \int_0^\infty \frac{x^{s-1}}{e^x-1}\,dx \qquad \operatorname{Re} s > 1}$$

⌈

$$\Gamma(s) = \int_0^\infty x^{s-1}e^{-x}\,dx \qquad \operatorname{Re} s > 0$$

$$\underset{\substack{x=ny\\ dx = n\,dy}}{=} n^s \int_0^\infty y^{s-1} e^{-ny}\,dy$$

$$\frac{\Gamma(s)}{n^s} = \int_0^\infty y^{s-1}e^{-ny}\,dy$$

$$\left|\sum_{n\geq 1} \frac{\Gamma(s)}{n^s}\right| = \left|\sum_{n\geq 1}\int_0^\infty y^{s-1}e^{-ny}\,dy\right|$$

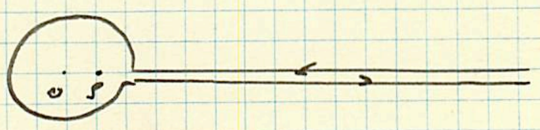
$$\leq \sum_{n\geq 1}\int_0^\infty |y^{s-1}|\, e^{-ny}\,dy \leq \sum_{n\geq 1}\int_0^\infty y^{\sigma-1}e^{-ny}\,dy$$

$$= \Gamma(s)\cdot\zeta(s) \qquad \text{absolute konv.}$$

$$\sum_{n\geq 1}\int_0^\infty y^{s-1}e^{-ny}\,dy = \int_0^\infty \underbrace{\sum_{n\geq 1} e^{-ny}}_{e^{-y}\frac{1}{1-e^{-y}} = \frac{1}{e^y-1}} y^{s-1}\,dy$$

$$= \int_0^\infty \frac{\cancel{e^{-y}}\, y^{s-1}}{e^y-1}\,dy$$ ⌋

Betrachte:



$0 < \rho < 2\pi$

$$I(s) = \int_C \frac{z^{s-1}}{e^z-1}\,dz$$

Lemma $\Omega \subseteq \mathbb{C}$ beschränktes Gebiet, dann ist $I(s) \in H(\Omega)$

⌈ $$\int_{|z|=\rho} \frac{z^{s-1}}{e^z-1}\,dz \in H(\Omega)$$

$$\left|\int_\rho^\infty \frac{x^{s-1}}{e^x-1}\,dx\right| \leq \int_\rho^\infty \frac{|x^{s-1}|}{|e^x-1|}\,dx = \int_\rho^\infty \frac{x^{\operatorname{Re} s-1}}{e^x-1}\,dx$$

$$\left(\sigma_0 := \sup_{s\in\Omega} \operatorname{Re} s\right) \leq \int_\rho^\infty \frac{x^{\sigma_0-1}}{e^x-1}\,dx < \infty$$ ⌋

Theorem $\boxed{\operatorname{Re} s > 1 \Rightarrow \zeta(s) = \frac{I(s)\,\Gamma(1-s)}{2\pi i \cdot e^{i\pi s}}}$

Γ $I(s) = \int_C \frac{z^{s-1}}{e^z-1}\,dz = -\int_\rho^\infty \frac{x^{s-1}}{e^x-1}\,dx + \int_{|z|=\rho} \frac{z^{s-1}}{e^z-1}\,dz$

~~$z = \log|z| + \arg z \cdot i$~~ $+ \int_\rho^\infty \frac{(x e^{2\pi i})^{s-1}}{e^x-1}\,dx$

$\int_{|z|=\rho} \frac{z^{s-1}}{e^z-1}\,dz$: $\left|\frac{e^z-1}{z}\right| \xrightarrow[|z|\to 0]{} 1$

$|e^z-1| > A|z|$ ρ klein

$|z^{s-1}| = |e^{(\log|z| + i\arg z)(s-1)}| \underset{s=\sigma+i\tau}{=} e^{\log|z| - \tau \arg z} \le |z|^{\sigma-1} e^{2\pi|\tau|}$

$\left|\int_{|z|=\rho} \frac{z^{s-1}}{e^z-1}\right| \le \int_{|z|=\rho} \frac{|z^{s-1}|}{|e^z-1|}|dz| \le \frac{\rho^{\sigma-1}\cdot e^{|\tau|2\pi}}{A\rho} 2\pi\rho \xrightarrow[\rho\to 0]{} 0$

$I(s) = -\int_0^\infty \frac{x^{s-1}}{e^x-1}\,dx + e^{2\pi i(s-1)}\int_0^\infty \frac{x^{s-1}}{e^x-1}\,dx$

$= \Gamma(s)\,\zeta(s)\,(e^{2\pi i(s-1)} - 1) = \Gamma(s)\,\zeta(s)\underbrace{(e^{2\pi i s} - 1)}_{2i \sin\pi s \cdot e^{\pi i s}}$

$\Gamma(s)\,\Gamma(1-s) = \frac{\pi}{\sin \pi s}$

$I(s) = \frac{\zeta(s)}{\Gamma(1-s)} \frac{\pi}{\sin\pi s} \cdot 2i \sin\pi s\, e^{\pi i s} = \frac{\zeta(s)}{\Gamma(1-s)} 2\pi i\, e^{\pi i s}$

Schliessl. $\zeta(s) = \frac{I(s)\,\Gamma(1-s)}{e^{\pi i s}\,2\pi i}$ $\operatorname{Re} s > 1$ $\lrcorner$

Pole von $\zeta(s)$ $\Gamma(1-s)$ Pole in $s = 1, 2, 3, \ldots$

~~$\zeta(s)$ ek~~

$\zeta(s)$ in $s = 2, 3, 4 \ldots$ keine Pole

$\Rightarrow I(s)$ null Pol in $s = 1$

$\Gamma(z) = \frac{1}{z} + \ldots$ $\Gamma(1-s) = -\frac{1}{s-1} + \ldots$

$\operatorname{Res} \zeta(s)|_{s=1} = \lim_{s\to 1} \frac{(s-1)\,\Gamma(1-s)}{e^{\pi i s}\,2\pi i} = 1$

$I(1) = \int_{|z|=\rho} \frac{1}{e^z-1}\,dz = 2\pi i \operatorname{Res} \frac{1}{e^z-1}\Big|_{z=0} = 2\pi i$

Lemma $\boxed{\zeta(-2m) = \begin{cases} 0 & m > 0 \\ -\frac{1}{2} & m = 0 \end{cases}}$

Γ zu zeigen : $I(1) = 0$

$\frac{1}{e^z-1} = \frac{1}{z} - \frac{1}{2} + B_1 z - B_2 \frac{z^3}{4!} \ldots$

$|z| = \rho$ ρ klein

$$I(-2m) = \int_{|z|=\varepsilon} z^{-2m-1} \left(\frac{1}{z} - \frac{1}{2} + B_1 z \cdots \right) dz$$

$$\int_{|z|=\varepsilon} z^k \, dz = 0 \quad \text{falls} \quad k \neq -1 \quad k \in \mathbb{Z}$$

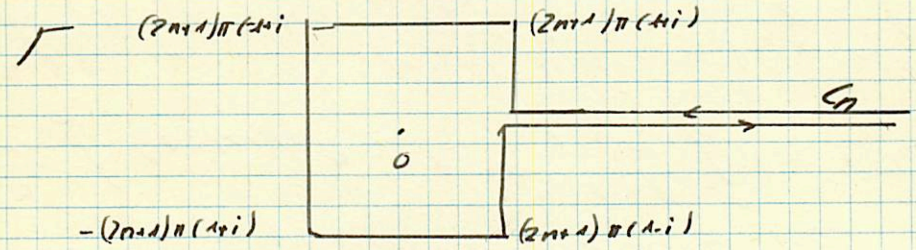
$$I(-2m) = 0 \quad m \neq 0 \qquad I(0) \quad -\frac{1}{2} \int_{|z|=\varepsilon} \frac{1}{z} \, dz = -\frac{1}{2} 2\pi i$$

$$\zeta(0) = -\frac{1}{2}$$

Die Funktionalgleichung

Theorem $\boxed{I(s) = 4\pi i \sin\frac{\pi}{2}s \, (2\pi)^{s-1} \zeta(1-s) \quad \text{Res}}$

$$= \Gamma(s)\zeta(s)\left(e^{2\pi i(s)} - 1\right)$$



$$\int_{C_n - C} \frac{z^{s-1}}{e^z - 1} dz = 2\pi i \sum_{\substack{m=-n \\ m \neq 0}}^{n} \underbrace{\operatorname{Res} \frac{z^{s-1}}{e^z - 1} \Big|_{z = 2m\pi i}}_{(2m\pi i)^{s-1} = (2m\pi e^{\pi i/2})^{s-1}}$$

$$= \sum_{\substack{m=-n \\ m \neq 0}}^{n} (2m\pi e^{i\pi/2})^{s-1} = \sum_{k=1}^{n} (2m\pi)^{s-1} e^{i\pi(s-1)} \left(e^{i\frac{\pi}{2}(s-1)} + e^{-i\frac{\pi}{2}(s-1)} \right.$$

$$= 2\pi^{s-1} \sum_{m=1}^{n} m^{s-1} e^{i\frac{\pi}{2}(s-1)} \cos\frac{\pi}{2}(s-1)$$

$$I(s) = \underbrace{\int_{C_n} \frac{z^{s-1}}{e^z - 1} dz}_{\xrightarrow{n \to \infty} 0} + \sum_{n=1}^{n} 4\pi i \sin\frac{\pi}{2}s \, e^{\pi i s} (2m\pi)^{s-1}$$

Funktionalgl. $\quad \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{-\frac{(1-s)}{2}} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-$

Der Primzahlsatz

(*) $\pi(x)\,\frac{\lg x}{x} \xrightarrow{x\to\infty} 1$ $\qquad \pi(x) = \sum_{p\le x} 1$

Bew 1 a) (*) $\iff \frac{\psi(x)}{x} \xrightarrow{x\to\infty} 1$

$\psi(x) := \sum_{p^m\le x} \lg p$

b) $\psi(x) \le \tilde{C}x \quad \tilde{C}>0$

c) i) $\left.\begin{array}{l}\zeta(s) = \prod_p (1-p^{-s})^{-1}\\ \zeta(s) \ne 0\end{array}\right\}$ Re $s>1$

ii) $\zeta(s) - \frac{1}{s-1}$ holom Re $s>0$

iii) $\zeta(s) \ne 0$ Re $s \ge 1$

2 Satz $f : [0,\infty[\to \mathbb{R}$ (lok. integrierbar)
$|f(x)| < k$

$\int_0^\infty f(t)e^{-zt}\,dt =: F(z)$ (Laplacetrafo)

ist holom. Funkt auf Re $z>0$

Beh $F(z)$ holom. Re $z\ge 0$

$\Rightarrow \int_0^\infty f(t)\,dt$ existiert und $\underset{\text{Erweiterung}}{=} F(0)$

3 Korollar $g : [1,\infty[\to \mathbb{R}_0^+$
monoton steigend

$g(x) \le Cx$

$s\int_1^\infty g(x)\,x^{-s-1}\,dx =: G(s)$ Mellin Wellen transf.

holom. auf Re $s>1$

Beh $G(s) - \frac{c}{s-1}$ holom. auf Re $s\ge 1$ $\quad \exists c>0$

$\Rightarrow \frac{G(x)}{x} \xrightarrow{x\to\infty} c$

4. $g := \psi \quad c := 1 \xrightarrow{3} \frac{\psi(x)}{x} \xrightarrow{x\to\infty} 1$

1a) $\psi(x) = \sum_{p\le x}\left[\frac{\lg x}{\log p}\right]\lg p \le \sum_{p\le x}\frac{\lg x}{\lg p}\log p$

$= \lg x\;\pi(x)$ ①

$1<y<x$ $\quad \pi(x) \le y + \sum_{y<p\le x} 1 \le y + \sum_{y<p\le x}\frac{\log p}{\log y} \le y + \frac{\psi(x)}{\lg y}$

$y := \frac{x}{\lg^2 x}$: $\quad \pi(x) \le \frac{x}{\lg^2 x} + \frac{\psi(x)}{\lg x - 2\lg\log x}$ ②

$\frac{\psi(x)}{x} \overset{①}{\le} \frac{\lg x\;\pi(x)}{x} \overset{②}{\le} \underbrace{\frac{1}{\lg x}}_{\substack{\to 0\\ x\to\infty}} + \frac{\psi(x)}{x}\underbrace{\frac{\lg x}{\lg x - 2\lg\lg x}}_{\substack{\to 1\\ x\to\infty}}$

1b) $p \mid \binom{2n}{n}$ $\quad p\in\,]n,2n]$

$\prod_{n<p<2n} p \le \binom{2n}{n} < 2^{2n}$

$n := 2^{k-1}$: $\sum_{2^{k-1}<p\le 2^k}\lg p < 2^k\cdot\lg 2$

$\sum_{p\le 2^k}\lg p < (2^k + 2^{k-1} + \ldots + 1)\lg 2 < 2^{k+1}\lg 2$

$2^k\le x<2^{k+1}$: $\sum_{p\le x}\lg p < \sum_{p\le 2^{k+1}}\lg p < 2^{k+2}\lg 2 <$

$< x\cdot\underbrace{4\lg 2}_{c}$

$\psi(x) = \Big(\sum_{p\le x} + \sum_{p^2\le x} + \ldots + \sum_{p^{[\lg_2 x]}\le x}\Big)\lg p$

$= \sum_{p\le x} + \sum_{p\le x^{1/2}} + \ldots + \sum_{p\le x^{\frac{1}{[\lg_2 x]}}}\Big)\lg p$

$< cx + cx^{1/2} + \ldots + cx^{\frac{1}{[\lg_2 x]}}$

$< cx + \lg_2 x\cdot c\,x^{1/2} < \tilde{c}x$

1c) $\zeta(s) = \sum_{n\ge 1} n^{-s}$

$(1-2^{-s})\zeta(s) = \sum_{n\ge 1} n^{-s} - \sum_{n\ge 1}(2n)^{-s} = \sum_{\substack{n\ge 1\\ 2\nmid n}} n^{-s}$

$\vdots$

$\prod_{p\le N}(1-p^{-s})\,\zeta(s) = \sum_{\substack{n\ge 1\\ p\nmid n\\ p\le N}} n^{-s} =: \Sigma$

$|\Sigma| \ge 1 - \underbrace{\sum_{n\ge N}|n^{-s}|}_{\to 0\ N\to\infty} > 0$ $\quad$ N genügend groß

$\zeta(s)\ne 0$ $\quad \mathrm{Re}(s)>1$

$\lim_{N\to\infty}\prod_{p\le N}(1-p^{-s})\,\zeta(s) = \lim_{N\to\infty}\sum_{\substack{n\ge 1\\ p\nmid n\ p\le N}} n^{-s} = 1$

iii) Ann. $\zeta(1+iy_0)=0 \qquad y_0\in\mathbb{R}\setminus\{0\} \qquad x>1$

$$\lg|\zeta(x+iy_0)| = -\mathrm{Re}\sum_p \lg(1-p^{-s})$$
$$= \mathrm{Re}\sum_p \left(p^{-s}+\tfrac{1}{2}(p^2)^{-s}+\tfrac{1}{3}(p^3)^{-s}+\dots\right)$$
$$= \mathrm{Re}\sum_{n\geq} a_n n^{-s} \qquad a_n\geq 0$$
$$= \sum \underbrace{a_n}_{\geq 0}\ \underbrace{n^{-x}}_{\geq 0}\ \underbrace{\cos(y_0\lg n)}_{?}$$

$$(3+4\cos\varphi+\cos 2\varphi) = (1+\cos\varphi)^2\geq 0 \qquad \varphi\in\mathbb{R}$$
$$\varphi(x) := \zeta^3(x)\,\zeta^4(x+iy_0)\,\zeta(x+2iy_0)$$
$$\lg|\varphi(x)| = \sum \underbrace{a_n}_{\geq 0}\ \underbrace{n^{-x}}_{\geq 0}\ \underbrace{(3+4\cos(y_0\lg n))+\cos(2y_0\lg n)}_{\geq 0}\geq 0$$

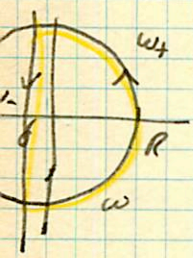
$$\varphi(x)\ \underset{x\searrow 1}{\not\to}\ 0$$

$$\zeta(1+iy_0)\neq 0$$

2. $F_\lambda(z) := \int_0^\lambda f(t)e^{-zt}\,dt \qquad$ ganz $\quad \lambda\geq 0$

$$|F(0)-F_\lambda(0)| \xrightarrow[\lambda\to\infty]{} 0$$

$$|F(z)-F_\lambda(z)|_{z=0}| = \frac{1}{2\pi}\left|\int_\omega \frac{F(z)-F_\lambda(z)}{z}\,dz\right|$$



$0<d=d(R)$ genug klein :

$F(z)$ holomorph auf $\mathrm{Re}\, z=:x>-2d$

$\omega_+ : x>0$

$\omega_- : x<0$

$$\omega_+ : \quad |F(z)-F_\lambda(z)| \leq \int_\lambda^\infty |f(t)|e^{-xt}\,dt$$
$$\leq k\cdot\frac{e^{-x\lambda}}{x}$$

$$\omega_- : \quad |F_\lambda(z)| \leq \int_0^\lambda k e^{-xt}\,dt = k\,\frac{e^{-x\lambda}-1}{-x} < \frac{e^{-x\lambda}}{|x|}k$$

$$|F(z)-F_\lambda(z)|_{z=0}| = |(F(z)-F_\lambda(z))e^{\lambda x}(1+\tfrac{z^2}{R^2})|_{z=0}|$$

uf kreis
$\frac{1}{z}+\frac{z}{R^2}=\frac{\bar z}{z}+\frac{z}{R^2}$
$=\frac{\bar z + z}{R^2}$

$$\leq \frac{1}{2\pi}\int_\omega |F(z)-F_\lambda(z)|\,e^{\lambda x}\left|\frac{1}{z}+\frac{z^2}{R^2}\right||dz|$$

$$\leq \frac{1}{2\pi}\int_{\omega_+} |F(z)-F_\lambda(z)|\,e^{\lambda x}\,\frac{2x}{R^2}\,|dz| \quad \Big\}\leq \frac{1}{2\pi}\cdot\pi R\,\frac{2k}{R^2} = \frac{1}{R}\cdot k$$

$\leq B_1(R,d)$ auf ·

$$+\frac{1}{2\pi}\int_{\omega_-'} |F_\lambda(z)|\,e^{\lambda x}\,\frac{2x}{R^2}\,|dz| \quad \Big\}\leq \frac{k}{R}$$

$$+\frac{1}{2\pi}\int \overbrace{|F(z)|\left|\frac{1}{z}+\frac{z}{R^2}\right|}^{+}\,e^{\lambda x}\,|dz| \quad \Big\}+\leq B(R)\ \text{auf}\ \omega_-'\cap\omega_- \times$$

3. term $\le \frac{k}{2\pi}\left(\underbrace{2\pi \arcsin \frac{\delta}{R}}_{\delta} B + \frac{2RB_1}{e^{\lambda\delta}}\right)$

$\varepsilon > 0$: $R > \frac{k}{\varepsilon}$

$\delta < \pi \frac{\varepsilon}{kB(R)}$

$\forall \lambda > \lambda_0 : \frac{k \cdot RB(R,\delta)}{e^{\lambda\delta}\pi} < \varepsilon$

3. $g(x), c$ $\quad f(t) := e^{-t} g(e^t) - c$

$F(z) := \int_0^\infty \left[e^{-t} f(e^t) \cdot e^{-zt} - c e^{-zt}\right] dt$

$x := e^t$, $dt = \frac{dx}{x}$

$= \int_1^\infty x^{-1} g(x) x^{-z} \frac{dx}{x} - \frac{c}{z}$

$= \int_1^\infty g(x) x^{-(z+1)-1} dx - \frac{c}{z}$

$= \frac{1}{z+1}\left[G(z+1) - \frac{c}{z} - c\right]$

$G(z) - \frac{c}{z-1}$ holom. auf $\operatorname{Re} z \ge 1$

$F(z)$ holom auf $\operatorname{Re} z \ge 0$

$\underset{\Longrightarrow}{\text{Satz}} \int_0^\infty f(t)\, dt$ exist.

$= \int_0^\infty \frac{g(x) - cx}{x^2} dx$

g wächst monoton

Ann $g(y) \ge (c+2d) y$ $\qquad$ y gross

$g(x) \ge (c+d) x$ $\qquad x < \underbrace{\left(\frac{c+2d}{c+d}\right)}_{\xi > 1} y$

$\int_y^{\xi y} \frac{g(x) - cx}{x^2} dx > \int_y^{\xi y} \frac{d}{x} dx = d \lg \xi > 0$

$\Longrightarrow \int_0^\infty f(t)$ würde diverg.! ↯

Ann $g(y) < (c-2d) y$ $\qquad \frac{c-2d}{c-d}$

$\dots \Longrightarrow \int_{\vartheta y}^{y} \frac{d}{x} dx = d \lg \vartheta < 0$ $\qquad \vartheta < 1$

$\Longrightarrow \int_0^\infty f(t)$ würde diverg. ↯

4. $s \int_1^\infty \psi(x)\, x^{-s-1} dx = s \cdot \sum_{n \ge 1} \int_n^{n+1} \psi(x)\, x^{-s-1} dx$

$= s \cdot \sum_{n \ge 1} \psi(n) \frac{1}{s}\left[\frac{1}{n^s} - \frac{1}{(n+1)^s}\right]$

$= \sum_{n \ge 2} \underbrace{\left[\psi(n) - \psi(n-1)\right]}_{\lg p \quad n = p^m} \frac{1}{n^s} = \sum_p \left[p^{-s} + (p^{-s})^2 \dots\right] l$

$$= \sum_p \frac{p^{-s}}{1-p^{-s}} \ln p$$

$$-\frac{d}{ds} \ln \zeta(s) \underset{\substack{\uparrow \\ \text{Prod.darst}}}{=} + \sum_p \frac{p^{-s}}{1-p^{-s}} \ln p \qquad \operatorname{Re} s > 1$$

$$-\frac{d}{ds} \lg \zeta(s) = -\frac{\zeta'(s)}{\zeta(s)}$$

$$\zeta(s) = \frac{1}{s-1} + a(s) \qquad a(s) \text{ analyt.}$$

$$\zeta'(s) = -\frac{1}{(s-1)^2} + a'(s)$$

d.h. $\quad -\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{s-1}$ analytisch in Umgeb. von 1

d.h. nahen Situation von Corollar

$$\Rightarrow \frac{\psi(x)}{x} \xrightarrow{x\to\infty} 1$$

Eine Extremaleigenschaft der Riemannschen ζ Funktion

Berllin ~1950

$\Lambda := \{ (\lambda_n)_{n \in \mathbb{Z}_+},\quad k \leq l \Rightarrow 0 < \lambda_k \leq \lambda_l \}$

$\Lambda_+ := \{ \lambda \in \Lambda : \inf\limits_{n \in \mathbb{Z}_+} \frac{\lambda_n}{n} > 0 \}$

Für $\lambda \in \Lambda$ definiere $\quad \zeta_\lambda(s) := \sum\limits_{n \in \mathbb{Z}_+} \frac{1}{\lambda_n^s} = \zeta(s)$

$$f(z) = f_\lambda(z) := \prod_{n \in \mathbb{Z}_+} \left(1 + \frac{z^2}{\lambda_n^2}\right)$$

Für $\lambda \in \Lambda_+$ konvergiert $\zeta(s)$ für $\sigma > 1$ und f_λ ist ganze Funktion vom Exponentialtypus

$(f(\lambda) = O(e^{A|z|}))$

$\Gamma \quad \delta := \inf\limits_{n \in \mathbb{Z}_+} \frac{\lambda_n}{n} > 0$

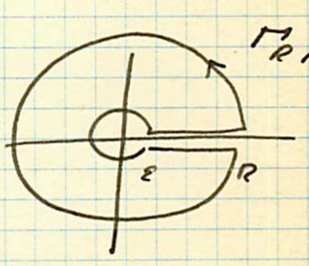
$$|f(z)| \leq \prod_{n \in \mathbb{Z}_+} \left(1 + \frac{|z|^2}{\lambda_n^2}\right) \leq \prod_{n \in \mathbb{Z}_+} \left(1 + \left(\frac{\frac{|z|}{\delta}}{n}\right)^2\right) = \frac{\sinh \pi \frac{|z|}{\delta}}{\pi \frac{|z|}{\delta}}$$

$$< C e^{A|z|} \quad \lrcorner$$

$\lambda \in \Lambda_+ \quad$ Für $1 < \sigma < 2$ betrachte $\displaystyle\int_0^\infty \log f(x) \frac{dx}{x^{1+s}}$

$$= \underbrace{-\frac{1}{s} \frac{1}{x^s} \log f(x) \Big|_0^\infty}_{0} + \frac{1}{s} \int_0^\infty \frac{1}{x^s}\, d \log f(x)$$

$x = \infty$: f Exponentialtypus $\sigma > 1$
$x = 0$: f gerade $\sigma < 2$



Betrachte folgenden Integrationsweg

$$\lim_{\substack{R \to \infty \\ \varepsilon \to 0}} \int_{\Gamma_{R\varepsilon}} \frac{1}{x^s}\, d\log f(x) = \int_0^\infty \frac{1}{x^s}\, d \log f(x)$$

$$+ \int_\infty^0 \frac{1}{x^s e^{2\pi i s}}\, d \log f(x)$$

$$= \int_0^\infty \frac{1}{x^s}\, d \log f(x) \left[\frac{e^{2\pi i s} - 1}{e^{2\pi i s}} \right]$$

andererseits

$$\lim_{\substack{R \to \infty \\ \varepsilon \to 0}} \int_{\Gamma_{R\varepsilon}} \dots = 2\pi i \sum_{n \in \mathbb{Z}_+} \left(\frac{1}{\lambda_n^s e^{\frac{\pi i s}{2}}} + \frac{1}{\lambda_n^s e^{\frac{3\pi i s}{2}}} \right)$$

$$= 2\pi i\, \zeta(s) \frac{e^{\pi i s} + 1}{e^{3\pi i s/2}}$$

zusammen: $$\boxed{\int_0^\infty \log f(x) \frac{dx}{x^{1+s}} = \frac{1}{s}\, 2\pi i\, \zeta(s) \frac{\frac{e^{\pi i s}+1}{e^{\frac{3\pi i s}{2}}}}{\frac{e^{2\pi i s} - 1}{e^{2\pi i s}}} = \frac{\zeta(s)}{s} \pi \frac{2i e}{e^{\pi i s} - 1}}$$

$$= \frac{\pi \zeta(s)}{s \sin \frac{\pi s}{2}}$$

$k > 0$

$C_k := \{ \varphi_\lambda : \lambda \in \Lambda \quad \sigma > 1 \Rightarrow |\varphi(s)| < \infty$; so dass (α),(β),(γ)}

(α) $\varphi(s) - \frac{1}{s-1}$ ganz

(β) $\varphi(-2\mathbb{Z}_+) = 0$

(γ) $|\varphi(s)| \leq \text{const} \frac{\Gamma(|s|)}{(2\pi k)^{|s|}}$

$D_k := \bigcap_{0<\varepsilon<k} (C_{k-\varepsilon} - C_{k+\varepsilon})$

Satz

$$\# D_k = \begin{cases} \infty & \text{falls } 0 < k < \frac{1}{2} \\ 2 & \text{falls } k = 1 \\ 0 & \text{sonst} \end{cases}$$

und $D_1 = \{ \varphi_{\mathbb{Z}_+}, \varphi_{\mathbb{Z}_+ - \frac{1}{2}} \}$

$\varphi_{\mathbb{Z}_+} = \zeta = \zeta(\cdot, 1) \qquad s \to \zeta(s)$

$\varphi_{\mathbb{Z}_+ - \frac{1}{2}} = \zeta(\cdot, \frac{1}{2}) \qquad s \mapsto (2^s - 1)\zeta(s)$

Lemma

$\lambda \in \Lambda \quad k > 0$

$\varphi_\lambda \in D_k \iff \exists$ reelle Zahlen $a > 0$ und p.

so dass die Funkt. f_λ d. Bed. $f_\lambda(x) = a|x|^p e^{\pi|x|}$

$+ O(e^{\pi(1+\varepsilon-2k)|x|})$

$|x| \to \infty \quad x \in \mathbb{R}$

genügt für $\varepsilon > 0$ aber nicht für $\varepsilon < 0$

Bem $0 < k < \frac{1}{2} \Rightarrow \# D_k = \infty$

Bsp. $\cosh \pi k z \cosh \pi(1-k) z \; (1 + \frac{z^2}{t})$ $0 < t$

d.h. 1. Teil ✓

Prop.

f ganze Funkt. von Exponentialtypus mit $f(0) = 1$ und es seien gegeben reelle Zahlen $a > 0, p, d > 0$, so dass $f(x) - a|x|^p e^{\pi|x|} = O(e^{-d|x|})$ $|x| \to \infty$ $x \in \mathbb{R}$

dann ist entweder $f(z) = \frac{e^{\pi z} - e^{-\pi z}}{2\pi z}$

$= \prod_{n \in \mathbb{Z}_+} (1 + \frac{z^2}{n^2})$

oder $f(z) = \frac{e^{\pi z} + e^{-\pi z}}{2} = \prod_{n \in \mathbb{Z}_+} \left(1 + \frac{z^2}{(n - \frac{1}{2})^2}\right)$

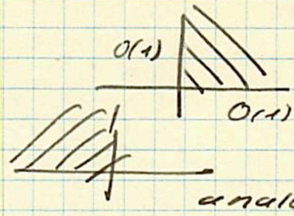
Γ $h|_{\mathrm{Re} z > 0}(z) = f(z) - a z^p e^{\pi z}$

$h|_{\mathrm{Re} z < 0}(z) = f(z) - a(-z)^p e^{-\pi z}$

Bem $\boxed{\begin{array}{ll} h \in \Theta(H) \cap C(\bar{H}) & \\ h(z) = O(e^{A|z|}) & \text{Expon. typus} \\ h(z) = O(e^{-d|x|}) & (|x| \to \infty \quad x \in \mathbb{R}) \\ \text{Dann} \quad h = 0 & \end{array}}$

Γ

$h(z)\, e^{(d+iA)z}$



P.L. $\Rightarrow |h(z)| \le C e^{\,r(-d\cos\theta + A\sin\theta)}$

$0 \le \theta \le \frac{\pi}{2}$

$h(z) = e^{(-d+iA)z}$

analog $\frac{\pi}{2} \le \theta \le \pi$

zusammen $|h(re^{i\theta})| \le C e^{\,r(-d|\cos\theta| + A\sin\theta)}$

$B > 0$ $\quad |h(z)e^{-iBz}| \le C\, e^{\,r(-d|\cos\theta| + (A+B)\sin\theta)}$

wähle θ so dass $\tan\theta = \frac{d}{A+B}$

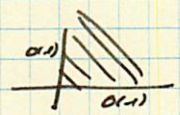
$\overset{P.L.}{\Rightarrow} \quad |h(re^{i\theta})| \le C e^{-r\sin\theta} \qquad 0 \le \theta \le \pi$

$B \to \infty$ $\quad \lrcorner$

$f(x) - f(-x) = h(x) - h(-x) = O(e^{-d|x|}) \qquad (|x| \to \infty \quad x \in \mathbb{R})$

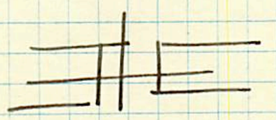
Bem $\quad f(z) - f(-z) = 0$

$h(z)\, e^{(d+iA)z}$



$h(z) = O(e^{Ay - dx}) \qquad \begin{array}{l} x \ge 1 \\ y \ge 0 \end{array}$

analog



$\{|x| \ge \delta\} \cap \{|y| \le 1\}$

$h(x) = O(e^{-d|x|}) \quad (|x| \to \infty)$

wegen $\quad h^{n}(x) = \int_{|z-x|=1} \frac{h(z)}{(z-x)^{n+1}}\, dz\, \frac{n!}{2\pi i}$

$h^{n}(x) = O(e^{-d|x|})$

Betrachte linearen Diff op. $\quad L := x^2\frac{d^2}{dx^2} - 2px\frac{d}{dx} - (\pi^2x^2 - p$

$(Lf)(x) = Lh(x) = O(e^{-d|x|}x^2) \quad (|x| \to \infty)$

da $|x|^p e^{-\pi|x|}$, $|x|^p e^{+\pi x}$ Lösungen von $Lu = 0$ sind

Nach obiger Bem: $\quad Lf = 0$

$f(z) = x^p(c_1 e^{\pi x} + c_2 e^{-\pi x})$

$f \in \Theta(\mathbb{C}) \quad f \not\equiv 0 \Rightarrow p \in \mathbb{Z}$

f gerade $\quad |c_1| = |c_2| =: c$

entweder $\quad c_1 = -c_2 \quad \overset{f(0)=1}{\Longrightarrow} \quad p = -1 \quad c = \frac{1}{2\pi}$

oder $\quad c_1 = +c_2 \quad \Longrightarrow \quad p = 0 \quad c = \frac{1}{2}$ $\quad \lrcorner$

($p = 2\zeta(0)$
$a = e^{2\zeta'(0)}$
$\zeta(0) = -\frac{1}{2}$
$\zeta'(0) = -\frac{1}{2}\log 2\pi$

Bew lemma $\quad \varphi_\lambda \in C_H \overset{(\alpha)!}{\Longrightarrow} \lambda \in \Lambda_+ \quad$ (Übung $\varphi_\lambda - \xi$)

$\varphi_\lambda \in C_H$ Aufgrund d. Eigenschaft. (α), (β) und mit

$$\int_0^\infty \log f(x) \frac{dx}{x^{1+s}} = \frac{\pi k \varphi(s)}{s \sin\frac{\pi s}{2}} =: \psi(s)$$

folgt. $\psi \in \mathcal{O}(\sigma < 2,\ s \notin \{0, 1\})$ mit

Hauptteilen in $s = 1$ und $s = 0$:

$$\boxed{\frac{\pi}{s-1}} \quad \text{und} \quad \operatorname*{res}_{s=0} \frac{\pi \varphi(s) s^k}{s \sin\frac{\pi s}{2}} = \begin{cases} 0 & k \geq 2 \\ \frac{\pi \varphi(0)}{\pi/2} = 2\varphi(0) & k = 1 \\ \lim_{s\to 0}\left(\frac{\pi \varphi(s)}{s \sin\frac{\pi s}{2}} - \frac{2\varphi(0)}{s^2}\right) s = \end{cases}$$

$$= \lim_{s \to 0} \frac{\pi \varphi(s) s - 2\varphi(0) \sin\frac{\pi s}{2}}{s \cdot \sin\frac{\pi s}{2}} \qquad \frac{\pi}{2}\cos\frac{\pi s}{2}$$

$$\boxed{\frac{2\varphi(0)}{s^2} + \frac{2\varphi'(0)}{s}} \qquad = \lim_{s\to 0} \frac{\pi \varphi'(s) s + \pi \varphi(s) - 2\varphi(0)}{\sin\frac{\pi s}{2} + \frac{s\pi}{2} \cos\frac{\pi s}{2}}$$

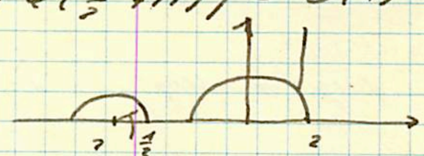
$$= \frac{2\pi \varphi'(0)}{2\frac{\pi}{2}} = 2\varphi'(0)$$

$\psi_0(s) := \Gamma(2-s)\,(2\pi k)^s$

Mit den Ungl. $\quad |\Gamma(\tfrac{1}{2} - it)| \geq \sqrt{\pi}\, e^{-\frac{|t|}{2}\pi}$

$\quad |\varphi(\tfrac{3}{2} + it)| = O(1) \quad (|t| \to \infty)$

und $|\gamma|$



~~$\left|\frac{\psi}{\psi_0}\right| (\tfrac{3}{2} + it) =$~~

$s \in B_{\frac{1}{2}}(-2n) \quad n \in \mathbb{Z}_+$

$$|\varphi(s)| \leq \underbrace{\varphi(-2n)}_{0} + \int_{-2n}^{s} |\varphi'(s)|\,|ds|$$

$$|\varphi(s)| \leq |s + 2n| \max_{s \in B_{\frac{1}{2}}(-2n)} |\varphi'(s)| \leq |s + 2n| \int_{|\cdot + 2n| = 1} \frac{|\varphi(s)|}{|\cdot|^2} \frac{1}{2\pi}$$

$$|\psi(s)| \leq \text{const}\, |\Gamma(2-s)(2\pi k)^s|$$

Beweisen Ungl.

$$\int_{-\infty}^{\infty} |\Gamma(2 + \sigma - it)|\, dt \leq \text{const}\ \sigma^3 \Gamma(\sigma) \qquad \sigma \geq 2 \qquad \text{(Stirling)}$$

$$\int_{-\infty}^{\infty} |\psi(-\sigma + it)|\, dt \leq \text{const}\ \frac{\sigma^3 \Gamma(\sigma)}{(2\pi k)^\sigma} \qquad \sigma > 2$$

$$\psi(\sigma + it) = \int_0^\infty \log f(x) \frac{dx}{x^{1+\sigma+it}} = \int_{-\infty}^{\infty} \log f(e^x) \frac{de^x}{e^{x(1+\sigma+it)}} =$$

$$= \int_{-\infty}^{\infty} \log f(e^x)\, e^{-\sigma x} e^{-ixt}$$

$$\log f(e^x) e^{-\sigma x} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \cancel{\log f(x)\, e^{ixt}}\ \psi(\sigma + it)\, e^{ixt}\, dt$$

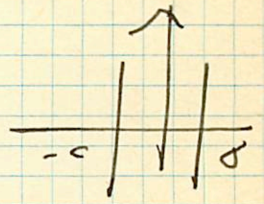
$$\log f(x) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \psi(s)\, x^s \, dit = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \psi(s) x^s ds \quad (1<\sigma$$

$$\psi(s) = \frac{\pi}{s-1} + O(1) \quad (s \to 1)$$

$$\psi(s) = \frac{2\varphi(0)}{s^2} + \frac{2\varphi'(0)}{s} + O(1) \quad s \to 0$$

$$x^s = e^{s\log x} = 1 + s\log x + O(|s|^2) \qquad (s \to 0)$$

$$x^s = x\, x^{s-1} = x + O(|s-1|) \qquad (s \to 1)$$



$$\operatorname{res}_{s=1} (\psi(s) x^s = \pi x$$

$$\operatorname{res}_{s=0} (\psi(s) x^s = 2\varphi'(0) + 2\varphi(0)\log x \qquad p := 2\varphi(0), \quad a := e^{2\varphi'(0)}$$

also $\cancel{f}$ $\log f(x) - \log a x^p e^{\pi x} = \frac{1}{2\pi i}\int_{-c-i\infty}^{c+i\infty} \psi(s) x^s ds \qquad c>0$

$$|\log f(x) - \log a x^p e^{\pi x}| = O\left(\frac{c^3\Gamma(c)}{(2\pi k x)^c}\right) \qquad (c \to \infty)$$

$$c := 2\pi k x \qquad x \to \infty$$

$$|\log f(x) - \log a x^p e^{\pi x}| = O\left(\frac{c^3 c^{c-\frac{1}{2}} e^{-c}}{c^c}\Big|_{c=2\pi k x}\right) = O(e^{-2\pi(k-} \quad x \to \infty$$

Benützen folgende Ungl $\quad 1-x \le e^{-x}, \quad e^x \le 1+2x \quad 0<x<d>0 \quad \forall\delta$

$$-\text{const}\, e^{-2\pi(k-\delta)x} \le \log\frac{f(x)}{a x^p e^{\pi x}} \le \text{const}\, e^{-2\pi(k-\delta)x}$$

$$1 - \text{const}\, e^{-2\pi(k-\delta)x} \le \frac{f(x)}{a x^p e^{\pi x}} \le 1 + \text{const}\, e^{-2\pi(k-\delta)x}$$

erhalten $\quad f(x) = a|x|^p e^{\pi|x|} + O(e^{\pi(1+\delta-2k)|x|}) \quad (x \to \infty)$ x gerade

Eine Richtung d. Lemmas

für Umkehrung:

$$f(z) = 1 + \varphi(2) z^2 + \frac{\varphi(2)^2 - \varphi(4)}{2} z^4 + \ldots$$

Methode von Eul
für reziproke Quadrate