SOME SUMMATION IDENTITIES

MONISHWARAN MAHESWARAN

ABSTRACT. Most of these identities found in the summer of 2017 while attending the Harvard Summer school.

Acknowledgement: thanks to Oliver Knill for several discussions and assistance in writing this down.

1. A sum which evaluates to an integer

Theorem 1. *For any polynomial p with integer coefficients, the sum*

$$f(p) = \sum_{k=0} p(k)/2^k$$

is an integer.

Proof. Summation by parts

$$\sum_{k=0}^{n} f_k[g_{k+1} - g(k)] = [f_{n+1}g_{n+1} - f_0 g_0] - \sum_{k=0}^{n} g_{k+1}[f_{k+1} - f_k]$$

for $g_k = 2^{-k}$ and $f_k = p(k)$ gives in the limit $n \to \infty$ the formula

$$-\sum_{k=0}^{\infty} p(k)/2^{k+1} = -p(0) - \sum_{k=0}^{\infty} (p(k+1) - p(k))/2^{k+1} \ .$$

Multiply by -2 to get

$$\sum_{k=0}^{\infty} p(k)/2^k = 2p(0) + \sum_{k=0}^{\infty} q(k)/2^k \ ,$$

where $q(k) = p(k+1) - p(k)$ is a polynomial of degree $n - 1$. By induction on the degree of the polynomial using that for degree polynomials p the result is 1 proves the result. □

Remark. My own proof was more complicated and got simplified by Oliver Knill using summation by parts. The sequence $a_n = f((1+x)^n)$ is known as the sequence A207047. The sequence $b_n = f(x^n)$ is the sequence A002050. It counts the number of simplices in a Barycentric subdivision of a n simplex.

Date: July 20, 2017, Last update August 20, 2017.

2. More sums which evaluate to an integer

Theorem 2. *For every m and n, the sum*

$$f(m,n) = \frac{1}{e}\sum_{x=0}^{\infty}\frac{(x^n+1)^n}{x!}$$

is an integer

Proof. Define $g(y) = (1/e)\sum_{x=0}^{\infty}(x^m+y)^n/x!$ and use induction, by differentiation with respect to y. For example, $f(0,n) = 2^n$ and $f(m,0) = 1$. □

3. A sum with a factorial

Theorem 3.

$$\sum_{x=1}^{\infty} 1/(2^x x!) = e^{1/2} - 1$$

Proof. Use Taylor $e^y = \sum_{x=0}^{\infty} y^x/x!$. □

4. A sum leading to a integer polynomial

Theorem 4. $\frac{1}{e}\sum_{x=0}^{\infty}(x+y)^k/x!$ *is an integer polynomial in y.*

Proof. Use induction with respect to k. A special interesting case is $y = i$. □

5. A sum of rational functions

Theorem 5.

$$\sum_{n=0}^{\infty}\frac{1}{n^2+n+k} = \pi\frac{\tan(\sqrt{1-4k}\pi/2)}{\sqrt{1-4k}}$$

Not yet done.

Theorem 6.

$$\sum_{n=0}^{\infty}\frac{1}{n(2n+1)(2n-1)} = \log(4) - 1$$

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{n(2n+1)(2n-1)} = \log(2) - 1$$

Proof. Make a Taylor series

$$\sum_{n=0}^{\infty}\frac{x^n}{n(2n+1)(2n-1)} = \log(1-x) + (1+x)\mathrm{arctanh}(\sqrt{x})/\sqrt{x}$$

The case $x \to 1$ is a limit. □

6. A sum which evaluates to a logarithm

Theorem 7.
$$\sum_{k=1}^{\infty} \frac{n^{-nk}}{nk} = -\log(1-n^{-n})/n$$

Proof. It follows from the more general identity
$$\sum_{k=1}^{n} \frac{x^{nk}}{nk} = -\log(1-x^n)/n$$
from $x = 1/n$. The identity for a function is obtained by differentiation with respect to x:
$$\sum_{k=1}^{\infty} x^{nk} = x^{n-1}/(1-x^n)$$
which is a geometric sum. □

Remark. This example shows how it can be useful to generalize the problem. It has become easier by solving a more general sum.

7. A series with a Binomial

Theorem 8. $\sum_{k=1}^{n}(-1)^k \sin(ak) \left(\begin{array}{c} n \\ k \end{array} \right) = (-2)^n \sin(a/2)^n \sin((an+n\pi)/2)$

Proof. This follows from the geometric series
$$\sum_{k=1}^{n} e^{-iak} \left(\begin{array}{c} n \\ k \end{array} \right) = (1+e^{-ia})^n - 1$$
□

8. A curious ArcCot sum

Theorem 9. $\cot(\sum_{k=1}^{\infty} \operatorname{arccot}((1+k)^2/2) = 1/3$

Remark. This is not yet proven. Originally, I wrote it as $\cot[\sum_n \operatorname{arccot}((2/n^2)\sum_{k=1}^{n} k^3)$ but this simplifies to the above sum. One brutal way to attack the problem is to write arccot as a Taylor series $\operatorname{arccot}(x) = \pi/4 + (x-1)/2 + (x-1)^2/4...$ then sum each term.

9. A Fibonacci sum

Theorem 10. *If $F(k)$ is the k'th Fibonacci number, then* $\sum_{k=1}^{\infty} F(k)/2^k = 2$

Proof. The direct proof can be done using Binet's formula $F(k) = \phi^k - (-1/\phi)^k)/\sqrt{5}$, where $\phi = (1+\sqrt{5})/2$ is the golden ratio.
An other proof can be done by proving a more general formula
$$\sum_{k=1}^{\infty} F(k)x^k = x/(x^2+x-1)$$
and applying it for $x = 1/2$. The formula can be obtained by multiplying both sides with x^2+x+1 and using the identity $F(k+1) = F(k) + F(k-1)$. □

Added August 20:

Theorem 11. *If $F(k)$ is the k'th Fibonacci number, then* $\sum_{k=1}^{\infty} \frac{2k+F(k)}{3^k} = \frac{21}{20}$.

Proof. Add

$$\sum_{k=1}^{\infty} F(k)x^k = -x/(x^2+x-1)$$

and

$$\sum_{k=1}^{\infty} 2kx^k = 2x/(x-1)^2$$

to get

$$\frac{x\left(x^2+4x-3\right)}{(x-1)^2\left(x^2+x-1\right)}.$$

Evaluated at $1/3$ this gives $21/20$. □

Remarks.
1) More generally, along the same lines, one has

$$\sum_{k=1}^{\infty}(nk+F(k))/p^k = p\left(\frac{n}{(p-1)^2}+\frac{1}{p^2-p-1}\right)$$

2) Other identities are

$$\sum_{k=1}^{\infty} kF(k)/2^k = 10$$

$$\sum_{k=1}^{\infty} F(k)^2/3^k = 3/2$$

Theorem 12. *If $F(k)$ is the k'th Fibonacci number and $L(k)$ is the k'th Lucas number, then* $\sum_{k=1}^{\infty} \frac{F(k)L(k)}{3^k} = 3$.

Proof. Since $L(k)L(k) = F(2k)$, this is equivalent to

$$\sum_{k=1}^{\infty} F(2k)/3^k = 3$$

which is solved similarly computing the generating function

$$f(x) = \sum_{k=1}^{\infty} F(2k)x^k$$

and setting $x = 1/3$. The function $f(x)$ is $-\frac{2(5+3\sqrt{5})x}{\sqrt{5}(-2x+\sqrt{5}+3)((3+\sqrt{5})x-2)}$. One can get this by noticing that $G(n) = F(2n)$ is A001906 which has the recursion $G(n) = 3 * G(n-1) - G(n-2)$. □

10. Summation of a trigonometric function

Theorem 13. *With $p(x) = \sin^3(x)$ the following identity holds:*

$$S(n) = \sum_{k=0}^{n} \frac{p(3^k)}{3^k} = \frac{\sin(3)-\sin(3^{n+1})}{4\cdot 3^n}\ .$$

Proof. Start with the identity

$$\sin(3a) = 3\sin(a) - 4\sin^3(a)$$

to get

$$(3\sin(a) - \sin(3a))/4 = \sin^3(a) \ .$$

Now use this with $a = 3^k$:

$$\begin{aligned} S(n) &= \sum_{k=0}^{n} \sin^3(k)/3^k \\ &= \sum_{k=0}^{n} [3\sin(3^k) - \sin(3^{k+1})/4]/3^k \\ &= \frac{3}{4}[\sum_{k=0}^{n} \sin(3^k)/3^k - \sum_{k=0}^{n} \sin(3^{k+1})/3^{k+1}] \ . \end{aligned}$$

This can be written as a telescopic sum:

$$\frac{3}{4}([\frac{\sin(3^0)}{3^0} - \frac{\sin(3^2)}{3^2}] + [\frac{\sin(3^1)}{3^1} - \frac{\sin(3^2)}{3^2}] + \cdots + [\frac{\sin(3^n)}{3^n} - \frac{\sin(3^{n+1})}{3^{n+1}}]) \ .$$

This reduces to $(1/4)[\sin(3) - \sin(3^{n+1})]/3^n$. □

Remark. This looks first like a geometric series in disguise. But it is not. It is interesting however that if we replace 3 with some other integer, it does not work. The summation also does not work if sin is replaced by cos. Nor does it if we take $p(x) = \exp(x)$ or $\exp(ix)$.

11. More trigonometric sums

[This was updated August 25, 2017].

Theorem 14. $\sum_{x=1}^{n} \sin(a + bx) = \csc\left(\frac{b}{2}\right)\sin\left(\frac{bn}{2}\right)\sin\left(\frac{1}{2}(2a + bn + b)\right)$.

Proof. Write $\sin(a + bx) = \mathrm{Im}(\exp(i(a + bx))$ and use a geometric series. □

Theorem 15. $\sum_{x=1}^{\infty} \sin(x)/x = (1 + \pi)/2$.

Proof. (With the assistance of Oliver Knill). This is a Fourier series in disguise. Replace first x by n to get $\sum_{n=1}^{\infty} \sin(n)/n = (1 + \pi)/2$. Then fill in more generally a variable x in $\sum_{n=1}^{\infty} \sin(nx)/n$ so that we have a Fourier series with coefficients $b_n = 1/n$. Now write $\sin(nx) = Im(e^{inx})$ and use $z = e^{ix}$ to get the imaginary part of $\sum_{n=1}^{\infty} z^n/n = -\log(1 - z)$. We can use this go get an explicit expression $\sum_{n=1}^{\infty} \sin(nx)/n = \frac{1}{2}i\left(\log\left(1 - e^{ix}\right) - \log\left(1 - e^{-ix}\right)\right)$. To get the original sum, just fill in $x = 1$. □

Theorem 16. $\sum_{x=1}^{\infty} \sin(x)/x! = \sin(\sin(1))(\sinh(\cos(1)) + \cosh(\cos(1))$.

Proof. This is similar as before, we deal with the Fourier series $\sum_{n=1}^{\infty} \sin(nx)/n!$. And use $\sum_{n=1}^{\infty} z^n/n! = \exp(z)$ with $z = \exp(ix)$. This gives the explicit formula $\sum_{n=1}^{\infty} \sin(nx)/n! = \sin(\sin(x))(\sinh(\cos(x)) + \cosh(\cos(x)))$. □

12. Leibniz like sums

Theorem 17. $\sum_{k=0}^{\infty}(-1)^k/(3k+1) = (\sqrt{3}\pi + \log(8))/9$.

Proof. (With the assistance of Oliver Knill): Similarly as the sum $\sum_{k=0}^{\infty}(-1)^k/(1+2k) = \pi/4$ follows from making the Fourier expansion of $\operatorname{sign}(x)$, one can take the series

$$f(x) = \sum_{k=0}^{\infty} \sin(1+2k)/(1+3k)$$

and evaluate it at $\pi/2$. But the function does not appear to be elementary. Mathematica gives expressions using Hypergeometric functions. Evaluating it at $\pi/2$ is however simple. □

13. Involving the zeta function

These were added September 1, 2017 $\zeta(n)$ is the **Riemann zeta function**. $\Gamma_k(x)$ is the k'th **Poly gamma function**, the k'th derivative of the **Digamma function** $\Gamma_0(x) = \Gamma'(x)/\Gamma(x)$. $\gamma = 0.5777216...$ is the **Euler Gamma** function.

Theorem 18. $\sum_{n=2}^{\infty} n\zeta(n+1)/2^n = \pi^2/6$.

Proof. $\sum_{n=2}^{\infty} n\zeta(n+1)x^n = x\Gamma_1(1-x)$. Now plug in $x = 1/2$. □

Theorem 19. $\sum_{n=2}^{\infty} \zeta(n)/2^n = \log(2)$.
$\sum_{n=2}^{\infty}(-1)^k\zeta(n)/2^n = 1 - \log(2)$.
$\sum_{n=2}^{\infty}(1+(-1)^n)\zeta(n)/2^n = 1$

Proof. $\sum_{n=2}^{\infty} \zeta(n)x^n = -x\gamma - x\Gamma_0(1-x)$. Now plug in $x = 1/2$ or $x = -1/2$. Addition gives the third. The third one is equivalent to

$$\sum_{n=1}^{\infty} \zeta(2n)/4^n = 1/2 \ .$$

□

Remark (by Oliver Knill): one can write this out as

$$\sum_{n=2}^{\infty}\sum_{m=1}^{\infty} \frac{1}{m^n 2^n} = \sum_{n=2}^{\infty}\sum_{m=1}^{\infty} \frac{1}{(2m)^n} = \sum_{n=2}^{\infty}[\zeta(n) - \lambda(n)] \ ,$$

where λ is the Dirichlet λ-function, a special Dirichlet L-series. So, one has

$$\sum_{n=2}^{\infty}(\zeta(n) - \lambda(n)) = \log(2)$$

Generalizing gives

$$\sum_{n=2}^{\infty}(\zeta(n) - \lambda(n))x^n = (-x/2)(\gamma - \Gamma_0(1-x/2))$$

14. Involving the Gamma function

Theorem 20. $\sum_{n=1}^{\infty}(\Gamma'(n)/\Gamma(n))/2^n = \log(2) - \gamma$.

Proof. $\sum_n(\Gamma'(n)/\Gamma(n))x^n = \frac{x}{x-1}[\gamma + \log(1-x)]$ □

15. Double summation

This example was added September 8, 2017:

Theorem 21. $\sum_{n=1}^{\infty} \sum_{m=1}^{n} \frac{1}{(nm)^2} = \frac{7\pi^4}{360} = 7\zeta(4)/4$.

No justification yet. Related is $\sum_{n=1}^{\infty} \sum_{m=1}^{n} \frac{1}{((n+1)m)^2} = \frac{\pi^4}{120}$. There might be a relation also with $\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{nm)^2} = (\sum_{n=1}^{\infty} \frac{1}{n^2})^2 = \zeta(2)^2 = \pi^4/36$.

16. Polylog sums which evaluate to an integer

This example was added September 15, 2018:

Theorem 22. *For any integer k the sum*

$$\sum_{n=1}^{\infty} n^{k-1} \frac{(x-1)^k}{x^n}$$

is an integer polynomial.

For example, for $k = 15, x = 3$, it evaluates to 696933753434112.

Proof. For $k = 1$, the result is 1 as it reduces to a geometric series. One can now use induction. If k is increased by 1, and integrated with respect to x we get a polynomial by induction □

Remark. With the polylog $\text{Poly}(k, x) = \sum_{n=1}^{\infty} x^n/n^k$ one can write $\text{Poly}(-k, 1/x) = \sum_{n=1}^{\infty} n^k/x^n$. Like the logarithm $\log(x) = \text{Poly}(1, 1-1/x)$ also the Polylog has the property $\text{Poly}(-k, x) = -\text{Poly}(-k, 1/x)$. One has therefore the equality of polynomials

$$\text{Poly}(-k, x)(1-x)^{k+1} = -\text{Poly}(-k, 1/x)(1-x)^{k+1} .$$

The statement is equivalent that $\text{Poly}(-k, x)(1-x)^{k+1}$ is a polynomial.

17. Two summands

If $[x]$ is the floor function, then $[e^\pi] - [\pi^e] = 1$ and $[e^\pi - \pi^e] = 0$.

E-mail address: `mom359@g.harvard.edu`

Harvard Summer school