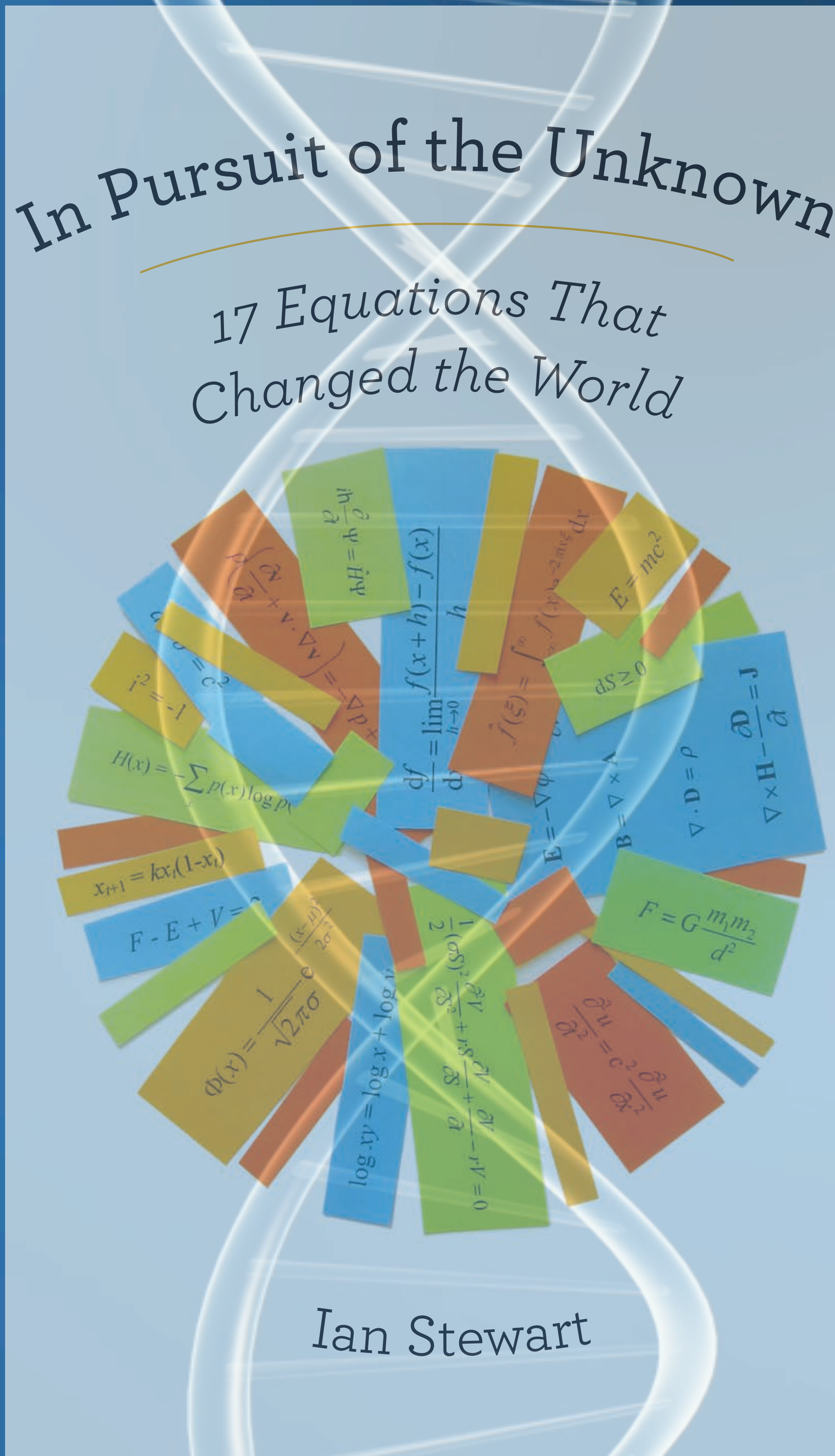
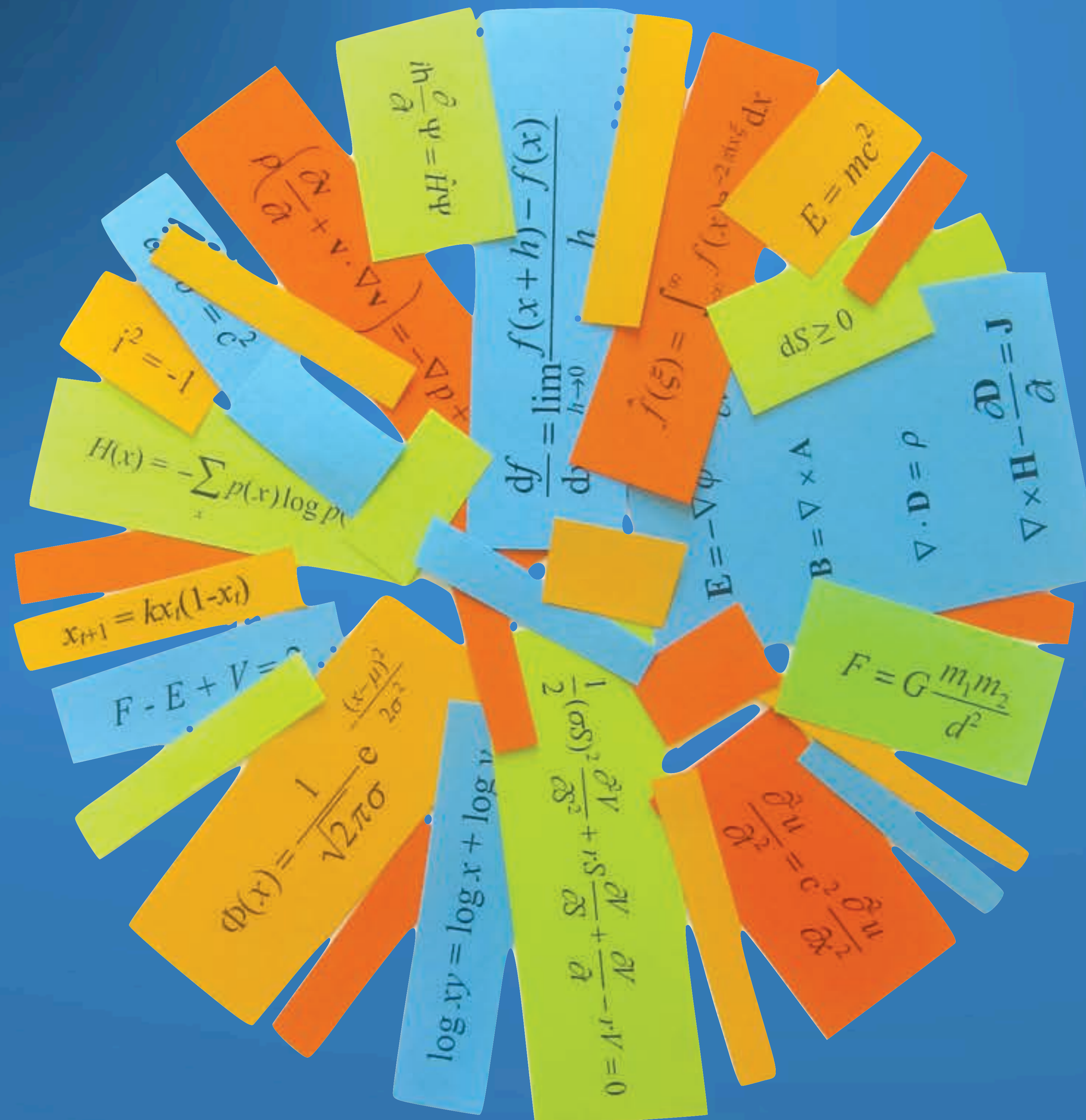
Partial Differential Equations

Oliver Knill,
Harvard University
October 7, 2019


In Pursuit of the Unknown
17 Equations That Changed the World
Ian Stewart

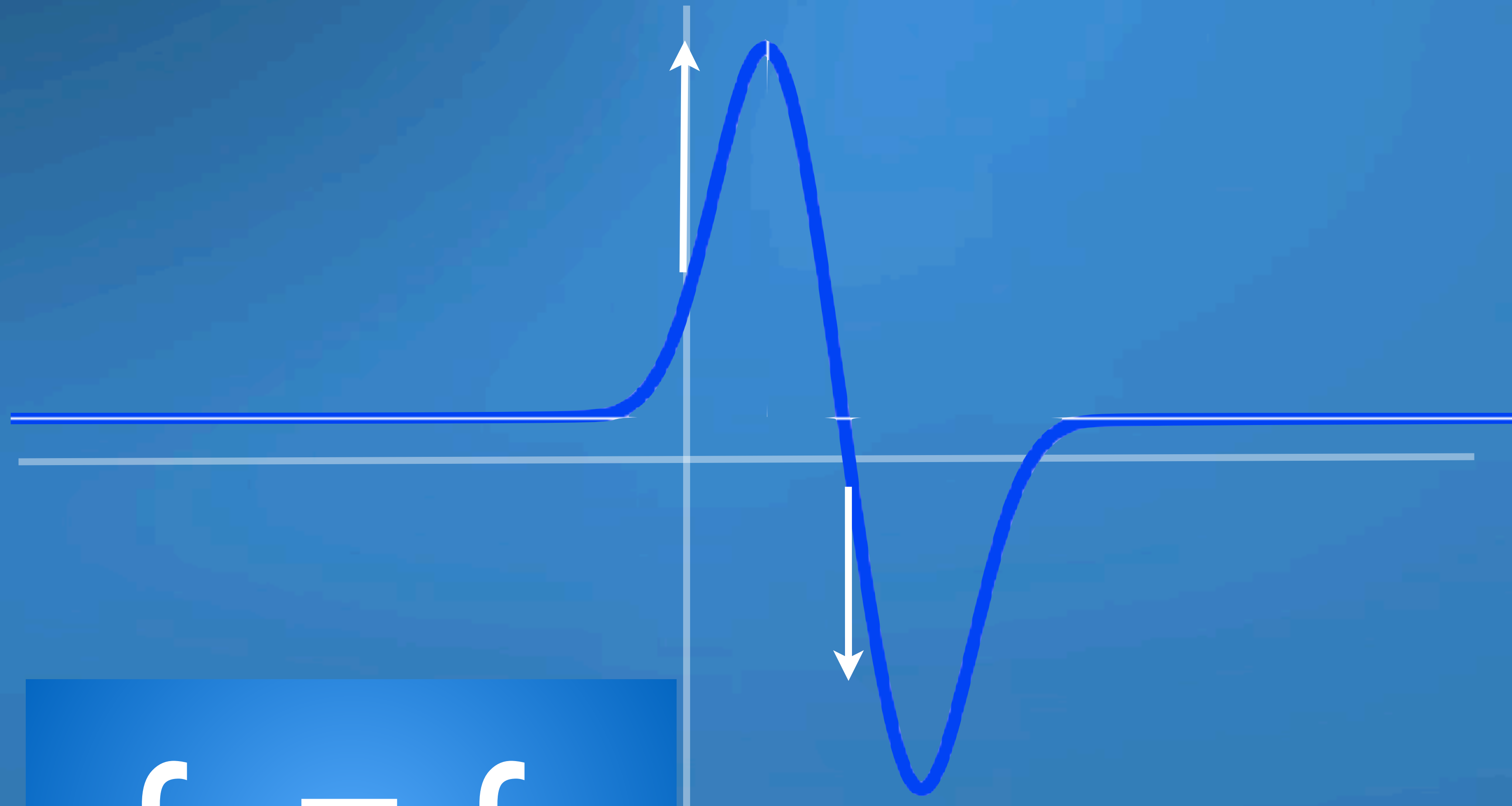
Ian Stewart


$E = mc^2$
$i^2 = -1$
$dS \geq 0$
$x_{t+1} = kx_t(1-x_t)$
$F = G\frac{m_1 m_2}{d^2}$
$\nabla \cdot \mathbf{D} = \rho$
$\Phi(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

Transport Equation

$$f_t = f_x$$

- Signal processing
- Advection
- Traveling waves

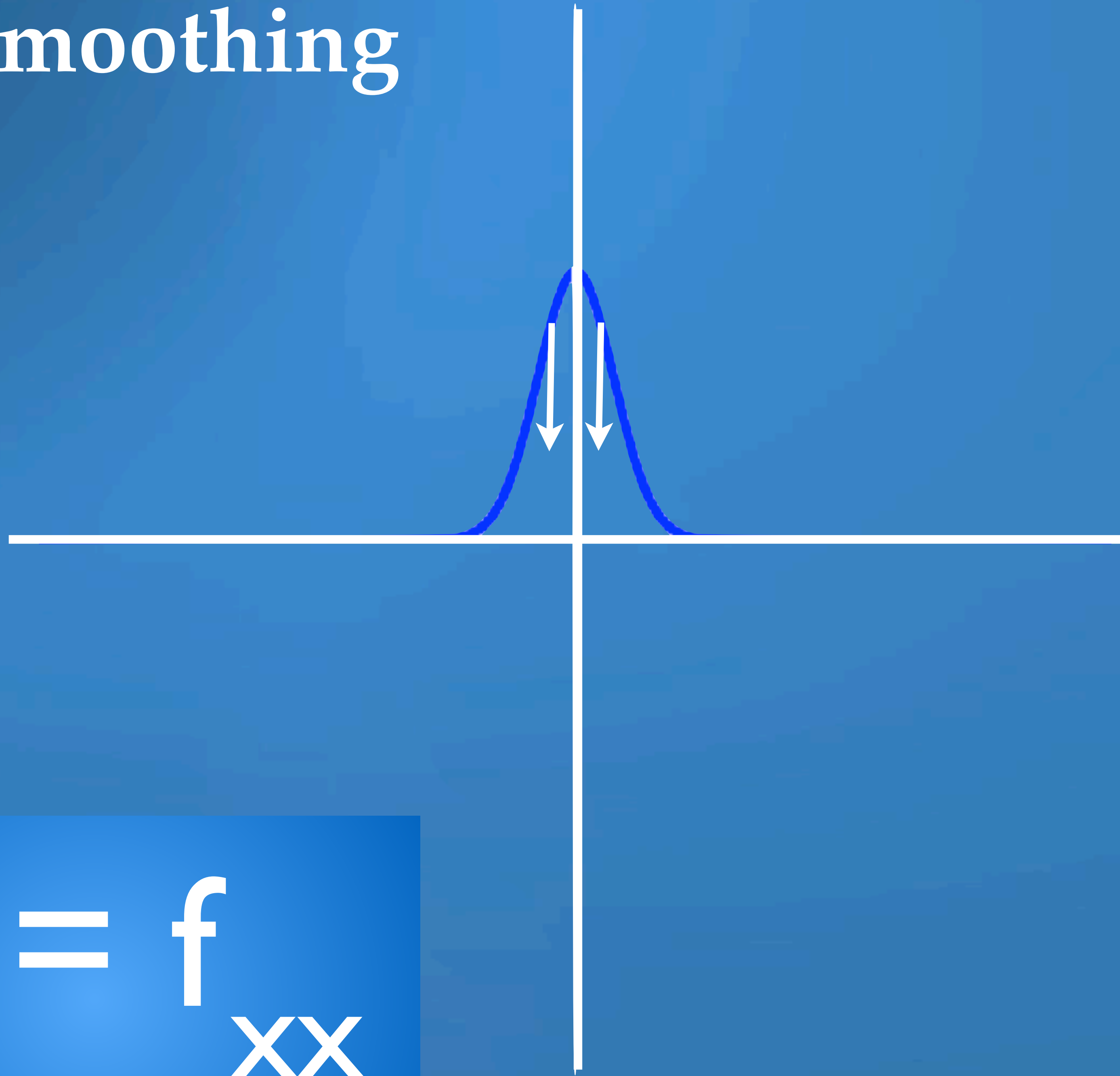


$$f_t = f_x$$

Heat Equation

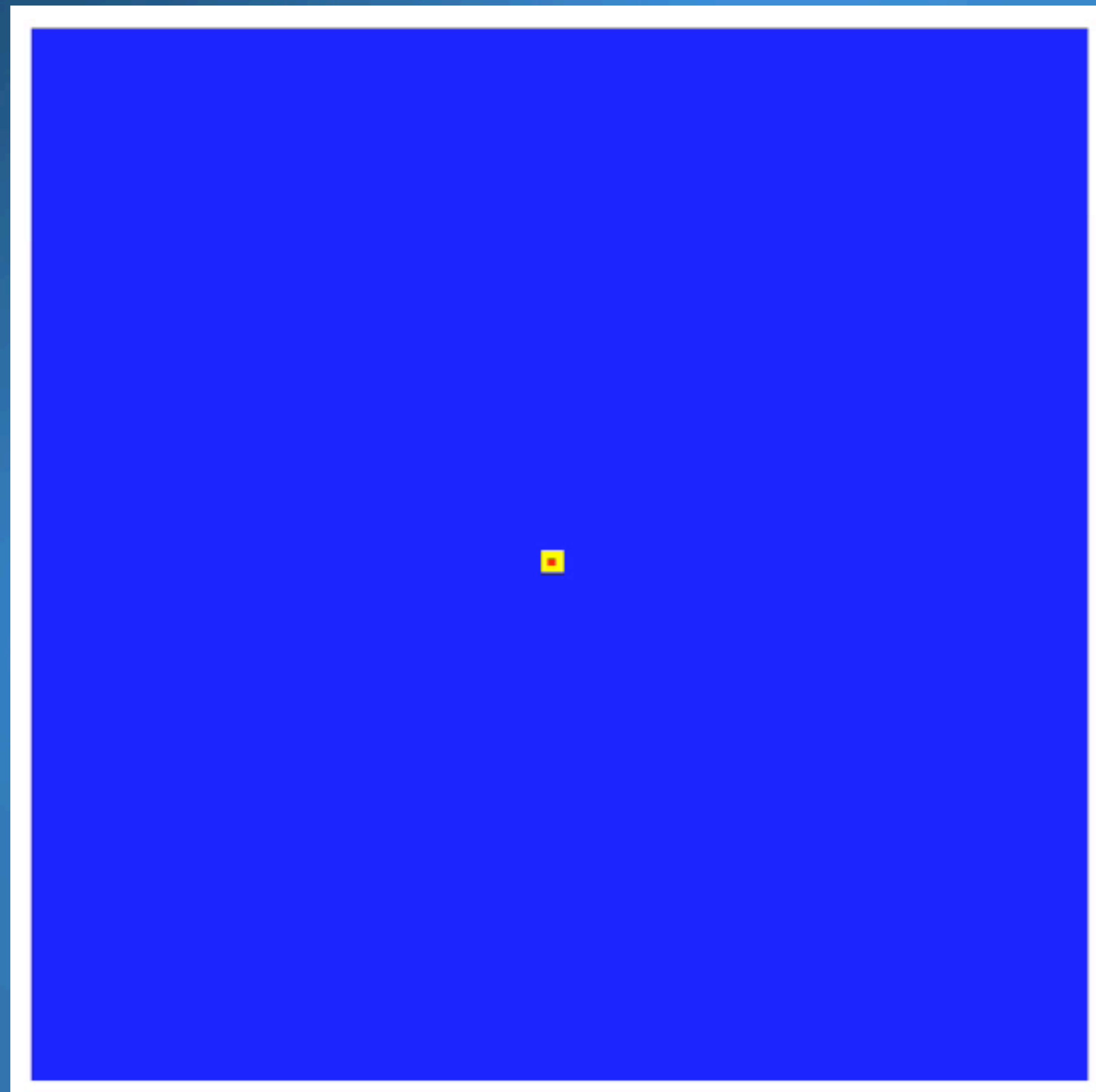
$$f_t = f_{xx}$$

- Heat propagation
- Diffusion
- Smoothing



$$f_t = f_{xx}$$

Evolution of Good



Nowak Automaton

Wave equation

- Light
- Sound
- Particles

$$f_{tt} = f_{xx}$$

8 Good vibrations

Wave Equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

second partial derivative
displacement
second partial derivative
with respect to time
speed
squared
with respect to space

What does it say?

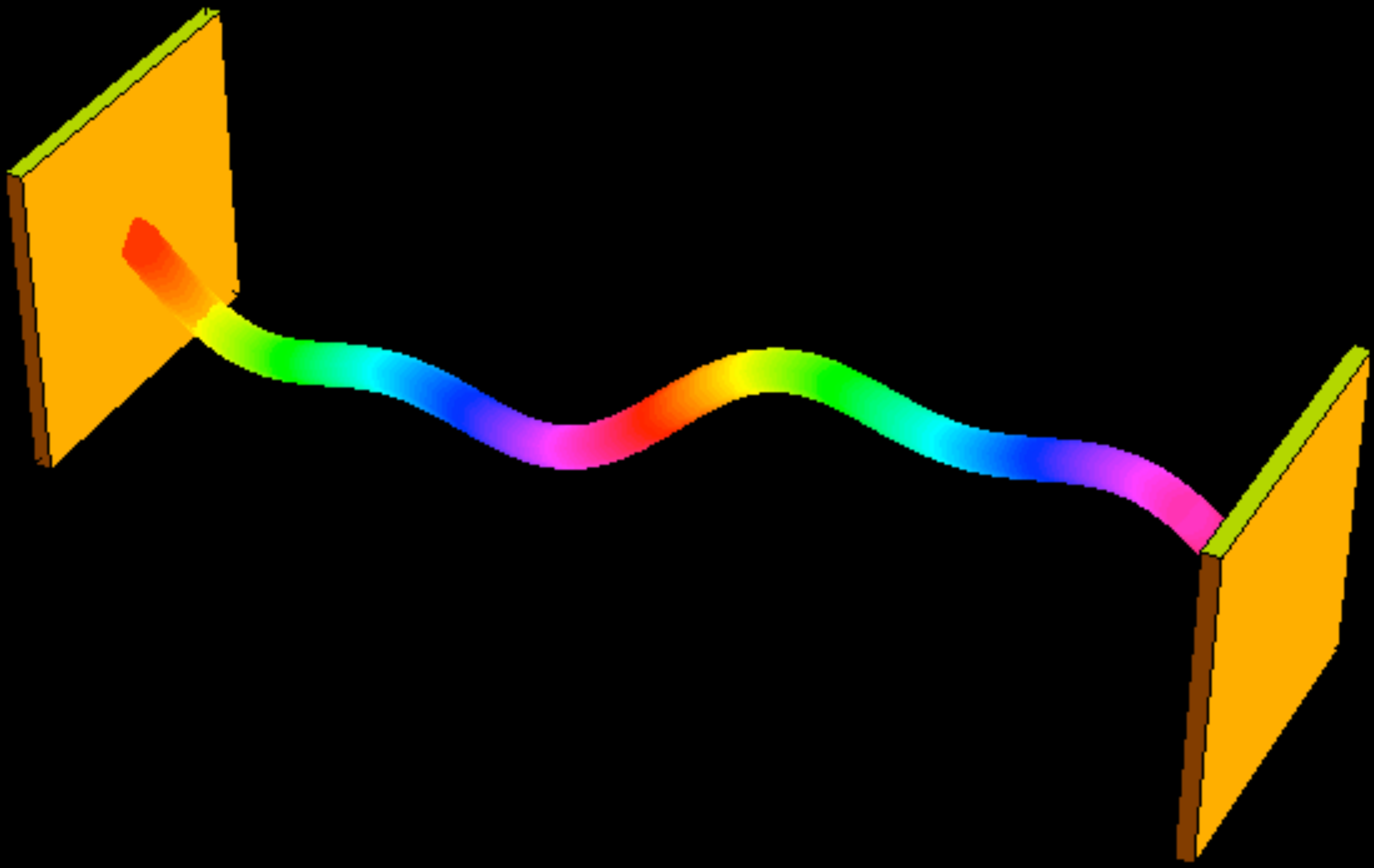
The acceleration of a small segment of a violin string is proportional to the average displacement of neighbouring segments.

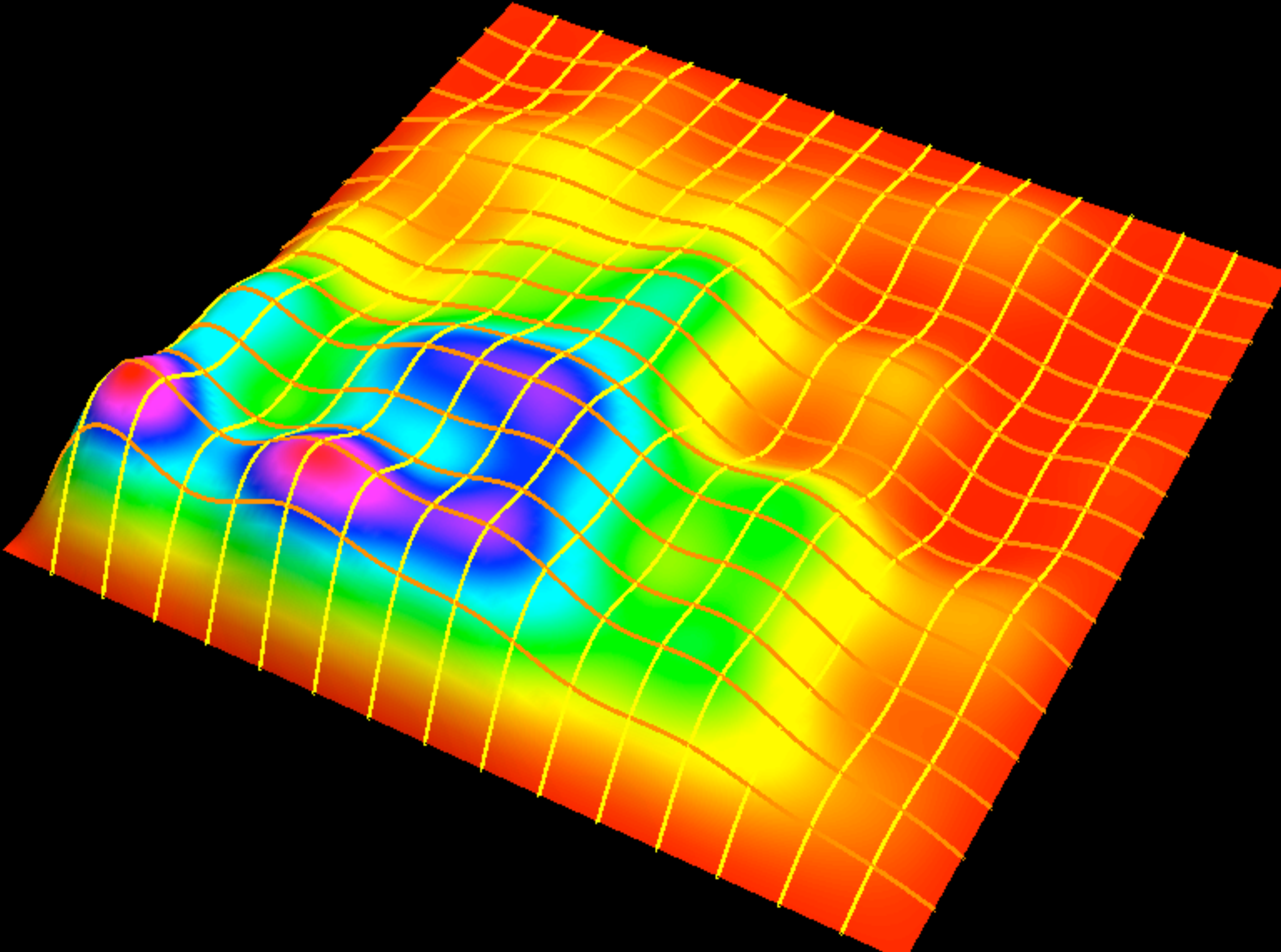
Why is that important?

It predicts that the string will move in waves, and it generalises naturally to other physical systems in which waves occur.

What did it lead to?

Big advances in our understanding of water waves, sound waves, light waves, elastic vibrations... Seismologists use modified versions of it to deduce the structure of the interior of the Earth from how it vibrates. Oil companies use similar methods to find oil. In Chapter 11 we will see how it predicted the existence of electromagnetic waves, leading to radio, television, radar, and modern communications.



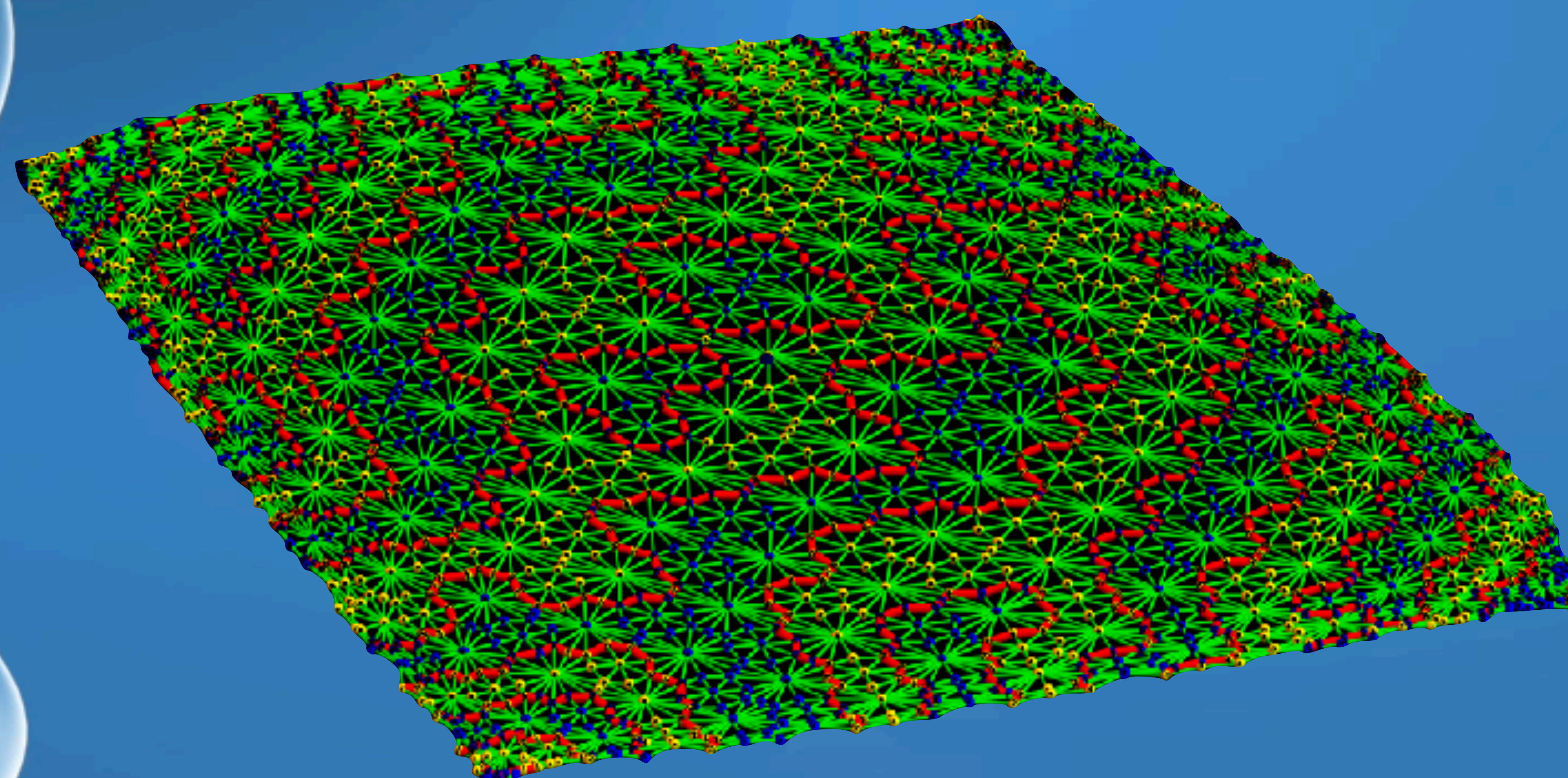


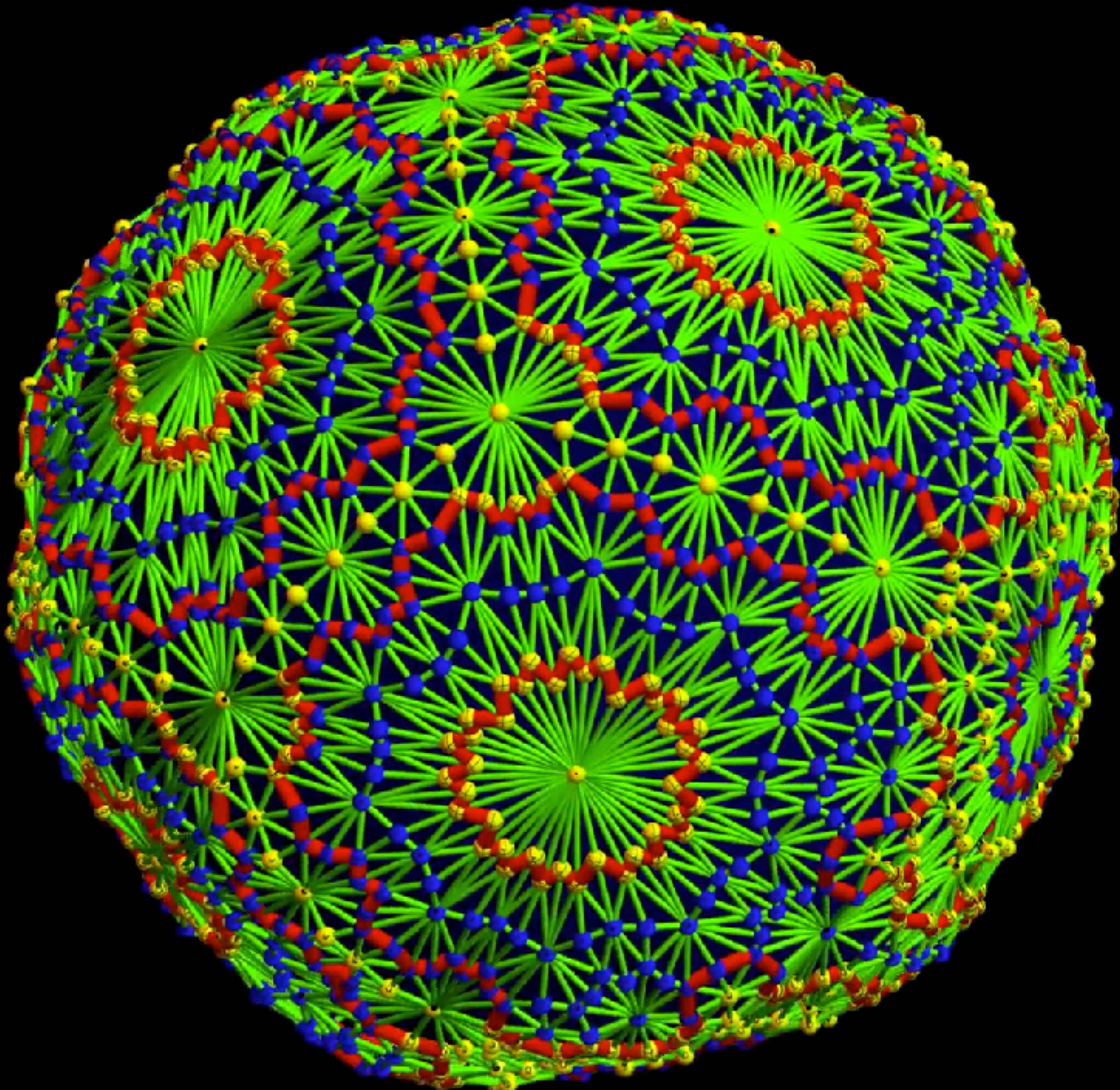
Laplace Equation

$$f_{xx} + f_{yy} = 0$$

Chladni Patterns

For networks





Burgers Equation

$$f_t + f f_x = 0$$

Schocks

Schrödinger equation

$$i f_t = f_{xx} + V(x) f$$

14 Quantum weirdness
Schrödinger's Equation

$$i\hbar \frac{\partial}{\partial t}\Psi = \hat{H}\Psi$$

square root of minus one (i); Planck's constant divided by 2π ($\hbar$); rate of change (∂) with respect to time (∂t); quantum wave function (Ψ); Hamiltonian operator ($\hat{H}$)

What does it say?

The equation models matter not as a particle, but as a wave, and describes how such a wave propagates.

Why is that important?

Schrödinger's equation is fundamental to quantum mechanics, which together with general relativity constitute today's most effective theories of the physical universe.

What did it lead to?

A radical revision of the physics of the world at very small scales, in which every object has a 'wave function' that describes a probability cloud of possible states. At this level the world is inherently uncertain. Attempts to relate the microscopic quantum world to our macroscopic classical world led to philosophical issues that still reverberate. But experimentally, quantum theory works beautifully, and today's computer chips and lasers wouldn't work without it.

Navier Stokes

$$f_t + f\,f_x = f_{xx} + F(u,p)$$

10 The ascent of humanity

Navier–Stokes Equation

density · velocity · pressure · stress · body forces

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v}\right) = -\nabla p + \nabla\cdot\mathbf{T} + \mathbf{f}$$

time derivative · dot product · gradient · divergence

What does it say?

It's Newton's second law of motion in disguise. The left-hand side is the acceleration of a small region of fluid. The right-hand side is the forces that act on it: pressure, stress, and internal body forces.

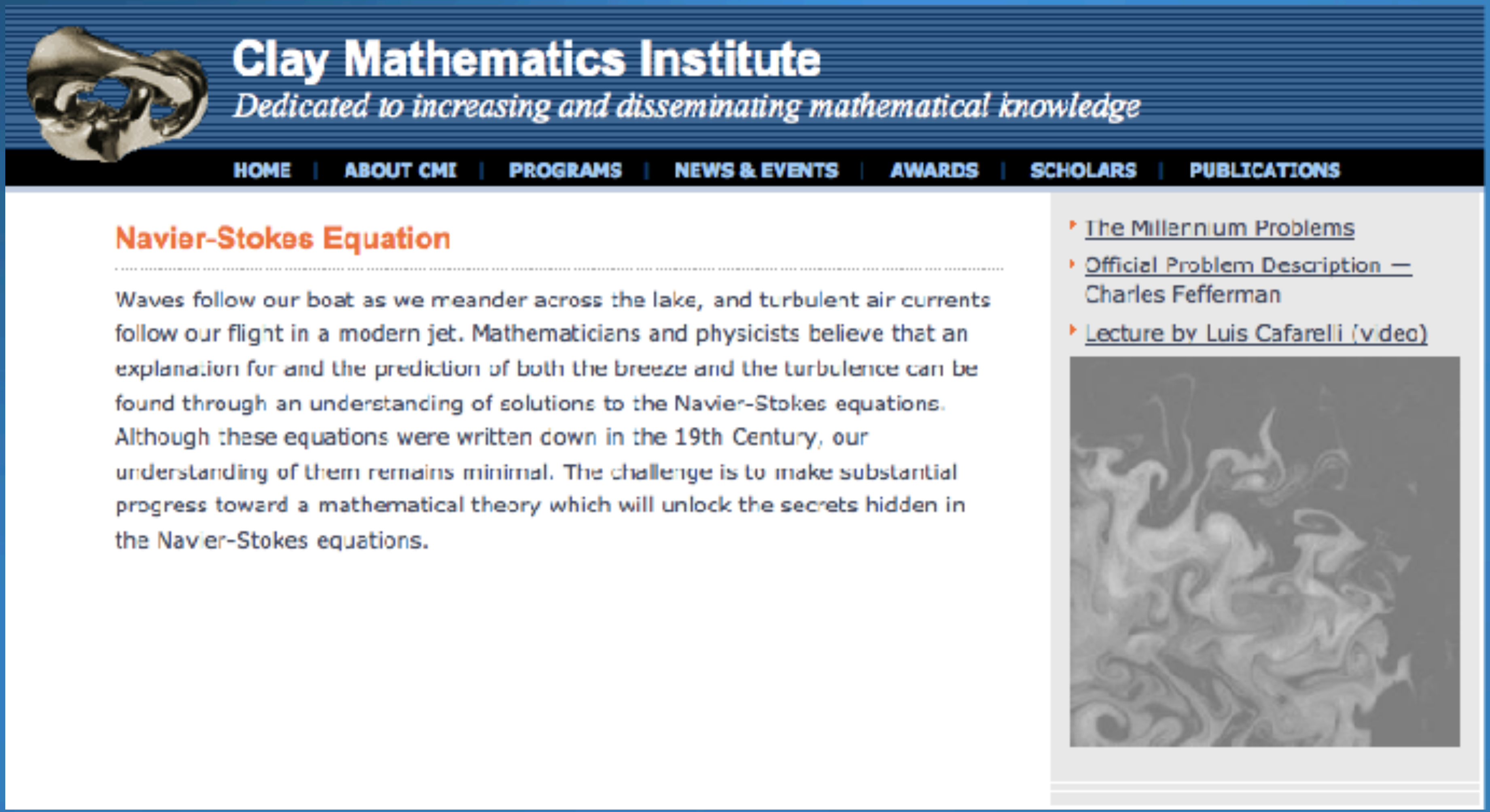
Why is that important?

It provides a really accurate way to calculate how fluids move. This is a key feature of innumerable scientific and technological problems.

What did it lead to?

Modern passenger jets, fast and quiet submarines, Formula 1 racing cars that stay on the track at high speeds, and medical advances on blood flow in veins and arteries. Computer methods for solving the equations, known as computational fluid dynamics (CFD), are widely used by engineers to improve technology in such areas.

1 Mio Dollars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Navier-Stokes Equation

Waves follow our boat as we meander across the lake, and turbulent air currents follow our flight in a modern jet. Mathematicians and physicists believe that an explanation for and the prediction of both the breeze and the turbulence can be found through an understanding of solutions to the Navier-Stokes equations. Although these equations were written down in the 19th Century, our understanding of them remains minimal. The challenge is to make substantial progress toward a mathematical theory which will unlock the secrets hidden in the Navier-Stokes equations.

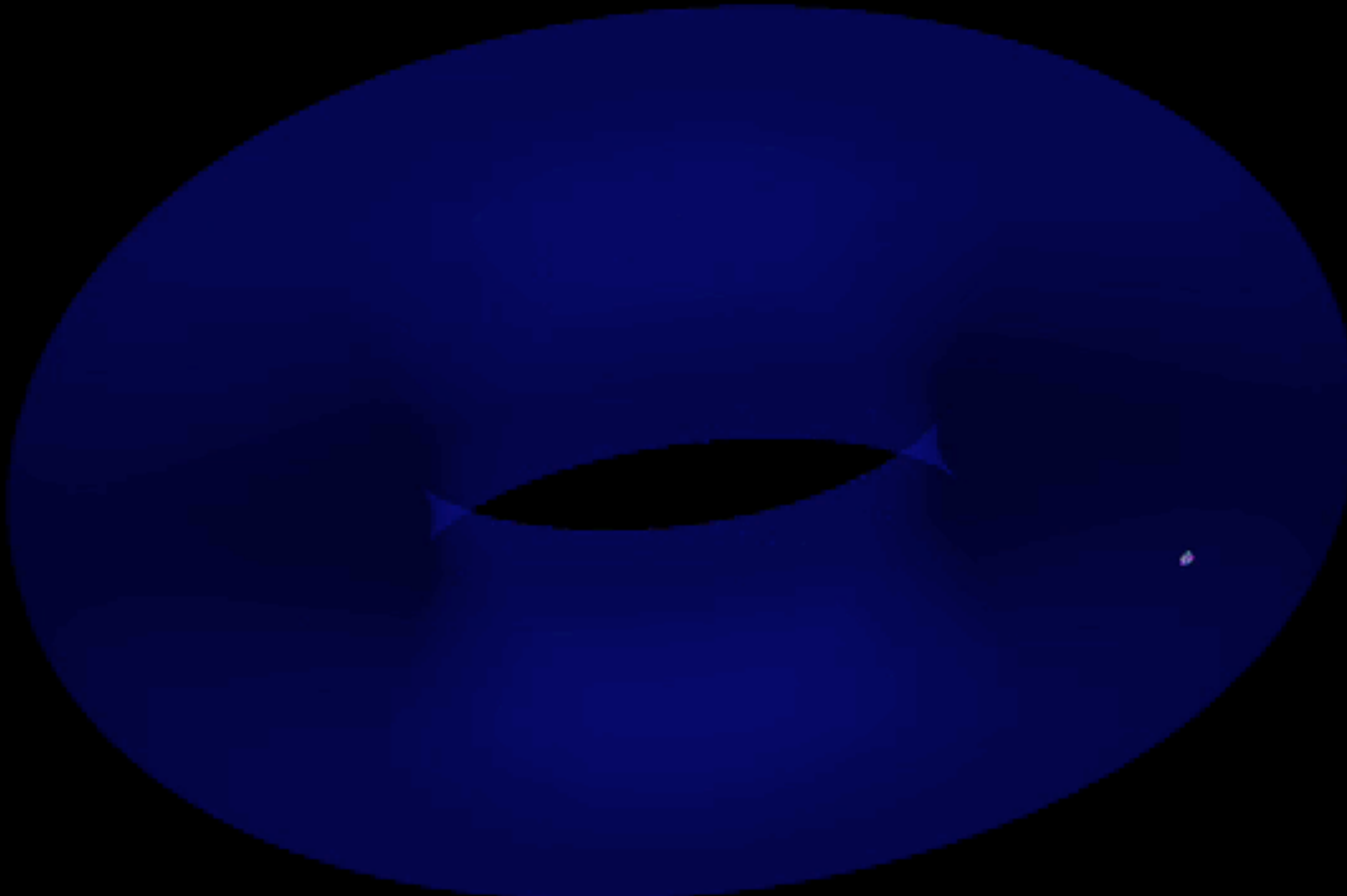
- The Millennium Problems
- Official Problem Description — Charles Fefferman
- Lecture by Luis Cafarelli (video)


MILLENNIUM PROBLEMS
Margaretta Taylor
Lobby

Sofia Kowalevsky

Eiconal Equation

$$f_x^2 + f_y^2 = 1$$



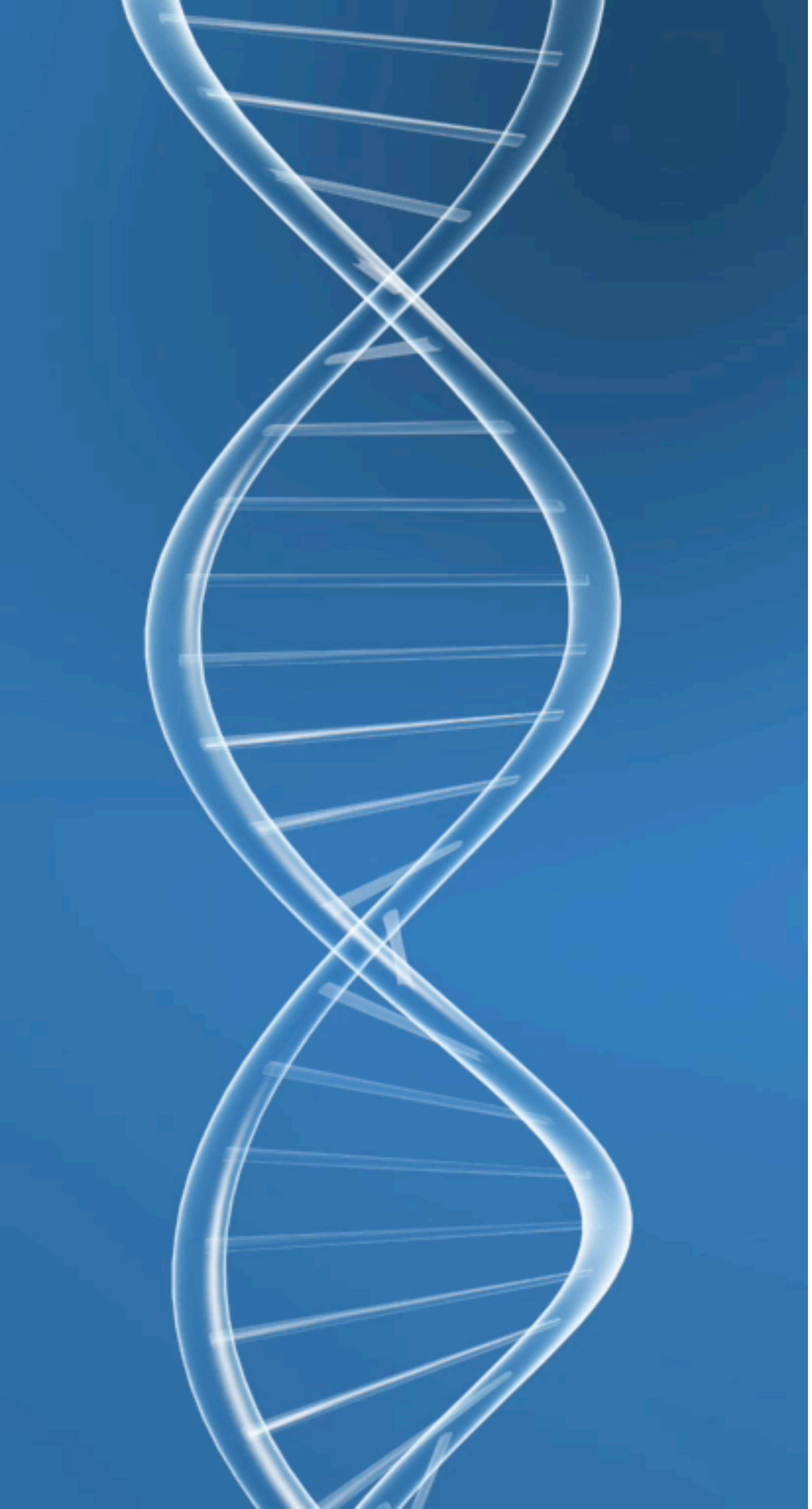
Maxwell equations

$$\mathrm{div}(B) = 0$$

$$\mathrm{div}(E) = 4\pi\rho$$

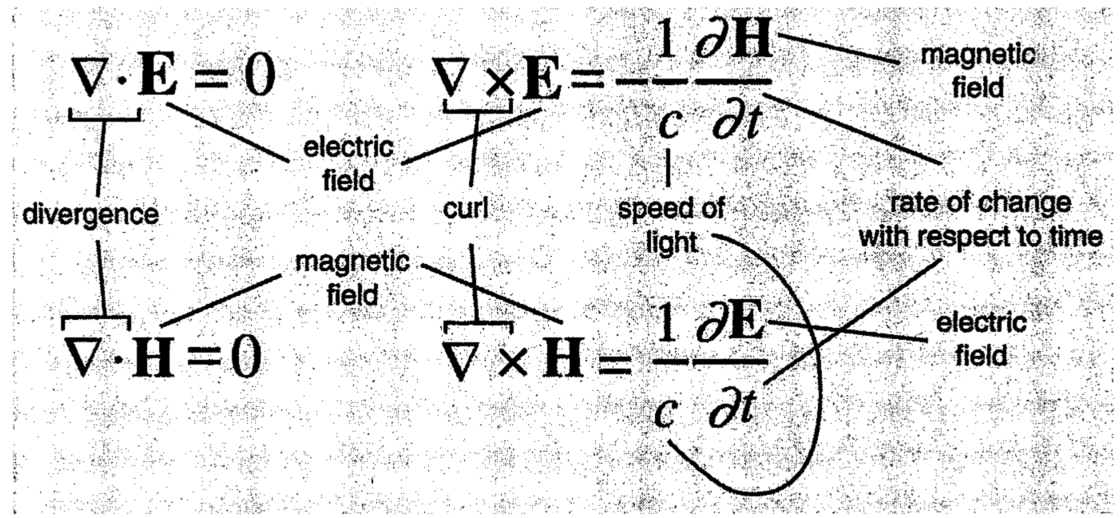
$$\mathrm{curl}(E) = -B'/c$$

$$\mathrm{curl}(B) = E'/c + 4\pi j/c$$



11 Waves in the ether

Maxwell's Equations



What do they say?

Electricity and magnetism can't just leak away. A spinning region of electric field creates a magnetic field at right angles to the spin. A spinning region of magnetic field creates an electric field at right angles to the spin, but in the opposite direction.

Why is that important?

It was the first major unification of physical forces, showing that electricity and magnetism are intimately interrelated.

What did it lead to?

The prediction that electromagnetic waves exist, travelling at the speed of light, so light itself is such a wave. This motivated the invention of radio, radar, television, wireless connections for computer equipment, and most modern communications.

Black Scholes

$$f_t + f f_x = f - x f_{xx}^2$$


UDA
WIN

17 The Midas formula

Black–Scholes Equation

rate of change of rate of change

volatility

price of financial derivative

rate of change

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} + \frac{\partial V}{\partial t} - rV = 0$$

price of commodity

with respect to

time

risk-free interest rate

What does it say?

It describes how the price of a financial derivative changes over time, based on the principle that when the price is correct, the derivative carries no risk and no one can make a profit by selling it at a different price.

Why is that important?

It makes it possible to trade a derivative before it matures by assigning an agreed 'rational' value to it, so that it can become a virtual commodity in its own right.

What did it lead to?

Massive growth of the financial sector, ever more complex financial instruments, surges in economic prosperity punctuated by crashes, the turbulent stock markets of the 1990s, the 2008–9 financial crisis, and the ongoing economic slump.

The End

O. Knill, October 7, 2019