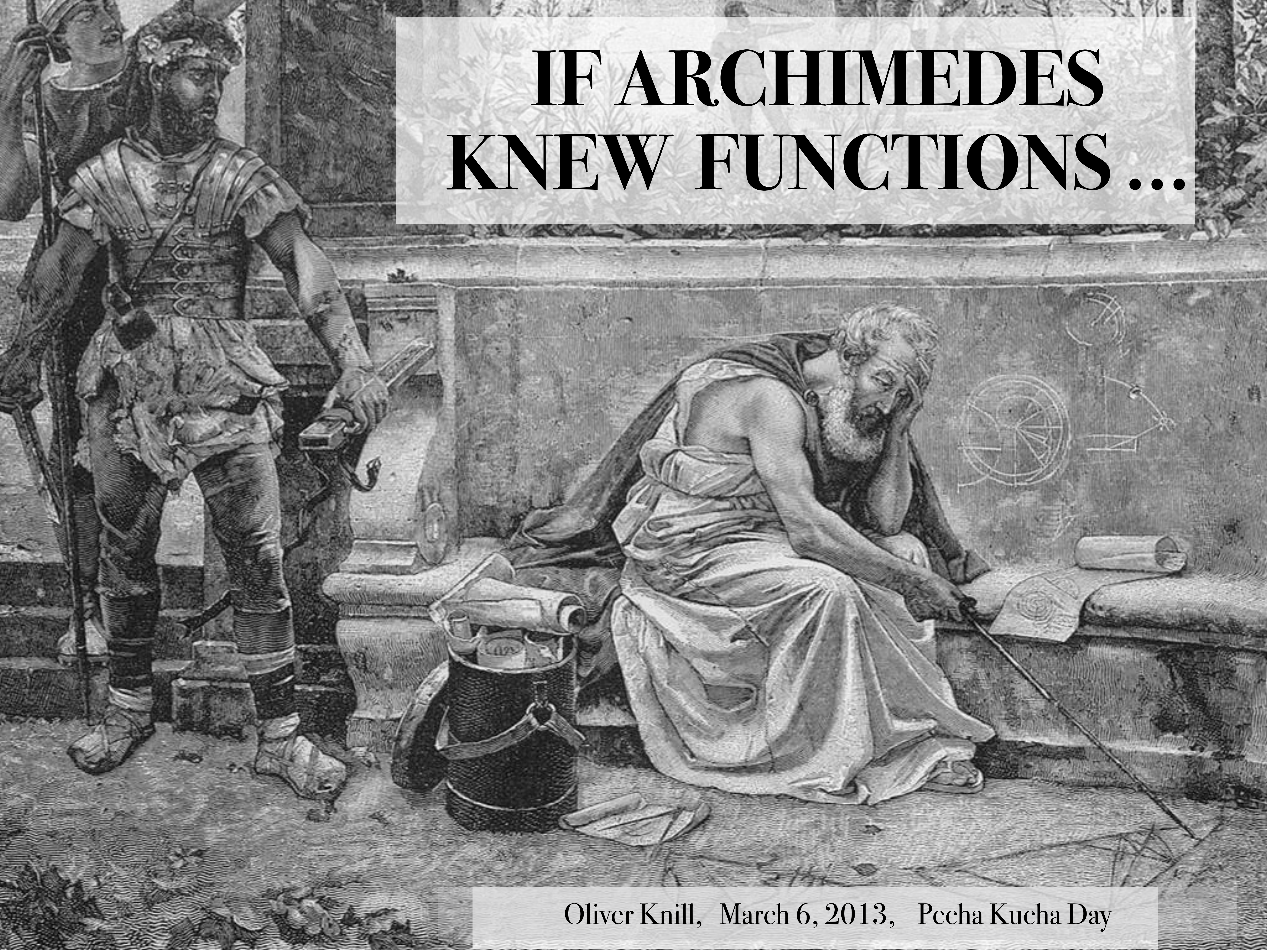


IF ARCHIMEDES KNEW FUNCTIONS ...

Oliver Knill, March 6, 2013, Pecha Kucha Day

20′000 -4000 years ago

S

| | | | | | | ...

1 2 3 4 5 6 7 ...

D

Ishango

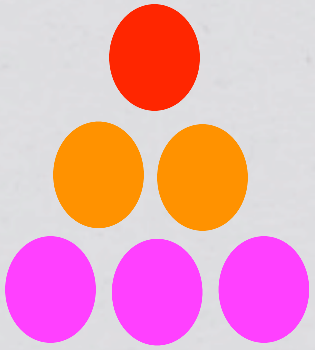
Plimpton 322

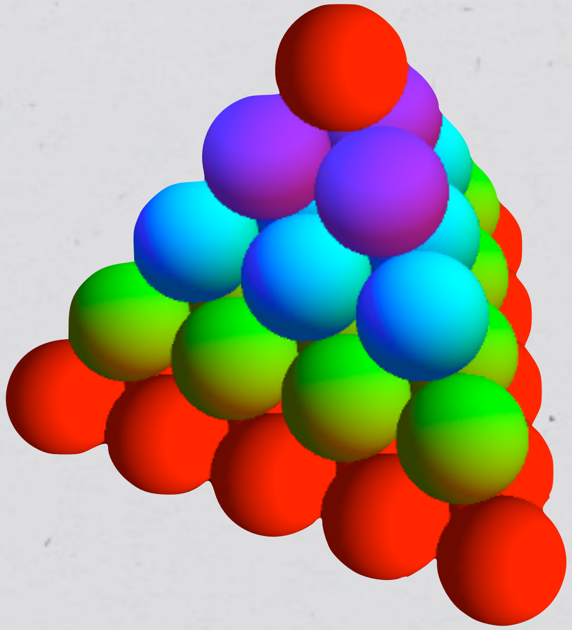
Differences and Sums

S

1	1	1	1	1	1	1
1	2	3	4	5	6	7
0	1	3	6	10	15	21
0	0	1	4	10	20	35
0	0	0	1	5	15	35

D





Polynomials

0	1	2	3	4	5	6		
1	1	1	1	1	1		$[1]$	1
0	1	2	3	4	5		$[x]$	x
0	0	1	3	6	10		$\frac{[x^2]}{2}$	$\frac{x(x-1)}{2}$
0	0	0	1	4	10		$\frac{[x^3]}{6}$	$\frac{x(x-1)(x-2)}{6}$

quantum deformation

Derivatives and Integrals

$$D\, f(x) = \quad f(x+1)-f(x)$$

$$S\, f(x) = \quad f(0)+f(1)+f(2)+\ldots+f(x-1)$$

$$D\, x^{n} = n\, x^{n-1}$$

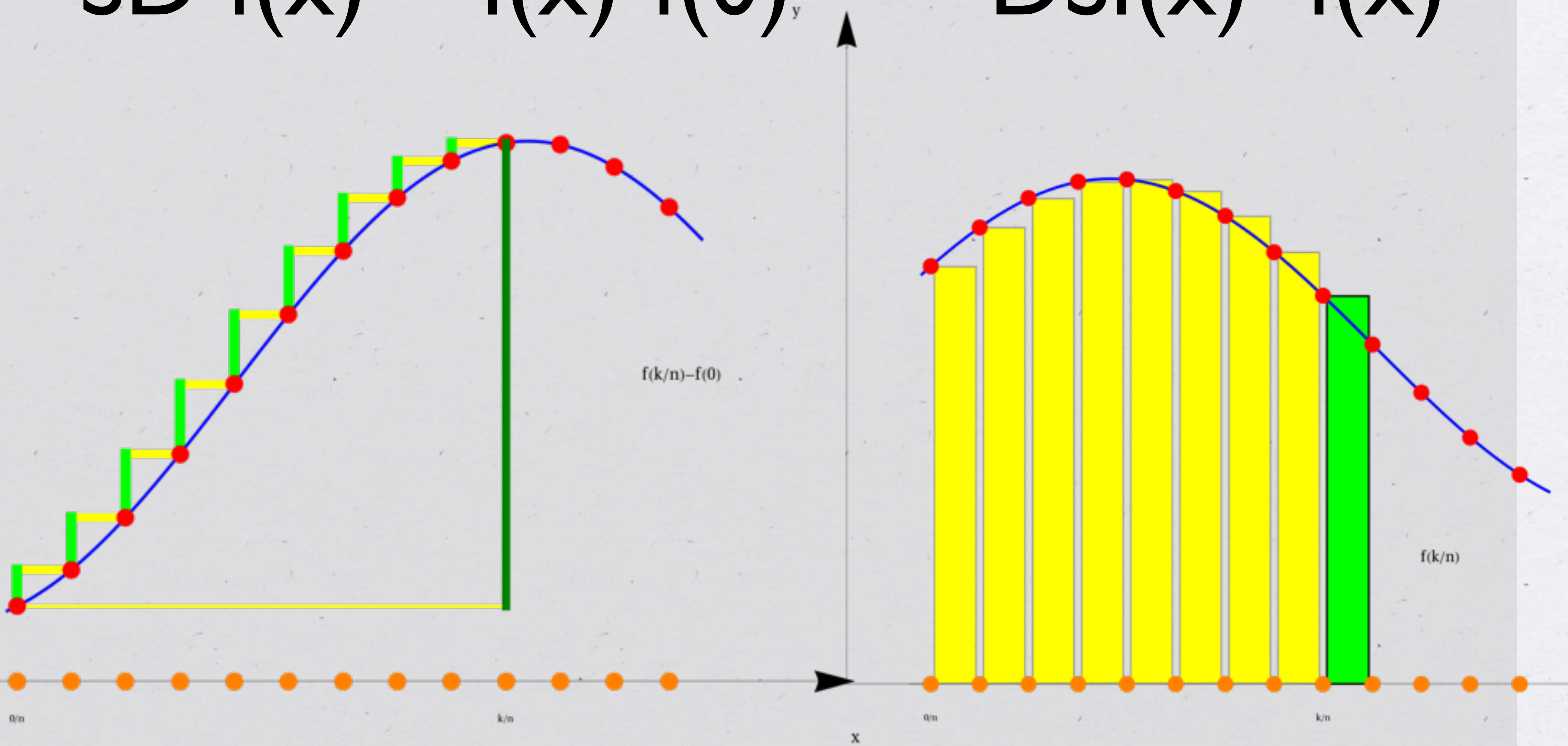
$$(x+1)x\,(x-1)(x-2)$$
$$-\quad x(x-1)(x-2)(x-3)$$
$$= 4\, x(x-1)(x-2)$$

$$D\, \exp(a\, x) = a\, \exp(ax)$$

$$(1+a)^{(x+1)}-(1+a)^{x}$$
$$= \quad a\,(1+a)^{x}$$

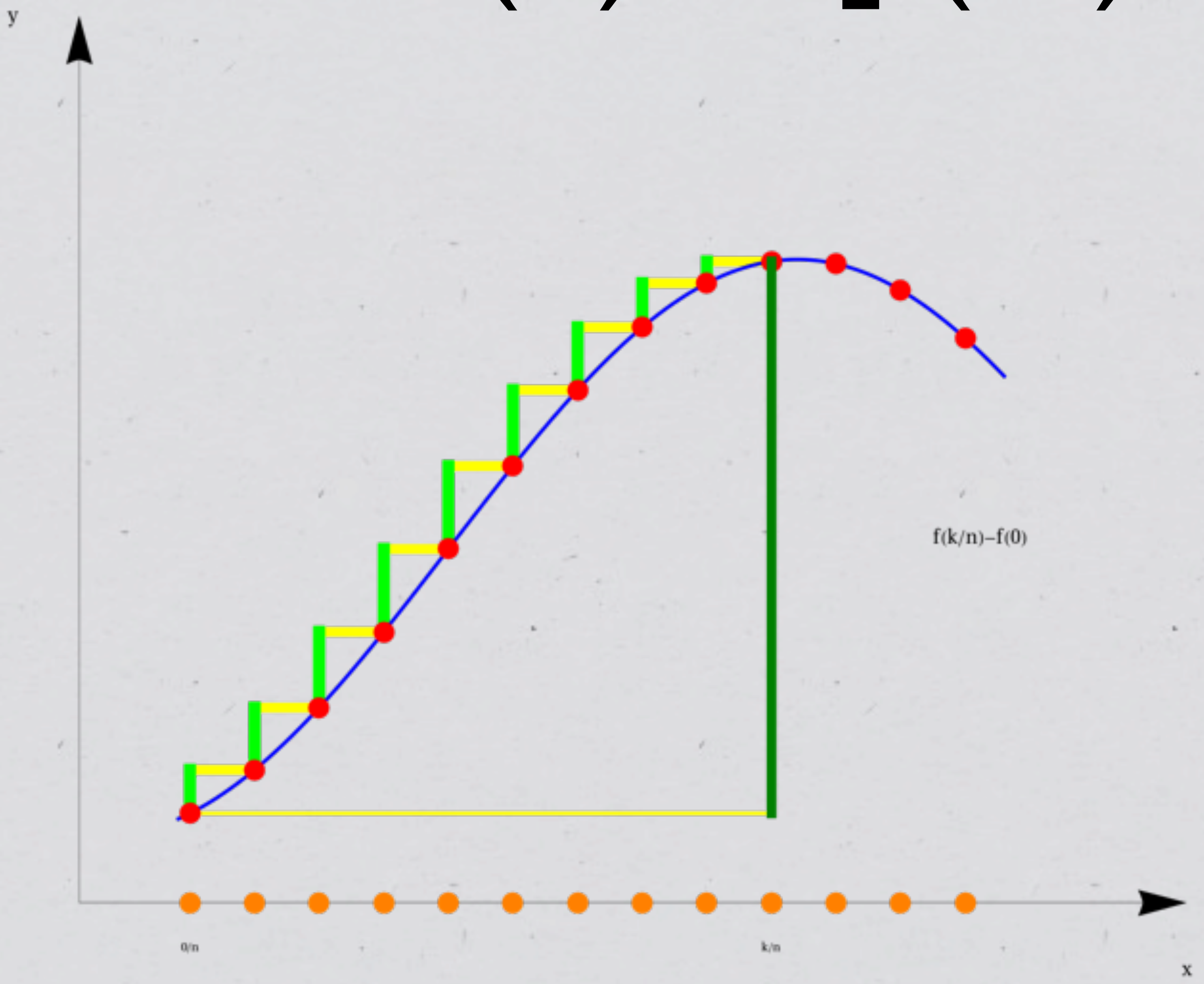
Fundamental Theorem

$SD\,f(x) = f(x)-f(0)$ $DSf(x)=f(x)$



Proof of Part I

$$SD\ f(n) = [f(n\ )- f(n-1)] +$$



$$[f(n-1)-f(n-2)] +$$

$$\ldots$$

$$[f(2)-f(1)]$$

$$+[f(1)-f(0)$$

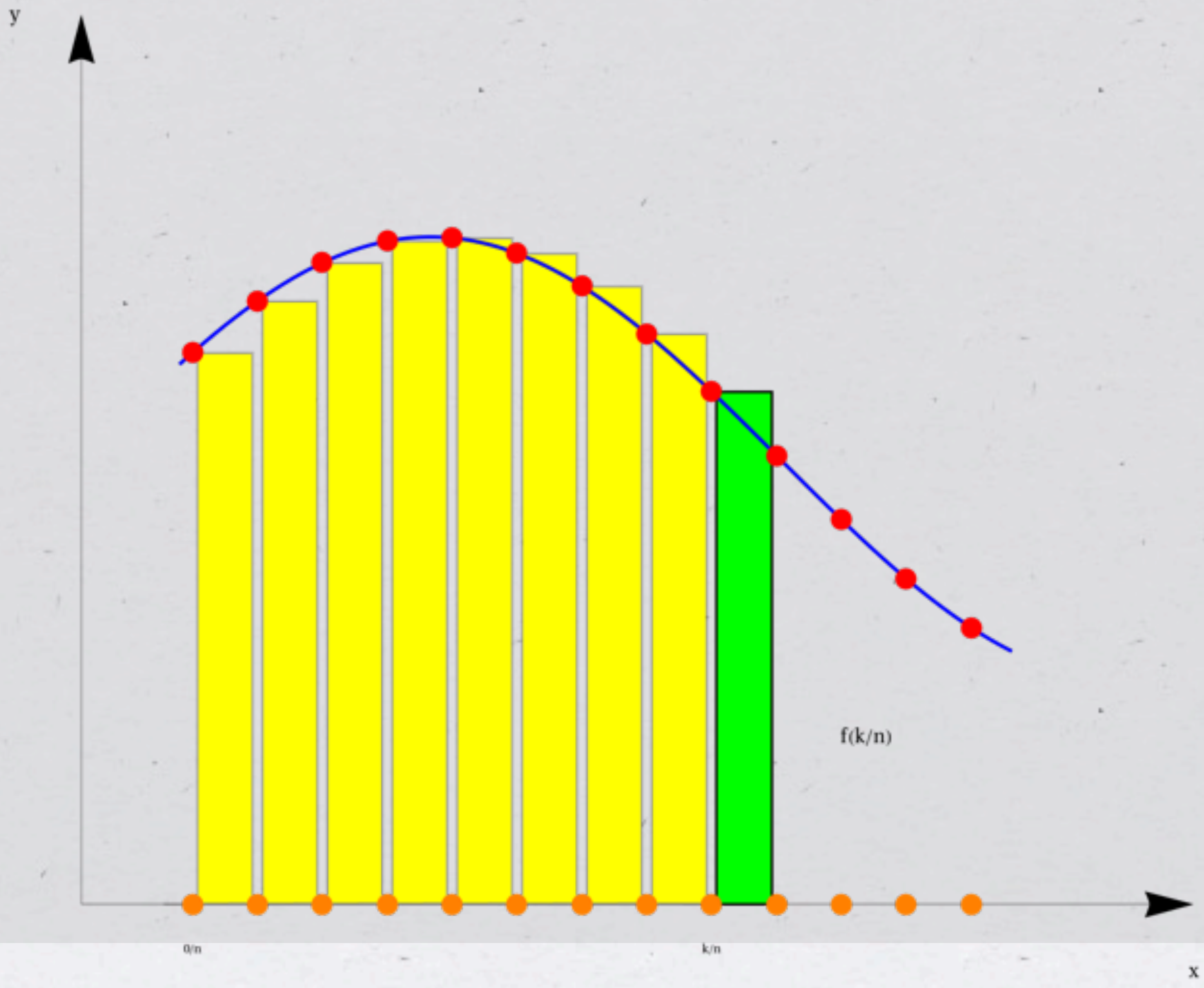
$$= f(n) -f(0)$$

Proof of part II

$$DSf(n) = [f(0)+f(1)+\ldots+ f(n-1)+ f(n)]$$

$$- [f(0)+f(1)+\ldots+ f(n-1)]$$

$$= f(n)$$



Generalization

$$D\ f(x) = \frac{[f(x+h)-f(x)]}{h}$$

$$S\ f(x) = [f(0)+f(h)+f(2h)+\ldots+f(x)]\ h$$

$$DSf(x)=f(x)$$

$$\frac{d}{dx}\int_0^x f(t)\ dt = f(x)$$

$$SD\ f(x) = f(x)-f(0)$$

$$\int_0^x f\,'(t)\ dt = f(x)-f(0)$$

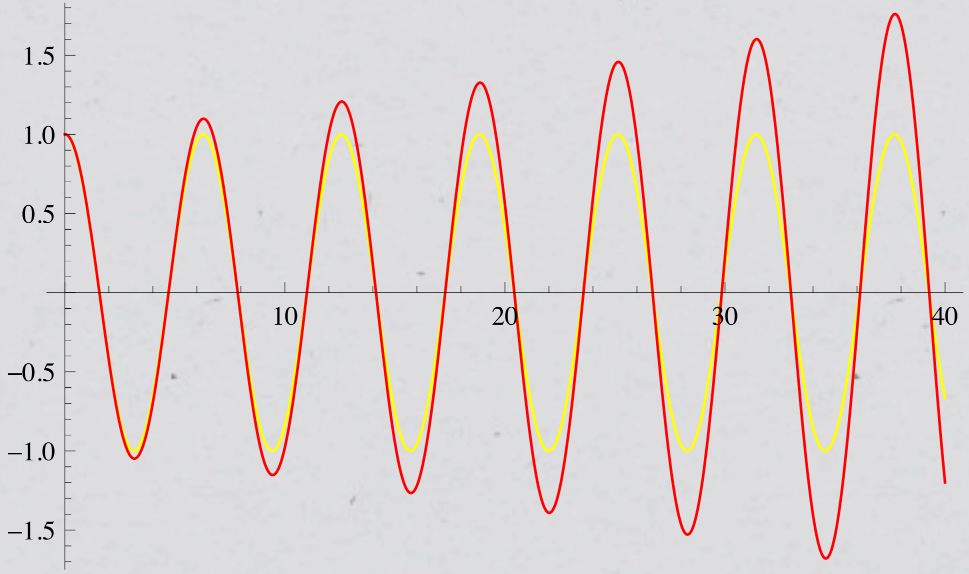
More Functions

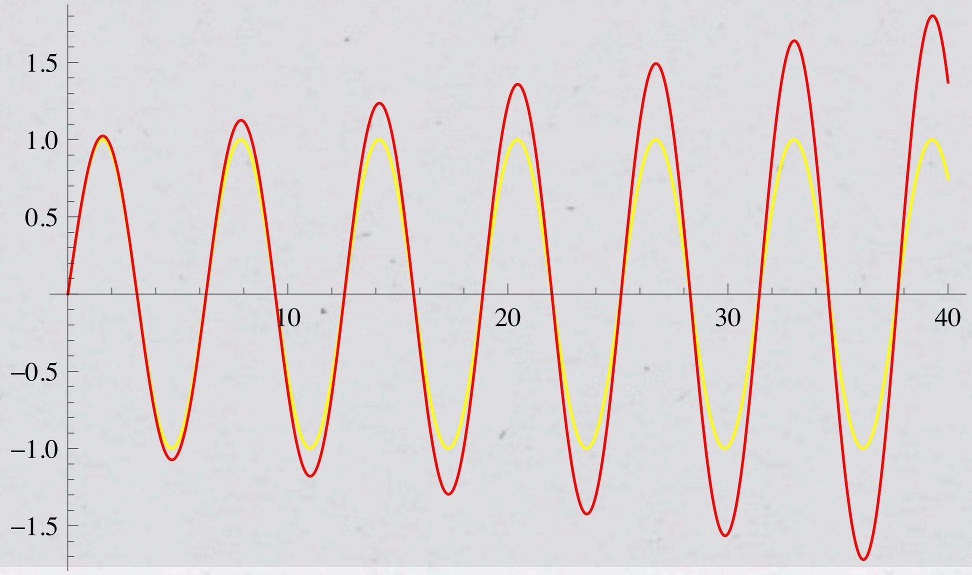
$$\exp(a\ x) = (1+a)^x$$

$$\exp(i\ x) = \cos(x) + i \sin(x)$$

$$D \sin(x) = \cos(x)$$

$$D \cos(x) = -\sin(x)$$





We can sum!

$$S\ x^{n} = x^{n+1}/(n+1)$$

$$S\ \sin(x) = 1-\cos(x)$$

$$S\ \cos(x) = \sin(x)$$

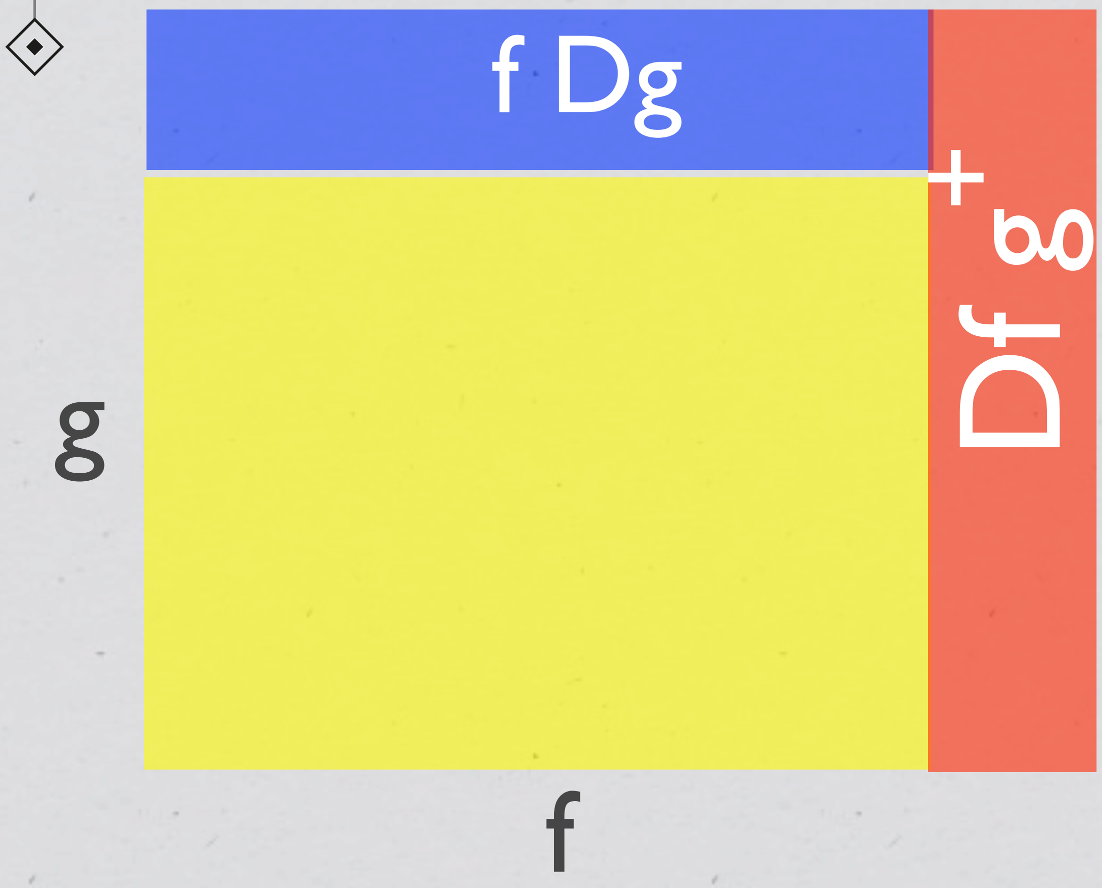
$$S\ \exp(x) = \exp(x)-1$$

Examples

$$1^2+2^2+3^2+\ldots+n^2 = \frac{n(n-1)(n-2)}{3}+\frac{n\ (n-1)}{2}$$

$$1+2+4+8+16 = \exp(5)-1 = 2^5-1$$

Leibniz rule



$$D(f\, g) = f\, Dg + Df\, g^{+}$$

Proof: $f(x+h)\, g(x+h)-f(x)g(x) =$

$$[(f(x+h)-f(x)]\, g(x+h) + f(x)\, [g(x+h)-g(x)]$$

Chain rule

$$Df(g(x)) = (\mathbf{D}f)(g(x)) \quad Dg(x)$$

Proof: $[f(g(x+h))-f(g(x))]/h=$

$$\frac{f(g(x)+H)-f(g(x))}{H}\,\frac{(g(x+h)-g(x))}{h}$$

if H is nonzero

QED

Taylor Theorem

$f\,'(x)=Df(x)=f(x+1)-f(x)$

$$f(x) = f(0)+f\,'(0)\,x + \frac{f\,''(0)\,x^2}{2!} + \ldots$$

Proof

$$f(1) = f(0) + (f(1)-f(0))\;1$$

$$f(2) = f(0) + (f(1)-f(0))\;2$$

$$+ (f(2)-2f(1)+f(0))\;\frac{(2\cdot 1)}{2!}$$

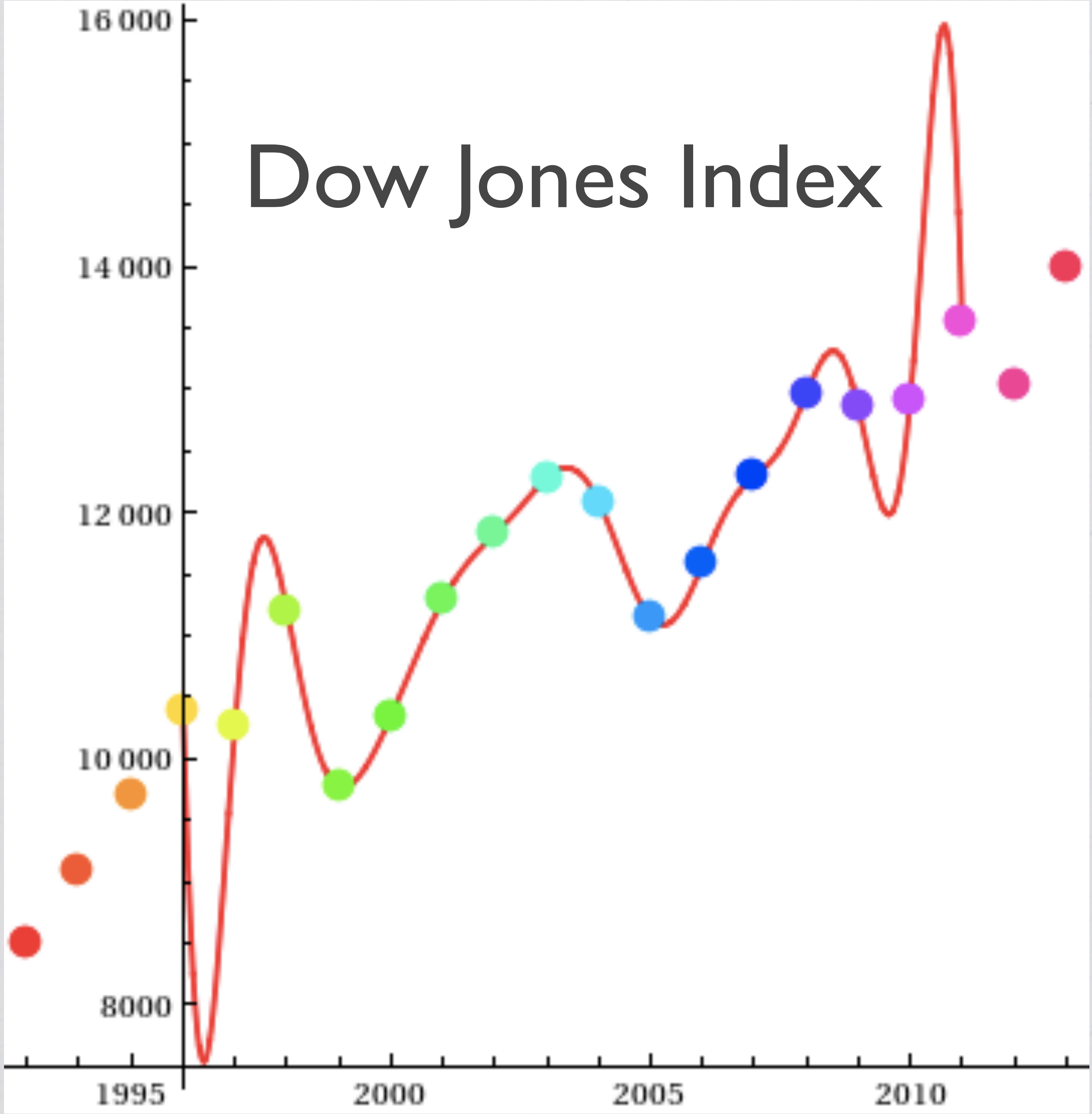
Newton 1665, Gregory 1670

Example:

$$\exp(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

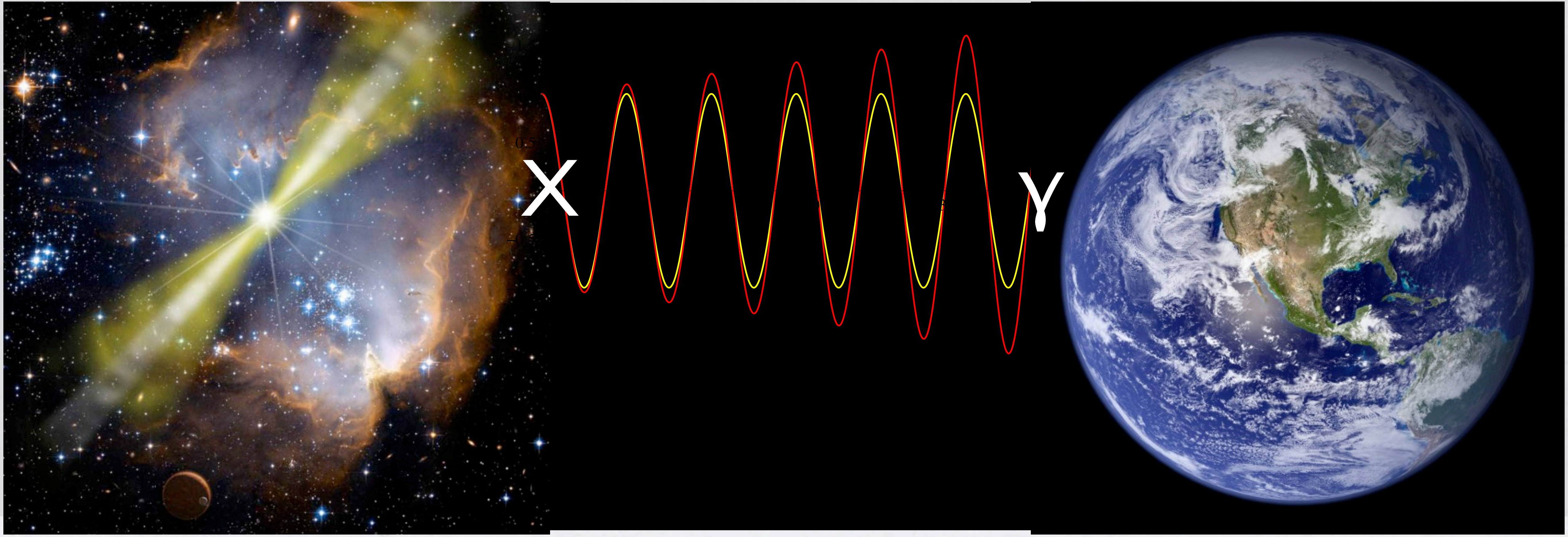
$$2^5 = 1+5+\frac{5*4}{2} + \frac{5*4*3}{6} + \frac{5*4*3*2}{24} + \frac{5*4*3*2*1}{120}$$

Application: Fitting

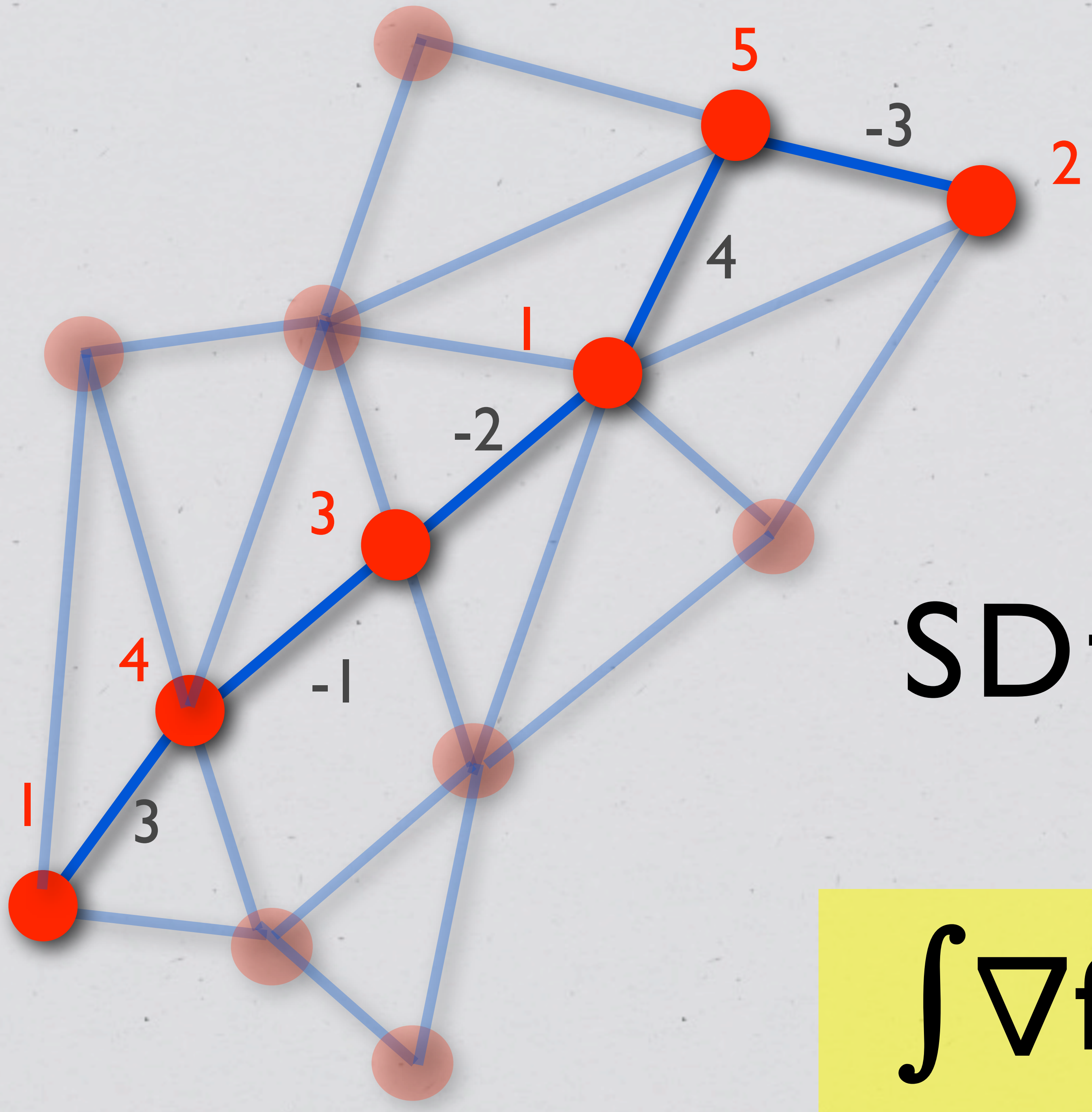

Dow Jones Index

Differential Equations

$Df \ = g$	$f(x)= f(0)+Sg(x)$
$D f = a f$	$f(x)= f(0) \exp(a x)$
$D^2 f = - f$	$f(x) = a \cos(x)+b\sin(x)$



Glimpse on Multivariable



$$Df = \nabla f$$

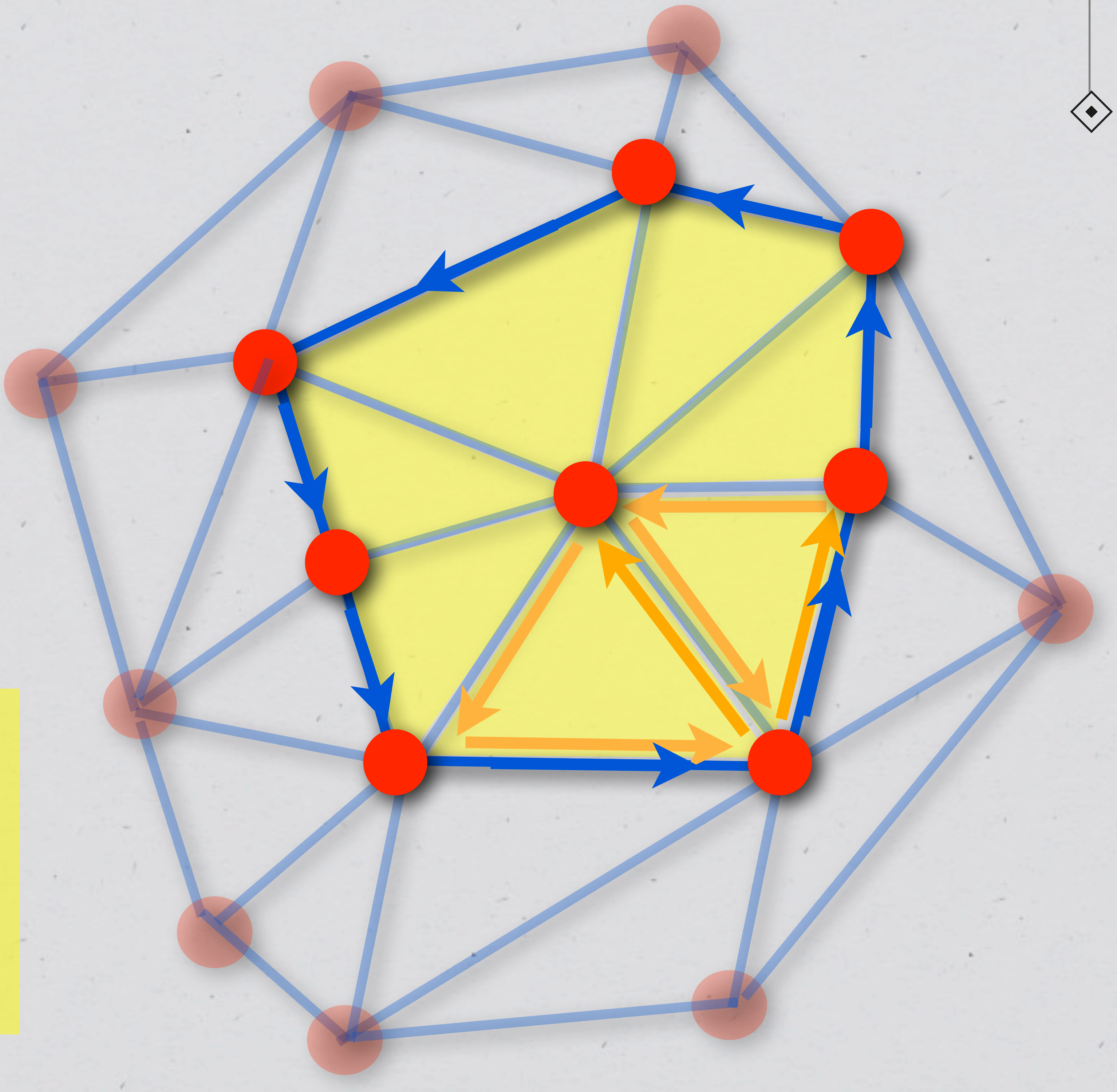
$$SDf(x) = f(x) - f(0)$$

$$\int \nabla f \, dr = f(b) - f(a)$$

Stokes Theorem

DF=curl(F)

$$\int_G dF = \int_{\delta G} F$$



Source: "Death of Archimedes", 1815, Thomas Degeorge (1786-1854)
"Circles Disturbed," edited by Apostolos Doxiadis and Barry Mazur.