

# Problem types in Calculus

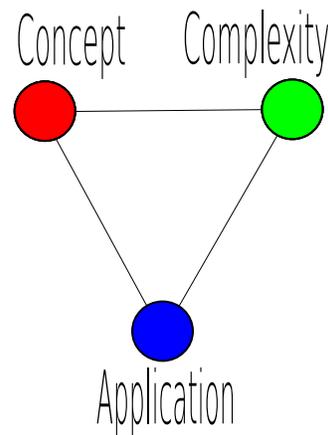
Oliver Knill

October 17, 2006

## Abstract

We discuss different type of problems in calculus and attach a vector  $(concept, complexity, applicability)$  to each problem. This can help to tell whether a problem is suited for homework, for a lecture, for an exam, for a review or for personal coaching. The "concept coordinate" measures how hard a student has to think when first seen it. The "complexity coordinate" is a measure of how much computation is needed. Complex problems do not need innovative thinking but are tedious. The third coordinate, the "applicability" measures how much it relates to other fields or real life. Experience shows that all three coordinates contribute to the "difficulty" of a problem. Extreme cases are

- application based courses (problems with large third coordinate dominate)
- inquiry based courses (mostly problems with large second coordinate)
- computer algebra system based (the third coordinate is large)



Each corner of this triangle has feverish followers: an extreme prototype of an inquiry based course is the "Moore method". It comes in many flavors but it in general bans computer algebra systems and avoids problems with applications.

Application based courses do not focus on proof based problems. They in general assume that all involved functions are smooth. Courses which use computer algebra systems or calculators can not allocate time to do applications or conceptual problems.

I feel that the common consensus today is a **culture of tolerance**: one can teach mathematics in many different ways. The strongest courses use the best of all: use computer algebra systems sporadically, include some conceptual proofs and tackle difficult problems as well as see also real life applications. The strength of the teacher is a major factor. A teacher who is computerphobic or inexperienced can not teach well a course based on computer algebra systems. A teacher who is unfamiliar with other sciences like physics, chemistry, engineering or economics has to prepare a lot in an application based course.

A more important issue is, where the problems are used. At Harvard for example, we encourage the use of computer algebra systems, do computer algebra system projects but most teachers ban computers from exams. We sporadically have students prove something or tackle conceptually difficult problems but we avoid them in general major problems from exams. There are exceptions. I personally like to move the conceptual part to True/False problems as much as possible. If a problem is original and needs ingenuity to solve, then it can be a matter of luck to get to the solution in the assigned exam time. Too original problems in section based courses also can have the risk that some sections have done such a problem and other sections not. Some True/False problems tough because proving or disproving a statement needs some real insight. Preparation for T/F problems helps to pinpoint conceptual misunderstandings. They also allow to see who really understand the material well, without making this a major issue.

**Drill problems**

Drill problems are straightforward problems which use the concepts. They usually are not difficult and mention no direct application.

[1,0,0]

Find the line integral of  $\vec{F}(x, y) = \langle -y, x^2 + y \rangle$  along the curve  $\vec{r}(t) = \langle \cos(t), \sin(t) \rangle$  from  $t = 0$  to  $t = 2\pi$ .

The same problem could be formulated in a more applied way. It becomes more difficult.

[1,0,5]

A duck swims along the unit circle in a pond, in which the water velocity produces a force field  $\vec{F}(x, y) = \langle -y, x^2 + y \rangle$ . How much work does the duck have to perform?

Here is a version in which the problem is made difficult by adding complexity

[1,5,0]

Find the line integral of  $\vec{F}(x, y) = \langle -y^4, x^3 + y^2 \rangle$  along the curve  $\vec{r}(t) = \langle \cos(t), \sin(t) \rangle$  from  $t = 0$  to  $t = 2\pi$ .

Finally, there is a version which makes the problem conceptually harder because an integral theorem is needed to solve it.

[5,0,0]

Find the line integral of  $\vec{F}(x, y) = \langle -x^5, y^6 \cos(y) + y \rangle$  along the curve  $\vec{r}(t) = \langle \cos(t), \sin(t) \rangle$  from  $t = 0$  to  $t = 2\pi$ .

## Homework problems

Problems with some purpose. With relations to other fields, like physics. Maybe proving something.

[4,2,8]

Show that a paraboloid  $z = x^2 + y^2$  has the property that straight lines with  $r'(t) = (0, 0, -1)$  pass through a given point.

[8,3,5]

Assume  $u$  is harmonic. Find the partial differential equation in polar coordinates.

## Lecture problems

Lecture problems should be fun, be not too trivial but also not complicated. They should allow discussion and possibly exploration.

## Practice exam problems

These can be real exam problems. Problems from which one can learn. Problems which combine different topics.

[10,5,7]

Let  $f(x, y)$  be the distance to the curve  $x^6 + y^6 = 1$ . Show that it satisfies the partial differential equation  $f_x^2 + f_y^2 + f_z^2 = 1$ .

## Review lecture problems

Straightforward. The solution involves no complicated computations and the problems are doable. There can be solution paths which lead to messy computations but such solution are not mentioned. So, the difficulty can be high but the problems should not be too complex.

[2,2,6]

An ice tray has height  $y$  and  $6 \times 2$  square compartments of size  $x \times x$ . Minimize the cost of the material if the volume is constant 9.

## Midterm exam problems

Also here. The solution should not lead to messy computations. In general no proofs. No or minor graphing. Different subjects can merge.

[5,1,4]

Find the points on the parameterized surface  $\vec{r}(u, v) = \langle (u + 1)^2, u + v, v \rangle$  where the distance to the origin has local extremum. Find the minimal distance.

## Final exam problems

Final exam problems can combine even more distant topics from different chapters.

[1,4,1]

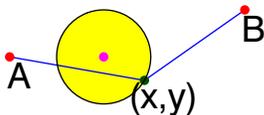
Find all critical points of a function  $f(x, y)$  and then compute the line integrals along the polygonal lines from different critical points.

## Final exam makeup problem

A makeup exam problem can be close to an already given final exam problem but because the exam is known, it can be a notch harder than the actual exam problem.

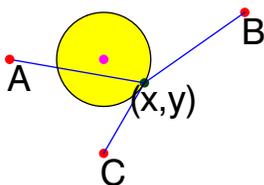
[3,3,4]

Original exam problem: A circular wheel with boundary  $g(x, y) = x^2 + y^2 = 1$  has the boundary point  $(x, y)$  connected to two points  $A = (-2, 0)$  and  $B = (3, 1)$  by rubber bands. The potential energy at position  $(x, y)$  is by Hooks law equal to  $f(x, y) = (x + 2)^2 + y^2 + (x - 3)^2 + (y - 1)^2$ , the sum of the squares of the distances to  $A$  and  $B$ . Our goal is to find the position  $(x, y)$  for which the energy is minimal. To find this position for which the wheel is at rest, minimize  $f(x, y)$  under the constraint  $g(x, y) = 1$ .



[3,5,4]

Makeup exam problem: a circular wheel with boundary  $g(x, y) = x^2 + y^2 = 1$  has the boundary point  $(x, y)$  connected to three points  $A = (-2, 0)$ ,  $B = (3, 1)$  and  $C = (0, -2)$  by rubber bands. The potential energy at position  $(x, y)$  is by Hooks law equal to  $f(x, y) = (x + 2)^2 + y^2 + (x - 3)^2 + (y - 1)^2 + x^2 + (y + 2)^2$ , the sum of the squares of the distances to  $A$ ,  $B$  and  $C$ . Our goal is to find the position  $(x, y)$  for which the energy is minimal, which is the position for which the wheel is at rest, minimize  $f(x, y)$  under the constraint  $g(x, y) = 1$ .



## Matching problems in exams

The matching problem should be relatively straight forward, unambiguous and test, whether the basic concepts are understood.

## Matching problems in classroom

These problems can be harder and even be close to ambiguous so that there are discussions.

## True False problems in exams

Can be tricky.

## True False problems in exams

Can be trickier.

## True False problems in classroom

Not too tricky, but should point to misconceptions.

## Hatsumon problems

Also called "Hook up" problems. Problems which can be used to jump start a topic.

[0,0,6]

Three bottles are filled by a constant rate with water. The water level  $f(t)$  is recorded. Match the water level growth functions with the bottles.

## Good problems

Since every problem should be a good problem, this name is funny. When we talk about "good problems", we mean problems which are a bit tricky but which hit the spot. They have a low complexity and have an application.

[2,0,10]

There has been a time, when you you were exactly 3 feet high. Source: Maria Terrell.

[5,0,10]

You sit in a boat in a pool. You have a bag of sand in the boat. You throw this bag from the boat into the water. Will the water level rise or go down or stay the same? Source: Maria Terrell.

[8,0,6]

When you accelerate a car from 0 to 10, your kinetic energy has grown by  $10^2/2 = 50$ . When you accelerate a car from 10 to 20, your kinetic energy has grown by  $20^2/2 - 10^2/2 = 150$ . You need three times as much gasoline, When you look at the later experiment in a moving coordinate system in which the car is initially at rest, you accelerate from 0 to 10 and have gained only 50 kinetic energy. But physical laws are independent of the chosen coordinate system, so that in both cases, the same amount of energy should be used. (The amount of gasoline we need to accelerate from 0 to 10 does not depend on the speed, with which the earth races around the sun for example).

## Proof problems

An inquiry base learning technique is to package a topic into a few dozen problems. The student tries to prove the theorems without external help, without interaction with other students, without using a book. These problems are in general not complex and usually do not have applications.

[0,4,0]

Prove that in any triangle, the angle bisectors intersect in a common point.

## More examples

[5,5,4]

Find algebraically the center of the circle passing through the points  $A = (16, -6)$ ,  $B = (9, 11)$ ,  $C = (-8, 4)$  by constructing algebraically the intersection of two perpendicular bisectors to two of the sides.

The geometric problem has an integer coordinate solution  $(4, -1)$  but leads to large fractions, when done algebraically. The complexity is perceived big for a student population used to calculators. The problem can be solved geometrically by sketching a construction on ruled paper and then verifying that the lattice point nearby has equal distance to the three corners  $A, B, C$ . An algebraic solution leads to conceptual hurdles: one has to know how to find the midpoints in a line segment algebraically, find the equation of a line perpendicular to a given line through a point and intersecting two lines (solving a system of two linear equations). A computer algebra system eliminates the difficulty handling large integer fractions. The applicability of the problem could be increased by formulating the problem differently: for example:

[5,5,8]

A nightguard has to supervise three machines located at  $A = (16, -6)$ ,  $B = (9, 11)$ ,  $C = (-8, 4)$ . Where does he have to be located so that the maximal time to reach one of the machines is minimized. Find the solution in a purely algebraic way which can be implemented in general.

Now, the student first has to translate the problem into a geometric problem. A major difficulty could be to find what the problem asks for. This can be frustrating. Especially, if the problem allows several different interpretations.