On Parameters for Communicating Mathematics

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Abstract. Mathematical ideas can be communicated in many different ways. As for teaching, especially in service courses, it is important as a teacher to be able to adapt the language, the notation, the abstraction level and the difficulty level to an ever changing and increasingly diverse student body. Having folks in the faculty who have the time, the sensitivity and the training to deal with these difficulties is more and more appreciated. Being aware of such issues can be especially challenging in the field of pure mathematics. I want to share here from my own experiences as a mathematics faculty who specializes more in teaching and support rather than in research. There will also be some comments on the role of non-ladder faculty in colleges, positions which come with some unexpected advantages. An example is the absence of publication pressure. The freedom to contribute to any particular area of mathematics allows to branch out into other fields, and explore lighter mathematical topics which are closer to topics taught in service courses. This is illustrated with an example where some new mathematics was created in this environment.

1. Introduction

1.1. In my experience, teaching is extremely rewarding and fulfilling if it works. Since it also happens in a surrounding, where one does not have control about everything, there can be quite a bit of drama in pedagogy. Teaching can be especially adventurous for creative teachers who like to experiment and try out new things or question existing practices or even try themselves in edutainment [**7**]. Pedagogy is also a subject, where theory and experience do not always line up. Similarly to a beautiful physical theory which can be "lost in math" [**37**], a paradigm for learning might not work for everybody as intended. The experiment is the actual classroom. There are many factors which contribute in the teaching process. Some aspects of this adventure are illustrated in [**27**]. The workshop on "Professional Norms in Mathematics" at Johns Hopkins illustrated in various ways how some parameters are difficult to control as they depend on overall norms in society, on expectations from the workplace, on funding situations, as well as on existing education structures in K-12 education.

1.2. Many of us mathematicians would of course like to axiomatize the teaching process, have theorems stating truths which are eternal and which can be adopted and mastered and which, if followed, should lead to successful teaching. But this goal can hardly ever be achieved. Teaching deals with many different humans embedded in a society subject to constraints, perception, opinion, social norms, fashion, anxieties, challenges and feelings. In my experience, a major reason why theory and practice in pedagogy do not always align is that models about student learning often do not take into account the teacher and the surroundings in which the model is applied. In particular, the personality of the teacher, the teaching culture at an institution, the classroom size, the student expectations and the preparation levels of the students can be very different from person to person or from place to place. And then there are mechanisms which are hard to control. Psychological and bias issues, sometimes entrenched over centuries, can surface. This happens especially in highly interactive classrooms. Such parts are often hard to predict and even harder to manage.

1.3. But it is far from hopeless. While teaching, there are some parameters which I'm able to control relatively easily without having to change society or previously entrenched education practices. I would like to focus on one or two such parameters. There is a particular important yard stick, which has been the hardest to control: it is the choice of the **abstraction level** in communicating mathematics. While abstraction can be hard to gauge, it can be managed with training, through experience and especially through experimentation and by getting lots of feedback. It also has a chance of being quantified. Maybe not as much as the parameters "complexity" and "difficulty" but it is easier to assess than "beauty" or "relevance".

1.4. One of the main points I want to make in this article is that "abstraction" is a crucial ingredient to be aware of when teaching mathematics. I also aim to illustrate why this parameter is hard to gauge. I have seen mathematicians with impeccable personal sensitivity, taste and skill that have got it wrong, as can be seen especially when observing talks in research seminars or conferences. I myself often get it wrong. The issue surfaces everywhere, even when I had been preparing for the current talk. I also want to illustrate it with some new mathematics later on. It will be up to the reader to judge whether I misjudged the abstraction level in the example I give.

1.5. Here is an other non-mathematical illustration: until shortly before the meeting, I had planned to use the talk title "abstraction diversity". It sounded good because this word addresses a core message of this talk: be aware of the abstraction spectrum and how to cultivate its variety. But when floating this to a colleague, she asked "what does abstraction diversity actually mean" and indeed, if one has not been told the story, the word does not say much. It does not work as a title because it is itself too abstract. The episode illustrates that "abstraction" comes in essential in any taxonomy about communication, even if it is not mathematical.

1.6. The subject of "parameters" allows me also to share a few general pedagogical remarks related to taxonomies. Taxonomies are practical tools which have proven to be useful. This is illustrated by the vastly popular **Bloom taxonomy** [**10, 4**], which when simplified splits things into **Factual**, **Conceptual**, **Procedural** and **Meta-cognitive** parts. In [**85**] the Bloom approach is called the "Swiss Army Knife of Curriculum Research". I myself learned about such taxonomies during a workshop organized by the Harvard Bok Center and later while directing a thesis in the "math in teaching program" in which taxonomies, meaning systematic categorizations of educational mathematical areas, were the central part.

1.7. Taxonomies in education are what coordinates are in geometry. One can use them to quantify and visualize teaching parameters. A mathematical lecture, a talk, a mathematical problem, a mathematical theory, a speech or a paper or book can be placed into such a parameter space. For me, as a teacher, such yard sticks have been extremely helpful, especially for lesson planning and in order to find the right approach for a specific audience. It is a trusty guide when preparing a presentation, a talk or a class. From personal interactions with teachers and just by looking on the shear amount of material which is available online, one can extrapolate that many other teachers feel similarly about this guidance tool.

1.8. One of the things which attracted me to teaching as an undergraduate student, and prompted me to become a course-assistant in calculus and computer science early on, is that there are so many different approaches which can work. Already in high school as well as college, I liked to observe and analyze the teaching styles of my teachers. I was fortunate to witness a wide spectrum of techniques which illustrated that brilliant teaching can be done in many different ways. Some classes were interactive, some less, some were improvised, some choreographed carefully, some would try new paths while others would follow more traditional ways, some would use props or make interactive experiments, even games played with the class, others would just present pure knowledge and focus on clarity. It is important to me that we should not only embrace diversity in society but also in classroom cultures.

1.9. In the third and last part of these talk notes, I would like to pass along some general words of wisdom or observations I have collected over the last 30 years, while learning mathematics and teaching, both as graduate student and as a non-ladder teaching faculty. I'm aware that the act of giving advice runs the risk of being perceived as condescending or even arrogant, but I believe that we have to honor not only the advice of highly successful people but also have to listen to ordinary folks like me who work in ordinary normal circumstances. In any case, I believe that sharing experiences is important and that the cross fertilization of (sometimes contradicting) ideas was one of the many wonderful parts of this workshop. It goes without saying that much what follows is based on personal experience and not on mathematics education.

1.10. Preparing this talk allowed me to start aligning some thought snippets I kept collecting over years and organize them, maybe to summarize in a book, once retired. Many of them concern with the role of the **"invisible university"**. [**18**] The original name "invisible university" relates to the complex topic of the profession of teaching educators [**69**] or teaching faculty in community colleges [**49**]. At the moment, there is no time for that yet, as I try to use my free time to pursue some mathematics research on my own. In my position, research has become a hobby but it remains an activity which builds strength for teaching and allows for relaxation. Every semester brings new challenges, new experiments and new students.

2. Acknowledgments

2.1. I would like to thank Emily Riehl for organizing this unique, rich and inspiring conference in Baltimore and for the kind invitation to participate. I consider myself a rather exotic creature in the landscape of pedagogy and certainly

am a bit of an outsider, so that the invitation came as a surprise, but I believe it is fitting in a time of increased recognition of diversity to listen also to a mostly invisible and quieter part of the teaching landscape. Yes, we are mostly invisible, but we are also proud of what we do and are also grateful to be able to work with undergraduate students, graduate students as well as other faculty and staff.

2.2. To add while revising this in October 2020, I also want to express sincere thanks to over hundred of constructive improvements suggestions contributed by referees who were reading the text. One of the main referee concerns had been that some of my statements had appeared without references to math education research. I tried to address this now locally, where relevant and labeled it clearly as opinion, where needed. It should again be stressed that this document is far from an education research paper. It contains sometimes rather personal opinions which are not backed up by data. I'm immensely grateful for the referees to identify these spots, make many corrections or point out places where things need to be clarified. This also required to expanded the bibliography at some points.

2.3. My perception of pedagogy is in constant flux. This is also a reason for why the subject is so attractive to me. Regular reflection about teaching makes the process also more rewarding and interesting to me. My perspectives are influenced from student feedback, other teachers and projects. Some of the best insights do not come from surveys or feedback forms but from actually teaching in the trenches or even from grading assignments and exams. [Added in proof in October 2020, it must be said that the temporary move to online teaching has already started to produce even larger shifts in my own perception of teaching and already produced more insight. It will certainly be important to assess and reflect on all this again in a few years.]

2.4. I benefited a lot from undergraduate students and teachers in the "math for teaching program" as they reached also in pedagogical areas. While teaching mathematics to an "artificial intelligence bot" in 2004, we gained some insight about teaching aspects [**43**]. The area of teaching machines is especially accessible because psychological parameters are absent. We do not yet have to inspire or motivate them. A thesis of Elizabeth Slavkovsky explored in 2013 the feasibility of 3D printing in the classroom [**73, 45**] and the work with Jose Luis Ramirez Herran on omni-vision in 2009 (i.e. [**44**]). Such projects were not only interesting for me in the context of pedagogy but also got to an exciting interplay of newer technology with teaching. It also changed the way I think about mathematical objects and how to present them. A project of Paul Hermany "Math puzzles, Games and Activities" from the Fall 2016 got me more exposed to taxonomies, a topic we will start with in the next section. Allen Lai (a Harvard undergraduate) worked with me for one semester on Gamification in Mathematics Education. It is a rather unexplored area with a lot of potential. This is an exciting world. Another angle came in from a project of Ethan Fenn, "The Uses of Spurious Proofs in Teaching Mathematics", a thesis written in the Fall 2017. It studies the use of paradoxes and fallacies to explain a topic also with the help of taxonomies. I mention these examples also to illustrate that teaching goes both ways.

3. Taxonomies

3.1. When thinking about parameters, **taxonomies** can be a useful tool. For a mathematician, these are parametrizations of teaching areas. They build coordinate systems in larger dimensional spaces of pedagogical parameters. To illustrate this, let me mention a handy tool which is useful when looking at any teaching communication. A topic for a class, a talk, project or text can be quantified in the ADC **parameter space**, where A stands for **abstraction** level, D stands for **difficulty** and C stands for **complexity**. There are only three parameters on purpose. It is simple to remember and can be applied without getting lost in too much meta-complexity.

3.2. The **complexity** level of a mathematical task can be quantified quite easily. It essentially measures on how long one has to work on the problem, if one knows how to do it. Computing the product of two 20-digit numbers is a complex task, it is not difficult and not abstract. It is just "tedious". Also search problems which require to try out many cases can require a long time. An example is the problem to place 10 non-interacting super queens (figures which can move both like queens or knights in chess) configurations on a 10×10 board. Up to symmetry, this problem has a unique solution.

3.3. The second parameter, the **difficulty level** of a problem can be measured by how long one has to work until one knows how to start. We sometimes say that a problem is "messy" if the complexity level is hard and "hard" or "tricky" if the difficulty level is high. These are different things. Deciding whether a Diophantine equation has a solution or not can be hard and the story of "Fermat's last theorem" shows this. An example of a doable but still hard problem is to prove that $(x, y) = (3, 5), (x, y) = (3, -5)$ are the only integer solutions on the Mordell curve $y^2 = x^3-2$. The problem was solved already by Fermat (he at least claimed to be able to prove it) but solving it needs an idea. When given the problem, one is stuck at first. There is no obvious way to start solving it. For the history of Mordell curve see [**58**]. For more about the above Mordell curve, see [**16**].

3.4. I have found that the third parameter, the **abstraction level** is a quantity which is the hardest to catch or even to estimate. This is the case not only when teaching a mathematics course, it is also the case when writing a mathematical article, when designing a program or when presenting a topic in a seminar. From my experience, the general rule is that one always over estimates the ability of the audience or user. Abstraction is also subjective because it very much depends on the training or exposure to such thoughts. An example: we all are familiar on how to go from integers to rational numbers by looking at pairs x/y of numbers (even so it can be tough topic to teach [**8**]). Fractions have been developed early by Babylonian or Egyptian mathematics [**31**]. There is an abstract version of this when doing a Grothendieck completion of a monoid to a group [**6**]. This construction by itself is neither difficult nor complex, it is the same idea as constructing fractions, but the more abstract and more general framework makes it less accessible at first.

3.5. If abstraction is so difficult to quantify, why do we like abstraction at all? One reason is that it leads to "elegance", "rigor" and "brevity". Calculus was developed first rather intuitively but errors in judgment required to go deeper. Both the process of building a foundation as well as the process of building rigor forced mathematicians to build in more abstraction. In calculus, abstraction increased especially during the 18th century and the abstraction has since increased even in an accelerated way [**33**]. There had been push-backs and the "new math" [**41**] or the "math wars" controversies [**48**] illustrate this. Abstraction is not bad in and of itself - the Grothendieck completion for example is extremely powerful in mathematics - but it needs training to master it. I found that teaching abstract material requires more skill and a feel of what a student is ready for.

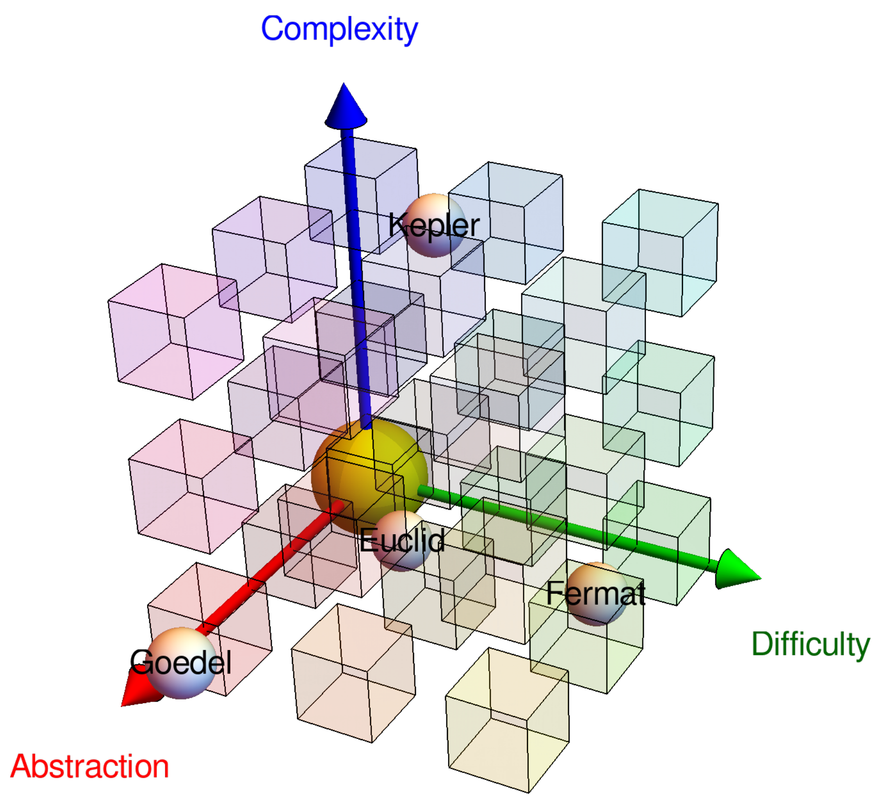

FIGURE 1. The abstraction-difficulty-complexity space is a simple taxonomy to label a mathematical problem. The simplicity of this scheme has purpose: taxonomies which are too complicated can not be observed and used well any more. While difficulty and complexity are relatively easy to measure for a teacher, I observe that the abstraction level is often difficult to gauge and often misjudged.

3.6. In my experience, taxonomies in general can help when preparing to teach. Of course, there are many other parameters which can matter when placing a mathematical problem in the landscape of all problems. The **amount of applicability** for example is an important one. In general however, I have found that the feasibility of applications is often misjudged badly, even by expert teachers. For example, using an application from physics, biology or sports requires background knowledge in that field. Even if everything is explained, the student has to be exposed to the topic already. This can be cultural: I can relate to soccer for example much more than to baseball because I grew up in Switzerland, where soccer is popular. Things can go worse then if more and more details are added for clarification: explaining one thing requires to introduce in general three more things, leading to an explosion of explanations which can bury the underlying math problem.

3.7. Relatively new is the urgent push of the use of "data" which add an other dimension to taxonomies. Our calculus courses for example were now required to satisfy a quantitative reasoning with data requirement (QRD). The committee approving courses requires that mathematics courses have to include problems with larger data sets. It was explicit in the recommendation that one has to deal with explicit messy data sets. I view this difficult because Mathematics is about elegance. It is poetry and beauty [**35, 83, 84, 68, 72, 2, 57**]. Still, the benefits of the QRD requirement were great. The discussions forced us to think about fundamental questions and also to reevaluate which topics need to be taught. Data science and calculus seem have little in common at first, but there are many interesting connections.

3.8. Another interesting parameter space is a taxonomy which encodes **"Knowledge, Algorithms and Concepts"**. We realized when teaching calculus to a bot [**43**], (which is now done on a large scale by essentially all AI related companies), that these are three fundamental parts of teaching. We called it there the "What-How-Why" parameter space. The idea is that we have to first transmit definitions and language and terminology, then learn procedures and algorithms and finally plant insight and understanding. The first two parts usually pose little difficulty to teach, the last one is difficult. The Bloom taxonomy is very close but contains more details. Bloom merges in also an assessment part. We actually thought of doing an assessment in each of the knowledge, algorithms and concepts parts separately.

3.9. Another simple taxonomy is the **"Theory-Example-Illustration"** space. I have often observed that good examples and pictures are essential to understanding a theory; but examples alone or illustrations alone do not work without theory. Examples are important to explain an abstract topic. Conversely, I have found that theory in the form "Theorem-Proof-Example" can be as off-putting as using examples only. Illustrations have pitfalls too. There is the problem to illustrating the obvious things well but then brush over more complicated things. The example in the next section can illustrate this. The reader can then judge whether the mix of theory, example and illustration work or not.

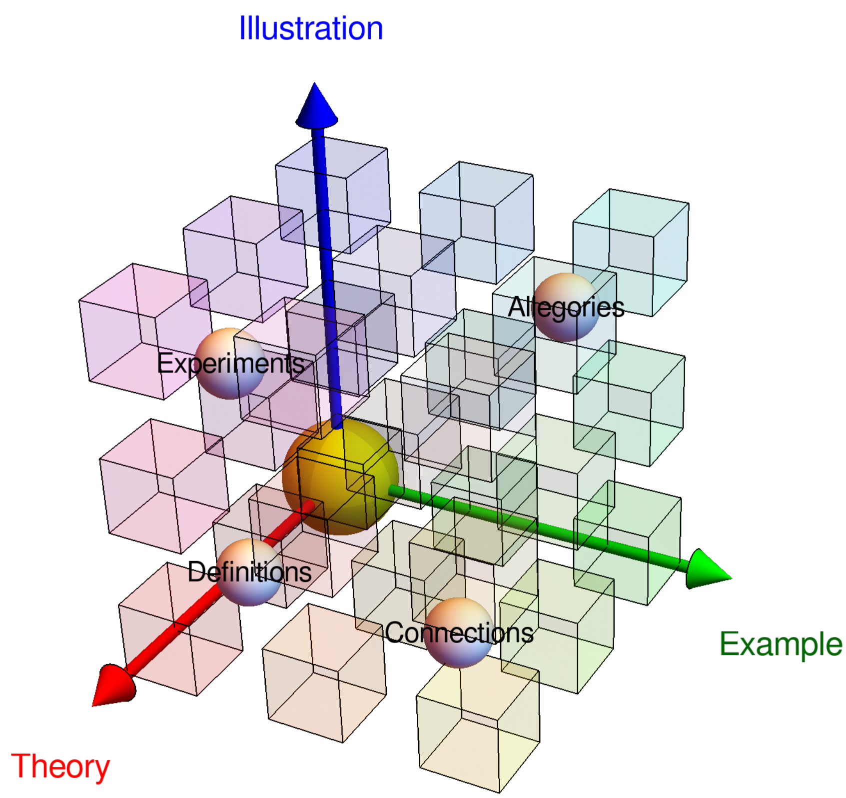

FIGURE 2. The **Theory, Example, Illustration** space is a three variable taxonomy to organize lessons. I feel that keeping it simple is key. We can not keep track of too many things when preparing for a lecture. But knowing about these parameters helps to keep a balance. A lecture, a book, a presentation needs all three ingredients.

4. Illustration

4.1. I brought models of the 6 positive curvature 2-graphs to the conference and presented part of the "Mickey Mouse sphere theorem" that was at that time being written down. The theorem says that every positive curvature d-graph is a d-sphere. It captures an aspect of of heavy sphere theorems [**20, 9**] in differential geometry: in the discrete setting, things are simpler. One reason is that the strong positive curvature assumption prevents projective spaces. The octahedron is a constant curvature 2-sphere with 6 vertices of curvature 1/3 and the icosahedron is a constant curvature 2-sphere with 12 vertices of curvature 1/6.

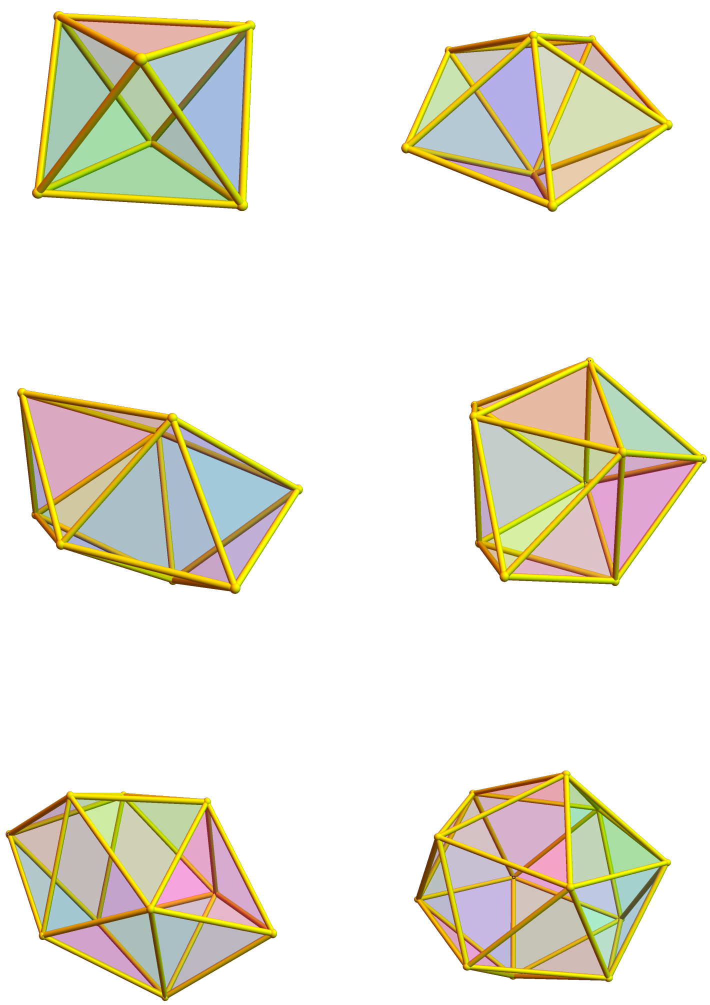

FIGURE 3. There are exactly 6 positive curvature 2-graphs. They are all 2-spheres.

4.2. I liked to mention this result in the "professional norms" set-up because I found that there is often little appreciation for simple things in math. Elementary topics are in danger to be dismissed because they are "not deep enough". It is the job position I find myself in which allows to do such things off the chart, away from mainstream and also ignore general perception and more importantly, general trends, opinion or fashion. The theorem also allowed me to demonstrate first-hand some teaching techniques, like bringing models to the classroom or to have students compute and count with such models in small groups during the class time. During the conference, the "class" was successfully computing the Euler characteristic of all 6 graphs by summing up the curvatures at the vertices. I like the Mickey Mouse world. The term "Mickey Mouse mathematics" appeared in [**75**].

4.3. In order to state the general Mickey Mouse sphere theorem in the higher dimensional case, one has to use clear definitions of what a discrete manifold and what a discrete sphere is. While one can illustrate the two-dimensional case easier, I try here to explain the general case in arbitrary dimensions. The proof is not difficult and also in higher dimensions still uses an argument we know when playing with magnetic sticks and balls. A given embedded wheel graph can be extended to an immersed two-dimensional surface, like when we are are knitting a surface. (By the way, I loved knitting as a kid. It is a beautiful mathematical process to constructing complex topological shapes from simple yarn. I believe that it was also one of the reasons to fall in love with geometry.) Once we have extended the surface and closed it, it has to be one of the 6 possible positive curvature 2-spheres. This shows that diameter is maximally 3, in arbitrary dimension. One can then cover the graph with two patches which are balls and conclude that we have a sphere.

4.4. In order to formulate the Mickey mouse sphere theorem, we use recursive definitions and use the notion of unit spheres $S(x)$ in a graph, which is the graph generated by all vertices directly connected to x. The empty graph 0 is declared to be the (-1)-sphere. The 1-point graph 1 is declared to be contractible. A d-graph is now a finite simple graph G for which every unit sphere is a $(d-1)$-sphere. A d-sphere is a d-graph G for which there exists a vertex x such that $G - x$, the graph in which x and connections to x are removed, is contractible. A graph G is declared to be contractible if there is a vertex x such that $S(x)$ and $G-x$ are both contractible. A graph is defined to have positive curvature if every wheel graph that is embedded in G has either five or six vertices.

4.5. The theorem is:

Every connected d-graph of positive curvature is a d-sphere.

4.6. There are exactly 6 positive curvature 2-graphs. It is a still unexplored question how many positive curvature d-graphs there are in dimension d. As the diameter of the graph is 2 or 3, there are only finitely many. There are positive curvature graphs in any dimension. One example is the d-polytope, which is obtained by making discrete suspensions (double cone extensions) over the $(d-1)$-cross polytope. The 1-cross-polytope is the cyclic graph with 4 vertices, the 2-cross-polytope is the octahedron with 6 vertices, the 3-cross polytope has 8 vertices etc.

4.7. As a meta remark, we want to encourage the reader to observe how I have used here the concepts of "theory", "example" and "illustration" to present a new topic. The illustrations have been placed in this document not just by chance but with the goal that they can serve a purpose. A picture might take away quite a bit of space but a picture can say more than 1000 words. On the other hand, when presenting theory, it can be important to be as clear as possible. Examples help to get intuition. I hope having been able to achieve this goal in the short exposition above. If not, then I again have misjudged the abstraction level of the presented material on the Mickey Mouse theorem so that I have to refer to [**42**] for more details.

5. Some pedagogical remarks

5.1. The following remarks are personal observations. They are not descriptions of analysis or reflections on studies in mathematics education. Indeed, they often tap into areas, where obtaining quantitative data would be rather difficult. The personal experiences of teachers can be different and depend on many parameters, especially if it concerns personalities.

Easier mathematical topics are not necessarily easier to teach.

5.2. At first sight, it appears that more elementary topics are easier to teach than advanced topics. The main reason for the fallacy is that the audience has to be factored in. Beginner topics are taught to less experienced target subjects and are in general also absorbed by a larger and more diverse population which has not yet chosen science as a calling. I myself have taught mostly to undergraduate students who need mathematics as a prerequisite for their degree. It is quite common however that students taking such service courses then get the taste for more advanced mathematics or even concentrate in a mathematical field. I myself taught as a substitute teacher in high-school for a couple of weeks, taught undergraduate to graduate courses in colleges. In retrospect, the high-school teaching was for me the most challenging to teach and the graduate level courses with graduate students in the classroom were the easiest to teach (even so the preparation for the later could be enormous). I myself like the challenge of explaining simpler things as well as observe mathematical grand masters doing this like [**25, 71**] teaching elementary algebra topics.

5.3. Also, after more than 30 years of teaching in colleges, teaching service courses remains for me a challenge and therefore has remained a rewarding activity. There are various reasons why things do not necessarily get easier with time: with more and more teaching experience, one has also to select and chose topics which are adequate and relevant and again and again adjust the correct and adequate level of abstraction, the difficulty and the complexity. One has also to invent new approaches if one wants as a teacher to remain excited about the topics one is teaching. For the later, new technologies like for example advances in web technologies, in 3D printing, in computer games, in computer algebra systems or computer graphics have been helpful to me.

5.4. And then there are parameters over which the teacher has less control: there is an ever changing level of student maturity, there is competition of other departments and other courses, there are changes in prerequisites as well as more diversity in learning. As more advanced a topic is also, as more mature and selected the student body is also. I noticed that quite many students on a higher level math course have acquired learning techniques which allow them to perform error correction or overcome bias assumptions. Especially for experts it can be difficult to hit the adequate difficulty level when teaching a subject. I have witnessed this when watching talks at conferences or seminars and myself spent a lot of time and effort to bring a topic to the right level.

5.5. We have to deal with ever changing **student preparation levels.** In a fast changing world, these changes can happen within a few years.

5.6. The level of preparation and maturity of a student are things over which we often have little control about when teaching a course. Yes, there are placement tests which filter students, but at most schools, they only serve as a recommendation which the student can override, often even against recommendations in personal advising sessions. It is the task of the teacher to read the preparation level of the class and adapt the course if necessary. This can be challenging if a specific goal needs to be reached. Science progresses fast. Just teaching less can not be done without good judgment about topic selection.

5.7. I noticed that there are college-relevant mathematical skills which could be taken for granted 10 or 20 years ago which are less present today. The PISA results [**62**] indicate only a slight decline of mathematics abilities in the US from 2003 to 2018 but an increase in science score points. These data do not back up my claim about having less college relevant math knowledge but they also test different things. In mathematics what can matter a lot are **geometric intuition** as well as **algebra mastery**. It would be wrong to blame the high schools, parents or the students for a lack of preparation. I speculate that a major reason is a diversification of knowledge which happened through rapid changes in information technology as well as in the culture of teaching media [**70, 54**]. Students might already have to learn to write computer programs early on, or then are exposed to statistic classes earlier. They might take a computer aided design course or a robotics course.

5.8. Students often struggle with fundamental mathematical techniques. Therefore, **mastering basic techniques** remains important.

5.9. I observed that in the last 20 years, basic algebra mastery for college students has dropped. I personally explain this to the tendency of modern education to focus on conceptional understanding and neglect also to include some drill. Looking at conceptional understanding is a good thing, but one has also not to forget that a solid mastery of basic algebra techniques frees up computing space in the brain for larger scale thinking. What happens is that a student with less practice has to spend precious thinking time on doing arithmetic, a more experience writer can do the computation "automatic" and in the same time think about also "why" this is done, make connections to other fields or reflect on whether the result "makes sense" or "explore what goes beyond". The reason is that can free up time to be more creative. Practice makes perfect in almost any human activity.

5.10. To put it more bluntly or even to provoke: spending some time with "drill" can pay off exponentially years later. More advanced topics are absorbed faster, and higher levels are reached faster later on. A student who has to look up how to differentiate a basic trigonometric function or can not simplify equations is swimming in oil, gets tired and can not enjoy the progress. Such a student can drown in a mess of errors. Therefore, I feel that some time has to be spent with skill training. Repetition is necessary in sports or music when learning a language or programming. It remains also a healthy habit in mathematics. The lack of basic algebra mastery appears to me as a major stumbling block to get ready for more advanced courses. [As expected, I was asked by referees to back up such a claim up by research in education or psychology. The only thing I can do here is to draw from experience and observing thousands of students personally and watching them doing mathematics. I also can draw on my own experience when learning mathematics, music, languages or sports or games. Activities start to become more fun for me when I'm better in it, and often getting better includes repetion. Practicing "finger etudes" when playing piano, learning vocabulary for languages, pushing weights in sports or do things over and over again when playing games so that reflexes kick in.]

5.11. It appears to be a challenge: how can one make repetitive exercises more fun and less routine? There are various approaches. One has tried for example to wrap it into games [**38, 51**]. I'm convinced that many parents who has had kids in the last 20-30 years must have tried this. One problem with games is that it is difficult to find the right level. I myself have sometimes difficulty with mastering a computer game. The reason is also lack of exercise. It sometimes requires that one has to spend enormous time to practice in order to be able to finish a game or that one gets drawn into parts of the game which have little educational benefit. As the gaming industry still grows and has moved into educational domains, we will learn more about this in the future. I myself advocate not to overthink it. The fun with practice can if the repetition becomes "meditation". Almost always when doing something over and over again, new variations, new patterns and even new insights appear. I myself experience that whether it is in music, in math or running long distance, there is something relaxing about repetition.

5.12. Sometimes grinding through an exercise and variations leads to a new idea, like in science, where break-through ideas often come only after trying out many wrong paths [**63**]. We would never dare to question that basic repetitive experiments in biology or chemistry are necessary to make a break-through. Modern students are often told that it is "not cool to memorize", that repetition can be done by a computer. While there is some truth in it, I think that this feels nihilistic. With the same logic, we could say that we do not have to exercise any more as we move around with cars, we do not need to play chess any more because computers win against us or we do not have to read any more because a bot can do that better, we do not need to learn a language any more because there apps which translate, we do not need to play a music instrument, because synthetic music has started to replace musicians.

5.13. Even in a time when an entire library fits in a smart phone, it is still important to know. **Knowledge helps to build connections**.

5.14. Since all information is easily accessible even from a smart watch, it can lessen the motivation to **know things**. Intelligent assistants like "Siri", "Alexa" or "Google" know how to compute and know many definitions of mathematics. Why do we need to learn any more? One reason can be that we live in a time, where knowledge is accumulated at a tremendous pace. As we discover more and more mathematics, it also becomes more and more important that we know some landmarks in that large and ever growing landscape. It allows us to build new connections and eventually to become more creative.

5.15. Being discouraged to learn and master techniques because of new technology is not a new phenomenon. When calculators appeared in school, I felt a decreased motivation to do arithmetic in the head or on paper. With computer algebra systems, it decreased my motivation to do complicated integrals by hand. This has led to a decline in the ability to do computations, as I can notice that myself. With the emergence of smarter computer algebra systems, I can develop and test most classroom examples also computer assisted and despite teaching calculus for a living, would have to practice to integrate more complicated quotients of trigonometric polynomials. While the help from computers has also freed up time to do more interesting things, true insight for me still comes from actually doing the process also by hand, at least in simpler examples.

5.16. The emergence of the now omnipresent internet makes it less likely that a student understands the need for a rudimentary encyclopedic knowledge. It does not make sense to lament about this. It is a fact that we can not only do computations with a computer, but also look-up knowledge very quickly these days. While this is great, still, knowing how to do an algorithm like a polynomial division or do do square root computation on paper has its benefits. Knowing basic things like the quadratic formula is a prototype 'knowledge pillar' that can help to foster interest in more advanced topics like Galois theory and also appreciate historical connections, like when algebra was done more geometrically and completing a square had geometric meaning. It is important to "feel comfortable". Knowing how to do things, sometimes makes one feel comfortable. The example of the Rubik cube [**39**] with speed cubing competitions illustrates that there can be also a lot of joy in mastering an algorithm fast. These competitions are still very popular.

5.17. In order to understand a topic well, one has to understand it also **well beyond the level** at which it is taught.

5.18. This is certainly my own opinion. The somehow idealistic statement can be challenged. The subject to teach can be so advanced for example that already mastering the minimum can be overwhelming. I personally feel that it is not enough to know the material well. In order to answer smart student questions, I want to be prepared in the subject well beyond what I expect to need. This might require also redoing some exercises myself. There are so many aspects to every subject: historical, cultural, there are relations to other fields. Teaching a subject requires to know as many other approaches as possible. This not only allows to chose the best way, it also allows the teacher to understand original new approaches found by students.

5.19. A teacher who teaches a course in computer science should have developed independent code. Only then, one can know how much time it can take to write code and what agony it can be if a computer program does not work yet. A piano teacher needs to be able to play the pieces which the pupil plays. The professors who taught me programming had been involved personally in projects building computers or developed computer algebra systems. All my personal piano teachers were able to play well the music pieces which I had to practice. Most of the time I felt that a good teacher should also be good at doing things and can demonstrate that regularly. This not only builds respect, it motivated me as a student.

5.20. Also important is that a student who wants to really learn a subject needs to learn it beyond the level which is needed. For example, I feel that a student who wants to master multi-variable calculus seriously should get at least to the integral theorems. While the later theorems are not used by most students later on, their understanding involves essentially every topic which appears in that course: curves, surfaces, fields, derivatives, integrals, area, volume, cross and dot products.

5.21. A mountain guide leading a group of tourists up the Swiss Matterhorn needs to master higher climbing levels. Such skills might be needed if an accident happens or the conditions become unexpectedly more difficult, for example due to weather. A tourist who wants to climb comfortably up that mountain also needs to have done some climbing before so that the climbing does not involve too much risk. There are climbing walls or boulder gardens, where one can practice various levels of climbing difficulty without danger. Being able to climb levels above what is actually needed when climbing the real mountain makes things more rewarding. The same could apply teaching. The beyond-understanding is sometimes only an ideal. I have a few times taught material that I just learned while preparing the class. I also climbed difficulty levels in the mountains which were close to the limits of my abilities and where under some less optimal conditions, the climb would have become dangerous.

5.22. It can be refreshing to **start with a blank slate**. It allows to see things from a new perspective.

5.23. When writing a new lecture or learning a new topic, it can be beneficial to avoid the library and internet from time to time and start thinking about the subject on a clean slate. When developing a lecture, it allows to ask again: what is important, why do we do what we do? Also as a student, it can help to just sit down and write down what one knows about a subject closing off any additional help for some time. Places, where one is stuck are places, where the topic has not yet been absorbed become then evident and one can with more focus revisit library or discuss it with others.

5.24. I found that starting from a blank slate can also be refreshing in research. Taking a blank paper and start developing something new can lead to surprises as it encourages to look at something old with new eyes. Mathematics education is so exciting because of the variety of teaching ideas which have been developed. This does not mean ignoring what has been done before or what others are doing.

5.25. Standing on the shoulders of giants is nice and can bring quick progress. However, when aiming for reaching some new ground, it can be helpful to climb down once in a while and try to walk without the help. Cross fertilization and building on previous work is of course needed. I myself most of the time start with some ideas which were given before or outlined by a book or syllabus. But then, after having it ready, it can help from time to time to go back to a blank slate and see whether some new possibility works better. The same happens with teaching, where ignoring previously written notes can lead to more ideas how to develop a lesson. It can also happen that one get reassured that the previous taken path was a good one.

5.26. The art of reaching the audience is finding the right **level of difficulty**, language and pre-knowledge.

5.27. Mathematics comes in different layers. Thousands of years ago, humans started to compute using sticks or pebbles. Today, the mathematical world is so huge that it is no more possible for a single person to overview all. While the nuts and bolts of mathematics like calculus or linear algebra are pretty well covered, there are also many nice popular books in mathematics which do a good job explaining more advanced topics. Examples of popular, beautiful or elegant mathematics are abundant [**65, 23, 21, 3, 83, 84, 24, 52, 77, 36, 23, 1, 13, 17, 14, 61, 32**]. We live in a great time of mathematical literature.

5.28. I also feel lucky to see that more and more also harder topics of mathematics are covered in popular culture. This can be movies which contain mathematics [**66**]. There are Mathematics exhibits like the "National museum of Mathematics in New York" or the permanent "Mathematica exhibit" at the Museum of Science in Boston. In my experience, pop culture can help in teaching. I personally collect movies also in order to be able to use it for illustrations. Sometimes, I have not found yet the right level of difficulty or abstraction in a class. A link o some pop culture can serve as a lubricant. I have been told by quite many teachers that they had been using my collection in their own class room.

5.29. We see also more and more appreciation for history and historical connections. Bringing in some historical remarks into a lecture can be helpful especially when covering more difficult topics. For Stokes theorem in multi-variable calculus, finding the right level of difficulty can be hard. Mentioning a bit of history about the theorem like [**40, 78, 33**] not only helps to smooth out the toughness of the topic but also can lead to a long term memory connections. It is nice to have a theorem linked to a story. In the case of Stokes theorem, one can mention for example that the result had been assigned as an exam problem by Stokes and that Maxwell was one of the students taking that exam. Maxwell was later able to write down marvelous equations which describe electromagnetic waves. When looking back myself and trying to remember some lectures from my college experience, the parts in which historical connections were woven in, are still the most memorable for me.

5.30. **Sensitivity** needs to be acquired and this can even for experienced teachers need constant work. It matters especially in interactive class rooms.

5.31. Teaching requires sensitivity and feel. Maybe less so in more advanced topics, where students are already prepared to absorb less refined material for example while reading research papers which by nature often are not yet polished or listening to talks which often include references to rather new results using background knowledge which is found elsewhere or which is not even published yet.

5.32. I believe that sensitivity can be trained and acquired, for example by taking feedback serious. There are the obvious sensitivity traps. Some include vocabulary. The word "trivial" for example can be devastating in mathematics. We also live in a more and more politically divided country where partisanship is more important than issues. This spills over into the classroom. We usually have students from all of the political spectrum in the same classroom.

5.33. Many different **teacher-student interactions** can work. It can depend on material, teachers and the students.

5.34. Today also some teaching methods are promoted from high school to colleges. An example is the concept of flipped classrooms, where teachers do no more teach but guide, where lectures are moved online and class time is used for discussion and work. It can make sense for some courses, but to push it universally is not a good idea in my opinion. [Added in proof of October 2020: we have now more experience in these techniques teaching happened online since March 2020. It will be important the next years to evaluate the effectiveness. My own perspectives from 2019 will certainly need revision. Our understanding of teaching might change during these months.]

5.35. As a student, I found seminars could become very intense and sometimes stressful. I found larger audience lectures more relaxing and still inspiring. Seminars were an important part of my college experience, but I liked very much also the calmer and brilliant delivery of lectures which did not feel like a competition and which allowed me to see how some great minds think and react, live and uncut. That I'm not alone with this can be backed up that there are many books which exist about meetings [**15, 79, 74**]. [Added in proof in October 2020: also here, the push to pure online teaching has changed things with respect to content delivery. There are some nice interactive tools like polls or collaboration jamboards which can be used quite elegantly in remote classrooms. How effective they turn out and well they can bring the students to the required level will need evaluation.]

5.36. There is a central and disputed issue of teaching and it comes up in many pedagogical discussions: how interactive should a classroom be? The answer is not easy. There are many opinions and the topic also came up at this conference. I myself found interactive classrooms in general harder to teach. My experience as a student (50 years) and as a teacher (30 years) has always been that managing an interactive classroom is difficult for teachers. If done well, it can be fantastic, but the margin of error is small. A teacher who can not lead, does not know the students, can not be proactive, can not read body language or the mood of the participants, who uses the wrong language or can not bridge different preparation levels which are present in the class, is likely to have a harder time and might fail to achieve the goal of a lecture. When advising graduate students I always encouraged to push for a high interaction level but only so far as it is comfortable. That comfort level can very much depend on the personality of the person.

5.37. It is possible for example that group work ends up in train wreck because of student dynamics. A student might dominate and do all the work for example. There are many more things which can go wrong and lack of sensitivity (both on the student and teacher side) often play a role. The off-shot is that most people think they are good in guiding discussions in interactive classrooms, while in reality, very few really can deliver. I have also seen teachers who can do that well (also during the conference). It needs strong psychological talent and experience to read all students and distribute the workload adequately, so that not only a few students dominate while the others just follow. [Added in proof in October 2020: in online settings, the group work actually can be organized pretty well using online platorms like zoom or blackboard. In smaller groups more difficult situations appear if some students decide not to turn video like if a technical problem prevents this. So far, it has worked smoothly.]

5.38. Maybe one reason why I personally found highly interactive class rooms difficult is that I'm a rather slow thinker myself. I need time to digest something and place it into the correct spot in the landscape of knowledge. Often, I'm confused at first. Of course, this happens more frequently in more advanced topics and I constantly try to learn new things. In most new topics which I encounter, I'm completely lost at first, sometimes have not the faintest idea at first what the topic is about. This is the same situation, a student who is learning algebra, probability or calculus is in at first. Now, there are always students (or colleagues) who have spent more time with a subject and can tell the answer immediately. This can be frustrating for students in a discussion session. I have noticed such issues in inquiry based learning (IBL) conferences, I have experimented with IBL techniques for decades. It still appears to me that teaching an effective IBL lecture is hard if one wants to reach the goal. As smaller the classes, the better this goal can be achieved. It still needs (and this is from experience when seeing teachers doing it) a considerable talent, leadership skill and charisma needed to make it work.

5.39. I have already been as a student been in flipped classroom (flipped classroom means for me "no teaching by the teacher at all") as a student, where an entire hour was wasted because a hyperactive other student wanted to perform and present ineffectively without a course assistant intervening. I have seen interactive practice lectures by other teachers, where I had problems to understand at first what the subject was about. An extreme case was a practice lecture, where a teacher handed out worksheets, asked us to work together on the chalk-boards and then report. The teacher maybe said two sentences in the entire lecture and otherwise just had students report. I believe that one of the main challenges of flipped classroom settings (meaning in the strict sense of no lecture at all in the classroom) are that teachers do not feel having to prepare anymore. The effect is that teaching culture can get destroyed. Little enthusiasm is transferred if the lesson serves cold coffee, possibly brewed years ago by somebody else. It can be as cheap as teaching verbatim from a book. In my own teaching in regular classrooms, I have always been successful with Socratic teaching methods. This means regular and frequent discussions in plenum, group discussions and work guided by teacher and course assistant, but also including some lecture. [Added in proof October 2020: also in online settings, having a good mix and variety seems to work well to foster some sort of community. Today, a variety of tools and formats are already available.]

5.40. **Humor needs time, reflection and work**. It can have huge advantages but its dependence on context bears risks.

5.41. Humor is a tool to ease tension and math anxiety. It can serve as an ice breaker. Humor needs training because humor is highly context sensitive. An established comedian has more leverage than a newcomer. The person itself can matter: I myself (as a Swiss) can make allow to make jokes about Switzerland or jokes about my relatively short body size (because I'm of shorter statue). It should be clear to everybody that jokes about religion, politics, body features, gender orientation or origin in general can be problematic. Even professional stand-up comedians or caricaturists have overstepped in this respect. An over-reaction can be to avoid or to forbid humor, especially in an educational or political setting. Humor however can be a lubricant which helps to ease tension and anxiety. It has been a life saver in diplomacy [**?**]. Mathematics especially has the reputation of being inaccessible and humorless. It can be difficult to use it well however. Forced humor for example almost never works.

5.42. A personal experience: in the spring of 2019 I had been asked by a Harvard comedy club to perform stand-up comedy in a faculty charity event. After they asked repetitively, I finally accepted, knowing that it will be a lot of work. I actually underestimated it even then. It needs a lot of time to come up with jokes and stories, it needs to be rehearsed and then performed well. I worked hard but it could have been much better. The experience made me also investigate a bit what the literature says: [**60**] investigates the social and cognitive benefits of humor and tries also to classify humor using a taxonomy: an example of a canned joke is a narrative, an example of irony is a self-deprecation, an example of a mockery is a parody. [**59**] stresses the links between positive emotions and education and also mentions humor as a survival tool for stressed leaders. I think it can be a survival tool also for stressed teachers and students.

6. Junior faculty

6.1. We live in a time, where the employers less and less commit to permanent workers. The word is "gig economy" [**67**]. Similarly as companies hire more and more contractors or temporary workers, universities hire more and more time-limited faculty and outsource operations like dining or infrastructure to third parties. The percentage of contingent faculty has risen from 21.7 percent in 1969 to 66.6 percent in 2009 and is now believed to be around 75 percent [**80**]. In [**11**], it is put more bluntly: "Thirty five years ago, nearly 75 percent of all college teachers were tenurable, one a quarter worked on an adjunct, part-time or non-tenurable basis. Today, those proportions are reversed".

6.2. It is a general trend that human resources are treated as less and less permanent. The reason is quite simple: it is just cheaper. Jobs change fast, skill requirements change fast. The modern worker has to constantly reinvent and adapt, possibly work two or three jobs at the same time. Both my grand fathers stayed in one profession, both my parents already reinvented themselves in their forties and branched out in other things. In our generation, reinvention comes earlier, like in the twenties or thirties with a second reinvention in sight with 50. This might accelerate even faster in the future. The situation is complicated [**12**].

6.3. In mathematics as well as in other sciences, more and more PhD's are produced which do not have the opportunity to get into academics. The prospects of getting a tenure is not always great. The document [**30**] only publishes the immediate hiring statistics after graduation and not long term tenure data. Many brilliant graduates lose in the lottery of getting a fixed and permanent position. I consider myself extremely lucky to have been able to work in the math profession. I would have much less chance today, where hundreds of applicants line up for a position like mine and larger committees make hiring decisions. Long gone are the times, when department chairs would pilgrim to AMS conferences to hunt for good PhDs and consider themselves lucky to get one. (Ulam [**81**] writes: "*The situation was completely reversed in the late 1950ies and early 1960ies, when a lone young man fresh from school, with a brand-new Ph.D, would be surrounded by chairmen looking for young professors.*" At that time, women were excluded from the profession. We should therefore not only with nostalgia look back upon this quote of Ulam.

6.4. At Harvard, graduate students and even undergraduate course assistants are now unionized. Some of the staff already belongs to unions. There are efforts to have tenured professors unionized but the group of non-permanent, so-called non-ladder faculty have not much protection yet, certainly no job security and there are also no plans for a union. We had in the fall of 2019 the situation that graduate students and some course assistants were on strike. Of course, the additional work-load had been pushed to the remaining teaching faculty. We internally supported the strike, but the story illustrates that non-ladder faculty are probably the least organized of a university today [**80**]. The unionization topic has come up also in the conference. It is a complex issue [**5**].

6.5. The terminology which is used for non-permanent teaching faculty ranges wide, it goes from "junk faculty", "contingent faculty" to "non-ladder faculty". At Harvard, part of this group are "preceptors", part of "teaching faculty". I like also the name "junior faculty" as it is a profession which keeps you young! We are sometimes also considered part of the "invisible university" as it is usually a postdoc position. We are present in all higher education and play an important role in teaching and administration. [Note added in proof: a referee has pointed out that the terminology VITAL faculty seems increasingly be used. It stands for Visitors, Instructors, TAs, Adjuncts, and Lecturers. This is actually a nice name and it feels very fitting because this group is a truely vital part for higher education.]

6.6. While the profession has still low prestige, it comes with upsides. There is no stress nor pressure to do research. There is a wide variety of work with like administrative, teaching, technology or management. There is flexible work time and a stimulating academic environment with many colleagues sharing similar passions and interests. For the most part, there is also autonomy.

6.7. The downsides are minor, but they need to be noted and one has to be aware of them when choosing the profession: first of all one is replaceable. But this is not unusual, even top leaders in a corporation can be replaced. The obvious difference is not having a "golden parachute". There is also little political power as there are usually no voting rights. The advantage is that less meetings that have to be attended. Teacher positions often come with low prestige. This is shared with assistant or postdoc positions. The low status has also advantages as it selects folks who do not care so much about titles and huge salaries but are excited about the subject and love what they are doing.

6.8. Also junior faculty suffer from the "fall of faculty" [**28, 47**]. Maybe it has been accelerated by electronic performance programs, but administration in most recent years gained more and more influence in universities. As faculty want to do less administration, they also lose control. The provocative book [**11**] starts with some thoughts of Albert Einstein, who in 1949 urged intellectual workers to secure their influence in the political field because: *"The intellectual worker, due to his lack of organization, is less well protected against arbitrariness and exploitation than a member of any other calling".* Since the time of Einstein much has changed. It does not only apply to the intellectual worker, but in general for all of work. It is a challenge and not easy to navigate this. Historians try to use the past to look ahead [**34**].

7. Personal Words of wisdom

7.1. As mentioned in the introduction, the following personal words are not meant to feel condescending but hope to be helpful. There is lots of professional advice coming from experts [**22, 50, 29**], but it could also be useful when it comes from ordinary folks like me.

7.2. Happiness is achievement minus expectation

This is a variant of "Happiness = Reality - Expectations" [**34**]. Great expectations, maybe fueled by early success, usually are crushed and lead to misery. Rich people need more riches and worry about losing it, beautiful people fret getting old, tenured faculty worry about getting good graduate students and if they get them, being able to provide them with jobs. Fame and fortune can also lead to trouble. Countless celebrity stories illustrate this. It does not mean that one should not try achieve the best possible, but it can also be helpful to temper expectations. There is a lot which is not under our control.

7.3. Be prepared to hack your value system.

What counts is not what other people think but how you feel when doing the work on a daily basis. If there are 2-3 hours a week, where one feels under appreciated and 100 hours a week are filled with fulfillment and achievements, then one is a lucky person. What is important is to be able to hack the value system. Kottke [**46**] once formulated a **theorem of Lebowski on machine super intelligence** "No AI will bother after hacking its own reward function.". Of course this was meant with a tongue in the cheek but there is some truth in it and some super intelligent mathematicians have shown that it is possible.

7.4. Mind the confirmation bias.

Confirmation bias is well documented. There is negative bias as well as positive bias. An example of positive bias is the **Dr Fox experiment**, which has been done as part of a PhD by J.E. Ware in 1974 [**82**]. There are consequences of confirmation bias: an unknown or unaffiliated mathematician has a harder time to publish stuff. This can also be a blessing as one is evaluated for the work and not whether it is a friendly colleague who referees the paper. It can be brutal as it is possible not even to get a referee report, but the advantages are there: having not to revise a paper frees up time to explore other paths. And also, there is little danger of being plagiarized or followed and raced to a new result. Doing some work away from the spot light is actually very relaxing.

7.5. Disappointments are opportunities

Having not reached the goals one has been dreaming of, also comes with advantages. I definitely do not suffer from the impostor syndrome [**86**]. I do not have to worry about hunting for recognition or being hunted for committee or editorial work. It actually is amazing how fast any disappointment fades away. I myself have passed lots of disappointments but always, there has been also something good about it. One of the advantages is "remaining hungry". The stakes are lower if success is not expected. Also, if one is at the bottom, it can only go up. A rich person always lives with the worries to be robbed. If one has less possessions, there is less to worry about.

7.6. Doing can be more fulfilling than managing

There is the "shop manager paradox": the daily job of a higher ranking manager can often be much more stressful and boring than the job of a person who actually creates and does things. The reason is that seeing things grow first hand can give satisfaction. Creating a dish or programming a piece of software can be more rewarding than supervise somebody to do it. This is of course a personal thing. Some are happier when doing stuff, others are better with telling others what to do and see things grow from a larger perspective. Some like to manage the achievements of the entire group, others enjoy doing it and contribute to the group.

7.7. Appreciate the small

A lot of satisfaction can come from teaching, even if it is a lecture has been done already many times before. Also the preparation can be fun. One might have to write a little program for illustration purposes or then have to work out administrative issues. It is the satisfaction which a mason gets when building a wall, or a designer has when creating a program or an actor when playing a scene. Some creative minds move on and become owners of a shop or become a movie director or even a movie producer. This comes with more prestige but it also can come with more frustration as the success can depend on factors on which are no more under direct influence any more. A software company owner needs good programmers, a director needs a good film crew and technicians etc. In general, the probability to do something meaningful and personally satisfying is bigger when scoring in small things but it is also less rewarded financially for example. This makes sense as it comes also with less risk.

7.8. Experiment and learn new things

To be at the bottom of the ladder makes you feel that there is opportunity to grow, to branch out, try other things. This is harder when not only your own venture is at stake but a larger structure must work. In a tenured situation with graduate students, it is hard to change the subject. There are obligations to a field of colleagues, to graduate students, expectations. Only few (usually only the best) can afford to completely change field. There are examples of fields medalists who have done that. Remarkably, it is also relatively easy for non-ladder faculty who are not hired as research mathematician but as teaching faculty. I myself enjoyed learning to work in new areas of mathematics and be in situations where the learning curve is still steep as things are new.

7.9. Be aware of the age myth.

There is a myth that mathematics is a "young man's game". It is twice wrong as mathematics is neither a game for men, nor a game for the young. The source of this misconception can be traced back to G.H. Hardy [**35**], who wrote in 1940 a rather depressive essay about a year after he had a heart attack (1939). Jean Dieudonné, considers a person to be a mathematician, if he or she has proven a nontrivial theorem, and points out that Kronecker, Cartan, Siegel, Weil, Leray or Gelfand have continued to prove fine theorems after the age of sixty. Many important textbooks (also containing original material) have been written later in life: In Ivor Grattan-Guinness book "The Rainbow of Mathematics" [**31**] features the ages of the authors. Galileo did great things with 74, Napier with 64, Bürgi with 68, Briggs with 63. An early study [**76**] already point out why there is a bias: many bright young mathematicians do not continue, possibly because they did not get a permanent job. In fact, many mathematicians do their best work in mature years if they have the chance. The amount of mathematics which has been produced still grows exponentially. Most theorems proven as a young mathematician will turn out to be already known. It needs time to get experience and see what is known and what is not. A more experienced mathematician can navigate the immense knowledge sea better. Finally, mathematics is probably the only science, where senior mentors help to lift their fledglings, often without appearing as coauthors. In other sciences, the first name of the paper might go to the principal investigator, even so the later might not have contributed much besides managing the project. In math, first works are often done on the shoulders of giants or mentors. This distorts the picture and suggests that creative work can only be done by the young.

7.10. What do you care what others think of you?

There are always others who are better. One can rub this in by giving away prizes, grades or student evaluations. Of course, evaluation is important, but once an evaluation system is in place, it is often taken as the gold standard. Especially damaging are the side effects of awards. It is something one is hardly aware of. While awards encourage a few, it can discourage the majority. There is much collateral damage coming from prizes. The effect is that in the immediate neighborhood of a prize winner, ambition is blown out. The oxygen has been sucked up by the nearby fire. This happens especially in early years, like in primary and middle school, maybe early high school. Later in high school or in college the interests have diversified already, so that it does not matter any more so much. Higher levels of depression are present in student populations [**19, 53**]. The increase from 2007 to 2018 among US undergraduates is dramatic [**55**]. And it is not only students who suffer from the "impostor syndrome" and ask themselves: "do I really belong in this school?" It can help already to be aware of such mechanisms [**86**].

7.11. We all have the same hardware

Why is that we observe large cognitive differences even so all humans have give or less the same brain structure? This has been investigated for a long time already like by Piaget [**64**] or more contemporary psychologists [**26**]. There is also the more computer scientist point of view [**56**]. A computer scientist would say that we all have the same CPU, RAM and clock-speed. How come then that some mathematicians are perceived to have multiple times better abilities than others even so they operate with the same hardware? Practical computer science tells us that not only the hardware but also the operating system matters for effectively running programs. The operating system is also software which can be tweaked. It suggests that not only mathematical ability but **mathematical talent** (what ever this means) can be improved through education. What we have not yet figured out is how to tweak the operating system more effectively, teach how to learn, and to teach how to search for new mathematics. It also leads to the question what does mathematical ability means. Certainly, there is not only a computer science but also social, historical, economic and cultural context.

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